Submodular Optimization with Submodular Cover and Submodular Knapsack Constraints (SCSC/ SCSK)

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Outline

Set functions $f : 2^V \rightarrow \mathbb{R}$

$$V = \left\{ \text{  } \right\}$$

- V is a finite "ground" set of objects.
- A set function $f : 2^V \rightarrow \mathbb{R}$ produces a value for any subset $A \subseteq V$.

Set functions $f : 2^V \rightarrow \mathbb{R}$

$$A = \left\{ \text{  }, \text{  }, \text{  }, \text{  }, \right\}$$

- For example, $f(A) = 22$,

Submodular Set Functions

- Special class of set functions.

$$f(A \cup v) - f(A) \geq f(B \cup v) - f(B), \text{ if } A \subseteq B \tag{1}$$

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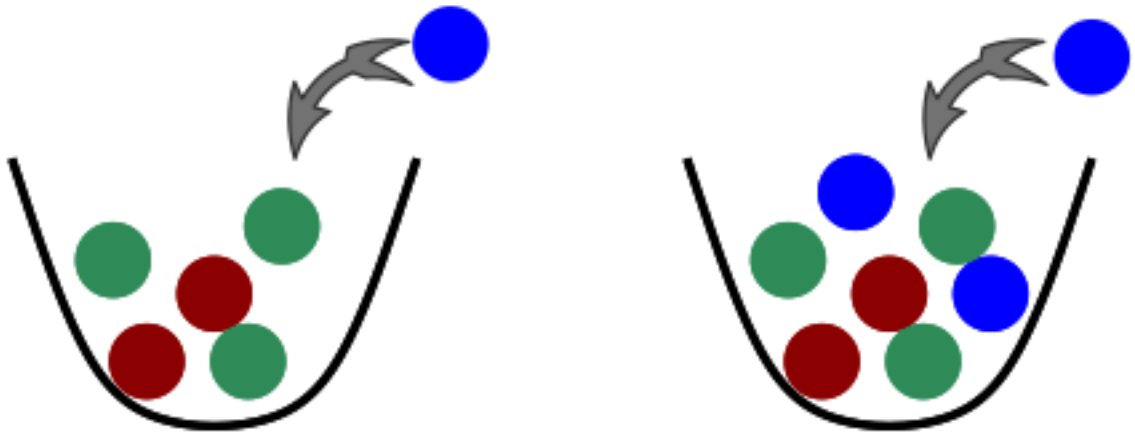

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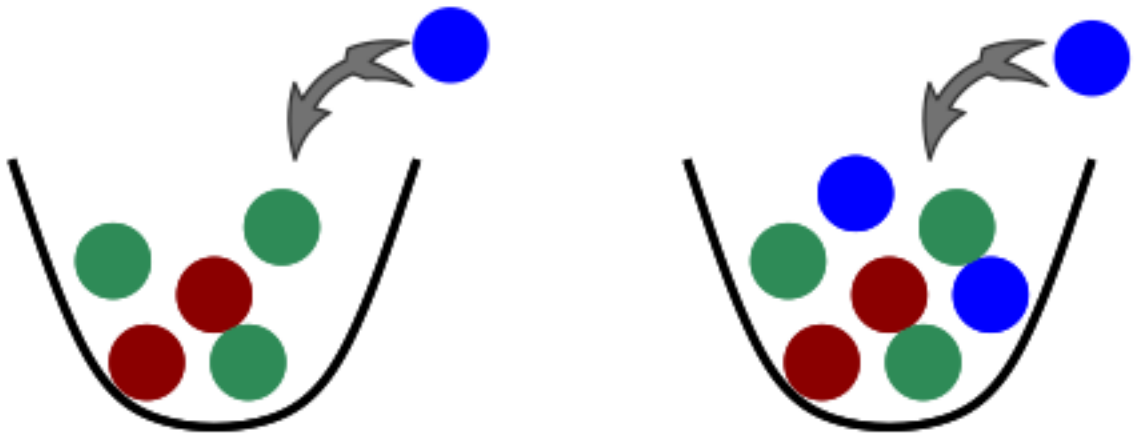

Gain $= 1$

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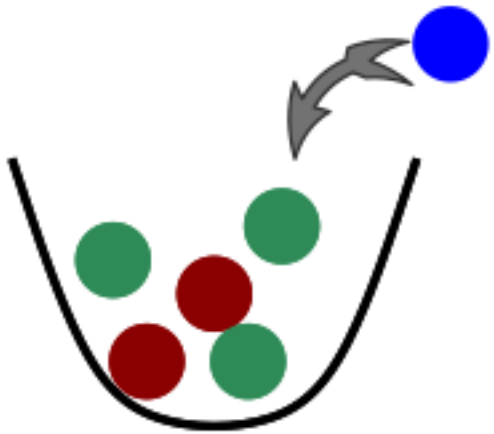

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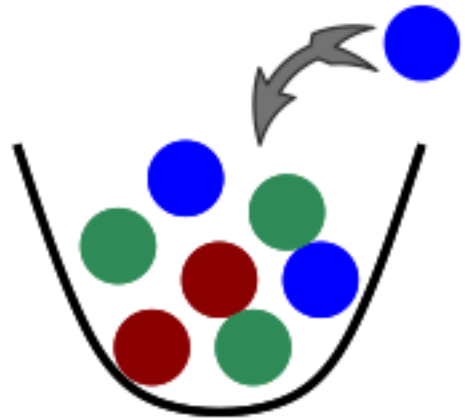

Gain $= 0$

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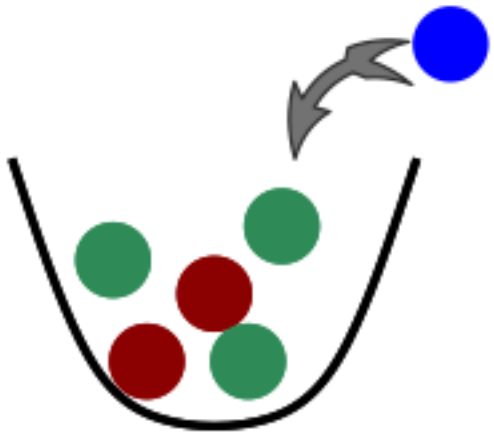

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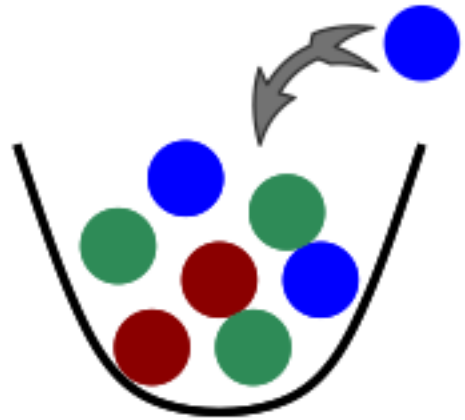

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- Monotonicity: $f(A) \leq f(B)$, if $A \subseteq B$.

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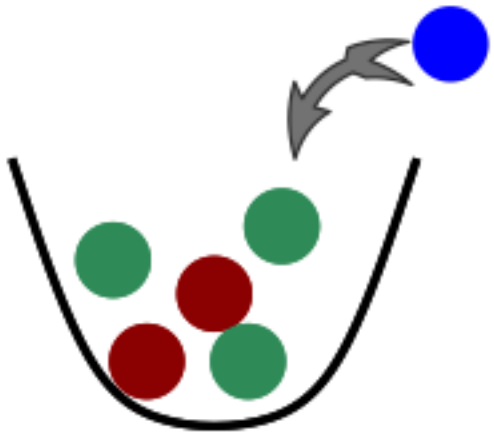

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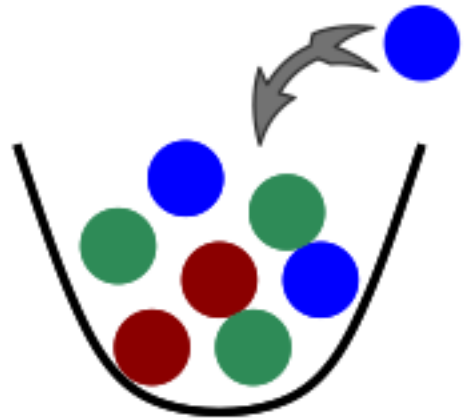

Gain $= 0$

- Monotonicity: $f(A) \leq f(B)$, if $A \subseteq B$.
- Modular function $f(X) = \sum_{i \in X} f(i)$ analogous to linear functions.

Two Sides of Submodularity

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Submodular Minimization

- Solve $\min\{f(X)|X \subseteq V\}$.
- Polynomial-time.
- Relation to convexity.
- Models cooperation.

$f($  $)-f($  $) \geq f($  $)-f($  $)$

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- Solve $\max\{g(X)|X \subseteq V\}$.
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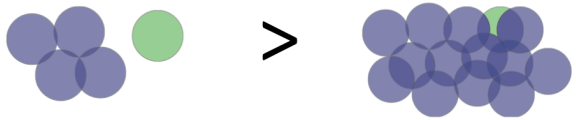
- Models diversity and coverage.



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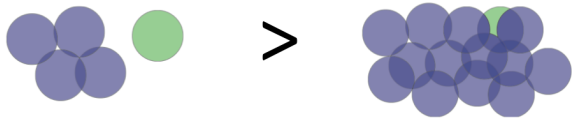

- Sometimes we want to simultaneously maximize coverage/ diversity (g) while minimizing cooperative costs (f).

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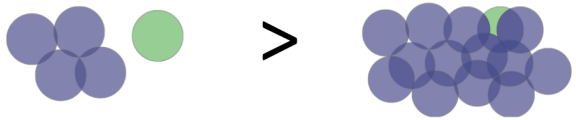

- Sometimes we want to simultaneously maximize coverage/ diversity (g) while minimizing cooperative costs (f).
- Often these naturally occur as budget or cover constraints (for example, maximize diversity subject to a budget constraint on the submodular cost).

Submodular Optimization with Submodular Constraints

- Historically: DS optimization

$$\min_{X \subseteq V} f(X) - \lambda g(X)$$

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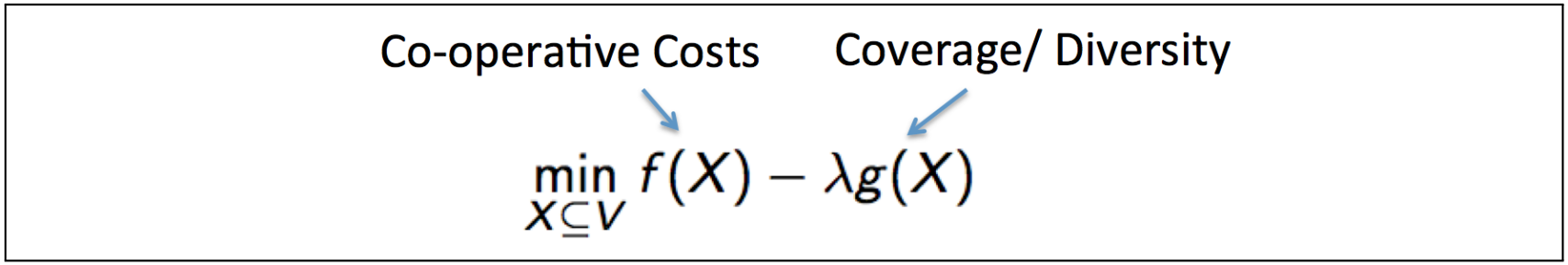

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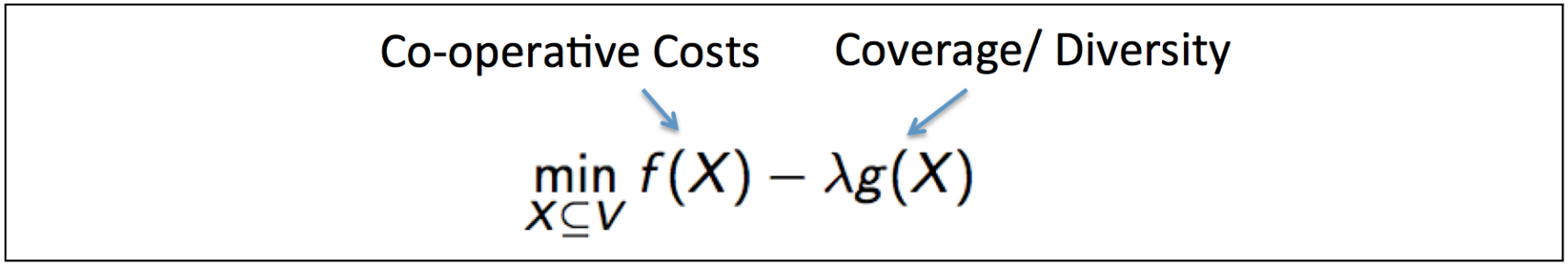

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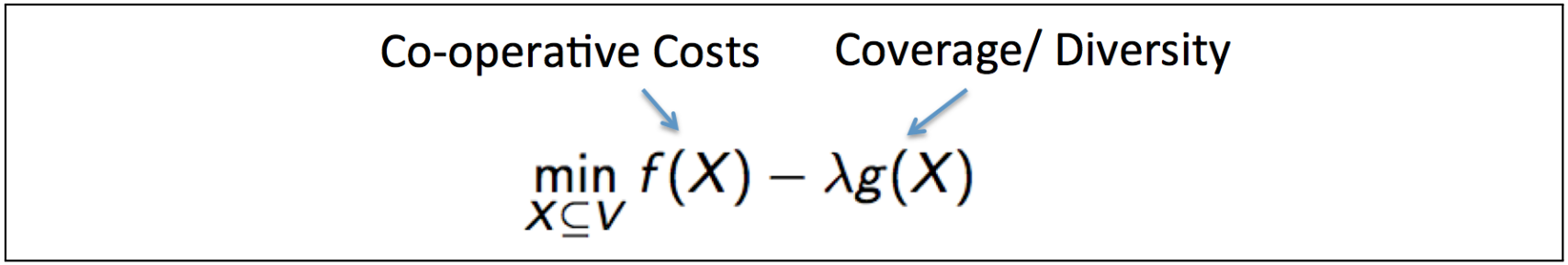

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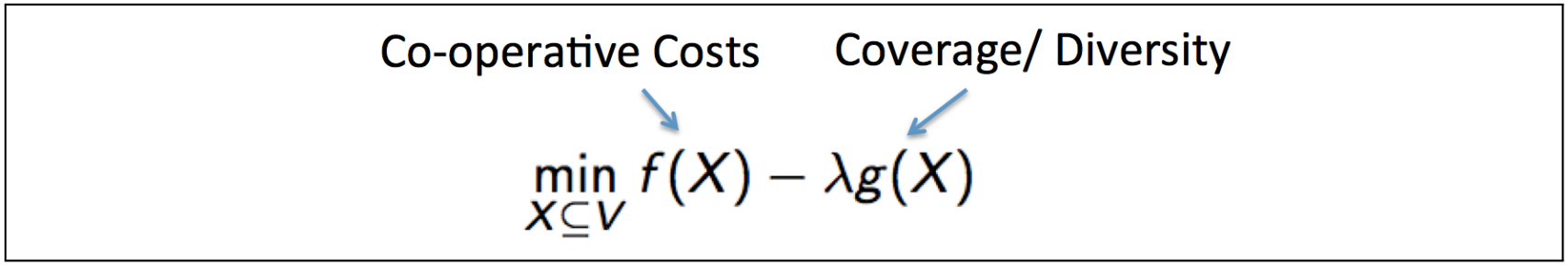

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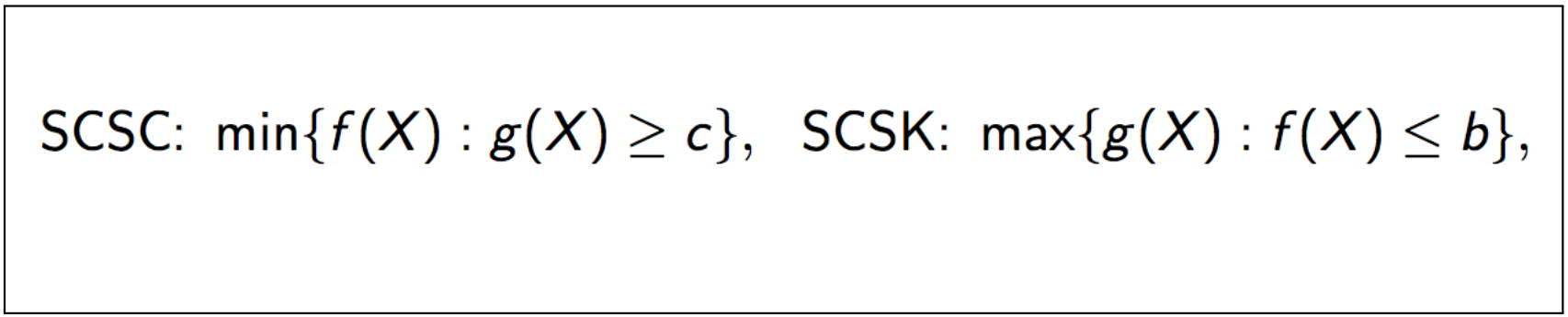

Submodular Optimization with Submodular Constraints

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Co-operative Costs Coverage/ Diversity

$$\min_{X \subseteq V} f(X) - \lambda g(X)$$

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Coverage/ Diversity

SCSC: $\min\{f(X) : g(X) \geq c\}$, SCSK: $\max\{g(X) : f(X) \leq b\}$,

Co-operative Costs

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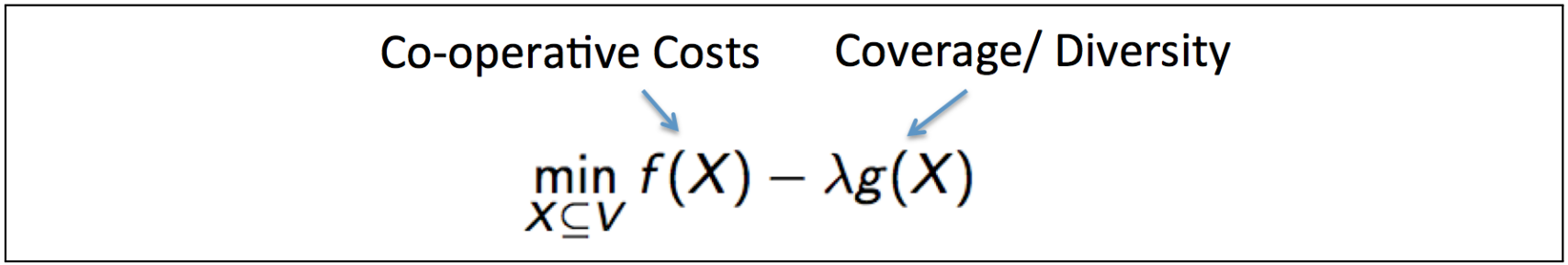

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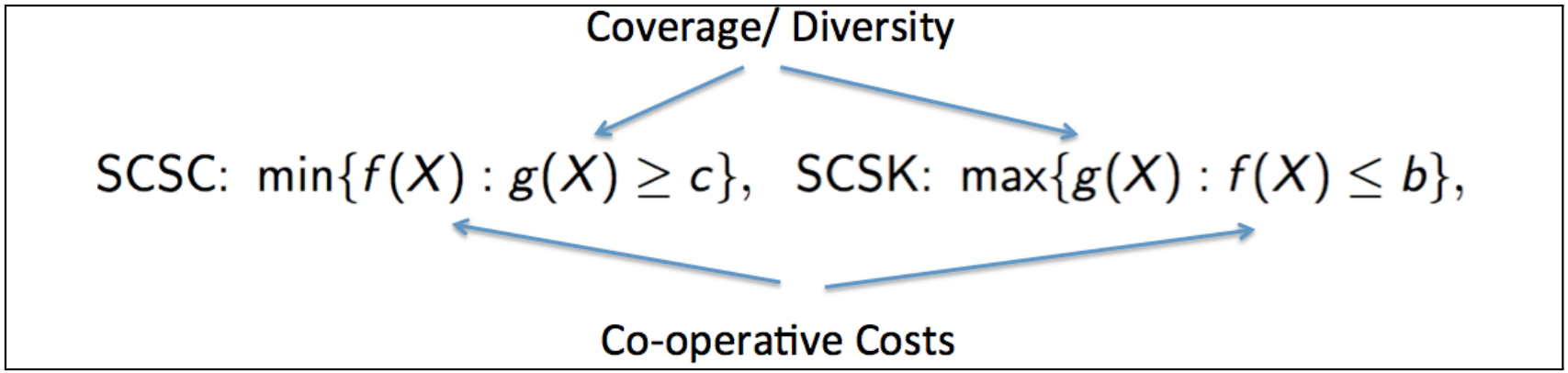

- While DS optimization is NP hard to approximate, SCSC and SCSK however, retain approximation guarantees!

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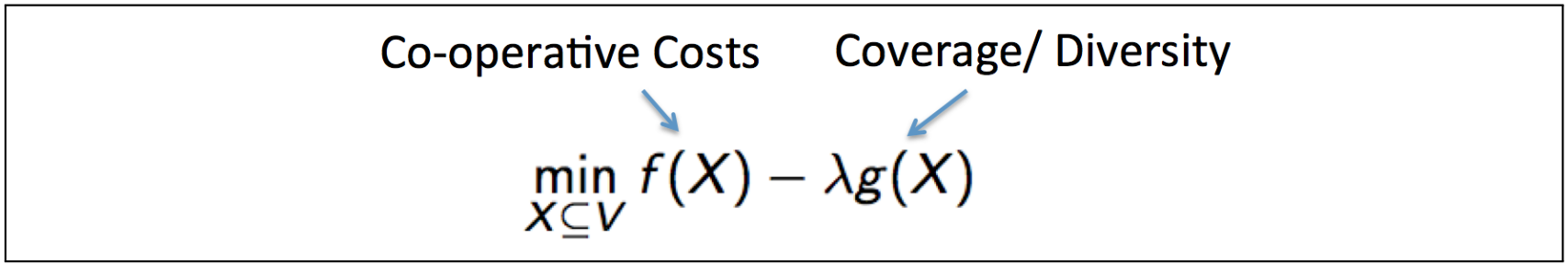

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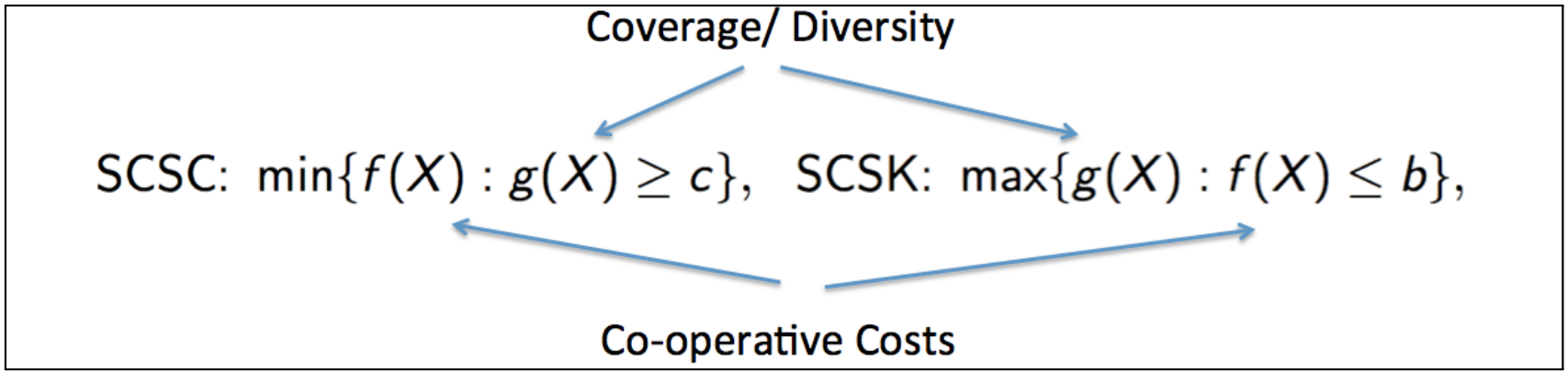

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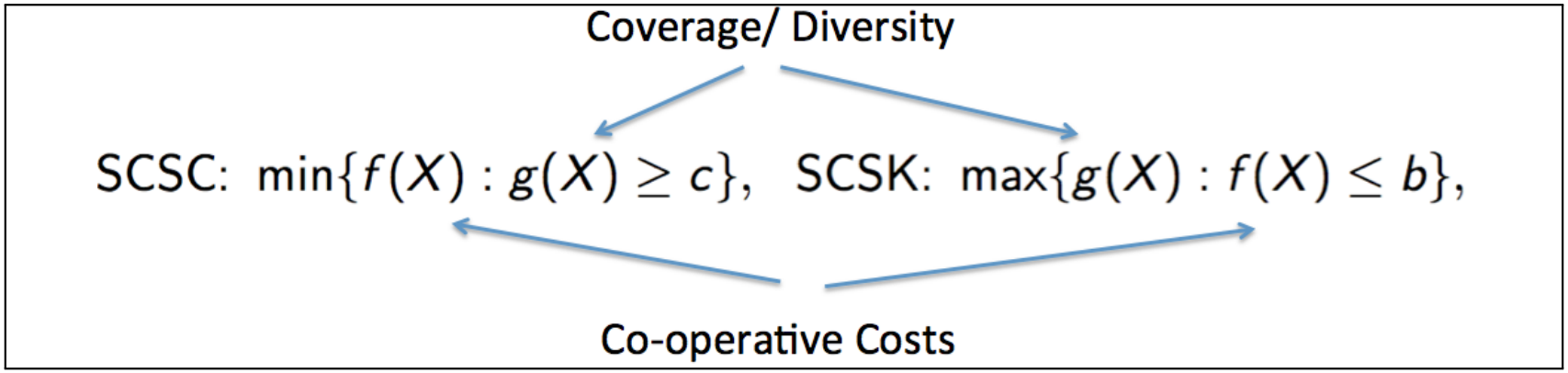
- Throughout this talk, assume f and g are monotone.

Our Main Contributions



- Show how SCSC/SCSK subsume a number of important optimization problems.
- Provide a unifying algorithmic framework for these.
- Provide a complete characterization of the hardness of these problems.
- Emphasize the scalability and practicality of some of our algorithms!

I - Submodular Set Cover and Submodular Knapsack

$$\text{SSC: } \min\{w(X) : g(X) \geq c\}, \quad \text{SK: } \max\{g(X) : w(X) \leq b\},$$

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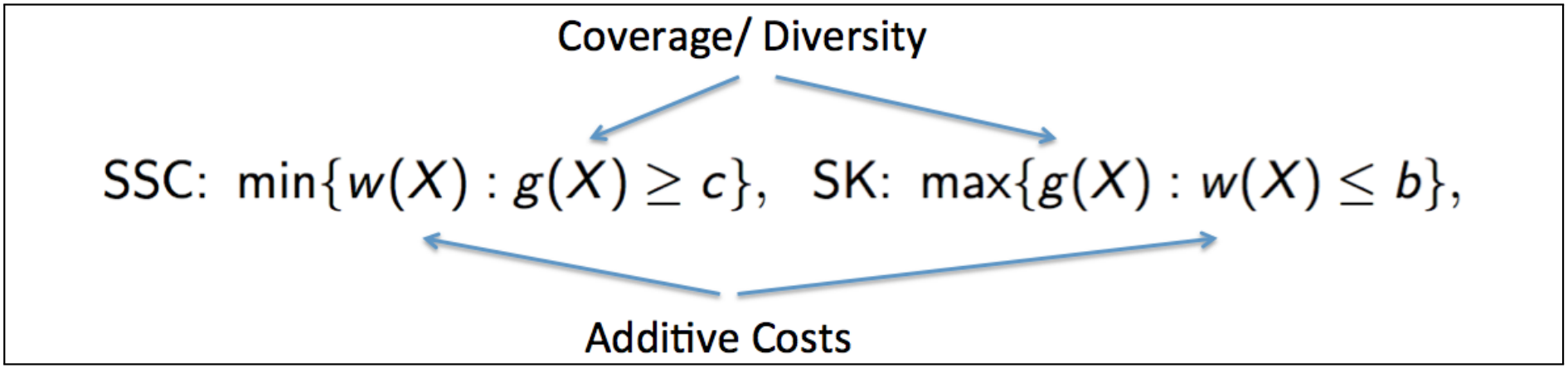

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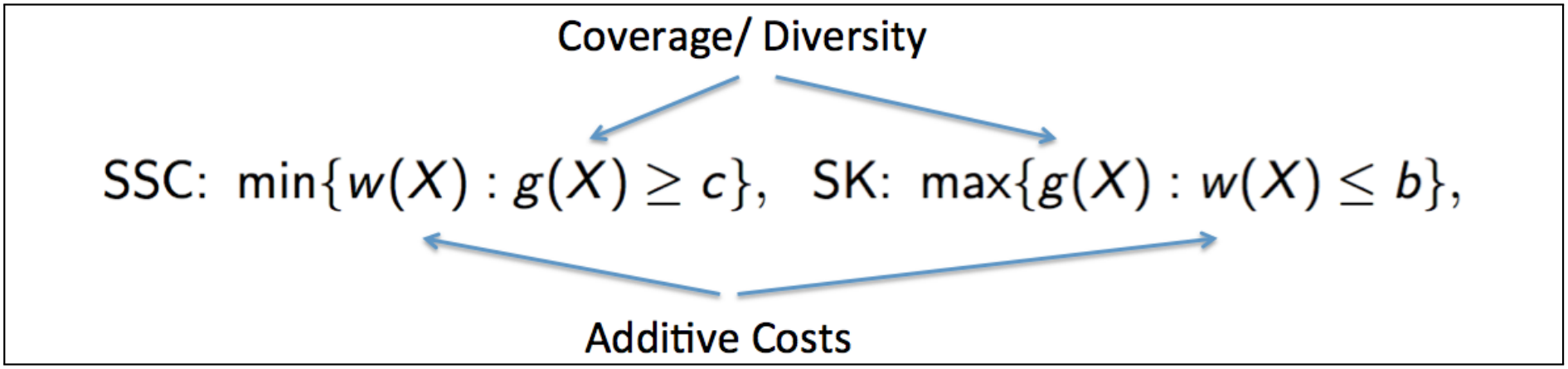

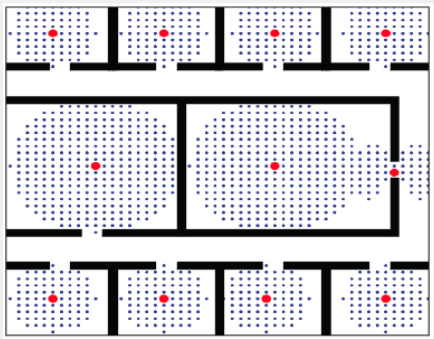

Sensor Placement
(Krause et al'08)

```
1 all_right how are_you doing
2 how are_you with yours
3 hi nadine my name is lorraine how are_you
4 good how are_you
5 hello hi how are_you
6 good thanks how are_you
7 uh how are_you
8 i'm good how are_you
9 fine how are_you
```

Data Subset Selection
(Wei et al'13)

Document Summarization
(Lin-Bilmes'11)

II - Submodular Cost with Modular Constraints

$$\text{SML: } \min\{f(X) : w(X) \geq c\}, \quad \text{SS: } \max\{w(X) : f(X) \leq b\},$$

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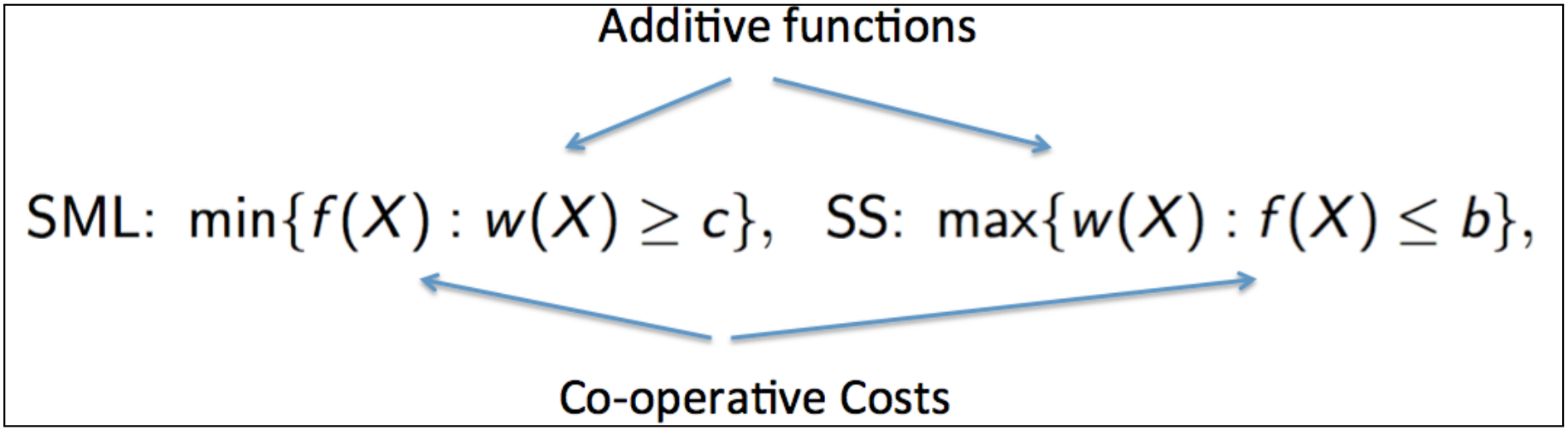

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Additive functions

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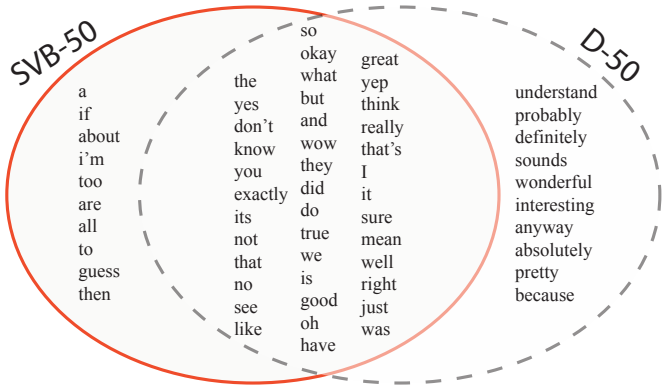
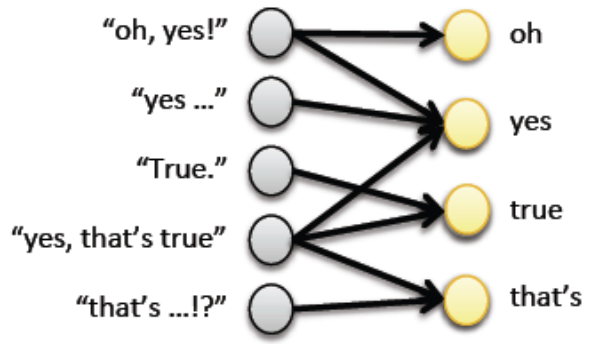
Co-operative Costs





Limited vocabulary speech corpus selection (Lin-Bilmes'11)

III - Most General Case: SCSC and SCSK

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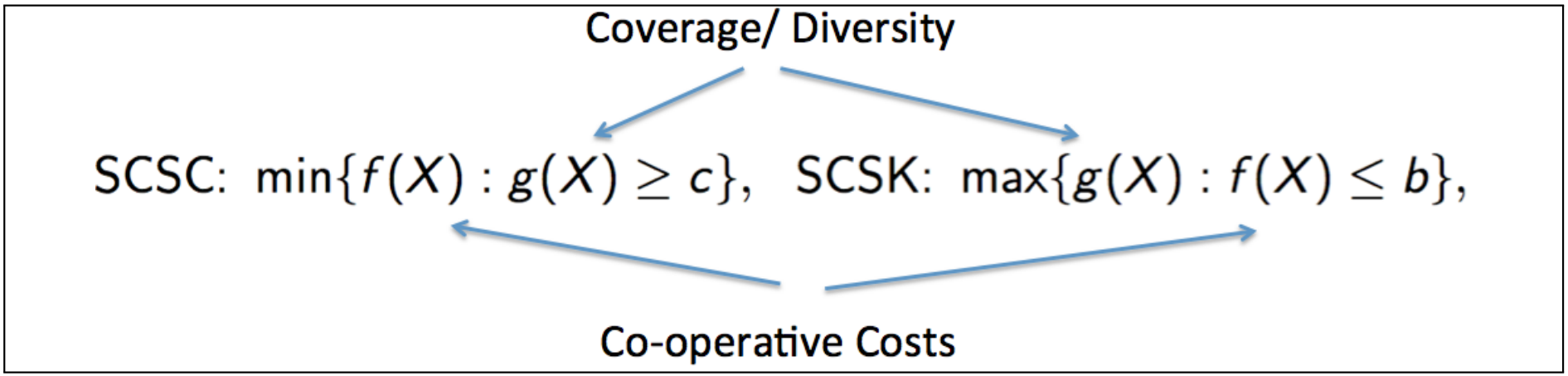

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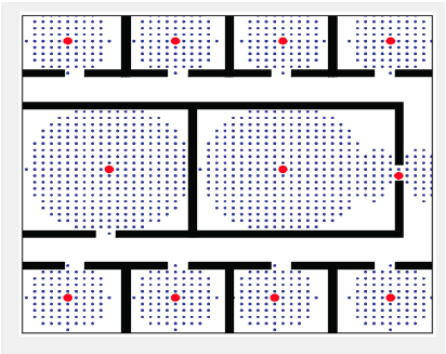
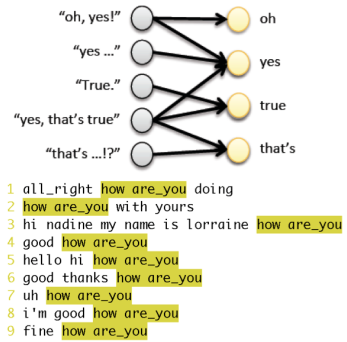
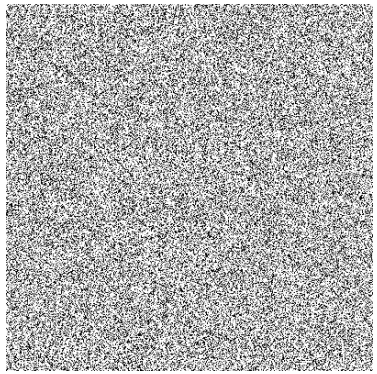


Sensor Placement with Submodular Costs (I-Bilmes'12)



Limited vocabulary and accoustically diverse speech corpus selection (Lin-Bilmes'11, Wei et al'13)



Privacy preserving communication (I-Bilmes'13)

Connections between SCSC and SCSK

- **Bi-criterion factors:**

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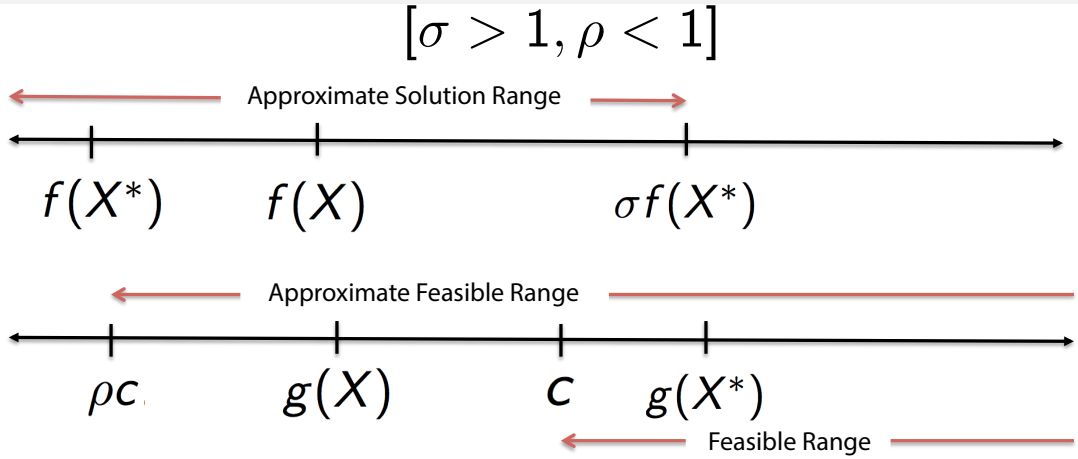
- $\min\{f(X) : g(X) \geq c\}$: $[\sigma, \rho]$ approximation for SCSC is a set $X : f(X) \leq \sigma f(X^*)$ and $g(X) \geq \rho c$.



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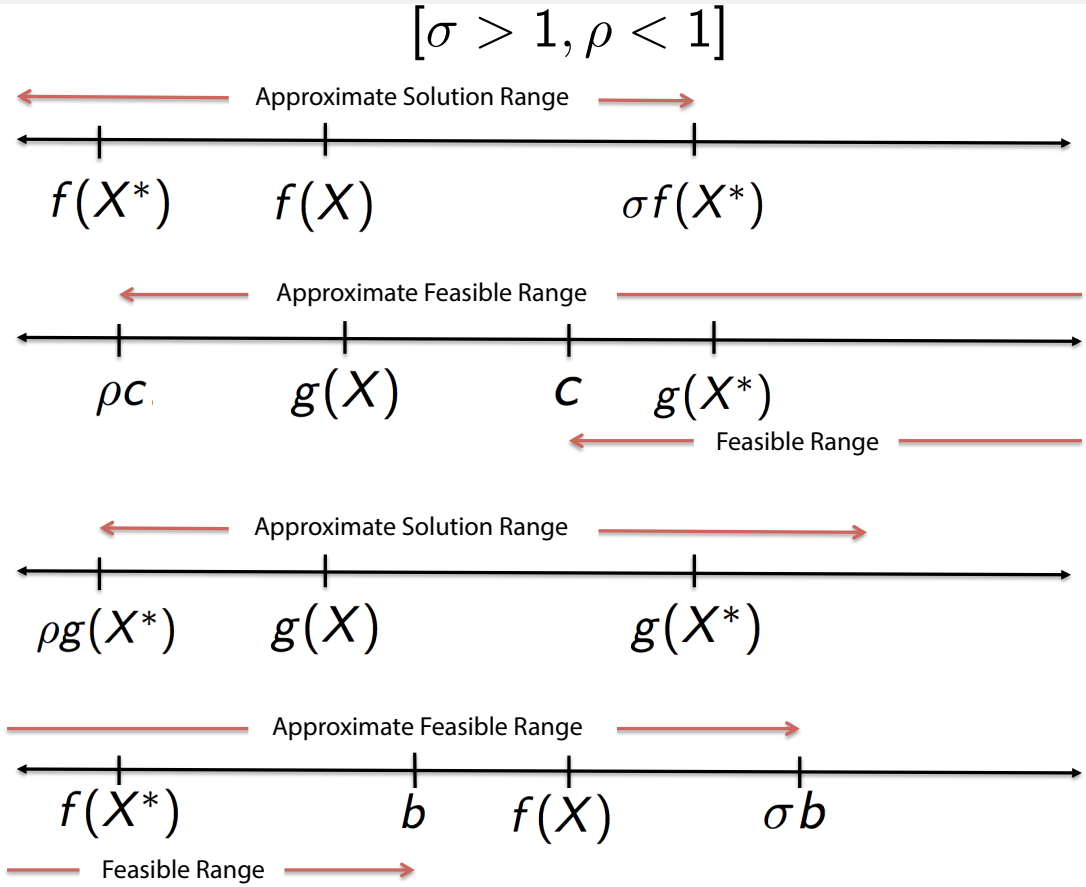
- $\max\{g(X) : f(X) \leq b\}$: $[\rho, \sigma]$ approximation for SCSK is a set $X : g(X) \geq \rho g(X^*)$ and $f(X) \leq \sigma b$.



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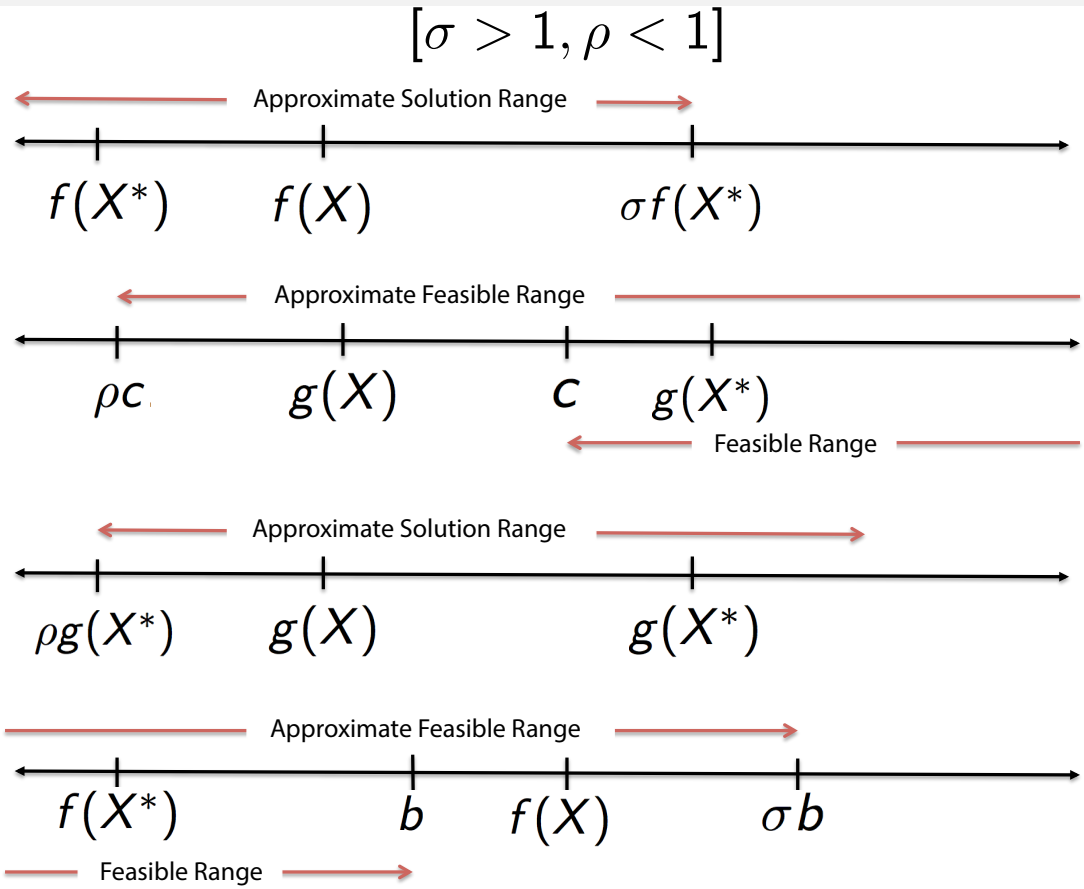
- $\max\{g(X) : f(X) \leq b\}$: $[\rho, \sigma]$ approximation for SCSK is a set $X : g(X) \geq \rho g(X^*)$ and $f(X) \leq \sigma b$.

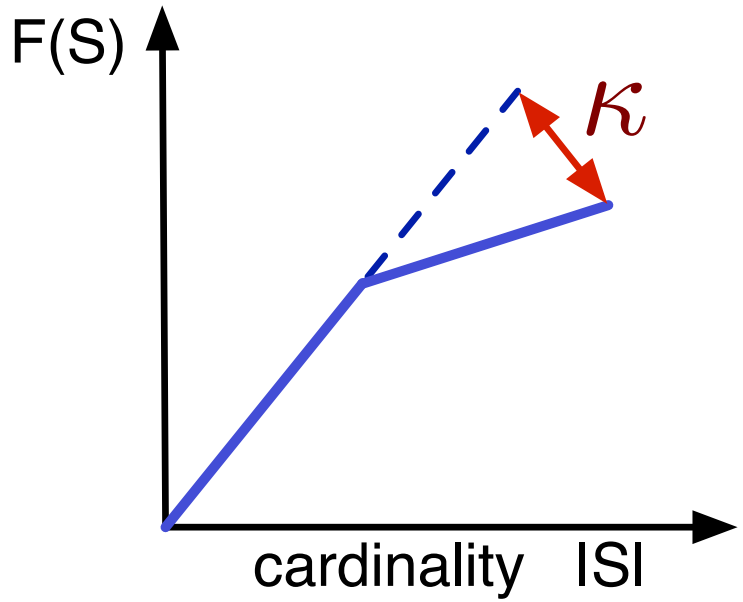


- **Theorem:** Given a $[\sigma, \rho]$ bi-criterion approx. algorithm for SCSC, we can obtain a $[(1+\epsilon)\rho, \sigma]$ bi-criterion approx. algorithm for SCSK, by running the algorithm for SCSC, $O(\log \frac{1}{\epsilon})$ times.
- The other direction also holds!

Curvature of a Submodular Function

- Curvature:

$$\kappa_f = 1 - \min_{j \in V} \frac{f(j|V \backslash j)}{f(j)} \quad \text{and} \quad \kappa_g = 1 - \min_{j \in V} \frac{g(j|V \backslash j)}{g(j)} \tag{2}$$



- Curvature is a fundamental "complexity" parameter of a submodular function.

Hardness (Lower bounds) of the problems

	Modular g	Submodular g	
	$(\kappa_g = 0)$	$(0 < \kappa_g < 1)$	$(\kappa_g = 1)$
Modular f $(\kappa_f = 0)$			
Submod f $(0 < \kappa_f < 1)$			
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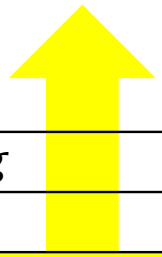
Hardness (Lower bounds) of the problems

Knapsack



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Modular f $(\kappa_f = 0)$	FPTAS		
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SSC/SK

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Submod f $(0 < \kappa_f < 1)$			
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Modular f $(\kappa_f = 0)$	FPTAS	$\frac{1}{\kappa_g}(1 - e^{-\kappa_g})$	$1 - 1/e$
Submod f $(0 < \kappa_f < 1)$	$\Omega(\frac{\sqrt{n}}{1+(\sqrt{n}-1)(1-\kappa_f)})$		
Submod f $(\kappa_f = 1)$	$\Omega(\sqrt{n})$		

SML/SS

Hardness (Lower bounds) of the problems

Knapsack

SSC/SK

	Modular g	Submodular g	
	$(\kappa_g = 0)$	$(0 < \kappa_g < 1)$	$(\kappa_g = 1)$
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Submod f $(0 < \kappa_f < 1)$	$\Omega(\frac{\sqrt{n}}{1+(\sqrt{n}-1)(1-\kappa_f)})$	$\Omega(\frac{\sqrt{n}}{1+(\sqrt{n}-1)(1-\kappa_f)})$	$\Omega(\frac{\sqrt{n}}{1+(\sqrt{n}-1)(1-\kappa_f)})$
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SML/SS

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SML/SS

SCSC/SCSK

- Hardness depends (mainly) on κ_f and not (so much) on that of κ_g.

Algorithmic framework

Algorithm 1 General algorithmic framework to address both Problems 1 and 2

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- Surrogate functions: modular upper/ lower bounds or Ellipsoidal Approximations.

Surrogate functions

- **Modular Lower Bounds:** Induced via orderings of elements:

Surrogate functions

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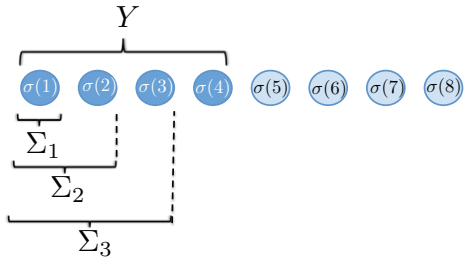
$$f(X) \leq h^{\sigma}_{Y}(X), \text{ where } h^{\sigma}_{Y}(\sigma(i)) = f(\Sigma_i) - f(\Sigma_{i-1})$$



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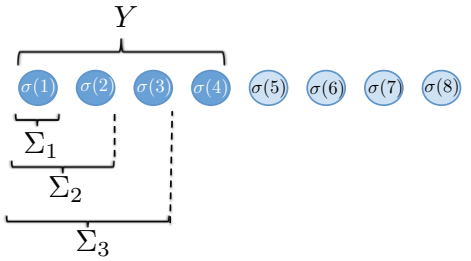

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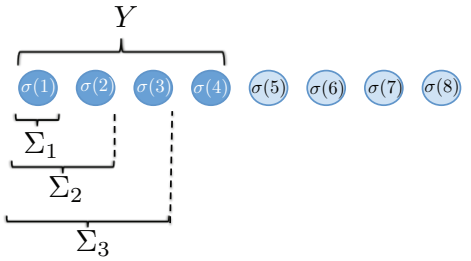
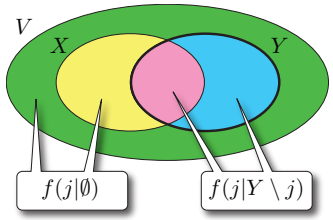


- **Modular upper bounds:**
  Upper bound-I

$$f(X) \leq m_{Y,1}(X) = f(Y) - \sum_{j \in Y \setminus X} f(j|Y \backslash j) + \sum_{j \in X \setminus Y} f(j|\emptyset)$$



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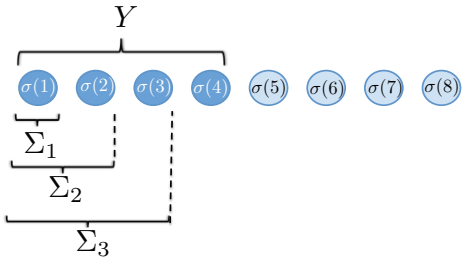

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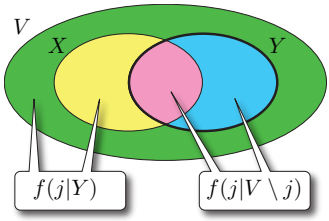
$$f(X) \leq m_{Y,2}(X) = f(Y) - \sum_{j \in Y \setminus X} f(j|V \setminus j) + \sum_{j \in X \setminus Y} f(j|Y)$$



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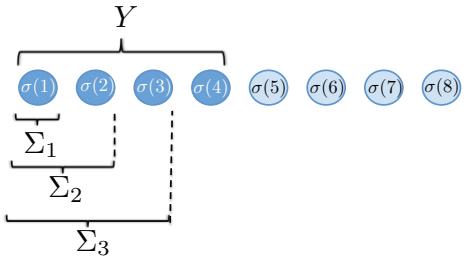

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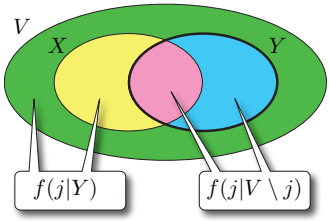

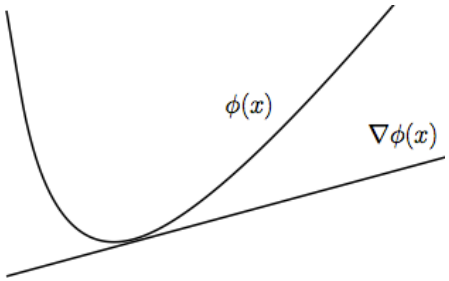

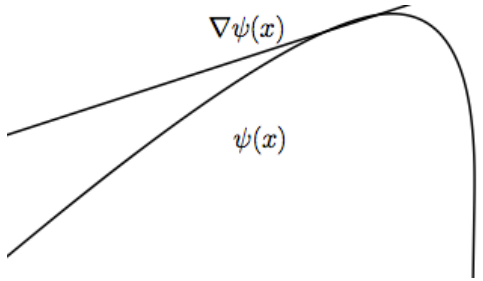

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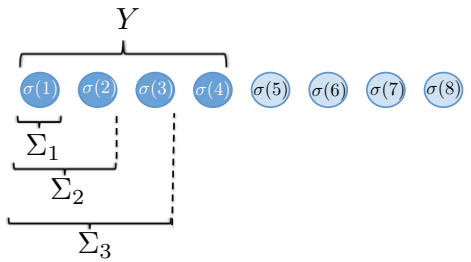

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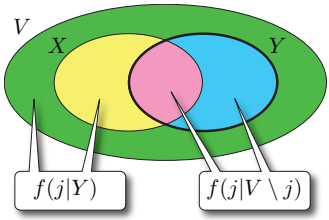
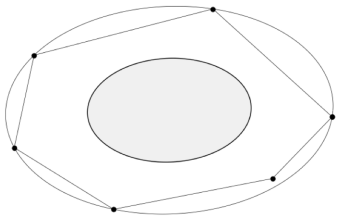
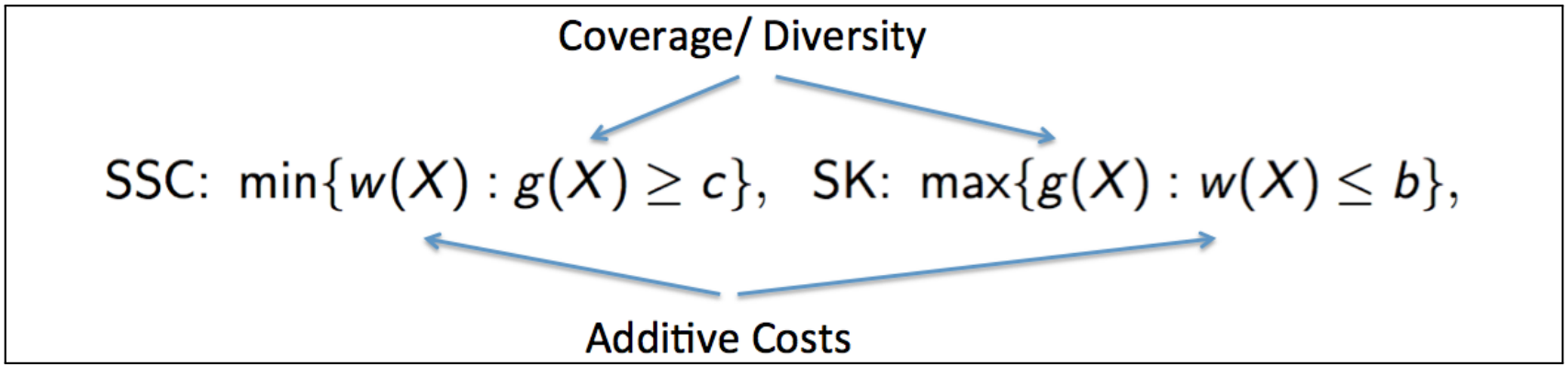


- **Approximations:** Ellipsoidal Approximation gives the *tightest* approximation to a submodular function.



Submodular Set Cover (SSC) and Submodular Knapsack (SK)



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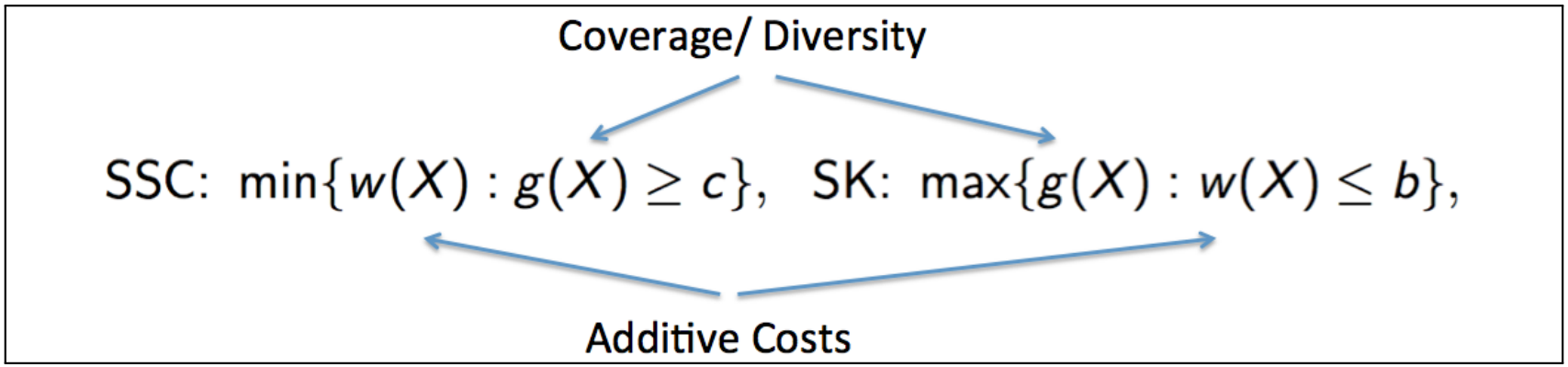

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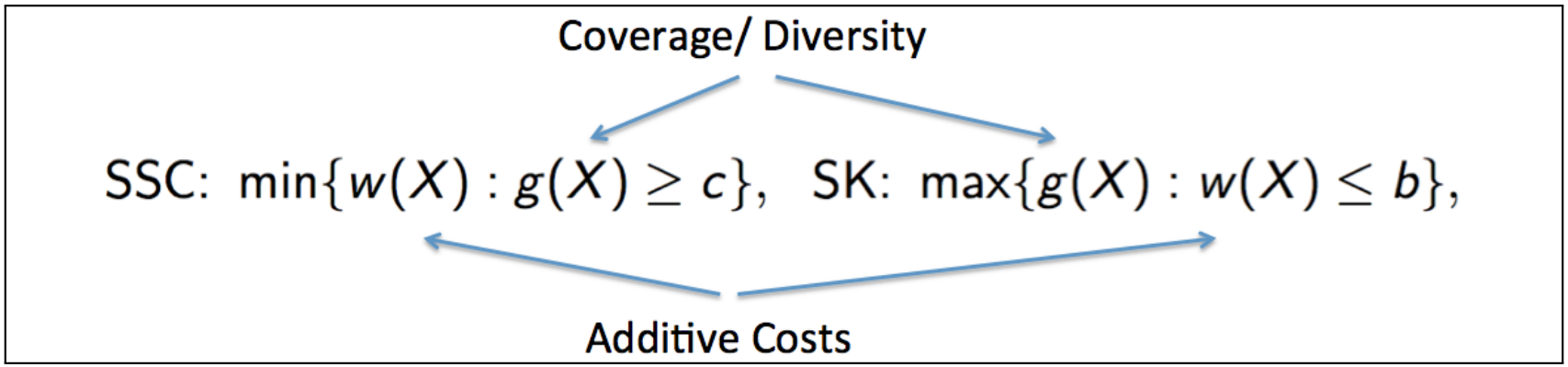
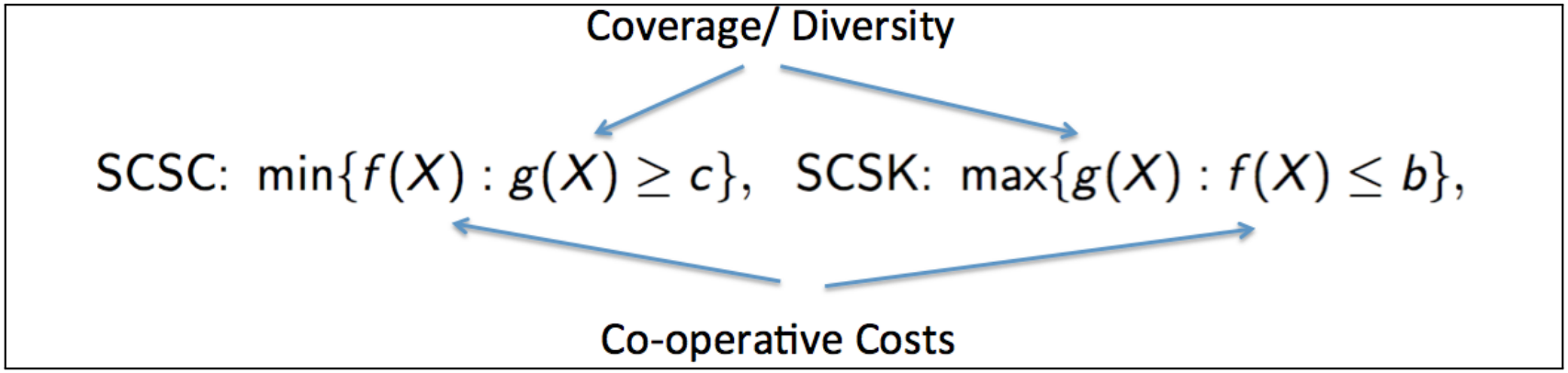


- Lemma: The greedy algorithm for SSC (Wolsey, 82) and SK (Nemhauser, 78) is special case of Algorithm 1 with g replaced by its modular lower bound.
- Approximation guarantees are constant factor – $1 - 1/e$ respectively.

Iterative Submodular Set Cover (ISSC)/Submodular Knapsack (ISK)



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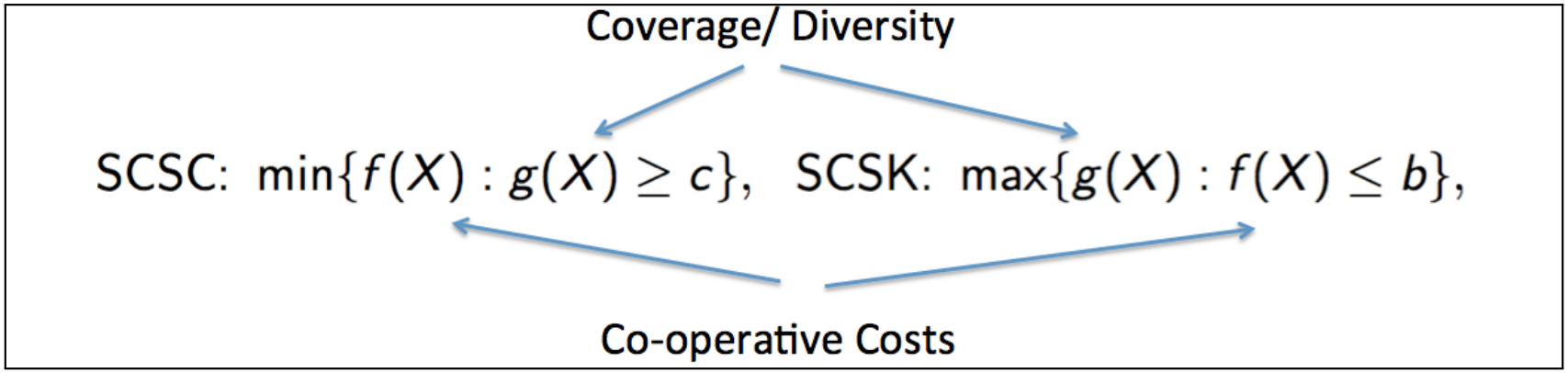

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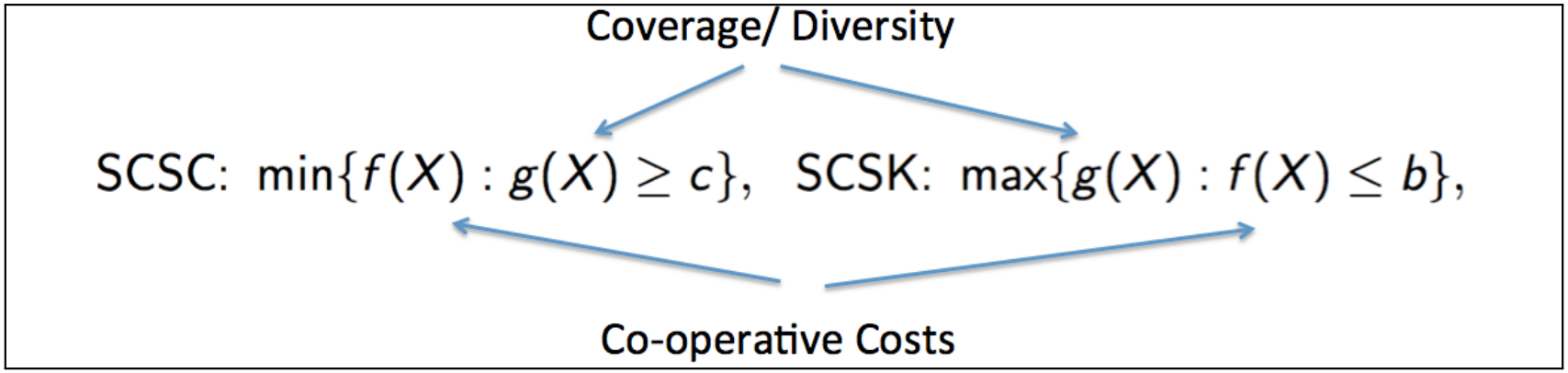

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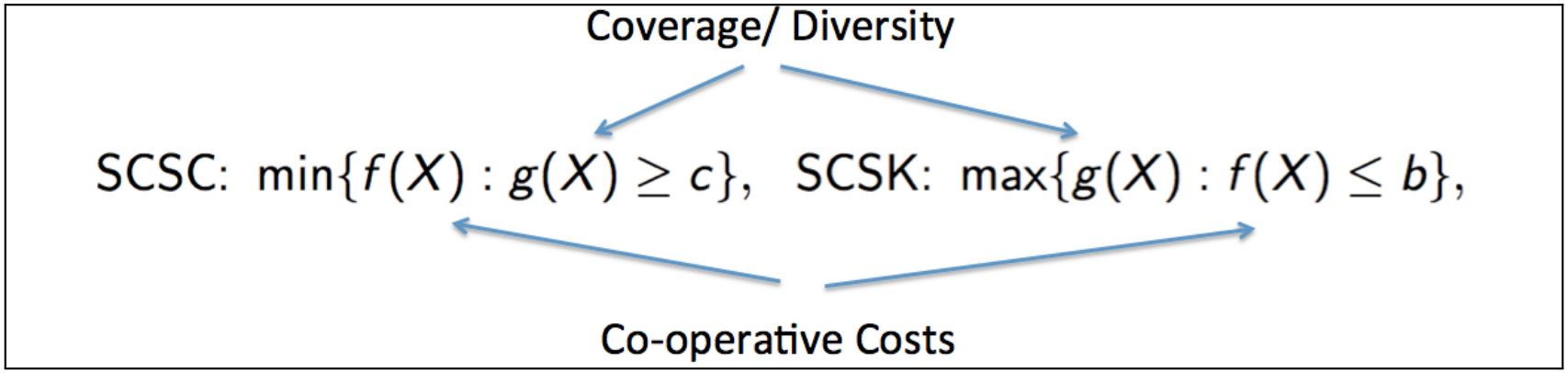

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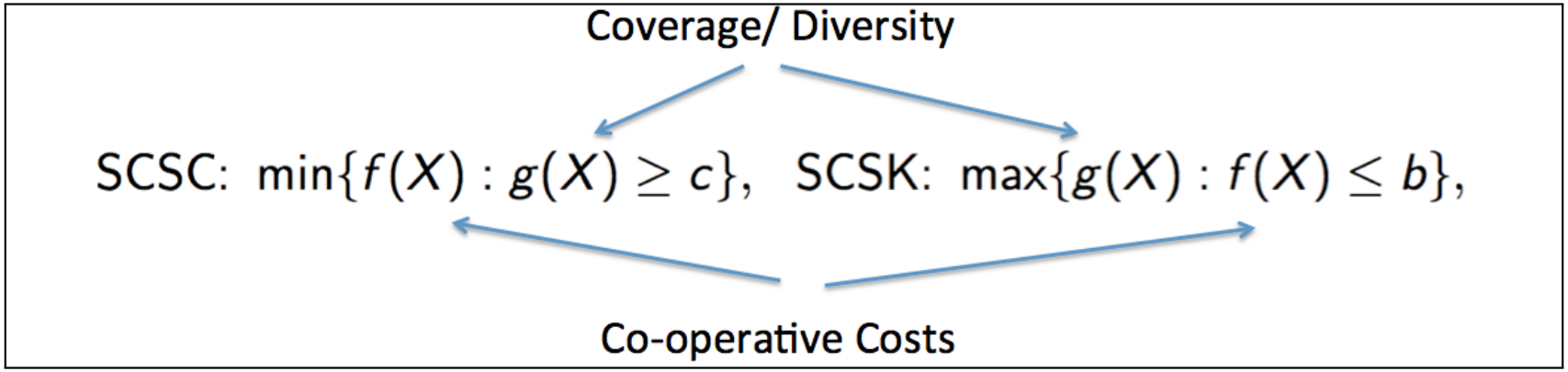
Ellipsoidal Approximation Submodular Set Cover (EASSC)/ Submodular Knapsack (EASK)



- Choose surrogate functions $\hat{f}_t$ as Ellipsoidal Approximation, in both SCSC and SCSK.

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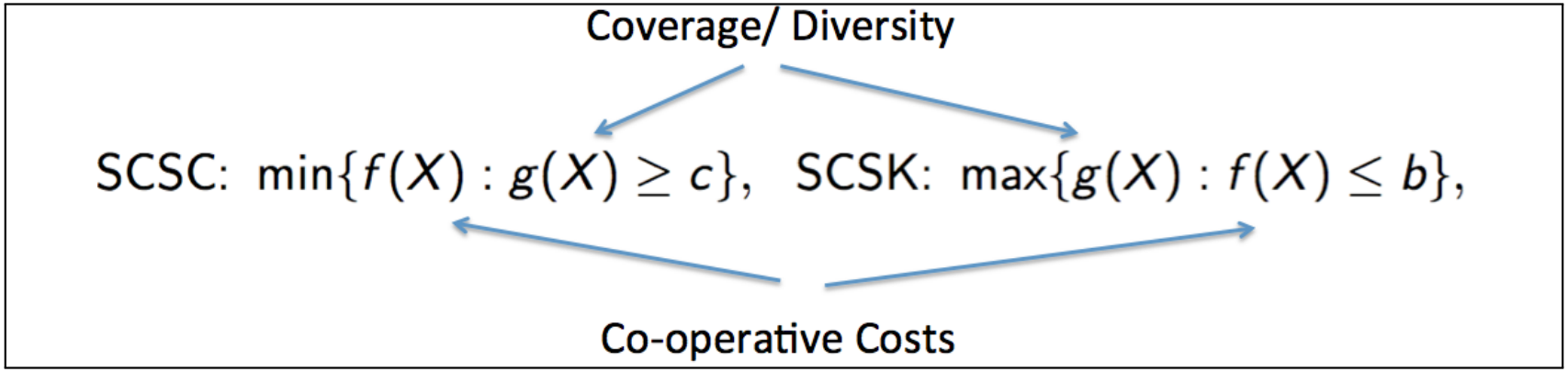

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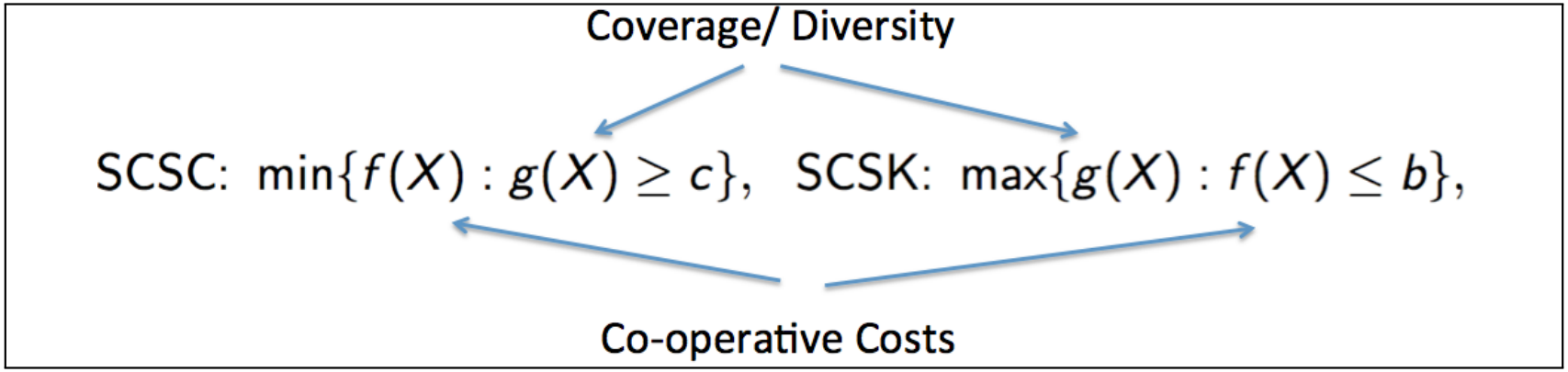

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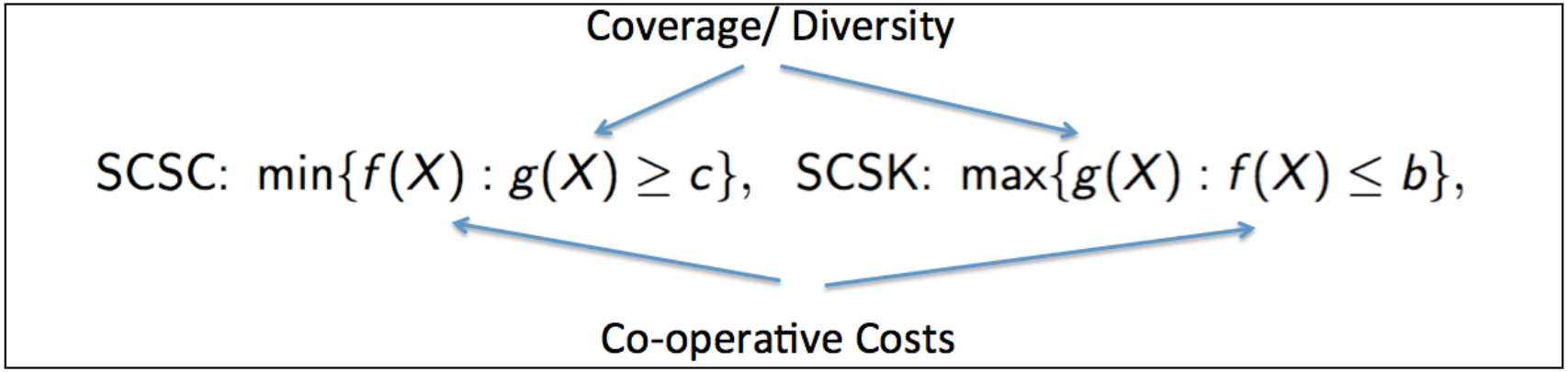

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- This algorithm matches the hardness of this problem upto log factors.

Limited Vocabulary data subset selection with Accoustic diversity

- **Accoustic Diversity:**

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2 how are_you with yours
3 hi nadine my name is lorraine how are_you
4 good how are_you
5 hello hi how are_you
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7 uh how are_you
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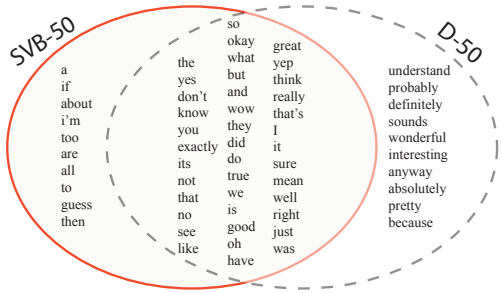

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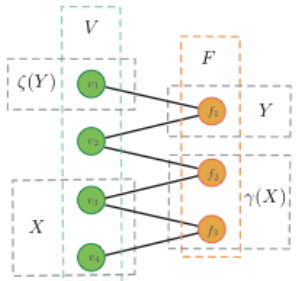
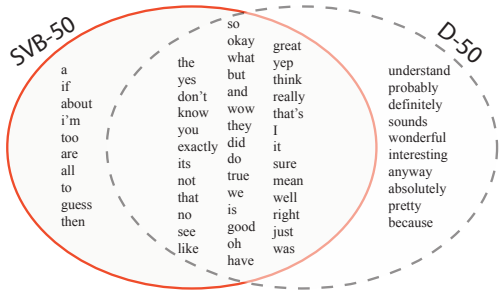


(a) Bipartite graph



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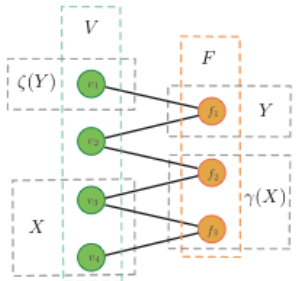

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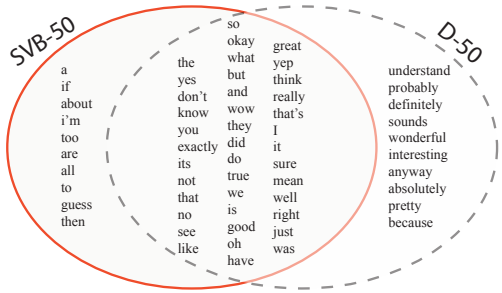

Bipartite Neighborhood function: $|\gamma(X)|$.

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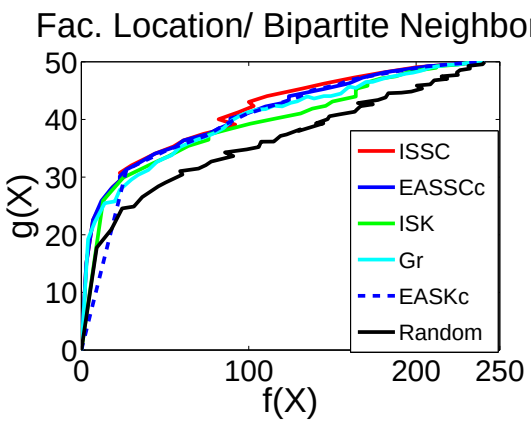

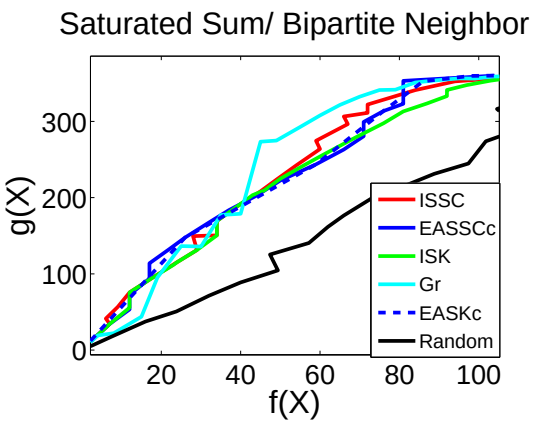

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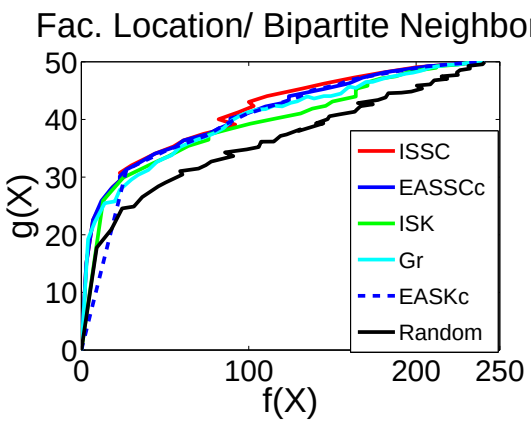

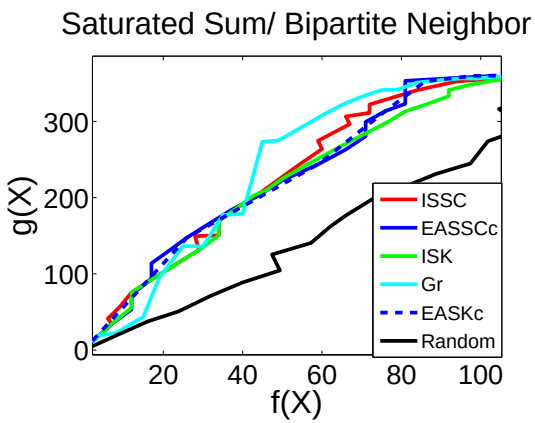

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  1. All the algorithms perform much better than random subset selection.
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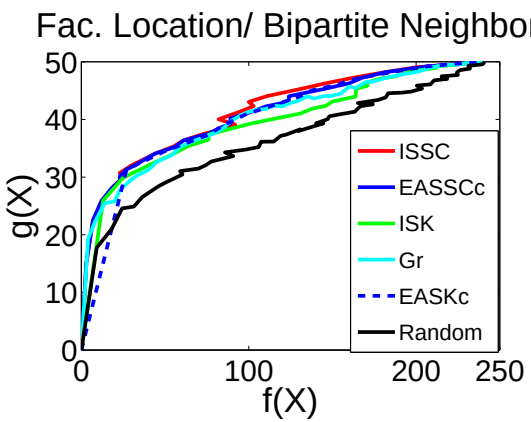

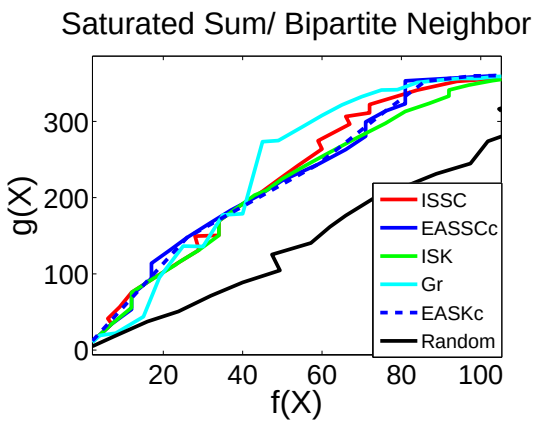

Conclusions/ Future Work

- We proposed some very efficient (scalable) algorithms and two tight algorithms for submodular optimization under submodular constraints.
- In the paper: Extensions to handle multiple constraints, and non-monotone submodular functions.
- Future Work: Investigate our new algorithms on different real world applications.

Thank You!

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Hardness Theorem

- Theorem: For any $\kappa > 0$, there exists submodular function f with curvature $\kappa_f = \kappa$ such that no polynomial time algorithm for SCSC and SCSK $\frac{\sigma}{\rho} = \frac{n^{1/2-\epsilon}}{1+(n^{1/2-\epsilon}-1)(1-\kappa)}$ for any $\epsilon > 0$.

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	Modular g	Submodular g	
	$(\kappa_g = 0)$	$(0 < \kappa_g < 1)$	$(\kappa_g = 1)$
Modular f $(\kappa_f = 0)$	FPTAS	$\frac{1}{\kappa_g}(1 - e^{-\kappa_g})$	$1 - 1/e$
Submod f $(0 < \kappa_f < 1)$	$\Omega(\frac{\sqrt{n}}{1+(\sqrt{n}-1)(1-\kappa_f)})$	$\Omega(\frac{\sqrt{n}}{1+(\sqrt{n}-1)(1-\kappa_f)})$	$\Omega(\frac{\sqrt{n}}{1+(\sqrt{n}-1)(1-\kappa_f)})$
Submod f $(\kappa_f = 1)$	$\Omega(\sqrt{n})$	$\Omega(\sqrt{n})$	$\Omega(\sqrt{n})$

Table : Summary of Hardness results for SCSC/ SCSK