# Estimating the Number of Modes in Multimode Waveguide Propagation Environment


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*Abstract*— **In this paper, we estimate the number of propagating modes in rectangular and circular multimode waveguides and express it as a closed-form function of excitation frequency and waveguide cross-sectional dimensions. The presented formulas can be used to quickly estimate the number of modes which can propagate in multimode waveguide communication channels such as pipes, ducts, and tunnels.**



***Keywords-waveguides, propagation***


## I. INTRODUCTION

In antenna and microwave engineering, waveguides are well known [1] and are usually carefully designed to support only a small number of selected modes. In the past, for example, special waveguides with a few low loss modes were used for long distance communications [2, 3]. Nowadays there are emerging applications where structures which were not designed to be RF waveguides are now considered as propagation channels and behave as highly overmoded waveguides.

Some examples include wireless propagation in tunnels and underground mines [4-6], heating, ventilation, and air conditioning (HVAC) ducts and lift shafts [7-8], pipe-inspecting robots wirelessly communicating through gas, drain, and other pipes [9-12], radio frequency identification (RFID) systems operating inside marine containers [13], drill pipes [14, 15], and ducts [16], RF communication systems inside aircraft fuselage [17-18], etc. In such cases, it is often useful to know the number of modes in order to understand the best way to model the channel using analytical methods, ray tracing, or electromagnetic simulation.

Some formulas for the number of modes exist for optical fibers [19] and lasers [20] and for rectangular cavities used as reverberation chambers [21-22] but to the best of our knowledge none have appeared in literature for waveguides. It is very convenient if the number of modes can be quickly estimated from some formula, especially for circular waveguides where the mathematics is complex because wavenumbers are determined by the zeros of the Bessel functions.

In this paper, we derive approximate formulas that can be used to calculate the number of modes in two most practical cases, rectangular and circular waveguides. The number of modes increases quadratically with the frequency. The quadratic coefficient is obtained using regression analysis on numerically computed mode cutoff frequencies. The same approach can theoretically be applied to waveguides of more complicated cross-section shape, such as elliptical.

## II. RECTANGULAR WAVEGUIDE

For a rectangular waveguide with cross-sectional dimensions $a$ (wide) and $b$ (narrow), the mode cutoff frequencies of both TE and TM modes are given by the well known formula [1]:

$$f_c^{rect} = \frac{c}{2}\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}\ , \tag{1}$$

where for TE modes either $n$ or $m$ must be positive, and for TM modes both $n$ and $m$ must be positive.

Let us first use an empirical approach to estimate the number of modes in rectangular waveguides. We numerically computed the total number of TE and TM modes in a square waveguide ($a$=$b$) as a function of normalized frequency for the first 1000 modes. We then carried out a regression analysis to find the best second order polynomial fit to these data. Using maximum likelihood estimates for regression coefficients, we obtain the following equation for predicted total number of modes in square waveguide:

$$N_{rect}\big|_{a=b} \approx 1.5711\left(\frac{f}{f_o^{rect}}\right)^2 - 0.1942\,\frac{f}{f_o^{rect}} + 0.2077 \tag{2}$$

The comparison between the exact number of modes in a square waveguide and the first term in formula (2) is given in Fig. 1. Note from Fig. 1 that formula (2) works very well even for small numbers of modes.

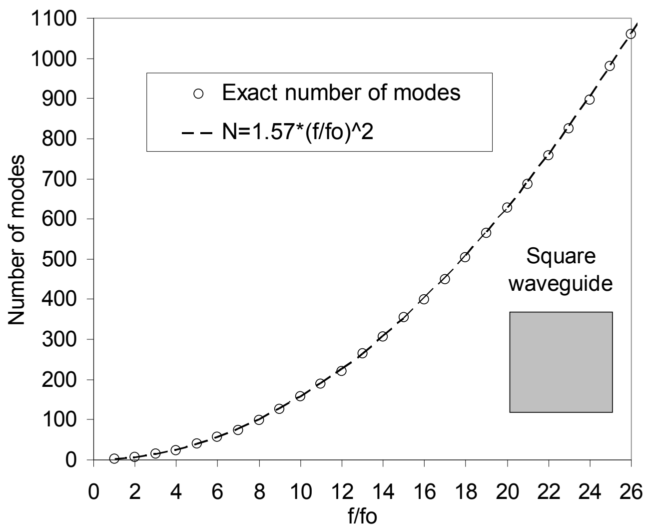


Figure 1. Comparison between the exact number of modes in square waveguide and the first term of approximation given by (2) as a function of normalized frequency.

It turns out that for rectangular waveguide an analytical solution for the number of modes is possible. One can recognize in (1) an equation for the ellipse with respect to mode indices n and m. For any excitation frequency *f*, only the modes whose cutoff frequencies are below the excitation frequency are excited ( $f_c^{rect} \le f$ ), which means that their mode indices must belong to the upper right quarter of this ellipse as shown in Fig. 2.

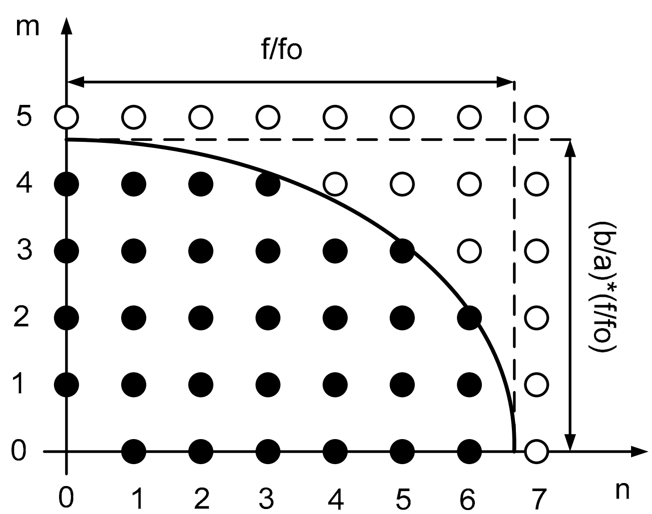


Figure 2. Calculating the number of modes in rectangular waveguide as number of points inside quarter ellipse in (n,m) coordinate system.

Hence, in heavily multimoded case the total number of modes in rectangular waveguide can be approximately calculated as one quarter of the area of the ellipse times two (to account for both TE and TM modes):

$$N_{rect} \approx \frac{\pi\, b}{2\, a}\left(\frac{f}{f_o^{\,rect}}\right)^2 , \tag{3}$$

where $f_o^{\,rect} = c/(2a)$ is the lowest cutoff frequency (mode TE10). In (3), we neglected the fact that for TE modes either *n* or *m* must be positive, and for TM modes both *n* and *m* must be positive which would add a first-order term to (3). The quadratic coefficient in (3) confirms the validity of regression approach used to obtain (2) for large number of modes, $f >> f_o^{\,rect}$ and $b=a$.

In the similar way, the validity of regression approach was confirmed for rectangular waveguides with various aspect ratios (2:1, 3:1, etc.). All quadratic coefficients obtained via regression for various waveguide aspect ratios agreed with the coefficient in equation (3).

## III. Circular Waveguide

For a circular waveguide of radius R, the mode cutoff frequencies are given by another well known formula:

$$f^{\,circ}{}_c = \frac{c}{2\pi R}\begin{cases} p'_{nm} & ,TE \\ p_{nm} & ,TM \end{cases} . \tag{4}$$

where $p'_{nm}$ and $p_{nm}$ are the m-th nulls of $J'_n(x)$ and $J_n(x)$, where $J_n(x)$ is the Bessel function of the first kind of order n [1]. For both TE and TM modes the indices must satisfy: $n \ge 0, m \ge 1$.

The mathematics of Bessel functions is complicated and an analytical solution for the number of modes is not possible even though some asymptotic approximations for Bessel functions and their roots are available [23]. For example, it is known that the successive roots of the Bessel function $J_n(x)$ are spaced $\pi$ apart at infinity (this also applies to its first derivative $J'_n(x)$ ). However, the spacing between the same (m-th) roots of the Bessel functions of successive orders n is irregular. This renders impossible an analytical derivation of the total number of modes vs. frequency in circular waveguides, at least in the same clear and straightforward way as for rectangular waveguides.

Thus, we use again an empirical approach to estimate the number of modes in circular waveguides. We know that the number of modes in a waveguide must increase quadratically with frequency, similar to the dependence given by (3). We computed the total number of TE and TM modes in circular waveguide numerically as a function of normalized frequency for the first 1000 modes and performed analogous maximum likelihood regression analysis to find the best second order polynomial fit to these data:

$$N_{circ} \approx 0.8507\left(\frac{f}{f_o^{circ}}\right)^2 + 0.4799\,\frac{f}{f_o^{circ}} - 0.3048\,, \qquad (5)$$

where $f_o^{circ} = 1.841\,c/(2\pi\,R)$ is the lowest cutoff frequency of the circular waveguide (mode TE11). Using equation (5), we can approximate the total number of modes in circular waveguide as

$$N_{circ} \approx 0.85\left(\frac{f}{f_o^{circ}}\right)^2 \qquad (6)$$

for a large number of modes ($f >> f_o^{circ}$). The comparison between the exact number of modes in circular waveguide and (6) is given in Fig. 3. Note from Fig. 3 that formula (6) works very well even for small number of modes.

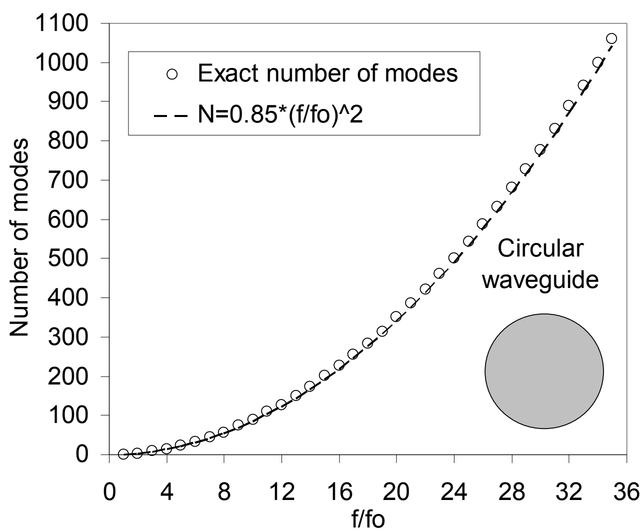


Figure 3. Comparison between the exact number of modes in circular waveguide and the approximation given by (6) as a function of normalized frequency.

Note that circular waveguides do not have any other cross-section characteristic except for radius (unlike rectangular or elliptical waveguides which can have different aspect and ellipticity ratios), and hence there can be only one formula for the number of modes which is already given by (6).

## IV. Discussion

Formulas (3) and (6) work well for both small and large numbers of modes and can help an engineer to analyze tradeoffs when, e.g., choosing which shape and size waveguide should be used for multimode communications. From (3) and (6), one can also easily find an approximation to the ratio between the total number of modes in rectangular and circular waveguides:

$$\frac{N_{rect}}{N_{circ}} \approx 0.63\,\frac{ab}{R^2}\;. \qquad (7)$$

The same empirical regression-based approach that we described in this paper can theoretically be applied to waveguides of more complicated cross-section shapes, such as elliptical where cutoff frequencies are determined by roots of Mathieu functions, which are even more complex than Bessel functions, and analytical solution is not possible. However, the regression analysis for elliptical waveguides would have to be repeated for different ellipticity ratios because quadratic coefficient in the expected mode formula will depend on the ellpiticity ratio. For ellipticity ratio of 1, the coefficient is expected to be 0.85, the same as in equation (6).

To give readers some idea about the practical accuracy of our formulas given by equations (3) and (6), let us consider two types of hollow metal HVAC ducts: rectangular and circular. In the United States, one of the common rectangular duct sizes is 40 cm x 20 cm (the cutoff frequency is 375 MHz) and one of the common circular duct diameters is 30.5 cm, or 12 in. (the cutoff frequency is 577 MHz). The 12 in. cylindrical duct is shown in Fig. 4.

Figure 4. Hollow circular (cylindrical) metal HVAC duct 30.5 cm (12 in.) in diameter, commonly used in buildings in the United States.

Tables 1 and 2 below give the number of modes that can propagate in those ducts at three frequencies which represent most of the current practical ISM bands: 900 MHz, 2.4 GHz, and 5.8 GHz. The exact number of modes was calculated from the numerically computed table of all modes, and the estimated number of modes was calculated using our simple approximate formulas given by equations (3) and (6).

TABLE I. NUMBER OF MODES IN RECTANGULAR 40 CM X 20 CM DUCT

| Frequency | Exact | Estimated |
|---|---|---|
| 900 MHz | 5 | 5 |
| 2.4 GHz | 33 | 32 |
| 5.8 GHz | 186 | 188 |

TABLE II. NUMBER OF MODES IN CIRCULAR 12 IN. DIAMETER DUCT

| Frequency | Exact | Estimated |
|---|---|---|
| 900 MHz | 2 | 2 |
| 2.4 GHz | 17 | 15 |
| 5.8 GHz | 91 | 86 |

The difference between the exact and the estimated number of modes decreases as the number of modes increases and becomes less than 1% when the number of modes approaches 1000.

## V. CONCLUSIONS

In this paper, we estimated the number of propagating modes in rectangular and circular multimode waveguides and expressed it as a closed-form function of excitation frequency and waveguide cross-sectional dimensions. An ability to quickly estimate the number of modes which can propagate in multimode waveguides is important for understanding and optimizing the properties of waveguide communication channels such as pipes, ducts, or tunnels. We hope that the analysis and the formulas presented in this paper will be useful for the propagation community.

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