Logistics

Cumulative Outstanding Reading

- Read chapters 2 and 3 from Fujishige's book.
- Read chapter 1 from Fujishige’s book.
Announcements, Assignments, and Reminders

- Homework 3, available at our assignment dropbox (https://canvas.uw.edu/courses/1039754/assignments), due (electronically) Monday (5/2) at 11:55pm.
- Homework 1, available at our assignment dropbox (https://canvas.uw.edu/courses/1039754/assignments), due (electronically) Friday (4/8) at 11:55pm.
- Weekly Office Hours: Mondays, 3:30-4:30, or by skype or google hangout (set up meeting via our discussion board (https://canvas.uw.edu/courses/1039754/discussion_topics)).
Spanning Sets

- We have the following definitions:

**Definition 9.2.4 (spanning set of a set)**

Given a matroid $M = (V, \mathcal{I})$, and a set $Y \subseteq V$, then any set $X \subseteq Y$ such that $r(X) = r(Y)$ is called a spanning set of $Y$.

**Definition 9.2.5 (spanning set of a matroid)**

Given a matroid $M = (V, \mathcal{I})$, any set $A \subseteq V$ such that $r(A) = r(V)$ is called a spanning set of the matroid.

- A base of a matroid is a minimal spanning set (and it is independent) but supersets of a base are also spanning.
- $V$ is always trivially spanning.
- Consider the terminology: “spanning tree in a graph”, comes from spanning in a matroid sense.

Dual of a Matroid

- Given a matroid $M = (V, \mathcal{I})$, a dual matroid $M^* = (V, \mathcal{I}^*)$ can be defined on the same ground set $V$, but using a very different set of independent sets $\mathcal{I}^*$.
- We define the set of sets $\mathcal{I}^*$ for $M^*$ as follows:

$$\mathcal{I}^* = \{ A \subseteq V : V \setminus A \text{ is a spanning set of } M \} \quad (9.11)$$

$$= \{ V \setminus S : S \subseteq V \text{ is a spanning set of } M \} \quad (9.12)$$

i.e., $\mathcal{I}^*$ are complements of spanning sets of $M$.
- That is, a set $A$ is independent in the dual matroid $M^*$ if removal of $A$ from $V$ does not decrease the rank in $M$:

$$\mathcal{I}^* = \{ A \subseteq V : \text{rank}_M(V \setminus A) = \text{rank}_M(V) \} \quad (9.13)$$

- In other words, a set $A \subseteq V$ is independent in the dual $M^*$ (i.e., $A \in \mathcal{I}^*$) if its complement is spanning in $M$ (residual $V \setminus A$ must contain a base in $M$).
- Dual of the dual: Note, we have that $(M^*)^* = M$. 

Prof. Jeff Bilmes
Dual of a Matroid: Bases

- The smallest spanning sets are bases. Hence, a base $B$ of $M$ (where $B = V \setminus B^*$ is as small as possible while still spanning) is the complement of a base $B^*$ of $M^*$ (where $B^* = V \setminus B$ is as large as possible while still being independent).
- In fact, we have that

**Theorem 9.2.4 (Dual matroid bases)**

Let $M = (V, \mathcal{I})$ be a matroid and $\mathcal{B}(M)$ be the set of bases of $M$. Then define

$$B^*(M) = \{V \setminus B : B \in \mathcal{B}(M)\}. \quad (9.11)$$

Then $B^*(M)$ is the set of basis of $M^*$ (that is, $B^*(M) = \mathcal{B}(M^*)$).

Example: cocycle matroid (sometimes “cut matroid”)

- The dual of the cycle matroid is called the cocycle matroid. Recall, $\mathcal{I}^* = \{A \subseteq V : V \setminus A$ is a spanning set of $M\}$
- It consists of all sets of edges the complement of which contains a spanning tree — i.e., an independent set can’t consist of edges that, if removed, would render the graph non-spanning.
The dual of a matroid is (indeed) a matroid

**Theorem 9.3.1**

Given matroid \( M = (V, \mathcal{I}) \), let \( M^* = (V, \mathcal{I}^*) \) be as previously defined. Then \( M^* \) is a matroid.

**Proof.**

- Clearly \( \emptyset \in \mathcal{I}^* \), so (I1') holds.
- Also, if \( I \subseteq J \in \mathcal{I}^* \), then clearly also \( I \in \mathcal{I}^* \) since if \( V \setminus J \) is spanning in \( M \), so must \( V \setminus I \). Therefore, (I2') holds.
- Next, given \( I, J \in \mathcal{I}^* \) with \( |I| < |J| \), it must be the case that \( \overline{I} = V \setminus I \) and \( \overline{J} = V \setminus J \) are both spanning in \( M \) with \( |\overline{I}| > |\overline{J}| \).
The dual of a matroid is (indeed) a matroid

**Theorem 9.3.1**

Given matroid $M = (V, I)$, let $M^* = (V, I^*)$ be as previously defined. Then $M^*$ is a matroid.

**Proof.**

- Now $J \setminus I \not\subseteq B_I$, since otherwise (i.e., assuming $J \setminus I \subseteq B_I$):

\[
|B_J| = |B_J \cap I| + |B_J \setminus I| \tag{9.1}
\]
\[
\leq |I \setminus J| + |B_J \setminus I| \tag{9.2}
\]
\[
< |J \setminus I| + |B_J \setminus I| \leq |B_I| \tag{9.3}
\]

which is a contradiction. *The last inequality on the right follows since $J \setminus I \subseteq B_I$ (by assumption) and $B_J \setminus I \subseteq B_I$ implies that $(J \setminus I) \cup (B_J \setminus I) \subseteq B_I$,* but since $J$ and $B_J$ are disjoint, we have that $|J \setminus I| + |B_J \setminus I| \leq |B_I|$.  

- Therefore, $J \setminus I \not\subseteq B_I$, and there is a $v \in J \setminus I$ s.t. $v \notin B_I$.

- So $B_I$ is disjoint with $I \cup \{v\}$, means $B_I \subseteq V \setminus (I \cup \{v\})$, or

Matroid Duals and Representability

**Theorem 9.3.2**

Let $M$ be an $\mathbb{F}$-representable matroid (i.e., one that can be represented by a finite sized matrix over field $\mathbb{F}$). Then $M^*$ is also $\mathbb{F}$-representable.

Hence, for matroids as general as matric matroids, duality does not extend the space of matroids that can be used.

**Theorem 9.3.3**

Let $M$ be a graphic matroid (i.e., one that can be represented by a graph $G = (V, E)$). Then $M^*$ is not necessarily also graphic.

Hence, for graphic matroids, duality can increase the space and power of matroids, and since they are based on a graph, they are relatively easy to use: 1) all cuts are dependent sets; 2) minimal cuts are cycles; 3) bases are one edge less than minimal cuts; and 4) independent sets are edges that are not cuts (minimal or otherwise).
Theorem 9.3.4

The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified in terms of the rank $r_M$ in matroid $M$ as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \quad (9.4)$$

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2. I.e., $|X|$ is modular, complement $f(V \setminus X)$ is submodular if $f$ is submodular, $r_M(V)$ is a constant, and summing submodular functions and a constant preserves submodularity.
- Non-negativity integral follows since $|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V)$. The right inequality follows since $r_M$ is submodular.
- Monotone non-decreasing follows since, as $X$ increases by one, $|X|$ always increases by 1, while $r_M(V \setminus X)$ decreases by one or zero.
- Therefore, $r_{M^*}$ is the rank function of a matroid. That it is the dual...
**Matroid restriction/deletion**

- Let \( M = (V, \mathcal{I}) \) be a matroid and let \( Y \subseteq V \), then
  \[
  \mathcal{I}_Y = \{ Z : Z \subseteq Y, Z \in \mathcal{I} \} \tag{9.7}
  \]
  is such that \( M_Y = (Y, \mathcal{I}_Y) \) is a matroid with rank \( r(M_Y) = r(Y) \).
- This is called the restriction of \( M \) to \( Y \), and is often written \( M|Y \).
- If \( Y = V \setminus X \), then we have that \( M|Y \) has the form:
  \[
  \mathcal{I}_Y = \{ Z : Z \cap X = \emptyset, Z \in \mathcal{I} \} \tag{9.8}
  \]
  is considered a deletion of \( X \) from \( M \), and is often written \( M \setminus X \).
- Hence, \( M|Y = M \setminus (V \setminus Y) \), and \( M|(V \setminus X) = M \setminus X \).
- The rank function is of the same form. i.e., \( r_Y : 2^Y \to Z_+ \), where \( r_Y(Z) = r(Z) \) for \( Z \subseteq Y \).

**Matroid contraction \( M/Z \)**

- Contraction by \( Z \) is dual to deletion, and is like a forced inclusion of a contained base \( B_Z \) of \( Z \), but with a similar ground set removal by \( Z \). Contracting \( Z \) is written \( M/Z \). Updated ground set in \( M/Z \) is \( V \setminus Z \).
- Let \( Z \subseteq V \) and let \( B_Z \) be a base of \( Z \). Then a subset \( I \subseteq V \setminus Z \) is independent in \( M/Z \) iff \( I \cup B_Z \) is independent in \( M \).
- The rank function takes the form
  \[
  r_{M/Z}(Y) = r(Y \cup Z) - r(Z) = r(Y|Z) \tag{9.9}
  \]
  \[
  = r(Y \cup B_Z) - r(B_Z) \tag{9.10}
  \]
- So given \( I \subseteq V \setminus Z \) and \( B_Z \) is a base of \( Z \), \( r_{M/Z}(I) = |I| \) is identical to \( r(I \cup Z) = |I| + r(Z) = |I| + |B_Z| \) but \( r(I \cup Z) = r(I \cup B_Z) \). This implies \( r(I \cup B_Z) = |I| + |B_Z| \), or \( I \cup B_Z \) is independent in \( M \).
- A minor of a matroid is any matroid obtained via a series of deletions and contractions of some matroid.
- In fact, it is the case \( M/Z = (M^* \setminus Z)^* \) (Exercise: show why).
**Matroid Intersection**

- Let $M_1 = (V, I_1)$ and $M_2 = (V, I_2)$ be two matroids. Consider their common independent sets $I_1 \cap I_2$.
- While $(V, I_1 \cap I_2)$ is typically not a matroid (Exercise: show graphical example.), we might be interested in finding the maximum size common independent set. That is, find $\max |X|$ such that both $X \in I_1$ and $X \in I_2$.

**Theorem 9.4.1**

Let $M_1$ and $M_2$ be given as above, with rank functions $r_1$ and $r_2$. Then the size of the maximum size set in $I_1 \cap I_2$ is given by

$$(r_1 \ast r_2)(V) \triangleq \min_{X \subseteq V} \left( r_1(X) + r_2(V \setminus X) \right) \tag{9.11}$$

This is an instance of the convolution of two submodular functions, $f_1$ and $f_2$ that, evaluated at $Y \subseteq V$, is written as:

$$(f_1 \ast f_2)(Y) = \min_{X \subseteq Y} \left( f_1(X) + f_2(Y \setminus X) \right) \tag{9.12}$$

**Convolution and Hall’s Theorem**

- Recall Hall’s theorem, that a transversal exists iff for all $X \subseteq V$, we have $|\Gamma(X)| \geq |X|$.
  - $\iff |\Gamma(X)| - |X| \geq 0, \forall X$
  - $\iff \min_X |\Gamma(X)| - |X| \geq 0$
  - $\iff \min_X |\Gamma(X)| + |V| - |X| \geq |V|$
  - $\iff \min_X \left( |\Gamma(X)| + |V \setminus X| \right) \geq |V|$
  - $\iff [\Gamma(\cdot) \ast |\cdot|](V) \geq |V|$
- So Hall’s theorem can be expressed as convolution. Exercise: define $g(A) = [\Gamma(\cdot) \ast |\cdot|](A)$, prove that $g$ is submodular.
- Note, in general, convolution of two submodular functions does not preserve submodularity (but in certain special cases it does).
**Matroid Union**

**Definition 9.4.2**

Let $M_1 = (V_1, \mathcal{I}_1), M_2 = (V_2, \mathcal{I}_2), \ldots, M_k = (V_k, \mathcal{I}_k)$ be matroids. We define the union of matroids as

$M_1 \cup M_2 \cup \cdots \cup M_k = (V_1 \uplus V_2 \uplus \cdots \uplus V_k, \mathcal{I}_1 \cup \mathcal{I}_2 \cup \cdots \cup \mathcal{I}_k)$,

where

$I_1 \cup I_2 \cup \cdots \cup I_k = \{I_1 \uplus I_2 \uplus \cdots \uplus I_k | I_1 \in \mathcal{I}_1, \ldots, I_k \in \mathcal{I}_k\}$ \hspace{1cm} (9.13)

Note $A \uplus B$ designates the disjoint union of $A$ and $B$.

**Theorem 9.4.3**

Let $M_1 = (V_1, \mathcal{I}_1), M_2 = (V_2, \mathcal{I}_2), \ldots, M_k = (V_k, \mathcal{I}_k)$ be matroids, with rank functions $r_1, \ldots, r_k$. Then the union of these matroids is still a matroid, having rank function

$r(Y) = \min_{X \subseteq Y} \left( |Y \setminus X| + r_1(X \cap V_1) + \cdots + r_k(X \cap V_k) \right)$ \hspace{1cm} (9.14)

for any $Y \subseteq V_1 \uplus \cdots \uplus V_2 \uplus \cdots \uplus V_k$.

**Exercise: Matroid Union, and Matroid duality**

Exercise: Fully characterize $M \cup M^*$. 
Matroids of three or fewer elements are graphic

- All matroids up to and including three elements (edges) are graphic.

(a) The only matroid with zero elements.
(b) The two one-element matroids.
(c) The four two-element matroids.
(d) The eight three-element matroids.

This is a nice way to visualize matroids with very low ground set sizes. What about matroids that are low rank but with many elements?

Affine Matroids

- Given an $n \times m$ matrix with entries over some field $\mathbb{F}$, we say that a subset $S \subseteq \{1, \ldots, m\}$ of indices (with corresponding column vectors $\{v_i : i \in S\}$, with $|S| = k$) is **affinely dependent** if $m \geq 1$ and there exists elements $\{a_1, \ldots, a_k\} \in \mathbb{F}$, not all zero with $\sum_{i=1}^{k} a_i = 0$, such that $\sum_{i=1}^{k} a_i v_i = 0$.

- Otherwise, the set is called **affinely independent**.

- Concisely: points $\{v_1, v_2, \ldots, v_k\}$ are affinely independent if $v_2 - v_1, v_3 - v_1, \ldots, v_k - v_1$ are linearly independent.

- Example: in 2D, three collinear points are affinely independent, three non-colinear points are affinely independent, and $\geq 4$ collinear or non-collinear points are affinely dependent.

**Proposition 9.5.1 (affine matroid)**

Let ground set $E = \{1, \ldots, m\}$ index column vectors of a matrix, and let $\mathcal{I}$ be the set of subsets $X$ of $E$ such that $X$ indices affinely independent vectors. Then $(E, \mathcal{I})$ is a matroid.

**Exercise:** prove this.
Euclidean Representation of Low-rank Matroids

Consider the affine matroid with $n \times m = 2 \times 6$ matrix on the field $\mathbb{F} = \mathbb{R}$, and let the elements be $\{(0, 0), (1, 0), (2, 0), (0, 1), (0, 2), (1, 1)\}$.

We can plot the points in $\mathbb{R}^2$ as on the right:

- Points have rank 1, lines have rank 2, planes have rank 3.
- Flats (points, lines, planes, etc.) have rank equal to one more than their geometric dimension.
- Any two points constitute a line, but lines with only two points are not drawn.
- Lines indicate collinear sets with $\geq 3$ points, while any two points have rank 2.
- Dependent sets consist of all subsets with $\geq 4$ elements (rank 3), or 3 collinear elements (rank 2). Any two points have rank 2.

As another example on the right, a rank 4 matroid

- All sets of 5 points are dependent. The only other sets of dependent points are coplanar ones of size 4. Namely:
  - $\{(0, 0, 0), (0, 1, 0), (1, 1, 0), (1, 0, 0)\}$,
  - $\{(0, 0, 0), (0, 0, 1), (0, 1, 1), (0, 1, 0)\}$, and
  - $\{(0, 0, 1), (0, 1, 1), (1, 1, 0), (1, 0, 0)\}$.
Euclidean Representation of Low-rank Matroids

- In general, for a matroid $\mathcal{M}$ of rank $m + 1$ with $m \leq 3$, then a subset $X$ in a geometric representation in $\mathbb{R}^m$ is dependent if:
  1. $|X| \geq 2$ and the points are identical;
  2. $|X| \geq 3$ and the points are collinear;
  3. $|X| \geq 4$ and the points are coplanar; or
  4. $|X| \geq 5$ and the points are anywhere in space.

- When they exist, loops are represented in a geometry by a separate box indicating how many loops there are.

- Parallel elements, when they exist in a matroid, are indicated by a multiplicity next to a point.

Theorem 9.5.2

Any matroid of rank $m \leq 4$ can be represented by an affine matroid in $\mathbb{R}^{m-1}$.

Euclidean Rep. of Low-rank Matroids: Conditions

- rank-1 (resp. rank-2, rank-3) flats correspond to points (resp. lines, planes).
- a set of parallel points (could be size 1) does not touch another set of parallel points (could be size 1).
- every line contains at least two points (not dependent unless $> 2$).
- any two distinct points lie on a line (often not drawn when only two)
- every plane contains at least three non-collinear points (not dependent unless $> 3$).
- any three distinct non-collinear points lie on a plane.
- If diagram has at most one plane, then any two distinct lines meet in at most one point.
- If diagram has more than one plane, then: 1) any two distinct planes meeting in more than two points do so in a line; 2) any two distinct lines meeting in a point do so in at most one point and lie in on a common plane; 3) any line not lying on a plane intersects it in at most one point.
- (see Oxley 2011 for more details).
Euclidean Representation of Low-rank Matroids

- Very useful for graphically depicting low-rank matrices but which still have rich structure. Also useful for answering questions.
- Example: Is there a matroid that is not representable (i.e., not linear for some field)? Yes, consider the matroid

```
1 2 3 4 5 6 7 8 9
```

Called the non-Pappus matroid. Has rank three, but any matric matroid with the above dependencies would require that \{7, 8, 9\} is dependent, hence requiring an additional line in the above.

Euclidean Representation of Low-rank Matroids: A test

- Is this a matroid?

- Check rank’s submodularity: Let \( X = \{1, 2, 3, 6, 7\}, \ Y = \{1, 4, 5, 6, 7\} \). So \( r(X) = 3 \), and \( r(Y) = 3 \), and \( r(X \cup Y) = 4 \), so we must have, by submodularity, that

\[
r(\{1, 6, 7\}) = r(X \cap Y) \leq r(X) + r(Y) - r(X \cup Y) = 2.
\]

- However, from the diagram, we have that since 1, 6, 7 are distinct non-collinear points, we have that \( r(X \cap Y) = 3 \)
**Euclidean Representation of Low-rank Matroids: A test**

- Is this a matroid?
- If we extend the line from 6-7 to 1, then is it a matroid?
- Hence, not all 2D or 3D graphs of points and lines are matroids.

**Matroid?**

- Consider the following geometry on $|V| = 8$ points with $V = \{a, b, c, d, e, f, g, h\}$.
- Note, we are given that the points $\{b, d, h, f\}$ are not coplanar. However, the following sets of points are coplanar: $\{a, b, e, f\}$, $\{d, c, g, h\}$, $\{a, d, h, e\}$, $\{b, c, g, f\}$, $\{b, c, d, a\}$, $\{f, g, h, e\}$, and $\{a, c, g, e\}$.
- Exercise: Is this a matroid? Exercise: If so, is it representable?
Other examples can be more complex, consider the following two matroids (from Oxley, 2011):

- Right: a matroid (and a 2D depiction of a geometry) over the field $\text{GF}(3) = \{0, 1, 2\} \mod 3$ and is "coordinatizable" in $\text{GF}(3)^3$.
- Hence, lines (in 2D) which are rank 2 sets may be curved; planes (in 3D) can be twisted.

Matroids, Representation and Equivalence: Summary

- Matroids with $|V| \leq 3$ are graphic.
- Matroids with $r(V) \leq 4$ can be geometrically represented in $\mathbb{R}^3$.
- Not all matroids are linear (i.e., matric) matroids.
- Matroids can be seen as related to projective geometries (and are sometimes called combinatorial geometries).
- Exists much research on different subclasses of matroids, and if/when they are contained in (or isomorphic to) each other.
Matroid Further Reading

- “The Coming of the Matroids”, William Cunningham, 2012 (a nice history)
- Crapo & Rota, “On the Foundations of Combinatorial Theory: Combinatorial Geometries”, 1970 (while this is old, it is very readable).
- Schrijver, “Combinatorial Optimization”, 2003

The greedy algorithm

- In combinatorial optimization, the greedy algorithm is often useful as a heuristic that can work quite well in practice.
- The goal is to choose a good subset of items, and the fundamental tenet of the greedy algorithm is to choose next whatever currently looks best, without the possibility of later recall or backtracking.
- Sometimes, this gives the optimal solution (we saw three greedy algorithms that can find the maximum weight spanning tree).
- Greedy is good since it can be made to run very fast $O(n \log n)$.
- Often, however, greedy is heuristic (it might work well in practice, but worst-case performance can be unboundedly poor).
- We will next see that the greedy algorithm working optimally is a defining property of a matroid, and is also a defining property of a polymatroid function.
Matroid and the greedy algorithm

- Let \((E, \mathcal{I})\) be an independence system, and we are given a non-negative modular weight function \(w : E \to \mathbb{R}_+\).

**Algorithm 1:** The Matroid Greedy Algorithm

1. Set \(X \leftarrow \emptyset\);
2. while \(\exists v \in E \setminus X \text{ s.t. } X \cup \{v\} \in \mathcal{I}\) do
3. \(v \in \text{argmax} \left\{ w(v) : v \in E \setminus X, X \cup \{v\} \in \mathcal{I} \right\} ;
4. \(X \leftarrow X \cup \{v\} ;

- Same as sorting items by decreasing weight \(w\), and then choosing items in that order that retain independence.

**Theorem 9.6.1**

Let \((E, \mathcal{I})\) be an independence system. Then the pair \((E, \mathcal{I})\) is a matroid if and only if for each weight function \(w \in \mathcal{R}_+^E\), Algorithm 1 leads to a set \(I \in \mathcal{I}\) of maximum weight \(w(I)\).
Matroids by bases

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

**Theorem 9.6.3 (Matroid (by bases))**

Let $E$ be a set and $B$ be a nonempty collection of subsets of $E$. Then the following are equivalent.

1. $B$ is the collection of bases of a matroid;
2. if $B, B' \in B$, and $x \in B' \setminus B$, then $B' - x + y \in B$ for some $y \in B \setminus B'$.
3. If $B, B' \in B$, and $x \in B' \setminus B$, then $B - y + x \in B$ for some $y \in B \setminus B'$.

Properties 2 and 3 are called “exchange properties.”

Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.

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Matroid and the greedy algorithm

**proof of Theorem 9.6.1.**

1. Assume $(E, \mathcal{I})$ is a matroid and $w : E \rightarrow \mathbb{R}_+$ is given.
2. Let $A = (a_1, a_2, \ldots, a_r)$ be the solution returned by greedy, where $r = r(M)$ the rank of the matroid, and we order the elements as they were chosen (so $w(a_1) \geq w(a_2) \geq \cdots \geq w(a_r)$).
3. $A$ is a base of $M$, and let $B = (b_1, \ldots, b_r)$ be any another base of $M$ with elements also ordered decreasing by weight, so $w(b_1) \geq w(b_2) \geq \cdots \geq w(b_r)$.
4. We next show that not only is $w(A) \geq w(B)$ but that $w(a_i) \geq w(b_i)$ for all $i$.
Matroid and the greedy algorithm

Proof of Theorem 9.6.1.

• Assume otherwise, and let \( k \) be the first (smallest) integer such that \( w(a_k) < w(b_k) \). Hence \( w(a_j) \geq w(b_j) \) for \( j < k \).

• Define independent sets \( A_{k-1} = \{a_1, \ldots, a_{k-1}\} \) and \( B_k = \{b_1, \ldots, b_k\} \).

• Since \( |A_{k-1}| < |B_k| \), there exists a \( b_i \in B_k \setminus A_{k-1} \) where \( A_{k-1} \cup \{b_i\} \in \mathcal{I} \) for some \( 1 \leq i \leq k \).

• But \( w(b_i) \geq w(b_k) > w(a_k) \), and so the greedy algorithm would have chosen \( b_i \) rather than \( a_k \), contradicting what greedy does.

Converse Proof of Theorem 9.6.1.

• Given an independence system \((E, \mathcal{I})\), suppose the greedy algorithm leads to an independent set of max weight for every non-negative weight function. We’ll show \((E, \mathcal{I})\) is a matroid.

• Emptyset containing and down monotonicity already holds (since we’ve started with an independence system).

• Let \( I, J \in \mathcal{I} \) with \(|I| < |J|\). Suppose to the contrary, that \( I \cup \{z\} \notin \mathcal{I} \) for all \( z \in J \setminus I \).

• Define the following modular weight function \( w \) on \( E \), and define \( k = |I| \).

\[
 w(v) = \begin{cases} 
 k + 2 & \text{if } v \in I, \\
 k + 1 & \text{if } v \in J \setminus I, \\
 0 & \text{if } v \in E \setminus (I \cup J) 
\end{cases} 
\]  \quad (9.15)
converse proof of Theorem 9.6.1.

- Now greedy will, after $k$ iterations, recover $I$, but it cannot choose any element in $J \setminus I$ by assumption. Thus, greedy chooses a set of weight $k(k + 2)$.
- On the other hand, $J$ has weight
  \[ w(J) \geq |J|(k + 1) \geq (k + 1)(k + 1) > k(k + 2) \]  
  (9.16)
  so $J$ has strictly larger weight but is still independent, contradicting greedy’s optimality.
- Therefore, there must be a $z \in J \setminus I$ such that $I \cup \{z\} \in \mathcal{I}$, and since $I$ and $J$ are arbitrary, $(E, \mathcal{I})$ must be a matroid.

Matroid and greedy

- As given, the theorem asked for a modular function $w \in \mathbb{R}_+^E$.
- This will not only return an independent set, but it will return a base if we keep going even if the weights are 0.
- If we don’t want elements with weight 0, we can stop once (and if) the weight hits zero, thus giving us a maximum weight independent set.
- We don’t need non-negativity, we can use any $w \in \mathbb{R}^E$ and keep going until we have a base.
- If we stop at a negative value, we’ll once again get a maximum weight independent set.
- Exercise: what if we keep going until a base even if we encounter negative values?
- We can instead do as small as possible thus giving us a minimum weight independent set/base.
Summary of Important (for us) Matroid Definitions

Given an independence system, matroids are defined equivalently by any of the following:

- All maximally independent sets have the same size.
- A monotone non-decreasing submodular integral rank function with unit increments.
- The greedy algorithm achieves the maximum weight independent set for all weight functions.

Convex Polyhedra

- Convex polyhedra a rich topic, we will only draw what we need.

**Definition 9.7.1**

A subset $P \subseteq \mathbb{R}^E$ is a polyhedron if there exists an $m \times n$ matrix $A$ and vector $b \in \mathbb{R}^m$ (for some $m \geq 0$) such that

$$P = \{ x \in \mathbb{R}^E : Ax \leq b \}$$

(9.17)

- Thus, $P$ is intersection of finitely many affine halfspaces, which are of the form $a_i^T x \leq b_i$ where $a_i$ is a row vector and $b_i$ a real scalar.
Convex Polytope

- A polytope is defined as follows

**Definition 9.7.2**

A subset $P \subseteq \mathbb{R}^E$ is a **polytope** if it is the convex hull of finitely many vectors in $\mathbb{R}^E$. That is, if $\exists x_1, x_2, \ldots, x_k \in \mathbb{R}^E$ such that for all $x \in P$, there exist $\{\lambda_i\}$ with $\sum_i \lambda_i = 1$ and $\lambda_i \geq 0 \ \forall i$ with $x = \sum_i \lambda_i x_i$.

- We define the convex hull operator as follows:

$$
\text{conv}(x_1, x_2, \ldots, x_k) \overset{\text{def}}{=} \left\{ \sum_{i=1}^k \lambda_i x_i : \forall i, \lambda_i \geq 0, \text{ and } \sum_i \lambda_i = 1 \right\}
$$

(9.18)

---

Convex Polytope - key representation theorem

- A polytope can be defined in a number of ways, two of which include

**Theorem 9.7.3**

A subset $P \subseteq \mathbb{R}^E$ is a polytope iff it can be described in either of the following (equivalent) ways:

- $P$ is the convex hull of a finite set of points.

- If it is a **bounded** intersection of halfspaces, that is there exits matrix $A$ and vector $b$ such that

$$
P = \{x : Ax \leq b\}
$$

(9.19)

- This result follows directly from results proven by Fourier, Motzkin, Farkas, and Carathéodory.
There are many ways to construct the dual. For example,

\[
\begin{align*}
\max \{c^T x \mid x \geq 0, Ax \leq b\} &= \min \{y^T b \mid y \geq 0, y^T A \geq c^T\} \\
\max \{c^T x \mid x \geq 0, Ax = b\} &= \min \{y^T b \mid y \geq 0, y^T A \geq c^T\} \\
\min \{c^T x \mid x \geq 0, Ax \geq b\} &= \max \{y^T b \mid y \geq 0, y^T A \leq c^T\} \\
\min \{c^T x \mid Ax \geq b\} &= \max \{y^T b \mid y \geq 0, y^T A = c^T\}
\end{align*}
\]
Linear Programming duality forms

How to form the dual in general? We quote V. Vazirani (2001)

*Intuitively, why is [one set of equations] the dual of [another quite different set of equations]? In our experience, this is not the right question to be asked. As stated in Section 12.1, there is a purely mechanical procedure for obtaining the dual of a linear program. Once the dual is obtained, one can devise intuitive, and possibly physical meaningful, ways of thinking about it. Using this mechanical procedure, one can obtain the dual of a complex linear program in a fairly straightforward manner. Indeed, the LP-duality-based approach derives its wide applicability from this fact.*

Also see the text “Convex Optimization” by Boyd and Vandenberghe, chapter 5, for a great discussion on duality.

---

Vector, modular, incidence

- Recall, any vector \(x \in \mathbb{R}^E\) can be seen as a normalized modular function, as for any \(A \subseteq E\), we have

  \[
x(A) = \sum_{a \in A} x_a
  \]  

  (9.26)

- Given an \(A \subseteq E\), define the the incidence vector \(\mathbf{1}_A \in \{0, 1\}^E\) on the unit hypercube as follows:

  \[
  \mathbf{1}_A \overset{\text{def}}{=} \left\{ x \in \{0, 1\}^E : x_i = 1 \quad \text{iff} \quad i \in A \right\}
  \]

  (9.27)

  equivalently,

  \[
  \mathbf{1}_A(j) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } j \in A \\ 0 & \text{if } j \notin A \end{cases}
  \]

  (9.28)
The next slide is review from lecture 6.

Matroid

Slight modification (non unit increment) that is equivalent.

Definition 9.8.3 (Matroid-II)

A set system \((E, \mathcal{I})\) is a Matroid if

\((I1')\) \(\emptyset \in \mathcal{I}\)

\((I2')\) \(\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}\) (down-closed or subclusive)

\((I3')\) \(\forall I, J \in \mathcal{I}, \text{ with } |I| > |J|\), then there exists \(x \in I \setminus J\) such that \(J \cup \{x\} \in \mathcal{I}\)

Note \((I1)\equiv(I1')\), \((I2)\equiv(I2')\), and we get \((I3)\equiv(I3')\) using induction.
Independence Polyhedra

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $1_I$.
- Taking the convex hull, we get the independent set polytope, that is

$$P_{\text{ind. set}} = \text{conv} \left\{ \bigcup_{I \in \mathcal{I}} \{1_I\} \right\} \quad (9.29)$$

- Since $\{1_I : I \in \mathcal{I}\} \subseteq P_{\text{ind. set}}$, we have

$$\max \{w(I) : I \in \mathcal{I}\} \leq \max \{w^\top x : x \in P_{\text{ind. set}}\}.$$

- Now take the rank function $r$ of $M$, and define the following polyhedron:

$$P_r^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \} \quad (9.30)$$

- Now, take any $x \in P_{\text{ind. set}}$, then we have that $x \in P_r^+$ (or $P_{\text{ind. set}} \subseteq P_r^+$). We show this next.

$$P_{\text{ind. set}} \subseteq P_r^+$$

- If $x \in P_{\text{ind. set}}$, then

$$x = \sum_i \lambda_i 1_i \quad (9.31)$$

for some appropriate vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$.
- Clearly, for such $x$, $x \geq 0$.
- Now, for any $A \subseteq E$,

$$x(A) = x^\top 1_A = \sum_i \lambda_i 1_i^\top 1_A \quad (9.32)$$

$$\leq \sum_i \lambda_i \max_{j : I_j \subseteq A} 1_{I_j}(E) \quad (9.33)$$

$$= \max_{j : I_j \subseteq A} 1_{I_j}(E) = \max_{I \in \mathcal{I}} |A \cap I| \quad (9.34)$$

$$= r(A) \quad (9.35)$$

- Thus, $x \in P_r^+$ and hence $P_{\text{ind. set}} \subseteq P_r^+$. 
Consider this in two dimensions. We have equations of the form:

\[ x_1 \geq 0 \text{ and } x_2 \geq 0 \]  
\[ x_1 \leq r(\{v_1\}) \]  
\[ x_2 \leq r(\{v_2\}) \]  
\[ x_1 + x_2 \leq r(\{v_1, v_2\}) \]

Because \( r \) is submodular, we have

\[ r(\{v_1\}) + r(\{v_2\}) \geq r(\{v_1, v_2\}) + r(\emptyset) \]

so since \( r(\{v_1, v_2\}) \leq r(\{v_1\}) + r(\{v_2\}) \), the last inequality is either touching (so inactive) or active.
Matroid Polyhedron in 2D

Matroid Polyhedron in 3D

\[ P^+_r = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \} \quad (9.42) \]

- Consider this in three dimensions. We have equations of the form:

\[ x_1 \geq 0 \quad and \quad x_2 \geq 0 \quad and \quad x_3 \geq 0 \quad (9.43) \]
\[ x_1 \leq r(\{v_1\}) \quad (9.44) \]
\[ x_2 \leq r(\{v_2\}) \quad (9.45) \]
\[ x_3 \leq r(\{v_3\}) \quad (9.46) \]
\[ x_1 + x_2 \leq r(\{v_1, v_2\}) \quad (9.47) \]
\[ x_2 + x_3 \leq r(\{v_2, v_3\}) \quad (9.48) \]
\[ x_1 + x_3 \leq r(\{v_1, v_3\}) \quad (9.49) \]
\[ x_1 + x_2 + x_3 \leq r(\{v_1, v_2, v_3\}) \quad (9.50) \]
Consider the simple cycle matroid on a graph consisting of a 3-cycle, $G = (V, E)$ with matroid $M = (E, I)$ where $I \in \mathcal{I}$ is a forest.

- So any set of either one or two edges is independent, and has rank equal to cardinality.
- The set of three edges is dependent, and has rank 2.

Two view of $P^+_r$ associated with a matroid
$\left(\{e_1, e_2, e_3\}, \{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\right)$. 
**Matroid Polyhedron in 3D**

$P^+_r$ associated with the “free” matroid in 3D.

**Another Polytope in 3D**

Thought question: what kind of polytope might this be?
So recall from a moment ago, that we have that

$$P_{\text{ind. set}} = \text{conv} \{ \bigcup_{I \in \mathcal{I}} \{1_I\} \} \subseteq P_r^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \} \quad (9.51)$$

- In fact, the two polyhedra are identical (and thus both are polytopes).
- We’ll show this in the next few theorems.

**Theorem 9.8.1**

*Let $M = (V, \mathcal{I})$ be a matroid, with rank function $r$, then for any weight function $w \in \mathbb{R}^V_+$, there exists a chain of sets $U_1 \subset U_2 \subset \cdots \subset U_n \subseteq V$ such that*

$$\max \{ w(I) | I \in \mathcal{I} \} = \sum_{i=1}^{n} \lambda_i r(U_i) \quad (9.52)$$

*where $\lambda_i \geq 0$ satisfy*

$$w = \sum_{i=1}^{n} \lambda_i 1_{U_i} \quad (9.53)$$
Proof.

- Firstly, note that for any such $w \in \mathbb{R}^E$, we have

\[
\begin{pmatrix}
w_1 \\
w_2 \\
\vdots \\
w_n
\end{pmatrix} = (w_1 - w_2) \begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix} + (w_2 - w_3) \begin{pmatrix}
1 \\
1 \\
\vdots \\
0
\end{pmatrix} +
\]

\[
\cdots + (w_{n-1} - w_n) \begin{pmatrix}
1 \\
1 \\
\vdots \\
0
\end{pmatrix} + (w_n) \begin{pmatrix}
1 \\
1 \\
\vdots \\
1
\end{pmatrix}
\]

(9.54)

- If we can take $w$ in decreasing order ($w_1 \geq w_2 \geq \cdots \geq w_n$), then each coefficient of the vectors is non-negative (except possibly the last one, $w_n$).

Define the sets $U_i$ based on this order as follows, for $i = 0, \ldots, n$

\[U_i \overset{\text{def}}{=} \{v_1, v_2, \ldots, v_i\}\] (9.55)

Define the set $I$ as those elements where the rank increases, i.e.:

\[I \overset{\text{def}}{=} \{v_i | r(U_i) > r(U_{i-1})\}\] (9.56)

Hence, given an $i$ with $v_i \notin I$, $r(U_i) = r(U_{i-1})$.

Therefore, $I$ is the output of the greedy algorithm for $\max \{w(I) | I \in I\}$. since items $v_i$ are ordered decreasing by $w(v_i)$, and we only choose the ones that increase the rank, which means they don’t violate independence.

And therefore, $I$ is a maximum weight independent set (even a base, ...
Maximum weight independent set via weighted rank

Proof.

1. Now, we define $\lambda_i$ as follows

$$
\lambda_i \overset{\text{def}}{=} w(v_i) - w(v_{i+1}) \quad \text{for} \quad i = 1, \ldots, n - 1 \quad (9.57)
$$

$$
\lambda_n \overset{\text{def}}{=} w(v_n) \quad (9.58)
$$

2. And the weight of the independent set $w(I)$ is given by

$$
w(I) = \sum_{v \in I} w(v) = \sum_{i=1}^{n} w(v_i) (r(U_i) - r(U_{i-1})) \quad (9.59)
$$

$$
w(I) = w(v_n)r(U_n) + \sum_{i=1}^{n-1} (w(v_i) - w(v_{i+1}))r(U_i) = \sum_{i=1}^{n} \lambda_i r(U_i) \quad (9.60)
$$

3. Since we took $v_1, v_2, \ldots$ in decreasing order, for all $i$, and since $w \in \mathbb{R}^E_+$, we have $\lambda_i \geq 0$