Submodularity beyond submodular energies: Coupling edges in graph cuts

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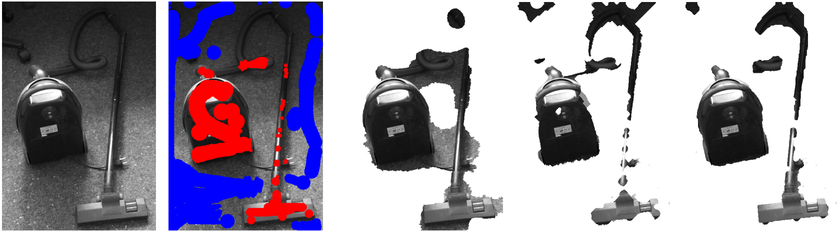
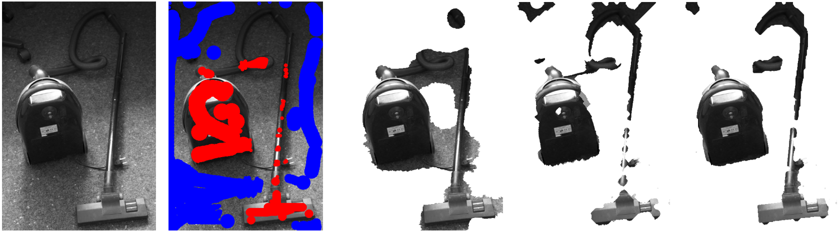
local pairwise random fields . . .



Random Walker Curvature reg. Graph Cut



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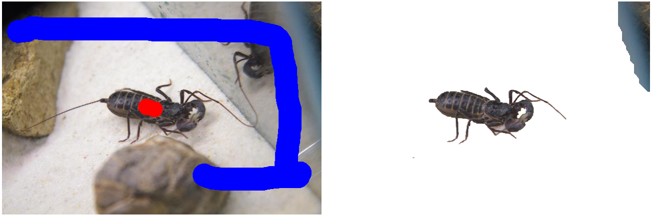
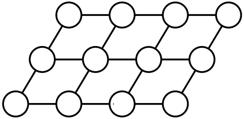


Markov Random Fields and Energies

$$p(\mathbf{x} \mid \mathbf{z}) \propto \exp(-E_\Psi(\mathbf{x}; \mathbf{z}))$$

$$\text{MAP} \quad \mathbf{x}^* = \arg\min_{\mathbf{x}} E_\Psi(\mathbf{x}; \mathbf{z})$$



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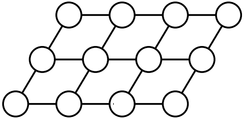

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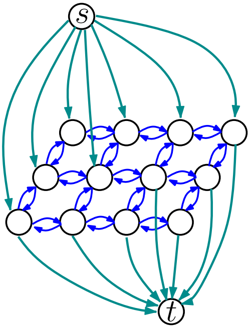

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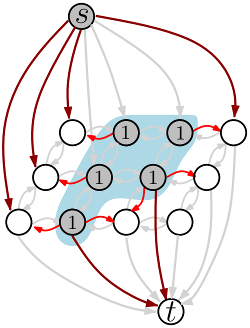

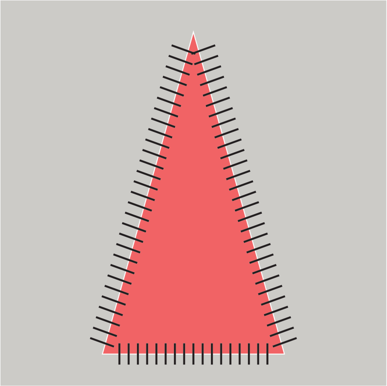

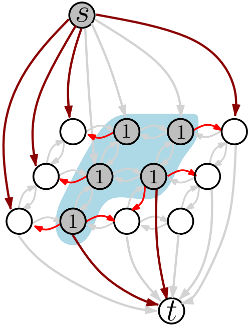
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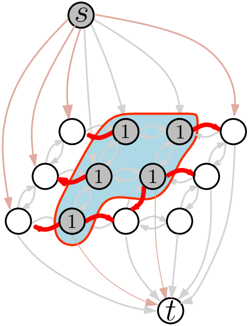
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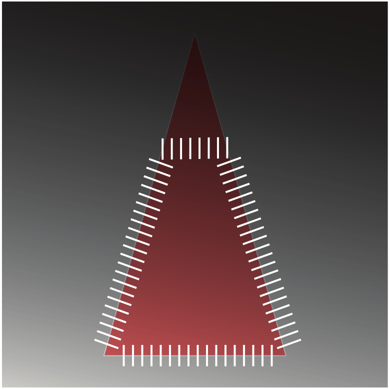

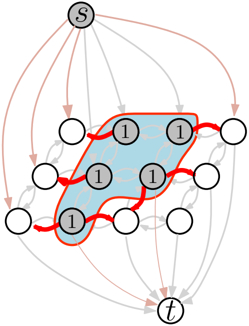
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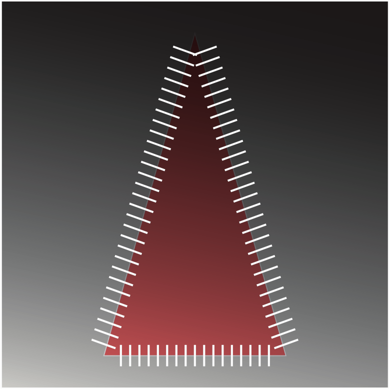
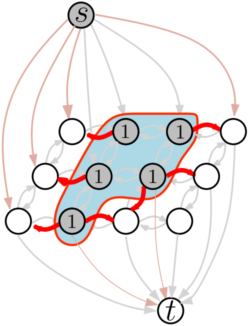
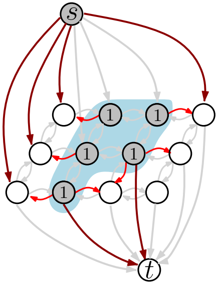




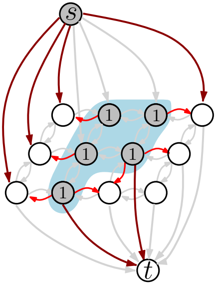
Couple edges globally

Richer Cuts: Cooperative Cuts



$$E(\mathbf{x}) = \sum_{e \in \Gamma\mathbf{x}} w(e)$$
$$= w(\Gamma\mathbf{x})$$

Richer Cuts: Cooperative Cuts

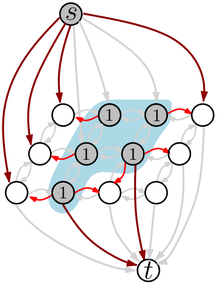


$$E(\mathbf{x}) = \sum_{e \in \Gamma\mathbf{x}} w(e) = w(\Gamma\mathbf{x})$$

$$E_f(\mathbf{x}) = f(\Gamma\mathbf{x})$$

submodular function on edges

Richer Cuts: Cooperative Cuts



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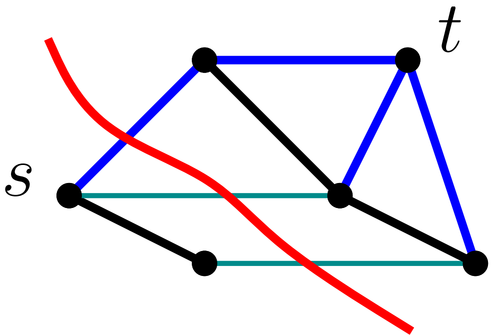
submodular function on edges

non-submodular & global energy

Coupling via Submodularity



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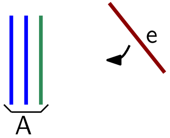
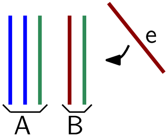
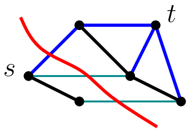


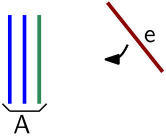


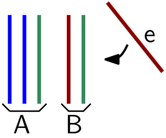
$$f(A \cup e) - f(A) \quad \geq \quad f(A \cup B \cup e) - f(A \cup B)$$



- Graph Cuts: LHS = RHS
  "it does not matter which other edges are cut"

Coupling via Submodularity





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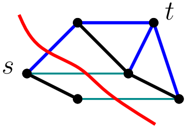
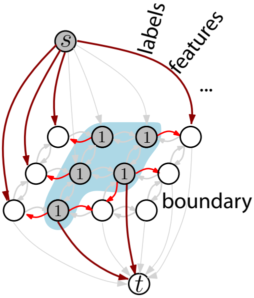


submodularity:

- reward co-occurrence
- structure

Generality



Special cases of cooperative cuts:

- (robust) P^n potentials (Kohli et al. '07,'09)
- label costs (Delong et al. '11)
- discrete versions of norm-based cuts (Sinop & Grady '07)
- ...

Optimization?

Optimization?

(s,t)-cut $\Gamma \subseteq \mathcal{E}$ with min cost $f(\Gamma)$.

Theorem

Minimum Cooperative Cut is NP-hard.

Optimization

$\Gamma_0 = \emptyset$;
repeat
    compute upper bound $\hat{f}_i \geq f$ based on Γ_{i-1};

until *convergence* ;

$$\hat{f}_i(\Gamma_{i-1}) = f(\Gamma_{i-1})$$

Optimization

$\Gamma_0 = \emptyset$;
repeat
  compute upper bound $\hat{f}_i \geq f$ based on Γ_{i-1};
  $\Gamma_i \in \text{argmin}\{ \hat{f}_i(\Gamma) \mid \Gamma \text{ a cut } \}$; // `Min-cut!`
  $i = i + 1$;
until *convergence* ;

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Worst-case approximation bound:

$$E_f(\mathbf{x}) \leq \frac{|\Gamma^*|}{1+(|\Gamma^*|-1)\nu} E_f(\mathbf{x}^*) \qquad \text{for } \nu = \frac{\min_{e\in\Gamma^*} \rho_e(\mathcal{E}\setminus e)}{\max_{e\in C^*} f(e)}$$

Image Segmentation

Random Walker

Curvature reg.

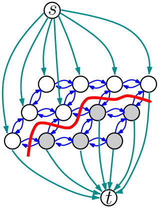
Graph Cut

Image Segmentation

Random Walker Curvature reg. Graph Cut

prefer congruous boundaries

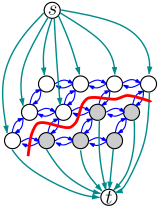
Selective Discount for Congruous Boundaries



$$E_w(\mathbf{x}) = \sum_{e \in \Gamma \cap \mathcal{E}_t} w_e \qquad + \lambda \sum_{e \in \Gamma \cap \mathcal{E}_n} w_e$$

$$E_f(\mathbf{x}) = \sum_{e \in \Gamma \cap \mathcal{E}_t} w_e \qquad + \lambda \boxed{f(\Gamma \cap \mathcal{E}_n)}$$

Selective Discount for Congruous Boundaries



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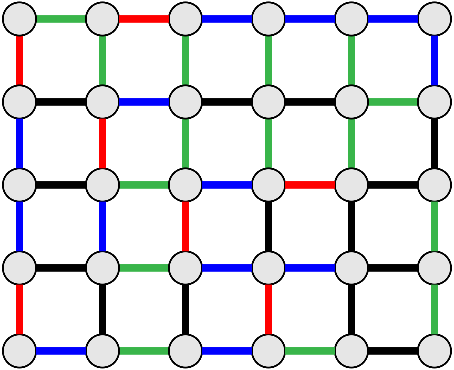
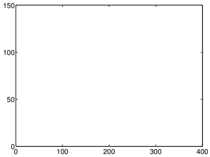
- discount for co-occurring similar edges
- no discount for dissimilar edges

Structured Discounts



groups S_i of edges

$$f(\Gamma) = \sum_i f_i(\Gamma \cap S_i)$$



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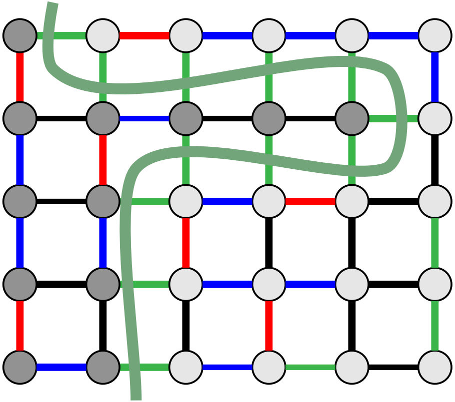

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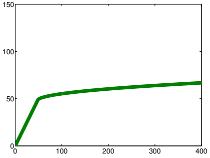

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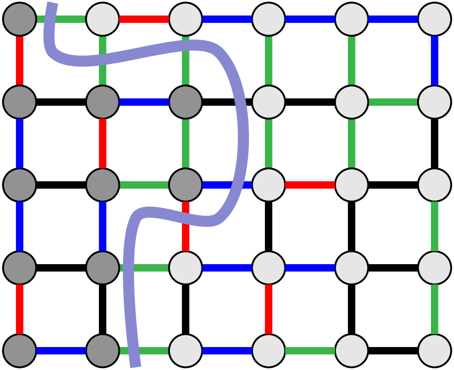

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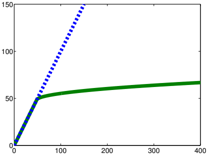
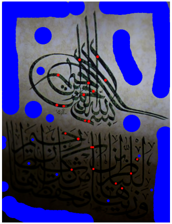


Some Results: Shading

	Graph Cut	CoopCut
	7.39%	2.23%

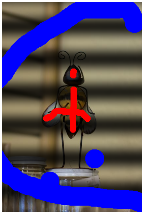
7.65%

3.50%



Some Results: Shading

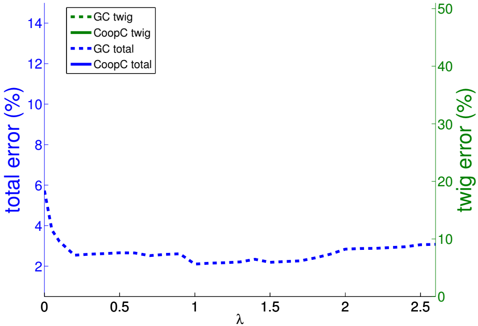
		gray	color	high-freq
Graph Cut:	no discount	14.03	3.41	2.56
CoopCut (1 group):	discount	11.58	2.95	1.49
CoopCut (15 groups):	structured discount	3.63	1.69	1.27



Graph Cut
5.08%

CoopCut
0.64%

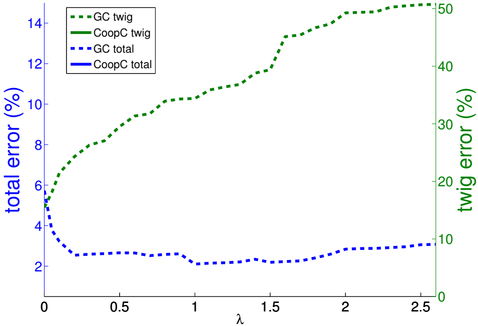
Shrinking bias



$$\sum_i \psi_i(x_i) \; + \; \lambda \sum_{e \in \Gamma \mathbf{x}} w_e$$

Graph Cut

Shrinking bias



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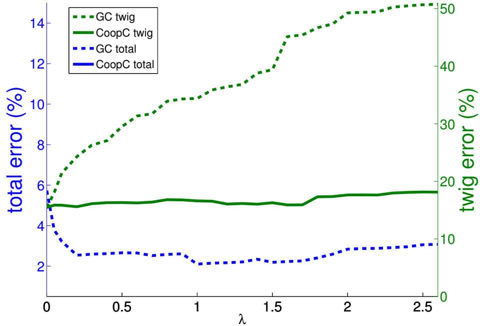

$$\sum_i \psi_i(x_i) \quad + \quad \lambda\, f(\Gamma \mathbf{x})$$

CoopCut

Graph Cut

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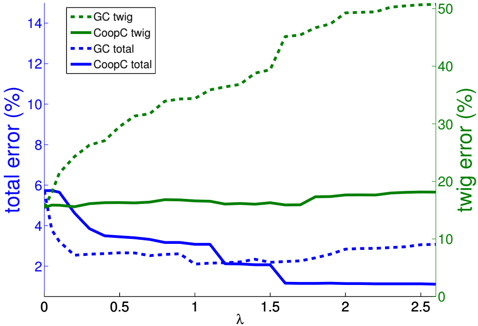

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CoopCut

Graph Cut

Summary: Coupling Edges in Graph Cuts

- global, non-submodular family of energies
- NP-hard, but ...
  - graph structure
  - indirect submodularity

  $\longrightarrow$ efficient approximation algorithm
- applications
  - guide segmentations via edge coupling