Optics of Liquid Crystals

Liquid crystal
- The elongated molecules have orientational order (like crystals) but lack positional order (like liquids).
- Nematic liquid crystals: The molecules tend to be parallel but their positions are random.
- The molecules change orientation when subjected to a force.
- Twisted nematic liquid crystals: A twist is imposed by external forces. See Figure 1, a thin layer of the liquid crystal material is placed between two glass plates polished in perpendicular directions.

Figure 1: Molecular orientation of the twisted nematic liquid crystal.

Optical Properties of Twisted Nematic Liquid Crystals
The twisted nematic liquid crystal is an optically inhomogeneous anisotropic medium that acts locally as a uniaxial crystal. This means the refractive index of the liquid crystal for an optical wave with polarization parallel to the long axis of the molecule is \( n_e \), while for an optical wave with polarization parallel to the short axis is \( n_o \). Normally \( n_e \) is greater than \( n_o \).

Consider a liquid crystal cell as shown in Figure 2. The twist angle of the molecules varies linearly with \( z \):
\[
\theta = \alpha z
\]

Figure 2 shows the special case of 90° twisting angle after a propagation distance of \( d \). The phase retardation per unit length (phase retardation coefficient) is:
\[
\beta = (n_e - n_o)k_0
\]

Figure 2: Propagation of light in a twisted nematic liquid crystal.
In the following, we will show that if the incident wave at \( z = 0 \) is linearly polarized in the x direction, then when \( \beta \gg \alpha \) (which means the twisting cannot happen too fast), the wave maintains its linear polarization, but the plane of polarization rotates in alignment with the molecular twist, so that the angle of rotation is \( \theta = \alpha z \). The liquid crystal then serves as a polarization rotator.

**Proof:**

Divide the width \( d \) of the cell into \( N \) incremental layers of equal widths \( \Delta z = d / N \). The \( m \)-th layer is a wave retarder whose slow axis (the optical axis shown in Figure 2) makes an angle \( \theta_m = m\Delta \theta \) with the x axis, where \( \Delta \theta = \alpha \Delta z \). It therefore has a Jones matrix in the x-y coordinate system as

\[
T_m = R(-\theta_m)T_r R(\theta_m),
\]

where

\[
T_r = \begin{bmatrix}
\exp(in_k \Delta z) & 0 \\
0 & \exp(in_o k \Delta z)
\end{bmatrix}
\]

is the Jones matrix of a phase retarder in the rotated coordinated system.

Let’s rewrite \( T_r \) in terms of the phase retardation coefficient:

\[
T_r = \exp(i\phi \Delta z) \begin{bmatrix}
\exp(i\beta \Delta z / 2) & 0 \\
0 & \exp(-i\beta \Delta z / 2)
\end{bmatrix},
\]

where \( \phi = (n_c + n_o)k_0 / 2 \). Since multiplying the Jones vector by a constant phase factor does not affect the state of polarization, we will ignore the \( \exp(i\phi \Delta z) \) term.

The overall Jones matrix of the device is the product

\[
T = \prod_{m=1}^{N} T_m = \prod_{m=1}^{N} R(-\theta_m)T_r R(\theta_m)
\]

Note that \( \prod_{m=1}^{N} T_m = T_N T_{N-1} \cdots T_2 T_1 \). Using the fact that \( R(\theta_m)R(-\theta_{m-1}) = R(\theta_m - \theta_{m-1}) = R(\Delta \theta) \), we obtain:

\[
T = R(-\theta_N)\left[ T_r R(\Delta \theta) \right]^{N-1} T_r R(\theta_1).
\]

Remember from our lecture that \( R(\Delta \theta) = \begin{bmatrix}
\cos \Delta \theta & \sin \Delta \theta \\
-\sin \Delta \theta & \cos \Delta \theta
\end{bmatrix} = \begin{bmatrix}
\cos \alpha \Delta z & \sin \alpha \Delta z \\
-\sin \alpha \Delta z & \cos \alpha \Delta z
\end{bmatrix} \) in this case. Therefore,

\[
T_r R(\Delta \theta) = \begin{bmatrix}
\exp(i\beta \Delta z / 2) & 0 \\
0 & \exp(-i\beta \Delta z / 2)
\end{bmatrix} \begin{bmatrix}
\cos \alpha \Delta z & \sin \alpha \Delta z \\
-\sin \alpha \Delta z & \cos \alpha \Delta z
\end{bmatrix}.
\]

When \( \alpha \ll \beta \), the second matrix can be approximated as an identity matrix (this is true for \( R(\theta_1) \) as well), while the phase change introduced by the first matrix cannot be ignored. Thus,

\[
T \approx R(-\theta_N)\left[ T_r R(\theta_1) \right]^{N-1} R(\theta_1) = R(-\alpha d)\left[ \exp(i\beta \Delta z / 2) \begin{bmatrix}
0 \\
0
\end{bmatrix} \right]^{N-1}
\]

\[
\rightarrow T = R(-\alpha d)\left[ \begin{bmatrix}
\exp(i\beta d / 2) \\
0
\end{bmatrix} \right]^{N-1}.
\]
Can you recognize what this is? This is a phase retarder of retardation $\beta d$ with the slow axis along the x direction, followed by a polarization rotator with rotation angle $\alpha d$. The rotation angle doesn’t have to be 90° as shown in Figure 2. It depends on $\alpha$ and $d$. If the original wave is linearly polarized along the x direction, the wave retarder provides only a phase shift for the optical wave; the device then simply rotates the polarization by an angle $\alpha d$ equal to the twist angle.