On fast approximate submodular minimization: Extended Version

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Abstract

We are motivated by an application to extract a representative subset of machine learning training data and by the poor empirical performance we observe of the popular minimum norm algorithm. In fact, for our application, minimum norm can have a running time of about $O(n^7)$ ($O(n^5)$ oracle calls). We therefore propose a fast approximate method to minimize arbitrary submodular functions. For a large sub-class of submodular functions, the algorithm is exact. Other submodular functions are iteratively approximated by tight *submodular* upper bounds, and then repeatedly optimized. We show theoretical properties, and empirical results suggest significant speedups over minimum norm while retaining higher accuracies.

1 Introduction

Submodularity has been and continues to be an important property in many fields, including combinatorics, operations research, game theory, and economics. A set function $f : 2^{\mathcal{V}} \to \mathbb{R}$ defined on subsets of a finite *ground set* $\mathcal{V}$ is submodular if it satisfies the inequality $f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$ for all $S, T \subseteq \mathcal{V}$. Submodular functions include entropy, graph cuts (defined as a function of graph nodes), potentials in many Markov Random Fields [3], various clustering objectives [23], covering functions (e.g., sensor placement objectives), and many more. One might consider submodular functions as being on the boundary between "efficiently", i.e., polynomial-time, and "not efficiently" optimizable set functions. They can be minimized in polynomial time, whereas subadditive functions (satisfying a less restrictive inequality $f(S) + f(T) \geq f(S \cup T)$) rule out **any** nontrivial approximation factor in polynomial time. Submodularity is gaining importance in machine learning too, but many machine learning data sets are so large that mere "polynomial-time" efficiency is not enough. Indeed, the submodular function minimization (SFM) algorithms with proven polynomial running time are practical only for very small data sets owing to exponents of 5 or higher in their running times. An alternative, often considered to be faster in practice, is the minimum-norm point algorithm [8]. Its worst-case running time however is still an open question.

Contrary to current wisdom, we in this paper demonstrate that for certain functions relevant in practice (see Section 1.1), the minimum-norm algorithm has an impractical empirical running time of about $O(n^7)$, requiring about $O(n^5)$ oracle function calls. To our knowledge, and interesting from an optimization perspective, this is worse than any results reported in the literature, where times of $O(n^{3.3})$ were obtained with simpler graph cut functions [22]. There is in fact an error tolerance parameter associated with the minimum-norm algorithm. As a result, in order to handle large data sets in a reasonable amount of time, one must use this tolerance parameter. To make running time reasonable, however, one must accept an accuracy so low that the solution is essentially useless. This negative empirical result about the minimum-norm point algorithm is a contribution in and of itself.

Since we found the minimum-norm algorithm to be either slow (when accurate), or inaccurate (when fast), in this work we take a different approach. We view the SFM problem as an instance of a

larger class of problems that *includes* NP-hard instances. This class admits worst case approximation algorithms, and we hence apply those instead of an exact method. Contrary to the possibly poor performance of "exact" methods, our approximate method is fast, is exact for a large class of submodular functions, and approximates all other functions with bounded deviation.

Our approach combines two ingredients: 1) the representation of functions by graphs; and 2) a recent generalization of graph cuts that combines edge-costs non-linearly. Representing functions as graph cuts is a popular basis for optimization, but cuts cannot efficiently represent all submodular functions. Contrary to previous constructions, including 2) leads to exact representations for *any* submodular function. To optimize an arbitrary submodular function f represented in our formalism, we construct a graph-representable tractable submodular upper bound $\hat{f}$ that is tight at a given set $T \subseteq \mathcal{V}$, i.e., $\hat{f}(T) = f(T)$, and $\hat{f}(S) \geq f(S)$ for all $S \subseteq \mathcal{V}$. We repeat this "submodular majorization" step and optimize, in at most a linear number of iterations. The resulting algorithm efficiently computes good approximate solutions for our motivating application and other difficult functions as well.

After describing our motivating application, we outline other approaches, describe our construction and optimization, and finally show our theoretical and empirical results.

1.1 Motivating application and the failure of the minimum-norm point algorithm

The proliferation of large real-world machine-learning data sets is both a blessing and a curse for machine learning algorithms. On the one hand, a large data set can accurately represent a complex task thereby giving an algorithm the opportunity to live up to its potential. On the other hand, the size of such data sets presents its own problem: many algorithms have complexities that scale at a level that renders them impractical for all but small problem sizes.

Our motivating problem is how to empirically evaluate new or expensive algorithms on large data sets without spending an inordinate amount of time doing so [20, 21]. We may wish to test various algorithms quickly, identifying the one that performs best. If a new idea ends up performing poorly, knowing this sooner rather than later will avoid futile work. Often the complexity of a training iteration is linear in the number of samples n but *polynomial* in the number c of classes or *types*. For example, for object recognition, it typically takes $O(c^k)$ time to segment an image into regions that each correspond to one of c objects, using an MRF with non-submodular k-interaction potential functions. In speech recognition, moreover, a k-gram language model with size-c vocabulary has a complexity of $O(c^k)$, where c is in the hundreds of thousands and k can be as large as six.

To reduce complexity one can reduce k, but this can be unsatisfactory since the nature and novelty of the algorithm might entail this very cost. An alternative is to extract and use a subset of the training data, one with small c. We would want any such subset to possess as much as possible of the richness and intricacy of the original data set as possible while simultaneously ensuring that c is bounded. We also may wish to impose a bias in favor of certain classes, say, on those between which it is most difficult to discriminate. For example, in computer vision, we may wish to choose a large subset of a face recognition database that limits the number of different people but that includes people looking as similar to each other as possible. In speech recognition, we may wish to choose a large number of utterances that, collectively, have a small confusable vocabulary [21].

One approach might be to choose a subset of classes of the appropriate size, and then choose all training samples that contain any of those classes. The resulting set of training samples will contain more than this set of classes, however. This would be problematic since, say in computer vision, much of the image would contain real objects that cannot be recognized, and, say, in speech recognition, would mean that the resulting corpus has a large out-of-vocabulary set. Alternatively, one could choose only those training samples that have no more than the selected set of classes, but this then might result in an extremely small set of training samples. We thus desire an approach that allows us to, on the one hand, choose a bounded set of classes, but on the other hand, chooses a subset of training samples that is very large.

This problem can be solved via SFM using the following *Bipartite neighborhoods* class of submodular functions: Define a bipartite graph $\mathcal{H} = (\mathcal{V}, \mathcal{U}, \mathcal{E}, w)$ with left/right nodes $\mathcal{V}$/$\mathcal{U}$, and a modular weight function $w : \mathcal{U} \to \mathbb{R}_+$. A function is *modular* if it is additive, i.e., $w(U) = \sum_{u \in U} w(u)$. Let the neighborhood of a set $S \subseteq \mathcal{V}$ be $\mathcal{N}(S) = \{u \in \mathcal{U} : \exists \text{ edge } (i, u) \in \mathcal{E} \text{ with } i \in S\}$. Then $f : 2^{\mathcal{V}} \to \mathbb{R}_+$, defined as $f(S) = \sum_{u \in \mathcal{N}(S)} w(u)$, is non-decreasing submodular. This function

class encompasses e.g. set covers of the form $f(S) = |\bigcup_{i \in S} U_i|$ for sets U_i covered by element i. We say f is the submodular function *induced* by modular function w and graph $\mathcal{H}$.

Let $\mathcal{U}$ be the set of types contained in a set of training samples $\mathcal{V}$. Moreover, let the aforementioned w measure the cost of a type $u \in \mathcal{U}$ (this corresponds e.g. to the "undesirability" of type u). Define also a modular function $m : 2^{\mathcal{V}} \to \mathbb{R}_+$, $m(S) = \sum_{i \in S} m(i)$ as the benefit of training samples (e.g., in vision, $m(i)$ might be the number of different objects in an image $i \in \mathcal{V}$, and in speech, this might be the length of utterance i). Then the above optimization problem can be solved by finding $\operatorname{argmin}_{S \subseteq \mathcal{V}} w(\mathcal{N}(S)) - \lambda m(S) = \operatorname{argmin}_{S \subseteq \mathcal{V}} w(\mathcal{N}(S)) + \lambda m(\mathcal{V} \setminus S)$ where λ is a tradeoff coefficient. As shown below, this can be easily represented and solved efficiently via graph cuts. In some cases, however, we prefer to pick certain subclasses of $\mathcal{U}$ *together*. We partition $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$ into blocks, and make it beneficial to pick items from the same block. Benefit restricted to blocks can arise from non-negative non-decreasing submodular functions $g : 2^{\mathcal{U}} \to \mathbb{R}_+$ restricted to blocks. The resulting optimization problem is $\min_{S \subseteq \mathcal{V}} \sum_i g(\mathcal{U}_i \cap \mathcal{N}(S)) + \lambda m(\mathcal{V} \setminus S)$; the sum over i expresses the obvious generalization to a partition into more than just two blocks. Unfortunately, this class of submodular functions is no longer representable by a bipartite graph, and general SFM must be used.

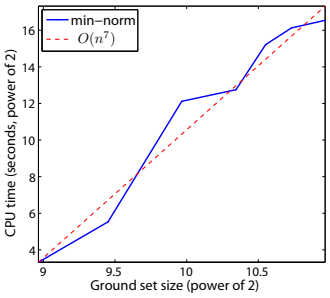


Figure 1: Running time of MN

With such a function, $f(S) = m(S) + 100\sqrt{w(\mathcal{N}(S))}$, we find that the empirical running time of the minimum norm point algorithm (MN) scales as $O(n^7)$, with $O(n^5)$ oracle calls (Figure 1). This rules out large data sets for our application, but is interesting with regard to the unknown complexity of MN.

1.2 Background on Algorithms for submodular function minimization (SFM)

The first polynomial algorithm for SFM was by Grötschel et al. [14], with further milestones being the first combinatorial algorithms [16, 27] ([22] contains a survey). The currently fastest strongly polynomial combinatorial algorithm has a running time of $O(n^5T + n^6)$ [24] (where T is function evaluation time), far from practical for large data sets. Thus, the minimum-norm algorithm [8] is often the method of choice.

Luckily, many sub-families of submodular functions permit specialized, faster algorithms. Graph cut functions fall into this category [1]. They have found numerous applications in computer vision [2, 13], begging the question as to which functions can be represented and minimized using graph cuts [10, 7, 32]. Živný et al. [33] show that cut representations[1] are indeed limited: even when allowing exponentially many additional variables, not all submodular functions can be expressed as graph cuts. Moreover, to maintain efficiency, we do not wish to add too many auxiliary variables, i.e., graph nodes. Other specific cases of relatively efficient SFM include graphic matroids [25] and symmetric submodular functions, minimizable in cubic time [26].

A further class of benign functions are those of the form $f(S) = \psi(\sum_{i \in S} w(i)) + m(S)$ for nonnegative weights $w : \mathcal{V} \to \mathbb{R}_+$, and certain concave functions $\psi : \mathbb{R} \to \mathbb{R}$. Fujishige and Iwata [9] minimize such a function via a parametric max-flow, and we build on their results in Section 4. However, restrictions apply to the effective number of breakpoints of ψ. Stobbe and Krause [29] generalize this class to arbitrary concave functions and exploit Nesterov's accelerated gradient descent. Whereas Fujishige and Iwata [9] decompose ψ into a minimum of modular functions, Stobbe and Krause [29] decompose it into a sum of truncated functions of the form $f(A) = \min\{\sum_{i \in A} w'(i), \gamma\}$ — this class of functions, however, is also limited (Appendix A). Truncations are expressible by graph cuts, as we show in Figure 3(c). Thus, if truncations could express any submodular function, then so could graph cuts, contradicting the results in [33]. This was proven independently in [30]. Moreover, the formulation itself of some representable functions in terms of concave functions can be challenging.

[1] Strictly speaking, their result is for representations where the inclusion of an element can be read off by the assignment of one corresponding node to a partition, as in Equation (1). However, to our knowledge, no other general representation is known so far. Thus, when we say "graph-representable", we mean Equation (1).

In this paper, by contrast, we propose a model that is exact for graph-representable functions, and yields an approximation for all other functions.

1.3 Notation and preliminaries

We denote the ground set of elements by $\mathcal{V}$, and subsets by $S, T, Q \subseteq \mathcal{V}$. Instead of $\{i\}$, we will also just write i when the context is clear. The boundary δS of a set $S \subseteq \mathcal{V}$ in a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is the set of edges leaving S, i.e., $\delta S = \{(u, v) \in \mathcal{E} | u \in S, v \notin S\}$. We also use the shorthand $\delta_s T = \delta(s \cup T)$. In a graph with edge weights w, we write the sum of edge weights for a set $A \subseteq \mathcal{E}$ as $w(A) = \sum_{e \in A} w(e)$. The letters m, w, ν always denote modular functions, i.e., sums of weights, and $n = |\mathcal{V}|$. Submodularity may be defined in terms of the *marginal cost* $\rho_f(S|T) = f(S \cup T) - f(T)$, i.e., the increase in cost if S is added to T. Submodular functions satisfy *diminishing marginal costs*: $\rho(i|S) \geq \rho(i|T)$ for any $S \subseteq T \subseteq \mathcal{V} \setminus i$. A set function is nondecreasing if $S \subseteq T \Rightarrow f(S) \leq f(T)$. Without loss of generality, we assume the cost function f to be *normalized*, i.e., $f(\emptyset) = 0$. Nondecreasing normalized submodular functions are also called *polymatroid rank functions* [6].

2 Representing submodular functions by generalized graph cuts

We begin with the representation of a set function $f : 2^{\mathcal{V}} \to \mathbb{R}$ by a graph cut, and then extend this to submodular edge weights. Formally, f is graph-representable if there exists a graph $\mathcal{G} = (\mathcal{V} \cup \mathcal{U} \cup \{s, t\}, \mathcal{E})$ with terminal nodes s, t, one node for each element i in $\mathcal{V}$, a set $\mathcal{U}$ of auxiliary nodes ($\mathcal{U}$ can be empty), and edge weights $w : \mathcal{E} \to \mathbb{R}_+$ such that, for any $S \subseteq \mathcal{V}$:

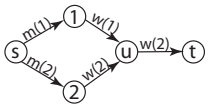


Figure 2: max

$$f(S) = \min_{U \subseteq \mathcal{U}} w(\delta(s \cup S \cup U)) = \min_{U \subseteq \mathcal{U}} \sum_{e \in \delta_s(S \cup U)} w(e). \tag{1}$$

Recall that any minimal (s, t)-cut partitions the graph nodes into the set $T_s \subseteq \mathcal{V} \cup \mathcal{U}$ reachable from s and the set $T_t = (\mathcal{V} \cup \mathcal{U}) \setminus T_s$ disconnected from s. That means, $f(S)$ equals the weight of the minimum (s, t)-cut that assigns S to T_s and $\mathcal{V} \setminus S$ to T_t, and the auxiliary nodes to achieve the minimum. The nodes in $\mathcal{U}$ act as auxiliary variables. As an illustrative example, Figure 2 represents the function $f(S) = \max_{i \in S} w(i) + \sum_{j \in \mathcal{V} \setminus S} m(j)$ for two elements $\mathcal{V} = \{1, 2\}$ and $w(2) > w(1)$, using one auxiliary node u. Note that this function is a sum of a submodular function and a modular function over the set complement. For any query set S, u might be joined with S ($u \in T_s$) or not ($u \in T_t$). If $S = \{1\}$, then $w(\delta_s(\{1, u\})) = m(2) + w(2)$, and $w(\delta_s(\{1\})) = m(2) + w(1) = f(S) < w(\delta_s(\{1, u\}))$. If $S = \{1, 2\}$, then $w(\delta_s(\{1, 2, u\})) = w(2) < w(\delta_s(\{1, 2\})) = w(1) + w(2)$, and indeed $f(S) = w(2)$. The graph representation (1) leads to the equivalence between minimum cuts and the minimizers of f:

Lemma 1. *Let S^* be a minimizer of f, and let $U^* \in \operatorname{argmin}_{U \subseteq \mathcal{U}} w(\delta_s(S^* \cup U))$. Then the boundary $\delta_s(S^* \cup U^*) \subseteq \mathcal{E}$ is a minimum cut in $\mathcal{G}$.*

Proof. Assume the Lemma is not true, and there is a cut $\delta_s(T)$ with $w(\delta_s(T)) < w(\delta_s(S^* \cup U^*))$. Let $T_{\mathcal{V}} = T \cap \mathcal{V}$. Then it holds that

$$f(T_{\mathcal{V}}) = \min_{U \subseteq \mathcal{U}} w(\delta_s(T_{\mathcal{V}} \cup U)) \leq w(\delta_s(T)) < w(\delta_s(S^* \cup U^*)) = f(S^*),$$

contradicting the optimality of S^*. □

The lemma is good news since minimum cuts can be computed efficiently. To derive S^* from a minimum cut, recall that any minimum cut is the boundary of some set $T_s^* \subseteq \mathcal{V} \cup \mathcal{U}$ that is still reachable from s after cutting. Then $S^* = T_s^* \cap \mathcal{V}$, so $S^* \subseteq T_s^*$ and $(\mathcal{V} \setminus S^*) \subseteq T_t^*$. A large sub-family of submodular functions can be expressed exactly in the form (1), but possibly with an exponentially large $\mathcal{U}$. For efficiency, the size of $\mathcal{U}$ should remain small. To express *any* submodular function with *few* auxiliary nodes, in this paper we extend Equation (1) as is seen below.

Unless the submodular function f is already a graph cut function (and directly representable), we first decompose f into a modular function and a nondecreasing submodular function, and then build up the graph part by part. This accounts for any graph-representable component of f. To approximate

the remaining component of the function that is not exactly representable, we use submodular costs on **graph edges** (in contrast with graph nodes), a construction that has been introduced recently in computer vision [17]. We first introduce a relevant decomposition result by Cunningham [4]. A polymatroid rank function is *totally normalized* if $f(\mathcal{V} \setminus i) = f(\mathcal{V})$ for all $i \in \mathcal{V}$. The marginal costs are defined as $\rho_f(i|S) = f(S \cup \{i\}) - f(S)$ for all $i \in \mathcal{V} \setminus S$.

Theorem 1 ([4, Thm. 18]). *Any submodular function f can be decomposed as $f(S) = m(S) + g(S)$ into a modular function m and a totally normalized polymatroid rank function g. The components are defined as $m(S) = \sum_{i \in S} \rho_f(i|\mathcal{V} \setminus i)$ and $g(S) = f(S) - m(S)$ for all $S \subseteq \mathcal{V}$.*

For unconstrained submodular minimization, we may assume that $m(i) < 0$ for all $i \in \mathcal{V}$. If $m(i) \geq 0$ for any $i \in \mathcal{V}$, then diminishing marginal costs, a property of submodular functions, imply that we can discard element i immediately. Such elements imply that $f(S) - f(S \setminus i) \geq f(\mathcal{V}) - f(\mathcal{V} \setminus i) = m(i)$ for all S containing i. Then $f(S) \geq m(i) + f(S \setminus i)$, and excluding i from S never increases the value of a SFM solution. To express such negative costs in a graph cut, we point out an equivalent formulation with positive weights: since $m(\mathcal{V})$ is constant, minimizing $m(S) = \sum_{i \in S} m(i)$ is equivalent to minimizing the shifted function $m(S) - m(\mathcal{V}) = -m(\mathcal{V} \setminus S)$. Thus, we instead minimize the sum of positive weights on the complement of the solution. We implement this shifted function in the graph by adding an edge (s, i) with *nonnegative* weight $-m(i)$ for each $i \in \mathcal{V}$. Every element $j \in T_t$ (i.e., $j \notin S$) that is not selected must be separated from s, and the edge (s, j) contributes $-m(j)$ to the total cut cost. If f is modular, then $f = m$ and the optimal solution is the empty cut separating t from $\mathcal{V}$, i.e., the set of all elements with negative weights as shown in Figure 3(a).

Having constructed the modular part of the function f by edges (s, i) for all $i \in \mathcal{V}$, we address its submodular part g. If g is a sum of functions, we can add a subgraph for each function. We begin with some example functions that are explicitly graph-representable with polynomially many auxiliary nodes $\mathcal{U}$ and then show a more general construction. The illustrations in Figure 3 include the modular part m as well. Moreover, if need be for non-normalized submodular functions, a positive constant can be represented by a direct edge between s and t that is always cut.

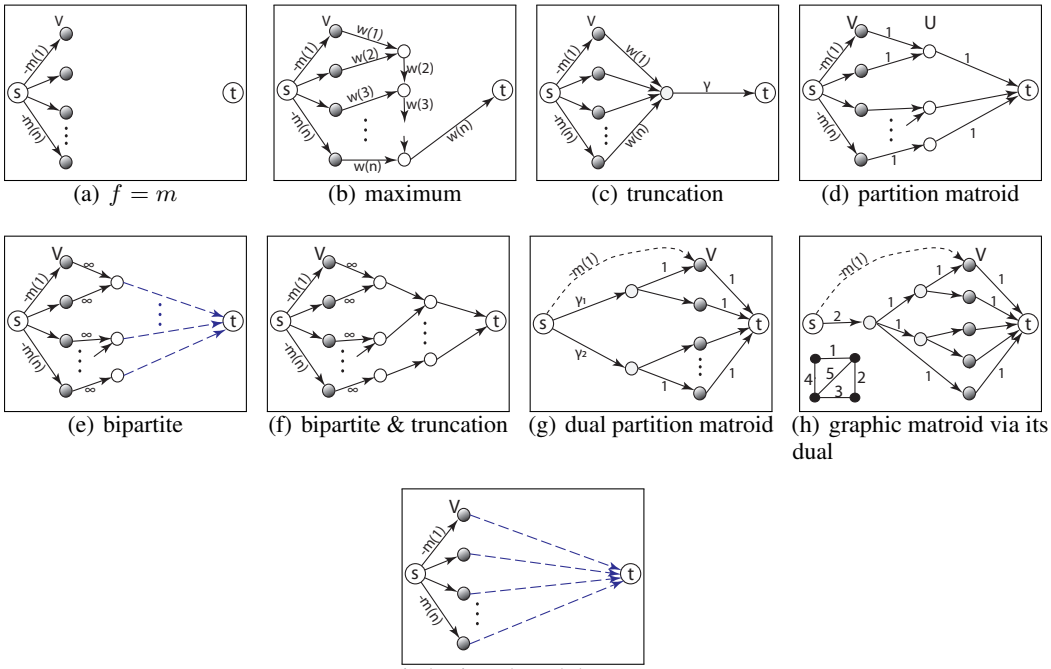
Maximum. The function $g(S) = \max_{i \in S} w(i)$ for nonnegative weights w is an extension of Figure 2. Without loss of generality, we assume the elements to be ordered by weight, so that $w(1) \leq w(2) \leq \ldots w(n)$. We introduce $n-1$ auxiliary nodes u_j, and connect them to form an imbalanced tree with leaves $\mathcal{V}$, as illustrated in Figure 3(b). The minimum way to disconnect a set S from t is to cut the single edge (u_{j-1}, u_j) with weight $w(j)$ of the largest element $j = \operatorname{argmax}_{i \in S} w(i)$.

Truncations. Truncated functions $f(S) = \min\{w(S), \gamma\}$ for $w, \gamma \geq 0$ can be modeled by one extra variable, as shown in Figure 3(c). If $w(S) > \gamma$, then the minimization in (1) puts u in T_s and cuts the γ-edge. This construction has been successfully used in computer vision [19]. Truncations can model piecewise linear concave functions of $w(S)$ [19, 29], and also represent negative terms in a pseudo-boolean polynomial involving the set $Q \subseteq \mathcal{V}$. Let $w(i) = c$ for all $i \in Q$, and $w(i) = 0$ otherwise. Then the term is a truncation with $\gamma = |Q|c - c$. Furthermore, these functions include rank functions $g(S) = \min\{|S|, k\}$ of uniform matroids, and rank functions of partition matroids. If $\mathcal{V}$ is partitioned into blocks $\{G_i\}_i \subset \mathcal{V}$, then the rank of the associated partition matroid counts the number of blocks that S intersects: $f(S) = |\{i|G_i \cap S \neq \emptyset\}|$ (Fig. 3(d)).

Bipartite neighborhoods. We already encountered bipartite submodular functions $f(S) = \sum_{u \in \mathcal{N}(S)} w(u)$ in Section 1.1. The bipartite graph that defines $\mathcal{N}(S)$ is part of the representation shown in Figure 3(e), and its edges get infinite weight. As a result, if $S \in T_s$, then all neighbors $\mathcal{N}(S)$ of S must also be in T_s, and the edges (u, t) for all $u \in \mathcal{N}(S)$ are cut. Each $u \in \mathcal{U}$ has such an edge (u, t), and the weight of that edge is the weight $w(u)$ of u.

Dual rank functions. A matroid with rank function $r : 2^{\mathcal{V}} \to \mathbb{R}_+$ has a dual matroid [28] with rank function $r^*(S) = |S| + r(\mathcal{V} \setminus S) - r(\mathcal{V})$. If we can represent r, then we can represent r^* by putting r (inverted) on the s-side of $\mathcal{V}$. Figure 3(g) shows the construction for the dual of a partition matroid, and Lemma 2 formalizes it.

Lemma 2. *Let $\mathcal{G}$ be a graph where the minimum cut between s and $S \subseteq \mathcal{V}$ has cost $r(S)$, and where there is an edge (j, t) with $w(j, t) = 1$ for each $j \in \mathcal{V}$. Then $\min_{U \subseteq \mathcal{U}} w(\delta(s \cup S \cup U)) = r^*(S) + const.$*



(a) $f = m$ (b) maximum (c) truncation (d) partition matroid

(e) bipartite (f) bipartite & truncation (g) dual partition matroid (h) graphic matroid via its dual

(i) basic submodular construction

Figure 3: Example graph constructions. Dashed blue edges can have submodular weights; auxiliary nodes are white. The bipartite graph can have arbitrary representations between $\mathcal{U}$ and t, 3(f) is one example. The duals 3(g),3(h) can also have edges (s, i) with weight $-m(i)$ that are not all shown.

Proof. For each element $j \in S$, the cut contains an edge (j, t); these edges contribute $|S|$ to the cost. The part of the cut separating s from $\mathcal{V} \setminus S$ costs $r(\mathcal{V} \setminus S)$. Thus, $\min_{U \subseteq \mathcal{U}} w(\delta(s \cup S \cup U)) = |S| + r(\mathcal{V} \setminus T) = r^*(T) + r(\mathcal{V})$, and $r(\mathcal{V})$ is constant. □

Sometimes the representation via the dual matroid can be simpler than a direct representation. Figure 3(h) shows the representation of a graphic matroid as the dual of its dual. Here, the $r(S)$ is the size of the largest subset of S that does not contain a cycle in the defining graph. For the dual, $r^*(S) = |S|$ if $|S| \leq 2$ and $S \neq \{1, 4\}, \{2, 3\}$. For any other set, $r^*(S) = 2$. We represent r^* via two truncations and place it between s and $\mathcal{V}$. It can be checked that the resulting graph represents r. It is unknown, however, if all graphic matroids can be represented in this way.

Of course, all the above constructions can also be applied to subsets $Q \subset \mathcal{V}$ of nodes. In fact, the decomposition and constructions above permit us to address arbitrary sums and restrictions of such graph-representable functions. These example families of functions already cover a wide variety of functions needed in applications. Minimizing a graph-represented function is equivalent to finding the minimum (s, t)-cut, and all edge weights in the above are nonnegative. Thus we can use any efficient min-cut or max-flow algorithm for any of the above functions.

2.1 Submodular edge weights

Next we address the generic case of a submodular function that is not (efficiently) graph-representable or whose functional form is unknown. We can still decompose this function into a modular part m and a polymatroid g. Then we construct a simple graph as shown in Figure 3(i). The representation of m is the same as above, but the cost of the edges (i, t) will be charged differently. Instead of a sum of weights, we define the cost of a set of these edges to be a non-additive function (i.e., a polymatroid rank function) on sets of edges. Each edge (i, t) is associated with exactly one ground set element $i \in \mathcal{V}$, and selecting i ($i \in T_s$) is equivalent to cutting the edge (i, t). Thus, the cost of edge (i, t) will model the cost $g(i)$ of its element $i \in \mathcal{V}$. Let $\mathcal{E}_t$ be the set of such edges (i, t) (i.e.,

edges adjacent to t), and denote, for any subset $C \subseteq \mathcal{E}_t$ the set of ground set elements adjacent to C by $V(C) = \{i \in \mathcal{V} | (i,t) \in C\}$. Equivalently, C is the boundary of $V(C)$ in $\mathcal{E}_t$: $\delta_s(V(C)) \cap \mathcal{E}_t = C$. We define the cost of C to be the cost of its adjacent ground set elements, $h_g(C) \triangleq g(V(C))$; this implies $h_g(\delta_s(S \cap \mathcal{E}_t)) = g(S)$. The equivalent of Equation (1) becomes

$$f(S) = \min_{U \subseteq \mathcal{U}} w(\delta_s(S \cup U) \setminus \mathcal{E}_t) + h_g(\delta_s(S \cup U) \cap \mathcal{E}_t) = -m(\mathcal{V} \setminus S) + g(S), \tag{2}$$

with $\mathcal{U} = \emptyset$ in Figure 3(i). This generalization from the standard sum of edge weights to a nondecreasing submodular function permits us to express many more functions, in fact *any* submodular function [5]. Such expressiveness comes at a price, however: *in general*, finding a minimum (s,t)-cut with such submodular edge weights is NP-hard, and even hard to approximate [18]. The graphs here that represent *submodular* functions correspond to benign examples that are not NP-hard. Nevertheless, we will use an approximation algorithm that applies to all such non-additive cuts and that is exact for the standard modular sum-of-weight costs. We describe the algorithm in Section 3. For the moment, we assume that we can handle submodular costs on edges.

The simple construction in Figure 3(i) itself corresponds to a general submodular function minimization. It becomes powerful when combined with parts of f that are explicitly representable. If g decomposes into a sum of graph-representable functions and a (nondecreasing submodular) remainder g_r, then we construct a subgraph for each graph-representable function, and combine these subgraphs with the submodular-edge construction for g_r. All the subgraphs share the same ground set nodes $\mathcal{V}$. In addition, we are in no way restricted to separating graph-representable and general submodular functions. The cost function in our application is a submodular function *induced* by a bipartite graph $\mathcal{H} = (\mathcal{V}, \mathcal{U}, \mathcal{E})$. Let, as before, $\mathcal{N}(S)$ be the neighborhood of $S \subseteq \mathcal{V}$ in $\mathcal{U}$. Given a nondecreasing submodular function $g_{\mathcal{U}} : 2^{\mathcal{U}} \to \mathbb{R}_+$ on $\mathcal{U}$, the graph $\mathcal{H}$ defines a function $g(S) = g_{\mathcal{U}}(\mathcal{N}(S))$. If $g_{\mathcal{U}}$ is nondecreasing submodular, then so is g [28, §44.6 g]. For any such function, we represent $\mathcal{H}$ explicitly in $\mathcal{G}$, and then add submodular-cost edges from $\mathcal{U}$ to t with $h_g(\delta_s(\mathcal{N}(S))) = g_{\mathcal{U}}(\mathcal{N}(S))$, as shown in Figure 3(e). If $g_{\mathcal{U}}$ is itself exactly representable, then we add the appropriate subgraph instead (e.g., Figure 3(f)).

3 Optimization

To minimize a function f, we find a minimum (s,t)-cut in its representation graph. Algorithm 1 applies to any submodular-weight cut; this algorithm is exact if the edge costs are modular (a sum of weights). In each iteration, we approximate f by a function $\hat{f}$ that is efficiently graph-representable, and minimize $\hat{f}$ instead. In this section, we switch from costs $f, \hat{f}$ of node sets S, T to equivalent costs w, h of edge sets A, B, C and back.

The approximation $\hat{f}$ arises from the cut representation constructed in Section 2: we replace the exact edge costs by approximate modular edge weights ν in $\mathcal{G}$. Recall that the representation $\mathcal{G}$ has two types of edges: those whose weights w are counted as the usual sum, and those charged via a submodular function h_g derived from g. We denote the latter set by $\mathcal{E}_t$, and the former by $\mathcal{E}_m$. For any $e \in \mathcal{E}_m$, we use the exact cost $\nu(e) = w(e)$. The submodular cost h_g of the remaining edges is upper bounded by referring to a fixed set $B \subseteq \mathcal{E}$ that we specify later. For any $A \subseteq \mathcal{E}_t$, we define

$$\hat{h}_B(A) \triangleq h_g(B) + \sum_{e \in A \setminus B} \rho_h(e | B \cap \mathcal{E}_t) - \sum_{e \in B \setminus A} \rho_h(e | \mathcal{E}_t \setminus e) \geq h_g(A). \tag{3}$$

This inequality holds thanks to diminishing marginal costs, and the approximation is tight at B, $\hat{h}_B(B) = h_g(B)$. Up to a constant shift, this function is equivalent [17] to the edge weights:

$$\nu_B(e) = \rho_h(e | B \cap \mathcal{E}_t) \quad \text{if } e \in \mathcal{E}_t \setminus B; \qquad \text{and} \qquad \nu_B(e) = \rho_h(e | \mathcal{E}_t \setminus e) \quad \text{if } e \in B \cap \mathcal{E}_t. \tag{4}$$

Plugging ν_B into Equation (2) yields an approximation $\hat{f}$ of f. In the algorithm, B is always the boundary $B = \delta_s(T)$ of a set $T \subseteq (\mathcal{V} \cup \mathcal{U})$. Then $\mathcal{G}$ with weights ν_B represents

$$\begin{aligned}
\hat{f}(S) &= \min_{U \subseteq \mathcal{U}} \; \nu_B(\delta_s(S \cup U) \cap \mathcal{E}_m) + \nu_B(\delta_s(S \cup U) \cap \mathcal{E}_t) \\
&= \min_{U \subseteq \mathcal{U}} \; w(\delta_s(S \cup U) \cap \mathcal{E}_m) + \sum_{(u,t) \in \delta_s(S \cup U) \cap B} \rho_g(u | \mathcal{V} \cup \mathcal{U} \setminus u) + \sum_{(u,t) \in \delta_s(S \cup U) \setminus B} \rho_g(u | T).
\end{aligned}$$

Algorithm 1: Minimizing graph-based approximations.

create the representation graph $\mathcal{G} = (\mathcal{V} \cup \mathcal{U} \cup \{s, t\}, \mathcal{E})$ and set $S_0 = T_0 = \emptyset$;
for $i = 1, 2, \ldots$ **do**
  compute edge weights $\nu_{i-1} = \nu_{\delta_s(T_{i-1})}$ (Equation 4);
  find the (maximal) minimum (s,t)-cut $T_i = \operatorname{argmin}_{T \subseteq (\mathcal{V} \cup \mathcal{U})} \nu_{i-1}(\delta_s T)$;
  if $f(T_i) = f(T_{i-1})$ **then**
    return $S_i = T_i \cap \mathcal{V}$;
  end
end

Here, we used the definition $h_g(C) \triangleq g(V(C))$. Importantly, the edge weights ν_B are always nonnegative, because, by Theorem 1, g is guaranteed to be nondecreasing. Hence, we can efficiently minimize $\hat{f}$ as a standard minimum cut. If in Algorithm 1 there is more than one set T defining a minimum cut, then we pick the largest (i.e., maximal) such set. Lemma 3 states properties of the T_i.

Lemma 3. *Assume $\mathcal{G}$ is any of the graphs in Figure 3, and let $T^* \subseteq \mathcal{V} \cup \mathcal{U}$ be the maximal set defining a minimum-cost cut $\delta_s(T^*)$ in $\mathcal{G}$, so that $S^* = T^* \cap \mathcal{V}$ is a minimizer of the function represented by $\mathcal{G}$. Then, in any iteration i of Algorithm 1, it holds that $T_{i-1} \subseteq T_i \subseteq T^*$. In particular, $S \subseteq S^*$ for the returned solution S.*

Lemma 3 has three important implications. First, the algorithm never picks any element outside the maximal optimal solution. Second, because the T_i are growing, there are at most $|T^*| \leq |\mathcal{V} \cup \mathcal{U}|$ iterations, and the algorithm is strongly polynomial. Finally, the chain property permits more efficient implementations. The proof of Lemma 3 relies on the definition of ν and submodularity .

Proof. We prove the result for the case of a bipartite graph in Figure 3(e) with node sets $\mathcal{V}, \mathcal{U}$, and cost $f(A) = m(A) + g(\mathcal{N}(A))$ for $A \subseteq \mathcal{V}$. The proof carries over to the other submodular-weight graphs by identifying $\mathcal{V}$ with the right hand side of the bipartite graph. For a graph with completely modular edge weights, $\hat{f} = f$, and T_1 is an optimal solution.

Let $T_i \subseteq \mathcal{V} \cup \mathcal{U}$ be the selected nodes at iteration i, i.e., the nodes reachable from s in the min-cut. We first observe that the T_i form a chain, i.e., $T_i \subseteq T_{i+1}$ for all i. The reason lies in the adaptive edge weights. Let ν_{i-1} be the weights based on T_{i-1} by which T_i was chosen. Assume there was a nonempty set $Q = T_i \setminus T_{i+1}$, with $Q_\mathcal{V} = Q \cap \mathcal{V}$, $Q_\mathcal{U} = Q \cap \mathcal{U}$. Let $\delta^+ Q$ be the outgoing edges from Q to t that are cut if Q is selected, and let $\delta^- Q$ be the incoming edges from s to Q that are cut if Q is not selected. Since Q is a part of the optimal T_i, it must hold that

$$0 \geq \nu_{i-1}(\delta_s T_i) - \nu_{i-1}(\delta_s(T_i \setminus Q)) \tag{5}$$
$$= \nu_{i-1}(\delta^+ T_i) + \nu_{i-1}(\delta^-(\mathcal{V} \setminus T_i)) - (\nu_{i-1}(\delta^+(T_i \setminus Q_\mathcal{U}))) - \nu_{i-1}(\delta^-(\mathcal{V} \setminus (T_i \setminus Q_\mathcal{V}))) \tag{6}$$
$$= \nu_{i-1}(\delta^+ Q_\mathcal{U}) - \nu_{i-1}(\delta^- Q_\mathcal{V}). \tag{7}$$

In the next iteration, the weight for any $(u,t) \in \delta^+ Q_\mathcal{U}$ cannot increase: $\nu_i(u,t) = g(\mathcal{U}) - g(\mathcal{U} \setminus u) \leq \nu_{i-1}(u,t)$ by the property of diminishing marginal costs. As a result, $\nu_i(\delta^+ Q_\mathcal{U}) \leq \nu_{i-1}(\delta^+ Q_\mathcal{U}) \leq \nu_{i-1}(\delta^- Q_\mathcal{V}) = -m(Q_\mathcal{V}) = \nu_i(\delta^- Q_\mathcal{V})$. Thus, cutting $\delta^+ Q_\mathcal{U}$ instead of $\delta^- Q_\mathcal{V}$ and including Q in T_{i+1} can never increase the cost: As in Equations (5) to (7), it then holds that

$$\nu_{i-1}(\delta_s(T_{i+1} \cup Q)) - \nu_{i-1}(\delta_s T_{i+1}) = \nu_i(\delta^+ Q_\mathcal{U}) - \nu_i(\delta^- Q_\mathcal{V}) \leq 0 \tag{8}$$

This contradicts the maximality and optimality of S_{i+1}, thus, Q must be empty and $S_i \subseteq S_{i+1}$.

Next, we will see that all the T_i are subsets of T^*. Initially, the solution is $T_0 = \emptyset$, so clearly $T_0 \subseteq T^*$. Assume that i is the first iteration where $T_i \setminus T^* \neq \emptyset$, and let $Q = T_i \setminus T^*$. As before, we define $Q_\mathcal{U} = Q \cap \mathcal{U}$ and $Q_\mathcal{V} = Q \cap \mathcal{V}$. We will show that $g(Q_\mathcal{U} \cup T^*_\mathcal{U}) - m(\mathcal{V} \setminus (Q_\mathcal{V} \cup T^*_\mathcal{V})) \leq g(T^*_\mathcal{U}) - m(\mathcal{V} \setminus T^*_\mathcal{V})$, and then T^* cannot be the maximal optimal solution, a contradiction. By diminishing marginal costs, it holds that

$$f(S^* \cup Q_\mathcal{V}) - f(S^*) = g(Q_\mathcal{U} \cup T^*_\mathcal{U}) - m(\mathcal{V} \setminus (Q_\mathcal{V} \cup T^*_\mathcal{V})) - g(T^*_\mathcal{U}) + m(\mathcal{V} \setminus T^*_\mathcal{V}) \tag{9}$$
$$= \rho_g(Q_\mathcal{U} | T^*_\mathcal{U}) + m(Q) \tag{10}$$
$$\leq \rho_g(Q_\mathcal{U} | T_{i-1} \cap \mathcal{U}) + m(Q), \tag{11}$$

since $T_{i-1} \subseteq T^*$ by assumption, and thus $Q \cap T_{i-1} = \emptyset$. Note that $\rho_g(Q_\mathcal{U}|T_{i-1} \cap \mathcal{U}) \leq \sum_{u \in Q_\mathcal{U}} \rho_g(u|T_{i-1} \cap \mathcal{U})$: number the elements in $Q_\mathcal{U}$ arbitrarily. Then

$$\rho_g(Q_\mathcal{U}|T_{i-1} \cap \mathcal{U}) = \sum_k \rho_g(u_k|(T_{i-1} \cap \mathcal{U}) \cup u_1 \cup \ldots \cup u_{k-1}) \leq \sum_k \rho_g(u_k|T_{i-1} \cap \mathcal{U}) \quad (12)$$

by diminishing marginal costs. Thus, by the definition of ν_{i-1},

$$\begin{aligned}
&g(Q_\mathcal{U} \cup T^*_\mathcal{U}) - m(\mathcal{V} \setminus (Q_\mathcal{V} \cup T^*_\mathcal{V}) - g(T^*_\mathcal{U}) + m(\mathcal{V} \setminus (T^*_\mathcal{V})) \\
&\leq \sum_{u \in Q_\mathcal{U}} \rho_g(u|T_{i-1} \cap \mathcal{U}) + m(Q) \\
&= \sum_{e \in \delta^+ Q_\mathcal{U} \setminus \delta^+ T_{i-1}} \rho_h(e|\delta^+ T_{i-1}) + m(Q_\mathcal{V}) \\
&= \nu_{i-1}(\delta^+ Q_\mathcal{U}) - \nu_{i-1}(\delta^- Q_\mathcal{V}) \\
&= \nu_{i-1}(\delta_s T_i) - \nu_{i-1}(\delta_s(T_i \setminus Q)) \\
&\leq 0.
\end{aligned}$$

The last part follows like Equations (5) to (7) and the optimality of T_i. Hence, including Q in T^* would be optimal, and contradicts the maximality of T^*. Thus, Q must be empty. Finally, if $T_i \subseteq T^*$, then $S_i = (T_i \cap \mathcal{V}) \subseteq (T^* \cap \mathcal{V}) = S^*$. □

3.1 Improvement via summarizations

The approximation $\hat{f}$ is loosest if the sum of edge weights $\nu_i(A)$ significantly overestimates the true joint cost $h_g(A)$ of sets of edges $A \subseteq \delta_s T^* \setminus \delta_s T_i$ still to be cut. This happens if the joint marginal cost $\rho_h(A|\delta_s T_i)$ is much smaller than the estimated sum of weights, $\nu_i(A) = \sum_{e \in A} \rho_h(e|\delta_s T_i)$. Luckily, many of the functions we are interested in that show this behavior strongly resemble truncations. Thus, to tighten the approximation, we summarize the joint cost of groups of edges by a construction similar to Figure 3(c). Then the algorithm can take larger steps and pick groups of elements.

We partition $\mathcal{E}_t$ into disjoint groups G_k of edges (u, t). For each group, we introduce an auxiliary node t_k and re-connect all edges $(u, t) \in G_k$ to end in t_k instead of t. Their cost remains the same. An extra edge e_k connects t_k to t, and carries the joint weight $\nu_i(e_k)$ of all edges in G_k; a tighter approximation. The weight $\nu_i(e_k)$ is also adapted in each iteration. Initially, we set $\nu_0(e_k) = h_g(G_k) = g(V(G_k))$. Subsequent approximations ν_i refer to cuts $\delta_s T_i$, and such a cut can contain either single edges from G_k, or the group edge e_k. We set the next reference set B_i to be a copy of $\delta_s T_i$ in which each group edge e_k was replaced by all its group members G_k. The joint group weight $\nu_i(e_k)$ for any k is then $\nu_i(e_k) = \rho_h(G_k \setminus B_i|B_i) + \sum_{e \in G_k \cap B_i} \rho_h(e|\mathcal{E}_t \setminus e) \leq \sum_{e \in G_k} \nu_i(e)$. Formally, these weights represent the upper bound

$$\hat{h}'_B(A) = h_g(B) + \sum_{G_k \subseteq A} \rho_h(G_k \setminus B|B) + \sum_{e \in (G_k \cap A) \setminus B, G_k \not\subseteq A} \rho_h(e|B) - \sum_{e \in B \setminus A} \rho_h(e|\mathcal{E}_t \setminus e) \leq \hat{h}(A),$$

where we replace G_k by e_k whenever $G_k \subseteq A$. In our experiments, this summarization helps improve the results while simultaneously reducing running time.

3.2 An Approximation Bound

The following lemma shows that the worst-case solution of Algorithm 1 is bounded. Here, we assume f is given in the graph-represented form $f(S) = g(S) - m(\mathcal{V} \setminus S)$.

Lemma 4. *Let $S^* \subseteq \mathcal{V}$, and $T^* \subseteq \mathcal{V} \cup \mathcal{U}$ be defined as in Lemma 3. For the solution S returned by Algorithm 1, it holds that*

$$f(S) \leq \frac{|\delta_s T^*|}{1 + (|\delta_s T^*| - 1)\beta(T^*)} f(S^*) \quad \leq \quad |\delta_s T^*| f(S^*),$$

where $\beta(T^*) = \min_{u \in T^*} \rho_g(u|\mathcal{U} \setminus u) / \max_{u \in T^*} g(u)$.

The proof follows from Lemma 1, submodularity and the definition of h_g via g, and is then based on Lemma 3 in [17].

4 Parametric constructions for special cases

For certain functions of the form $f(S) = m(S) + g(\mathcal{N}(S))$, the graph representation in Figure 3(e) admits a specific algorithm. We use approximations that are exact on limited ranges, and eventually pick the best range. For this construction, g must have the form $g(U) = \psi(\sum_{u\in U} \tilde{w}(u))$ for weights $\tilde{w} \geq 0$ and one piecewise linear, concave function ψ with a small (polynomial) number ℓ of breakpoints. Alternatively, ψ can be *any* concave function if the weights $\tilde{w}$ are such that $\tilde{w}(U) = \sum_{u\in U} \tilde{w}(u)$ can take at most polynomially many distinct values x_k; e.g., if $\tilde{w}(u) = 1$ for all u, then effectively $\ell = |\mathcal{U}| + 1$ by using the x_k as breakpoints and interpolating. In all these cases, ψ is equivalent to the minimum of at most ℓ linear (modular) functions.

We build on the approach in [9], but, whereas their functions are defined on $\mathcal{V}$, g here is defined on $\mathcal{U}$. Contrary to their functions and owing to our decomposition, the ψ here is nondecreasing. We define ℓ linear functions, one for each breakpoint x_k (and use $x_0 = 0$):

$$\psi_k(t) = (\psi(x_k) - \psi(x_{k-1}))(t - x_k) + \psi(x_k) = \alpha_k t + \beta_k. \tag{13}$$

The ψ_k are defined such that $\psi(t) = \min_k \psi_k(t)$. Therefore, we approximate f by a series $\hat{f}_k(S) = -m(\mathcal{V} \setminus S) + \psi_k(\tilde{w}(\mathcal{N}(S)))$, and find the exact minimizer S_k for each k. To compute S_k via a minimum cut in $\mathcal{G}$ (Fig. 3(e)), we define edge weights $\nu_k(e) = w(e)$ for edges $e \notin \mathcal{E}_t$ as in Section 3, and $\nu_k(u, t) = \alpha_k \tilde{w}(u)$ for $e \in \mathcal{E}_t$. Then $T_k = S_k \cup \mathcal{N}(S_k)$ defines a minimum cut $\delta_s T_k$ in $\mathcal{G}$. We compute $\hat{f}_k(S_k) = \nu_k(\delta_s T_k) + \beta_k + m(\mathcal{V})$; the optimal solution is the S_k with minimum cost $\hat{f}_k(S_k)$. This method is exact. To solve for all k within one max-flow, we use a parametric max-flow method [11, 15]. Parametric max-flow usually works with edges both from s and to t. Here, $\nu_k \geq 0$ because ψ is nondecreasing, and thus we only need t-edges which already exist in the bipartite graph $\mathcal{G}$.

This method is limited to few breakpoints. For more general concave ψ and arbitrary $\tilde{w} \geq 0$, we can approximate ψ by a piecewise linear function. Still, the parametric approach does not directly generalize to more than one nonlinearity, e.g., $g(U) = \sum_i g_i(U \cap W_i)$ for sets $W_i \subseteq \mathcal{U}$. In contrast, Algorithm 1 (with the summarization) can handle all of these cases. We point out that without indirection via the bipartite graph, i.e., $f(S) = m(S) + \psi(w(S))$ for a ψ with few breakpoints, we can minimize f very simply: The solution for ψ_k includes all $j \in \mathcal{V}$ with $\alpha_k \leq -m(j)/w(j)$. The advantage of the graph cut is that it easily combines with other objectives.

5 Experiments

In the experiments, we test whether the graph-based methods improve over the minimum-norm point algorithm in the difficult cases of Section 1.1. We compare the following methods:
MN: a re-implementation of the minimum norm point algorithm in C++ that is about four times faster than the C code in [8] ; Figure 5 compares both implementations and shows that our results are not due to a slow implementation. The MN algorithm terminates if $\left|x^\top x^* - x^\top x\right| < \epsilon \max_{j\in S} ||x_j||^2$ [31], where $x \in B(f)$ is the current iterate, $x^* \in \operatorname{argmin}_{y\in B(f)} x^\top y$, and $x_j, j \in S$ is a set of extreme points of $B(f)$, and $\epsilon \geq 0$ is the "accuracy" or tolerance parameter. In general, we set $\epsilon = 10^{-10}$ to allow some round-off error.
MC: a minimum cut with static edge weights $\nu(e) = h_g(e)$;
GI: the graph-based iterative Algorithm 1, implemented in C++ with the max-flow code of [3], (i) by itself; (ii) with summarization via $\sqrt{|\mathcal{E}_t|}$ random groups (GIr); (iii) with summarization via groups generated by sorting the edges in $\mathcal{E}_t$ by their weights $h_g(e)$, and then forming groups G_k of edges adjacent in the order such that for each $e \in G_k$, $h_g(e) \leq 1.1 h_g(G_k)$ (GIs);
GP: the parametric method from Section 4, using $|\mathcal{E}_t|$ equispaced breakpoints; based on C code from RIOT[2].

We also implemented the SLG method from [29] in C++ (public code is not available), but found it to be impractical on the problems here (see Figure 4), as the gradient computation of our function requires finding gradients of $|\mathcal{U}|$ truncation functions, which is quite expensive . Thus, we did not include it in the tests on the large graphs. We use bipartite graphs of the form described in Section 1.1, with a cost function $f(S) = m(S) + \lambda g(\mathcal{N}(S))$. The function g uses a square root, $g(U) = \sqrt{w(U)}$.

[2] `http://riot.ieor.berkeley.edu/riot/Applications/Pseudoflow/parametric.html`

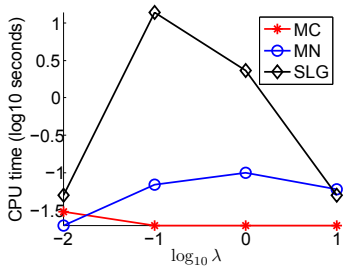


Figure 4: Running time of MN, MC and SLG with varying λ on a graph with $|\mathcal{V}| = 300$, $|\mathcal{U}| = 636$, and $f(S) = -m(S) + \lambda w(\mathcal{N}(S))$.

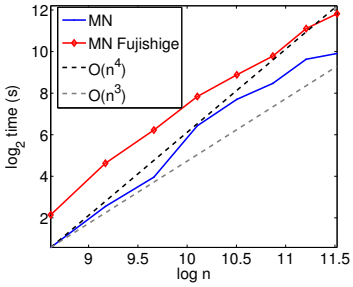


Figure 5: Comparison of our min-norm code with that by Fujishige on minimizing graphcut functions. The graphs were generated by GENRMF, and problem size refers to the number of nodes. The tolerance was set to 10^{-10}.

Solution quality with solution size. Running time and results depend on the size of S^*. Thus, we vary λ from 50 ($S^* \approx \mathcal{V}$) to 9600 ($S^* = \emptyset$) on a speech recognition data set [12]. In particular, we generated a bipartite graph, where the left nodes represent sentences, and right nodes represent words. A left node $u \in \mathcal{U}$ is connected to a right node $v \in \mathcal{V}$ only when the corresponding sentence contains the corresponding word. The bipartite graph represents a corpus subset extraction problem (Section 1.1), and has $|\mathcal{V}| = 54915$, $|\mathcal{U}| = 6871$ nodes, and uniform weights $w(u) = 1$ for all $u \in U$. The results look similar with non-uniform weights, but for uniform weights the parametric method from Section 4 always finds the optimal solution and thus allows us to report errors. Figure 6 shows the running times and the relative error $\mathrm{err}(S) = |f(S) - f(S^*)|/|f(S^*)|$ (note that $f(S^*) \leq 0$). If $f(S^*) = 0$, we report absolute errors. The running times were recorded on a machine with CPU 3.8GHz. Because of the large graph, we used the minimum-norm algorithm with accuracy 10^{-5}. Still, it takes up to 100 times longer than the other methods. It works well if S^* is large, but as λ grows, its accuracy becomes poor. In particular when $f(S^*) = f(\emptyset) = 0$, it returns large sets with large positive cost. In contrast, the deviation of the approximate edge weights ν_i from the true cost is bounded. All algorithms except MN return an optimal solution for $\lambda \geq 2000$. Updating the weights ν clearly improves the performance of Algorithm 1, as does the summarization (GIr/GIs perform identically here). With the latter, the solutions are very often optimal, and almost always very good.

Scaling: To test how the methods scale with the size $|\mathcal{V}|$, we sample small graphs from the big graph, and report average running times across 20 graphs for each size. As the graphs have non-uniform weights, we use GP as an approximation method and estimate the nonlinearity $\sqrt{w(U)}$ by a piecewise linear function with $|\mathcal{U}|$ breakpoints. All algorithms find the same (optimal) solution. Figure 7 shows that the minimum-norm algorithm with high accuracy is much slower than the other methods. Empirically, MN scales as up to $O(n^5)$ (note that Figure 1 is a *specific* worst-case graph), the parametric version approximately $O(n^2)$, and the variants of GI up to $O(n^{1.5})$. **Groups of Iwata's test function.** We also test running time and performance for a function that is supposed to be "ideal" for the minimum-norm algorithm [22]. For this function, suggested by Satoru Iwata, the elements are numbered $j = 1, \ldots, n \in \mathcal{V}$. Then $f_I(S) = |S||\mathcal{V} \setminus S| - \sum_{j \in S}(5j - 2n)$ [8]. We modify this function as follows: randomly assign the elements in $\mathcal{V}$ to m groups Q_i. The members of each Q_i

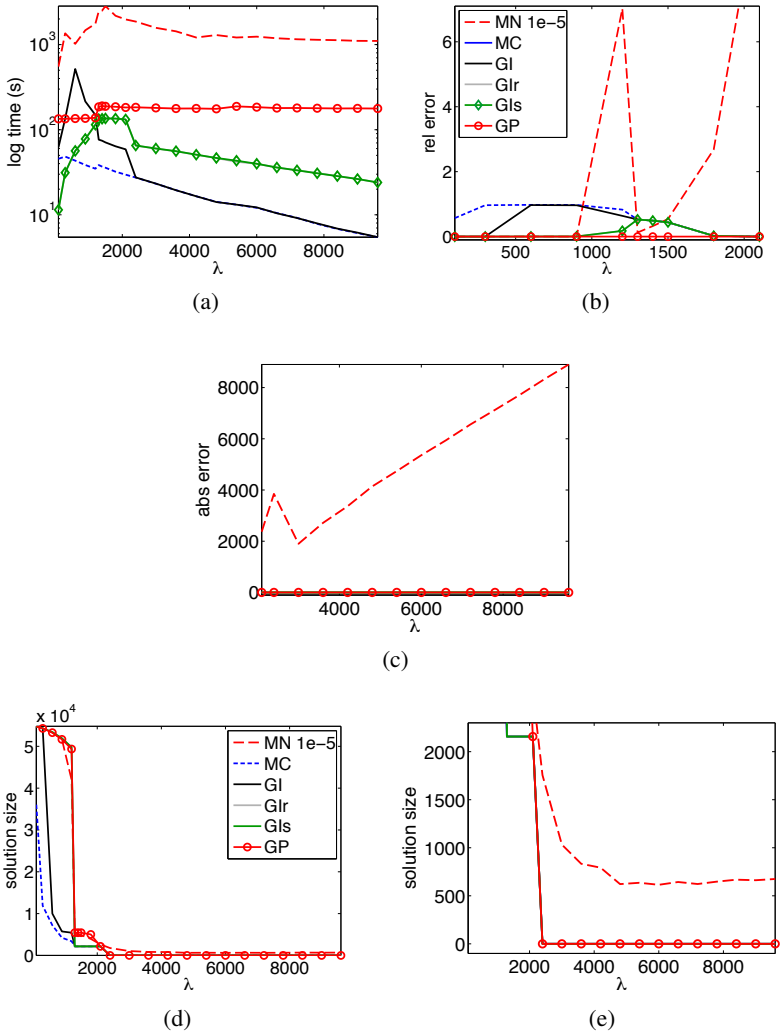


Figure 6: (a) Running time, (b) relative and (c) absolute error and (d) solution sizes with varying λ for a data set as described in Section 1.1, $|\mathcal{V}| = 54915$, $|\mathcal{U}| = 6871$, and $f(S) = -m(S) + \lambda\sqrt{|\mathcal{N}(S)|}$. Where $f(S^*) = 0$, we show absolute errors. (e) is a zoom in to (d) that shows that the minimum norm point algorithm returns solutions of size 500 and more when all other methods find the optimal (empty) set. (Figure best viewed in color.)

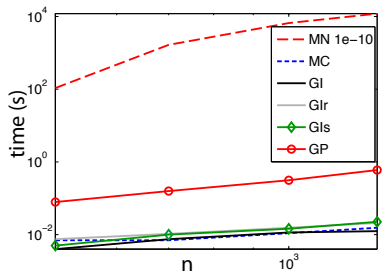
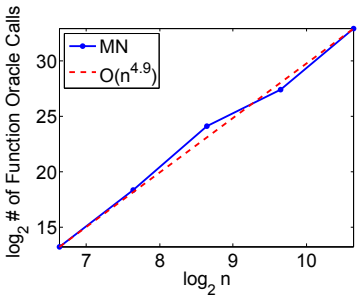


Figure 7: Running times with respect to $|\mathcal{V}|$, for $f(S) = -m(S) + \lambda\sqrt{w(\mathcal{N}(S))}$. Apart from MC, all solution qualities are similar. (Figure best viewed in color.)



(a)

Figure 8: Number of oracle function calls versus the ground set size for $f(S) = -|S| + 100\,(|\mathcal{N}(S)|)^{0.3}$. For each ground set size, a random graph was sampled from the big graph, and all edge weights were set to 1.

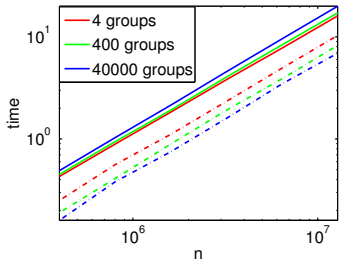


Figure 9: Groups with Iwata's test function. The solid lines are the running times of MN, the dashed lines of GI.

are now numbered 1 to $|Q_i|$, and we apply f_I group-wise: $f(S) = \sum_{i=1}^{m} f_I(S \cap Q_i)$. We compare Algorithm 1 to MN. Here, we implement the simplest version (Figure 3(i)), so that we actually do not need the graph explicitly, but for each element only compare whether $\nu(s, j) = -m(j) > \nu(j, t)$. Since the solutions form a chain, we never need to test any element again that was selected. Figure 9 shows that this implementation is even slightly faster than the minimum-norm algorithm, and scales similarly. In particular, the minimum-norm algorithm becomes slower as there are more groups, whereas Algorithm 1 becomes faster. Moreover, GI always finds the optimal solution.

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A The class of concave after modular functions

Theorem 2. *The class $\mathcal{F}_{conc}$ of submodular functions of the form $f(S) = \sum_i c_i \psi_i(w_i(S))$, where the w_i are nonnegative modular functions, $c_i \geq 0$, and the ψ_i are concave functions, is a strict subset of the class of all submodular functions.*

To show this result, we need another lemma.

Lemma 5. *Every submodular function that is a concatenation $f(S) = \psi(w(S))$ of a nondecreasing nonnegative modular function and a concave scalar function ψ can be represented as a graph cut (up to a constant).*

Before proving the lemma, we observe that it is sufficient to consider nondecreasing concave functions:

Proposition 1. *Every continuous concave function $\psi : [0, \theta] \to \mathbb{R}$ can be decomposed into a non-increasing linear function c and a nondecreasing concave function ϕ.*

This proposition a continuous analogue to Theorem 1.

Proof. (Proposition 1) If ψ is nondecreasing, we set $c \equiv 0$ and are done. Otherwise, we define a linear function $c(x) = \gamma x$ with $\gamma < f(x) - f(y) < 0$ for all $x \leq y \in [0, \theta]$. (If ψ is differentiable, we can use $\gamma = \psi'(\theta) < 0$). Then

$$\psi(x) = c(x) + (\psi - c)(x);$$

this implies that $\phi(x) = (\psi - c)(x)$. Owing to the definition of c, the function ϕ is nondecreasing: it holds that $\phi(x + h) - \phi(x) = \psi(x + h) - \psi(x) - c(h) \geq 0$ for any $x \in [0, \theta]$, and $0 \leq h \leq \theta - x$. As ϕ is the difference between a concave and a linear function, it is also concave. □

Proof. To prove Lemma 5, we construct a graph with terminal nodes s, t, and one node v_i for each element i in the ground set. We first use Proposition 1 to decompose f into a negative modular part, $m(S) = c(w(S))$, and a nondecreasing part $\phi(w(S))$. Without loss of generality, we can assume the submodular function f to be normalized. We represent the modular part by introducing edges (s, v_i) with weight $-c(w_i)$. (This will represent the function up to a shift, like the representation in Section 2.)

We observe that the only critical points of ϕ are those where it is actually evaluated; those are all the values that $w(S)$ can take on $2^{\mathcal{V}}$, at most 2^n many. Therefore we replace ϕ with a piecewise linear function φ that has a breakpoint at each value that $w(S)$ takes; and $\varphi(z) = \phi(z)$ at each such breakpoint z. (This function is merely the linear interpolation between the points $(z, \phi(z))$) Note that $\phi(w(S)) = \varphi(w(S))$ for all $S \subseteq \mathcal{V}$. We number the breakpoints $0 = z_0 < z_1 < \ldots < z_\ell$, and define the slopes $m_i = \frac{\phi(z_i) - \phi(z_{i-1})}{z_i - z_{i-1}}$. Then φ can equivalently be written as

$$\phi(x) = \sum_{i, z_i < x} m_i(z_i - z_{i-1}) + m_{i(x)}(x - z_{i(x)-1}), \tag{14}$$

where $i(x)$ is such that $x \in (z_{i(x)-1}, z_i(x)]$. Using Equation (14), we re-write φ as a sum of truncations. Beginning with $c_\ell = m_\ell$, we define constants

$$c_k = m_k - \sum_{i=k+1}^{\ell} c_i \geq 0.$$

Then we can write

$$\begin{aligned}
\sum_{k=1}^{\ell} c_k \min\{x, z_k\} &= \sum_{j < i(x)} c_j z_j + \sum_{j \geq i(x)} c_j x \\
&= \sum_{j < i(x)} (z_j - z_{j-1}) \sum_{i=j}^{\ell} c_i + \sum_{j \geq i(x)} c_j (x - z_{i(x)-1}) \\
&= \sum_{j < i(x)} (z_j - z_{j-1}) m_j + m_{i(x)}(x - z_{i(x)-1}) \\
&= \varphi(x).
\end{aligned}$$

We represent each function $c_k \min\{w(S), z_k\}$ by introducing an auxiliary node t_k and an edge (t_k, t) with weight $c_k z_k$ (as in Figure 3(c)). Moreover, we add edges (v_i, t_k) with weight $w(i)c_k$. Selecting S means to disconnect the corresponding nodes from t, that is on each truncation, we either cut the edges to t_k with weight $c_k w(S)$, or the edge (t_k, t) with weight $c_k z_k$. □

With this lemma and known results on representation capacities of graph cuts, the theorem follows straightforwardly.

Proof. (Thm. 2) By Lemma 5, all of the functions in $\mathcal{F}_{conc}$ can be represented as (potentially exponentially large) graph cuts. If all submodular functions were in $\mathcal{F}_{conc}$, then any submodular function could be represented as a graph cut, and this contradicts the results by [33] that some submodular functions cannot be represented by graph cuts.

Remark: Jan Vondrák has proved the same result by directly constructing a non-representable counterexample. □