Fast Semi-differential based Submodular Function Optimization

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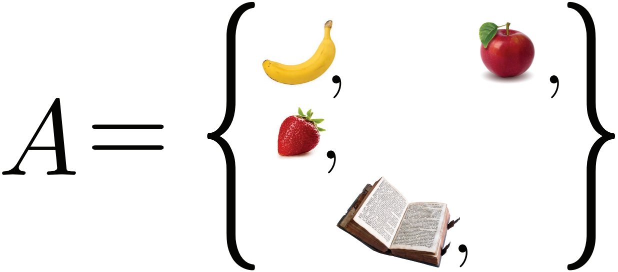
Outline

Set functions $f : 2^V \to \mathbb{R}$

$V =$

- V is a finite "ground" set of objects.

Set functions $f : 2^V \to \mathbb{R}$



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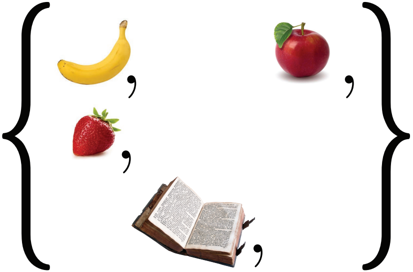
$A =$ 

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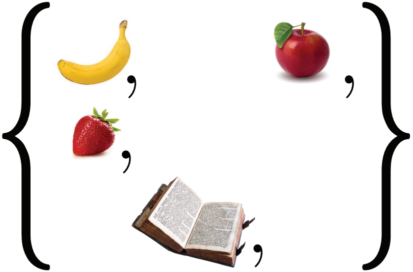
$A =$ 

- A set function $f : 2^V \to \mathbb{R}$ produces a value for any subset $A \subseteq V$.
- For example, $f(A) = 22$,
- General set function optimization can be really hard!

Submodular Set Functions

- Special class of set functions.

$$f(A \cup v) - f(A) \geq f(B \cup v) - f(B), \text{ if } A \subseteq B \tag{1}$$

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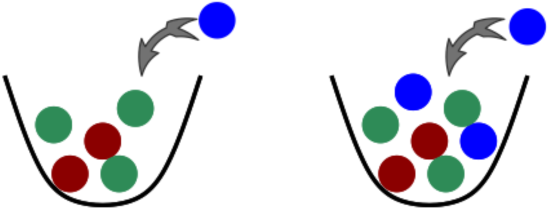

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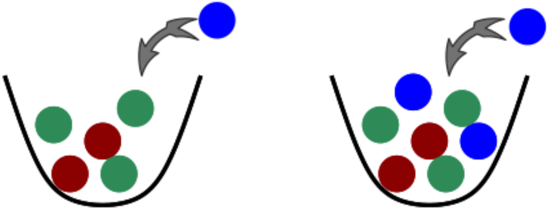

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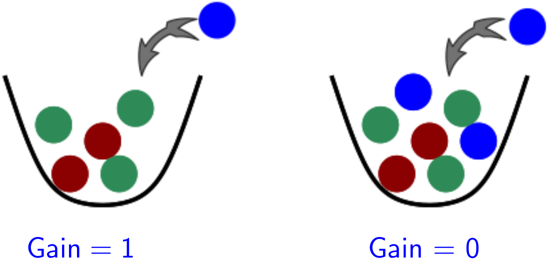

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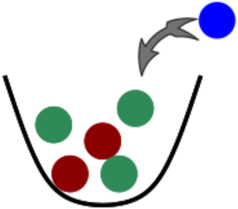

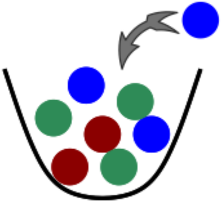

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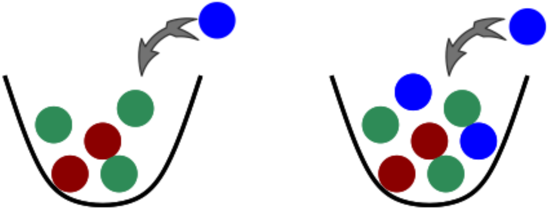

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- Monotonicity: $f(A) \leq f(B)$, if $A \subseteq B$.
- Modular function $f(X) = \sum_{i \in X} f(i)$ analogous to linear functions.

Submodular Function Maximization

$$\text{compute } A^* \in \operatorname*{argmax}_{A \in \mathcal{C}} f(A)$$

where f is submodular, and where $\mathcal{C}$ is constraint set over which a modular function can be optimized efficiently.

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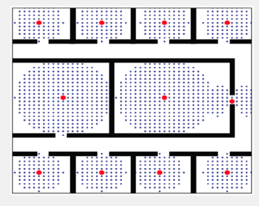

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(Krause et al, 2008)

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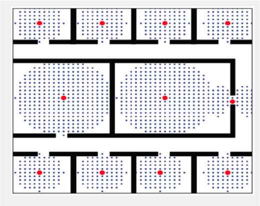

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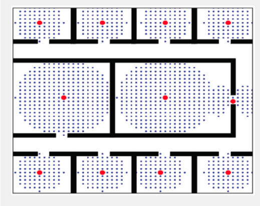

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Diversified Search (He et al 2012, Kulesza & Taskar, 2012)

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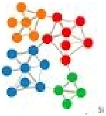

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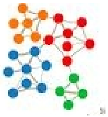

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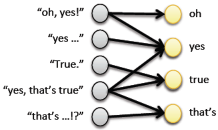

Corpus Data Subset Selection (Lin & Bilmes, 2011)

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Contribution: We present the first unifying framework for submodular minimization & maximization. Our framework is scalable to large data.

Convex/Concave and Semigradients

- A convex function ϕ has a subgradient h_y and linear lower bound:

$$\phi(y) + \langle h_y, x - y \rangle \leq \phi(x), \forall x.$$

- A concave function ψ has a supergradient g_y and linear upper bound:

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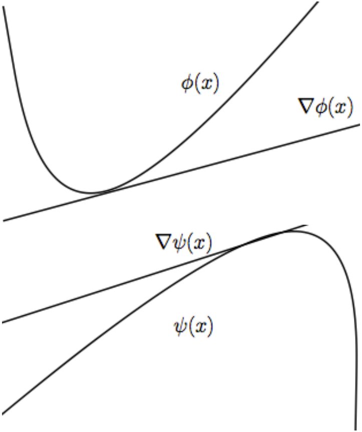

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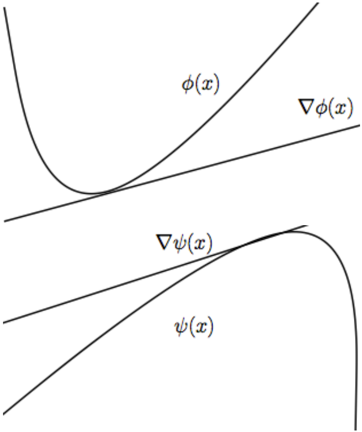

- Submodular functions have properties analogous to convexity and concavity.

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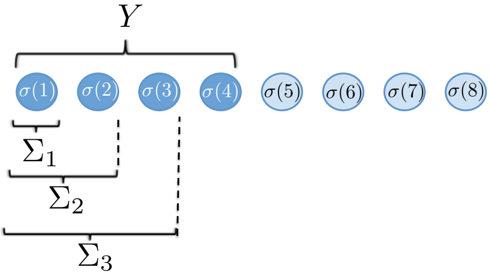

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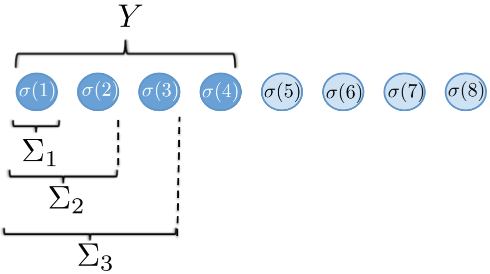

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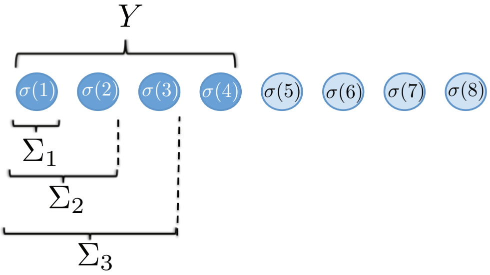

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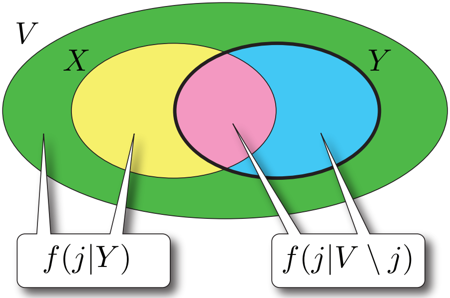
Grow:

$$\hat{g}_Y(j) = \begin{cases} f(j|Y) & \text{for } j \notin Y \\ f(j|V \backslash \{j\}) & \text{for } j \in Y \end{cases}$$



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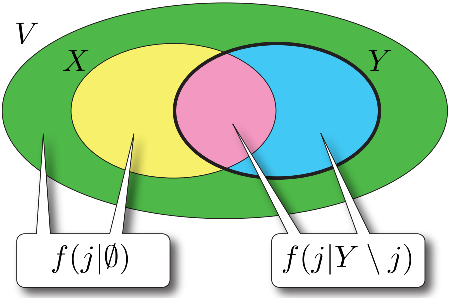

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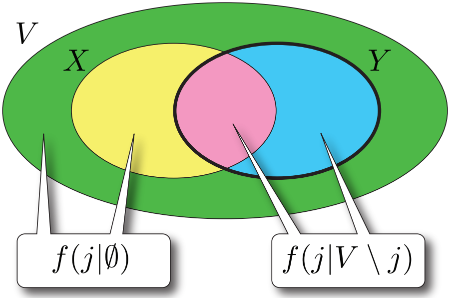

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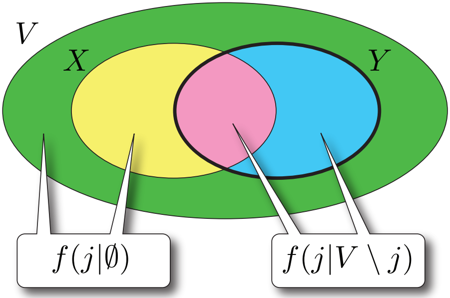

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Lemma: Algorithm 1 monotonically improves the objective function value for submodular maximization and minimization at every iteration.

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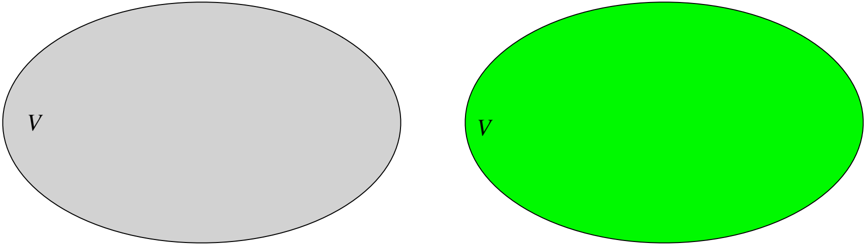
$$A \subseteq A_+ \subseteq X^* \subseteq B_+ \subseteq B$$

Illustrating Unconstrained Minimization



MMin-I   MMin-II

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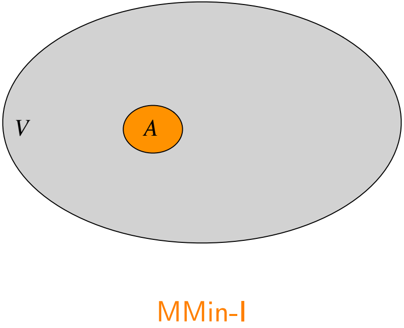

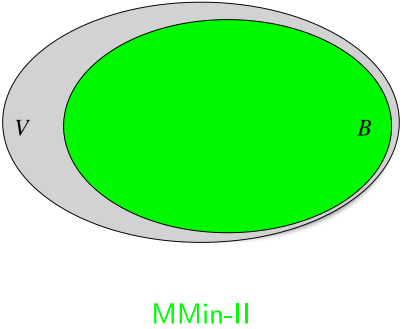

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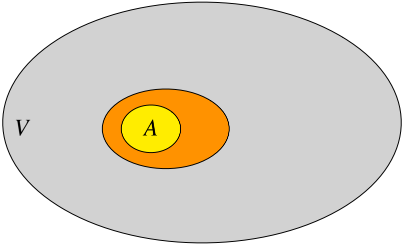

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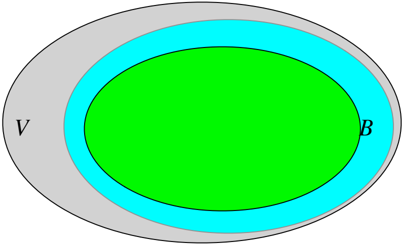

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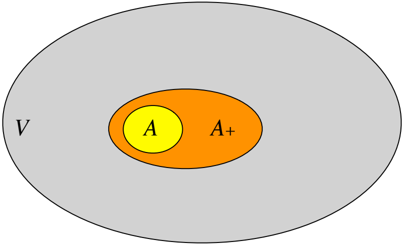

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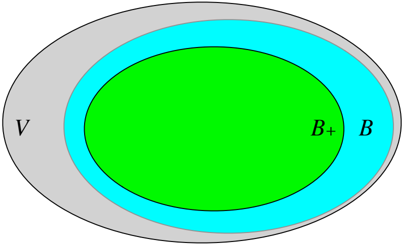
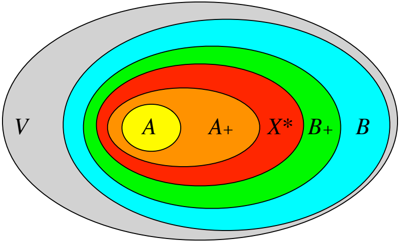


MMin-II

Illustrating Unconstrained Minimization



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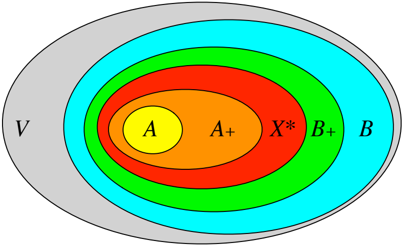
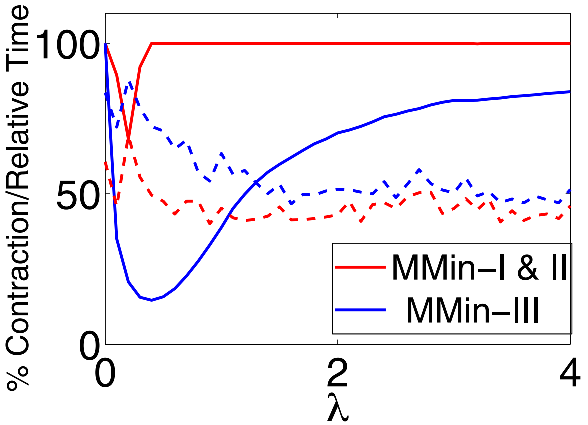


MMin-II

Empirical Results: Submodular Minimization

Test function: concave over modular, $\sqrt{w_1(X)} + \lambda w_2(V \backslash X)$.



Lattice reduction (solid line), and runtime reduction (dotted line).

Note: results for Bipartite Neighborhoods shown in paper.

Constrained Submodular Minimization

- Curvature of a monotone submodular function:

$$\kappa_f(X) \triangleq 1 - \min_j \frac{f(j|X \backslash j)}{f(j)}. \tag{2}$$

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Theorem

The solution $\widehat{X}$ returned by MMin-I satisfies:

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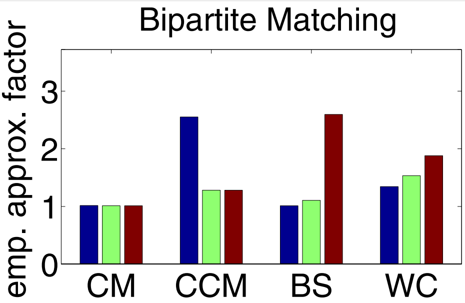

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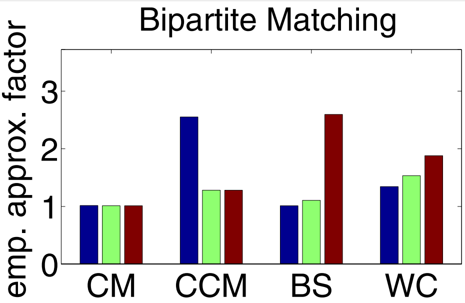

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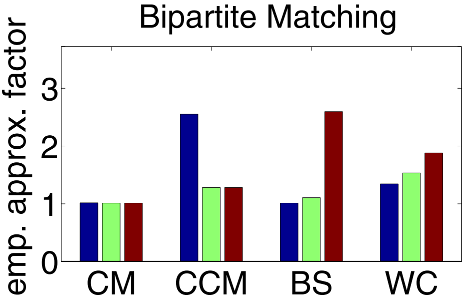

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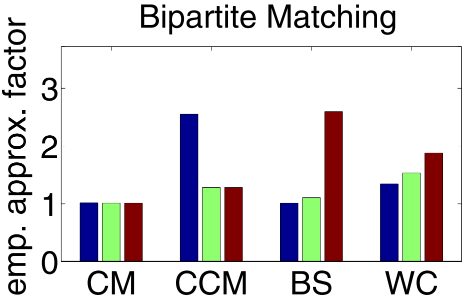

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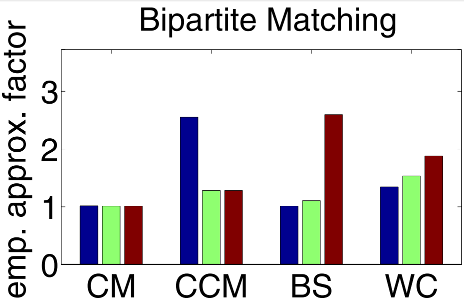

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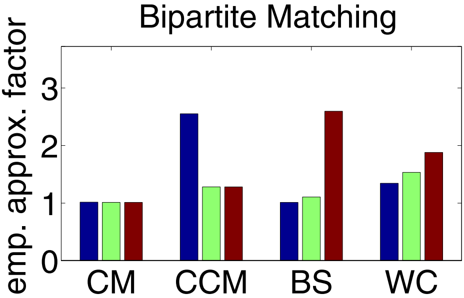

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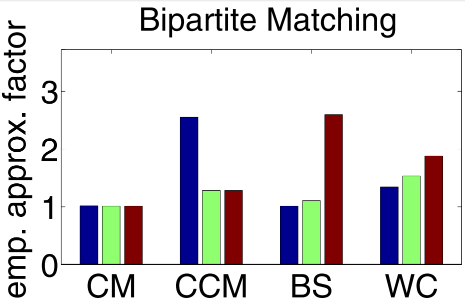

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- Submodular spanning tree & shortest path results given in paper.

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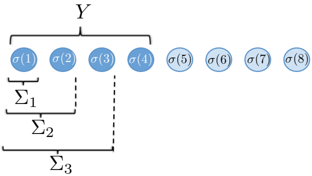
Constrained Maximization and Extensions



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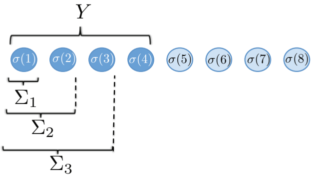

- Greedy subgradient for monotone submodular functions:

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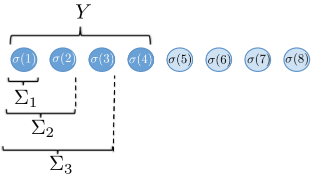

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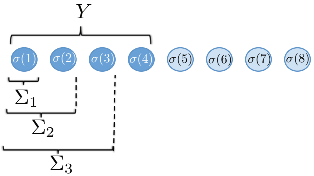

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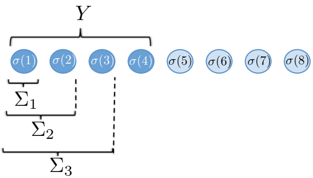

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Generality of Algorithm MMax: For every α-approximation algorithm, there exists a schedule of subgradients obtainable in poly-time, such that Algorithm 1 (MMax) achieves an approximation factor of at least α.

Summary

- Submodular functions in machine learning.
- A generic sub-gradient ascent [super-gradient descent] framework for submodular maximization [minimization].
- The first unifying framework for general submodular optimization.
- New theoretical results for unconstrained and constrained submodular minimization.
- A novel view as a framework for submodular maximization and subsuming number of existing algorithms.
- Empirical experimental validation.

Thank You

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Questions please.