Submodular Functions, Optimization, and Applications to Machine Learning
— Spring Quarter, Lecture 7 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee596b_spring_2016/

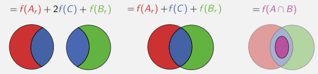
Prof. Jeff Bilmes

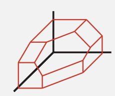
University of Washington, Seattle
Department of Electrical Engineering
http://melodi.ee.washington.edu/~bilmes

Apr 20th, 2016

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

$= f(A_r) + 2f(C) + f(B_r)$ $\quad = f(A_r) + f(C) + f(B_r)$ $\quad = f(A \cap B)$





Cumulative Outstanding Reading

- Read chapters 2 and 3 from Fujishige's book.
- Read chapter 1 from Fujishige's book.

Announcements, Assignments, and Reminders

- Homework 2, available at our assignment dropbox (https://canvas.uw.edu/courses/1039754/assignments), due (electronically) Monday (4/18) at 11:55pm.
- Homework 1, available at our assignment dropbox (https://canvas.uw.edu/courses/1039754/assignments), due (electronically) Friday (4/8) at 11:55pm.
- Weekly Office Hours: Mondays, 3:30-4:30, or by skype or google hangout (set up meeting via our our discussion board (https://canvas.uw.edu/courses/1039754/discussion_topics)).

Class Road Map - IT-I

- L1(3/28): Motivation, Applications, & Basic Definitions
- L2(3/30): Machine Learning Apps (diversity, complexity, parameter, learning target, surrogate).
- L3(4/4): Info theory exs, more apps, definitions, graph/combinatorial examples, matrix rank example, visualization
- L4(4/6): Graph and Combinatorial Examples, matrix rank, Venn diagrams, examples of proofs of submodularity, some useful properties
- L5(4/11): Examples & Properties, Other Defs., Independence
- L6(4/13): Independence, Matroids, Matroid Examples, matroid rank is submodular
- L7(4/18): Matroid Rank, More on Partition Matroid, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation, Dual Matroid
- L8(4/20):
- L9(4/25):
- L10(4/27):
- L11(5/2):
- L12(5/4):
- L13(5/9):
- L14(5/11):
- L15(5/16):
- L16(5/18):
- L17(5/23):
- L18(5/25):
- L19(6/1):
- L20(6/6): Final Presentations maximization.

Finals Week: June 6th-10th, 2016.

Matroid

Independent set definition of a matroid is perhaps most natural. Note, if $J \in \mathcal{I}$, then J is said to be an independent set.

Definition 7.2.3 (Matroid)

A set system $(E, \mathcal{I})$ is a Matroid if

(I1) $\emptyset \in \mathcal{I}$

(I2) $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$

(I3) $\forall I, J \in \mathcal{I}$, with $|I| = |J| + 1$, then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$.

Why is (I1) is not redundant given (I2)? Because without (I1) could have a non-matroid where $\mathcal{I} = \{\}$.

Matroids - important property

Proposition 7.2.3

In a matroid $M = (E, \mathcal{I})$, for any $U \subseteq E(M)$, any two bases of U have the same size.

- In matrix terms, given a set of vectors U, all sets of independent vectors that span the space spanned by U have the same size.
- In fact, under (I1),(I2), this condition is equivalent to (I3). Exercise: show the following is equivalent to the above.

Definition 7.2.4 (Matroid)

A set system $(V, \mathcal{I})$ is a Matroid if

(I1') $\emptyset \in \mathcal{I}$ (emptyset containing)

(I2') $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ (down-closed or subclusive)

(I3') $\forall X \subseteq V$, and $I_1, I_2 \in \text{maxInd}(X)$, we have $|I_1| = |I_2|$ (all maximally independent subsets of X have the same size).

Partition Matroid

- Let V be our ground set.
- Let $V = V_1 \cup V_2 \cup \cdots \cup V_\ell$ be a partition of V into ℓ blocks (i.e., disjoint sets). Define a set of subsets of V as

$$\mathcal{I} = \{X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell\}. \tag{7.5}$$

  where $k_1, \ldots, k_\ell$ are fixed parameters, $k_i \geq 0$. Then $M = (V, \mathcal{I})$ is a matroid.
- Note that a k-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$.
- Parameters associated with a partition matroid: ℓ and $k_1, k_2, \ldots, k_\ell$ although often the k_i's are all the same.
- We'll show that property (I3') in Def **??** holds. First note, for any $X \subseteq V$, $|X| = \sum_{i=1}^{\ell} |X \cap V_i|$ since $\{V_1, V_2, \ldots, V_\ell\}$ is a partition.
- If $X, Y \in \mathcal{I}$ with $|Y| > |X|$, then there must be at least one i with $|Y \cap V_i| > |X \cap V_i|$. Therefore, adding one element $e \in V_i \cap (Y \setminus X)$ to X won't break independence.

Matroids - rank function is submodular

Lemma 7.2.3

The rank function $r : 2^E \to \mathbb{Z}_+$ of a matroid is submodular, that is $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$

Proof.

1. Let $X \in \mathcal{I}$ be an inclusionwise maximal set with $X \subseteq A \cap B$
2. Let $Y \in \mathcal{I}$ be inclusionwise maximal set with $X \subseteq Y \subseteq A \cup B$.
3. Since M is a matroid, we know that $r(A \cap B) = r(X) = |X|$, and $r(A \cup B) = r(Y) = |Y|$. Also, for any $U \in \mathcal{I}$, $r(A) \geq |A \cap U|$.
4. Then we have (since $X \subseteq A \cap B$, $X \subseteq Y$, and $Y \subseteq A \cup B$),

$$r(A) + r(B) \geq |Y \cap A| + |Y \cap B| \tag{7.5}$$
$$= |Y \cap (A \cap B)| + |Y \cap (A \cup B)| \tag{7.6}$$
$$\geq |X| + |Y| = r(A \cap B) + r(A \cup B) \tag{7.7}$$

□

A matroid is defined from its rank function

Theorem 7.2.3 (Matroid from rank)

Let E be a set and let $r : 2^E \to \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with r being its rank function if and only if for all $A, B \subseteq E$:

(R1) $\forall A \subseteq E \;\; 0 \leq r(A) \leq |A|$ *(non-negative cardinality bounded)*

(R2) $r(A) \leq r(B)$ *whenever* $A \subseteq B \subseteq E$ *(monotone non-decreasing)*

(R3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ *for all* $A, B \subseteq E$ *(submodular)*

- So submodularity and non-negative monotone non-decreasing, and unit increase is necessary and sufficient to define the matroid.
- Can name matroid as (E, r), E is ground set, r is rank function.
- Given above, unit increment (if $r(A) = k$, then either $r(A \cup \{v\}) = k$ or $r(A \cup \{v\}) = k + 1$) holds.
- From above, $r(\emptyset) = 0$. Let $v \notin A$, then by monotonicity and submodularity, $r(A) \leq r(A \cup \{v\}) \leq r(A) + r(\{v\})$ which gives only two possible values to $r(A \cup \{v\})$.

Matroids from rank

Proof of Theorem 7.2.3 (matroid from rank).

- Given a matroid $M = (E, \mathcal{I})$, we see its rank function as defined in Eq. **??** satisfies (R1), (R2), and, as we saw in Lemma 7.2.3, (R3) too.
- Next, assume we have (R1), (R2), and (R3). Define $\mathcal{I} = \{X \subseteq E : r(X) = |X|\}$. We will show that $(E, \mathcal{I})$ is a matroid.
- First, $\emptyset \in \mathcal{I}$.
- Also, if $Y \in \mathcal{I}$ and $X \subseteq Y$ then by submodularity,

$$r(X) \geq r(Y) - r(Y \setminus X) + r(\emptyset) \tag{7.1}$$

$$\geq |Y| - |Y \setminus X| \tag{7.2}$$

$$= |X| \tag{7.3}$$

implying $r(X) = |X|$, and thus $X \in \mathcal{I}$.

...

Matroids from rank

Proof of Theorem 7.2.3 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \dots, b_k\}$ (note $k \leq |B|$).
- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such b, $r(A + b) = r(A) = |A| < |A + b|$. Then

$$r(B) \leq r(A \cup B) \tag{7.4}$$
$$\leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) \tag{7.5}$$
$$= r(A \cup (B \setminus \{b_1\}) \tag{7.6}$$
$$\leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) \tag{7.7}$$
$$= r(A \cup (B \setminus \{b_1, b_2\})) \tag{7.8}$$
$$\leq \dots \leq r(A) = |A| < |B| \tag{7.9}$$

giving a contradiction since $B \in \mathcal{I}$. □

Matroids from rank II

Another way of using function r to define a matroid.

Theorem 7.3.1 (Matroid from rank II)

Let E be a finite set and let $r : 2^E \to \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with r being its rank function if and only if for all $A \subseteq E$, and $x, y \in E$:

(R1') $r(\emptyset) = 0$;

(R2') $r(X) \leq r(X \cup \{y\}) \leq r(X) + 1$;

(R3') *If* $r(X \cup \{x\}) = r(X \cup \{y\}) = r(X)$, *then* $r(X \cup \{x, y\}) = r(X)$.

Matroids by submodular functions

Theorem 7.3.2 (Matroid by submodular functions)

Let $f : 2^E \to \mathbb{Z}$ be a integer valued monotone non-decreasing submodular function. Define a set of sets as follows:

$$\mathcal{C}(f) = \Big\{ C \subseteq E : C \text{ is non-empty, is inclusionwise-minimal, and has } f(C) < |C| \Big\} \quad (7.10)$$

Then $\mathcal{C}(f)$ is the collection of circuits of a matroid on E.

Inclusionwise-minimal in this case means that if $C \in \mathcal{C}(f)$, then there exists no $C' \subset C$ with $C' \in \mathcal{C}(f)$ (i.e., $C' \subset C$ would either be empty or have $f(C') \geq |C'|$). Also, recall inclusionwise-minimal in Definition 7.9.5, the definition of a circuit.

Summarizing: Many ways to define a Matroid

Summarizing what we've so far seen, we saw that it is possible to uniquely define a matroid based on any of:

- Independence (define the independent sets).
- Base axioms (exchangeability)
- Circuit axioms
- Closure axioms (we didn't see this, but it is possible)
- Rank axioms (normalized, monotone, cardinality bounded, submodular)
- Matroids by submodular functions.

Maximization problems for matroids

- Given a matroid $M = (E, \mathcal{I})$ and a modular cost function $c : E \to \mathbb{R}$, the task is to find an $X \in \mathcal{I}$ such that $c(X) = \sum_{x \in X} c(x)$ is maximum.
- This seems remarkably similar to the max spanning tree problem.

Minimization problems for matroids

- Given a matroid $M = (E, \mathcal{I})$ and a modular cost function $c : E \to \mathbb{R}$, the task is to find a basis $B \in \mathcal{B}$ such that $c(B)$ is minimized.
- This sounds like a set cover problem (find the minimum cost covering set of sets).

Partition Matroid

- What is the partition matroid's rank function?
- A partition matroids rank function:

$$r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i) \tag{7.11}$$

which we also immediately see is submodular using properties we spoke about last week. That is:

1. $|A \cap V_i|$ is submodular (in fact modular) in A
2. $\min(\text{submodular}(A), k_i)$ is submodular in A since $|A \cap V_i|$ is monotone.
3. sums of submodular functions are submodular.

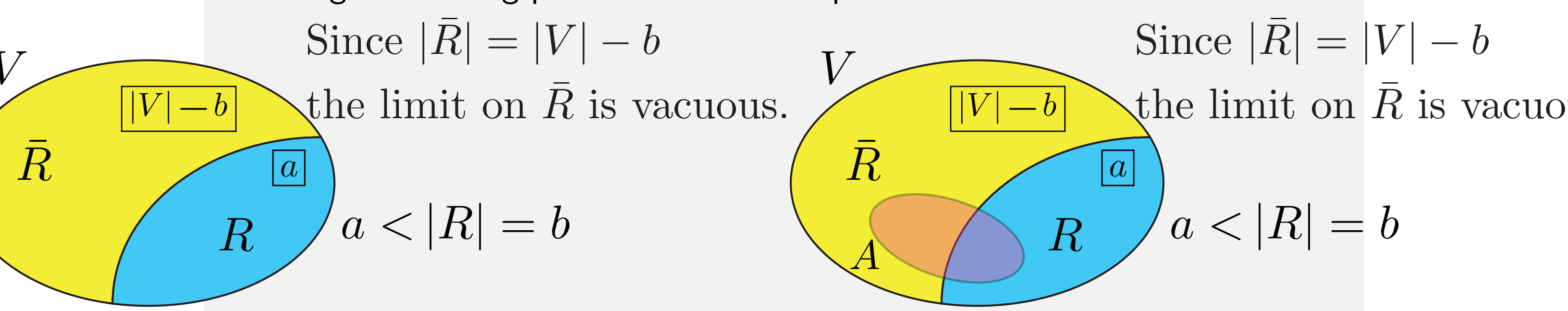
- $r(A)$ is also non-negative integral monotone non-decreasing, so it defines a matroid (the partition matroid).

From 2-partition matroid rank to truncated matroid rank

- Example: 2-partition matroid rank function: Given natural numbers $a, b \in \mathbb{Z}_+$ with $a < b$, and any set $R \subseteq V$ with $|R| = b$.
- Create two-block partition $V = (R, \bar{R})$, where $\bar{R} = V \setminus R$ so $|\bar{R}| = |V| - b$. Gives 2-partition matroid rank function as follows:

$$r(A) = \min(|A \cap R|, a) + \min(|A \cap \bar{R}|, |\bar{R}|) \tag{7.12}$$
$$= \min(|A \cap R|, a) + |A \cap \bar{R}| \tag{7.13}$$
$$= \min(|A \cap \bar{R}| + |A \cap R|, |A \cap \bar{R}| + a) \tag{7.14}$$
$$= \min(|A|, |A \cap \bar{R}| + a) \tag{7.15}$$

- Figure showing partition blocks and partition matroid limits.



Truncated Matroid Rank Function

- Define truncated matroid rank function. Start with 2-partition matroid rank $r(A) = \min(|A \cap R|, a) + \min(|A \cap \bar{R}|, |\bar{R}|)$, $a < b$. Define:

$$f_R(A) = \min\left\{ r(A), b \right\} \quad (7.16)$$

$$= \min\left\{ \min(|A|, |A \cap \bar{R}| + a), b \right\} \quad (7.17)$$

$$= \min\left\{ |A|, a + |A \cap \bar{R}|, b \right\} \quad (7.18)$$

- Defines a matroid $M = (V, f_R) = (V, \mathcal{I})$ (Goemans et. al.) with

$$\mathcal{I} = \{I \subseteq V : |I| \leq b \text{ and } |I \cap R| \leq a\}, \quad (7.19)$$

Useful for showing hardness of constrained submodular minimization. Consider sets $B \subseteq V$ with $|B| = b$.

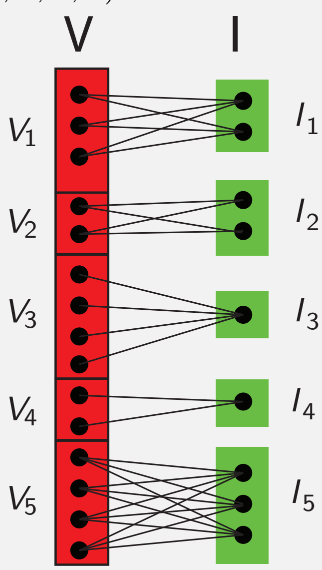
- For R, we have $f_R(R) = \min(b, a, b) = a < b$.
- For any B with $|B \cap R| \leq a$, $f_R(B) = b$.
- For any B with $|B \cap R| = \ell$, with $a \leq \ell \leq b$, $f_R(B) = a + b - \ell$.
- R, the set with minimum valuation amongst size-b sets, is hidden within an exponentially larger set of size-b sets with larger valuation.

Partition Matroid, rank as matching

- A partition matroid can be viewed using a bipartite graph.
- Letting V denote the ground set, and $V_1, V_2, \ldots$ the partition, the graph is $G = (V, I, E)$ where V is the ground set, I is a set of "indices", and E is the set of edges.
- $I = (I_1, I_2, \ldots, I_\ell)$ is a set of $k = \sum_{i=1}^{\ell} k_i$ nodes, grouped into ℓ clusters, where there are k_i nodes in the i^{th} group I_i.
- $(v, i) \in E(G)$ iff $v \in V_j$ and $i \in I_j$.

Partition Matroid, rank as matching

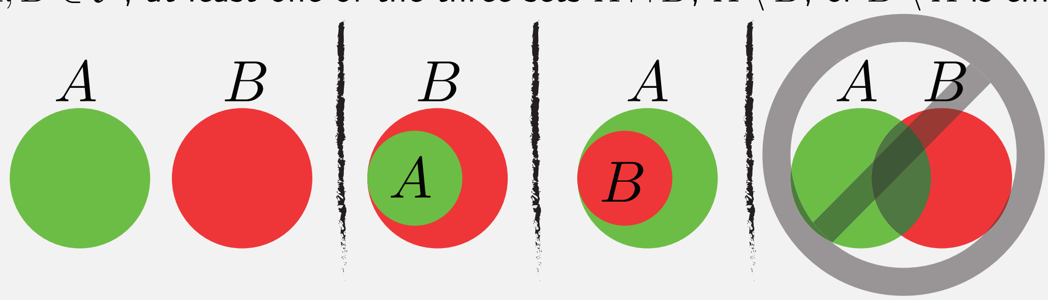
- Example where $\ell = 5$, $(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3)$.



- Recall, $\Gamma : 2^V \to \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of X is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.
- Here, for $X \subseteq V$, we have $\Gamma(X) = \{i \in I : (v, i) \in E(G) \text{ and } v \in X\}$.
- For such a constructed bipartite graph, the rank function of a partition matroid is $r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i) =$ the maximum matching involving X.

Laminar Family and Laminar Matroid

- We can define a matroid with structures richer than just partitions.
- A set system $(V, \mathcal{F})$ is called a laminar family if for any two sets $A, B \in \mathcal{F}$, at least one of the three sets $A \cap B$, $A \setminus B$, or $B \setminus A$ is empty.



- Family is laminar $\exists$ no two properly intersecting members: $\forall A, B \in \mathcal{F}$, either A, B disjoint ($A \cap B = \emptyset$) or comparable ($A \subseteq B$ or $B \subseteq A$).
- Suppose we have a laminar family $\mathcal{F}$ of subsets of V and an integer k_A for every set $A \in \mathcal{F}$. Then $(V, \mathcal{I})$ defines a matroid where

$$\mathcal{I} = \{I \subseteq E : |I \cap A| \leq k_A \text{ for all } A \in \mathcal{F}\} \tag{7.20}$$

- Exercise: what is the rank function here?

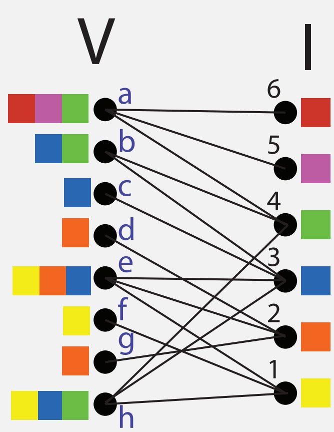
System of Representatives

- Let $(V, \mathcal{V})$ be a set system (i.e., $\mathcal{V} = (V_i : i \in I)$ where $\emptyset \subset V_i \subseteq V$ for all i), and I is an index set. Hence, $|I| = |\mathcal{V}|$.
- Here, the sets $V_i \in \mathcal{V}$ are like "groups" and any $v \in V$ with $v \in V_i$ is a member of group i. Groups need not be disjoint (e.g., interest groups of individuals).
- A family $(v_i : i \in I)$ with $v_i \in V$ is said to be a system of representatives of $\mathcal{V}$ if $\exists$ a bijection $\pi : I \to I$ such that $v_i \in V_{\pi(i)}$.
- v_i is the representative of set (or group) $V_{\pi(i)}$, meaning the i^{th} representative is meant to represent set $V_{\pi(i)}$.
- Example: Consider the house of representatives, $v_i =$ "Jim McDermott", while $i =$ "King County, WA-7".
- In a system of representatives, there is no requirement for the representatives to be distinct. I.e., we could have some $v_1 \in V_1 \cap V_2$, where v_1 represents both V_1 and V_2.
- We can view this as a bipartite graph.

Matroid Rank | More on Partition Matroid | System of Distinct Reps | Transversals | Transversal Matroid | Matroid and representation | Dual Matroid

System of Representatives

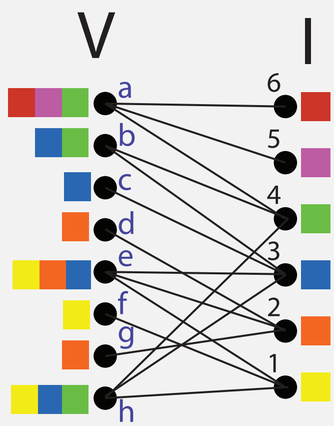
- We can view this as a bipartite graph. The groups of V are marked by color tags on the left, and also via right neighbors in the graph.
- Here, $\ell = 6$ groups, with $\mathcal{V} = (V_1, V_2, \ldots, V_6)$
$= \left(\{e, f, h\}, \{d, e, g\}, \{b, c, e, h\}, \{a, b, h\}, \{a\}, \{a\}\right)$.



- A system of representatives would make sure that there is a representative for each color group. For example,
- The representatives $(\{a, c, d, f, h\})$ are shown as colors on the left.
- Here, the set of representatives is not distinct. Why? In fact, due to the red and pink group, a distinct group of representatives is impossible (since there is only one common choice to represent both color groups).

System of Representatives

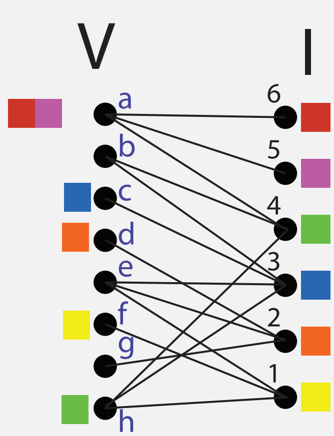
- We can view this as a bipartite graph. The groups of V are marked by color tags on the left, and also via right neighbors in the graph.
- Here, $\ell = 6$ groups, with $\mathcal{V} = (V_1, V_2, \dots, V_6)$ $= \Big(\{e, f, h\}, \{d, e, g\}, \{b, c, e, h\}, \{a, b, h\}, \{a\}, \{a\}\Big)$.



- A system of representatives would make sure that there is a representative for each color group. For example,
- The representatives $(\{a, c, d, f, h\})$ are shown as colors on the left.
- Here, the set of representatives is not distinct. Why? In fact, due to the red and pink group, a distinct group of representatives is impossible (since there is only one common choice to represent both color groups).

System of Representatives

- We can view this as a bipartite graph. The groups of V are marked by color tags on the left, and also via right neighbors in the graph.
- Here, $\ell = 6$ groups, with $\mathcal{V} = (V_1, V_2, \dots, V_6)$ $= \Big(\{e, f, h\}, \{d, e, g\}, \{b, c, e, h\}, \{a, b, h\}, \{a\}, \{a\}\Big)$.



- A system of representatives would make sure that there is a representative for each color group. For example,
- The representatives $(\{a, c, d, f, h\})$ are shown as colors on the left.
- Here, the set of representatives is not distinct. Why? In fact, due to the red and pink group, a distinct group of representatives is impossible (since there is only one common choice to represent both color groups).

System of Distinct Representatives

- Let $(V, \mathcal{V})$ be a set system (i.e., $\mathcal{V} = (V_k : i \in I)$ where $V_i \subseteq V$ for all i), and I is an index set. Hence, $|I| = |\mathcal{V}|$.
- A family $(v_i : i \in I)$ with $v_i \in V$ is said to be a system of distinct representatives of $\mathcal{V}$ if $\exists$ a bijection $\pi : I \leftrightarrow I$ such that $v_i \in V_{\pi(i)}$ and $v_i \neq v_j$ for all $i \neq j$.
- In a system of distinct representatives, there is a requirement for the representatives to be distinct. We can re-state (and rename) this as a:

Definition 7.5.1 (transversal)

Given a set system $(V, \mathcal{V})$ and index set I for $\mathcal{V}$ as defined above, a set $T \subseteq V$ is a transversal of $\mathcal{V}$ if there is a bijection $\pi : T \leftrightarrow I$ such that

$$x \in V_{\pi(x)} \text{ for all } x \in T \tag{7.21}$$

- Note that due to $\pi : T \leftrightarrow I$ being a bijection, all of I and T are "covered" (so this makes things distinct automatically).

Transversals are Subclusive

- A set $T' \subseteq V$ is a partial transversal if T' is a transversal of some subfamily $\mathcal{V}' = (V_i : i \in I')$ where $I' \subseteq I$.
- Therefore, for any transversal T, any subset $T' \subseteq T$ is a partial transversal.
- Thus, transversals are down closed (subclusive).

When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
- Given a set system $(V, \mathcal{V})$ with $\mathcal{V} = (V_i : i \in I)$, and $V_i \subseteq V$ for all i. Then, for any $J \subseteq I$, let

$$V(J) = \cup_{j \in J} V_j \tag{7.22}$$

so $|V(J)| : 2^I \to \mathbb{Z}_+$ is the set cover func. (we know is submodular).
- We have

Theorem 7.6.1 (Hall's theorem)

Given a set system $(V, \mathcal{V})$, the family of subsets $\mathcal{V} = (V_i : i \in I)$ has a transversal $(v_i : i \in I)$ iff for all $J \subseteq I$

$$|V(J)| \geq |J| \tag{7.23}$$

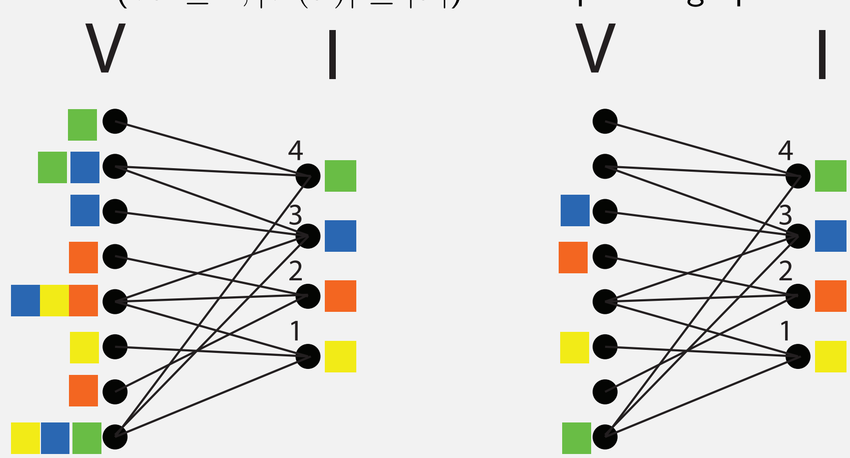
Matroid Rank | More on Partition Matroid | System of Distinct Reps | Transversals | Transversal Matroid | Matroid and representation | Dual Matroid

When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
- Given a set system $(V, \mathcal{V})$ with $\mathcal{V} = (V_i : i \in I)$, and $V_i \subseteq V$ for all i. Then, for any $J \subseteq I$, let

$$V(J) = \cup_{j \in J} V_j \tag{7.22}$$

so $|V(J)| : 2^I \to \mathbb{Z}_+$ is the set cover func. (we know is submodular).
- Hall's theorem ($\forall J \subseteq I, |V(J)| \geq |J|$) as a bipartite graph.



When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
- Given a set system $(V, \mathcal{V})$ with $\mathcal{V} = (V_i : i \in I)$, and $V_i \subseteq V$ for all i. Then, for any $J \subseteq I$, let

$$V(J) = \cup_{j \in J} V_j \tag{7.22}$$

so $|V(J)| : 2^I \to \mathbb{Z}_+$ is the set cover func. (we know is submodular).
- Moreover, we have

Theorem 7.6.2 (Rado's theorem (1942))

If $M = (V, r)$ is a matroid on V with rank function r, then the family of subsets $(V_i : i \in I)$ of V has a transversal $(v_i : i \in I)$ that is independent in M iff for all $J \subseteq I$

$$r(V(J)) \geq |J| \tag{7.24}$$

- Note, a transversal T independent in M means that $r(T) = |T|$.

More general conditions for existence of transversals

Theorem 7.6.3 (Polymatroid transversal theorem)

If $\mathcal{V} = (V_i : i \in I)$ is a finite family of non-empty subsets of V, and $f : 2^V \to \mathbb{Z}_+$ is a non-negative, integral, monotone non-decreasing, and submodular function, then $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that

$$f(\cup_{i \in J}\{v_i\}) \geq |J| \text{ for all } J \subseteq I \tag{7.25}$$

if and only if

$$f(V(J)) \geq |J| \text{ for all } J \subseteq I \tag{7.26}$$

- Given Theorem 7.6.3, we immediately get Theorem 7.6.1 by taking $f(S) = |S|$ for $S \subseteq V$. *In which case, Eq. 7.25 requires the system of representatives to be distinct.*
- We get Theorem 7.6.2 by taking $f(S) = r(S)$ for $S \subseteq V$, the rank function of the matroid. *where, Eq. 7.25 insists the system of representatives is independent in M, and hence also distinct.*

Submodular Composition with Set-to-Set functions

- Note the condition in Theorem 7.6.3 is $f(V(J)) \geq |J|$ for all $J \subseteq I$, where $f : 2^V \rightarrow \mathbb{Z}_+$ is non-negative, integral, monotone non-decreasing and submodular, and $V(J) = \cup_{j \in J} V_j$ with $V_i \subseteq V$.
- Note $V(\cdot) : 2^I \rightarrow 2^V$ is a set-to-set function, composable with a submodular function.
- Define $g : 2^I \rightarrow \mathbb{Z}$ with $g(J) = f(V(J)) - |J|$, then the condition for the existence of a system of representatives, with quality Equation 7.25, becomes:

$$\min_{J \subseteq I} g(J) \geq 0 \tag{7.27}$$

- What kind of function is g?

Proposition 7.6.4

g as given above is submodular.

- Hence, the condition for existence can be solved by (a special case of) submodular function minimization, or vice verse!

More general conditions for existence of transversals

first part proof of Theorem 7.6.3.

- Suppose $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that Eq. 7.25 (i.e., $f(\cup_{i \in J}\{v_i\}) \geq |J|$ for all $J \subseteq I$) is true.
- Then since f is monotone, and since $V(J) \supseteq \cup_{i \in J}\{v_i\}$ when $(v_i : i \in I)$ is a system of representatives, then Eq. 7.26 (i.e., $f(V(J)) \geq |J|$ for all $J \subseteq I$) immediately follows.

...

More general conditions for existence of transversals

Lemma 7.6.5 (contraction lemma)

Suppose Eq. 7.26 ($f(V(J)) \geq |J|, \forall J \subseteq I$) is true for $\mathcal{V} = (V_i : i \in I)$, and there exists an i such that $|V_i| \geq 2$ (w.l.o.g., say $i = 1$). Then there exists $\bar{v} \in V_1$ such that the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 7.26.

Proof.

- When Eq. 7.26 holds, this means that for any subsets $J_1, J_2 \subseteq I \setminus \{1\}$, we have that, for $J \in \{J_1, J_2\}$,

$$f(V(J \cup \{1\})) \geq |J \cup \{1\}| \tag{7.28}$$

and hence

$$f(V_1 \cup V(J_1)) \geq |J_1| + 1 \tag{7.29}$$

$$f(V_1 \cup V(J_2)) \geq |J_2| + 1 \tag{7.30}$$

...

Matroid Rank | More on Partition Matroid | System of Distinct Reps | Transversals | Transversal Matroid | Matroid and representation | Dual Matroid

More general conditions for existence of transversals

Lemma 7.6.5 (contraction lemma)

Suppose Eq. 7.26 ($f(V(J)) \geq |J|, \forall J \subseteq I$) is true for $\mathcal{V} = (V_i : i \in I)$, and there exists an i such that $|V_i| \geq 2$ (w.l.o.g., say $i = 1$). Then there exists $\bar{v} \in V_1$ such that the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 7.26.

Proof.

- Suppose, to the contrary, the consequent is false. Then we may take any $\bar{v}_1, \bar{v}_2 \in V_1$ as two distinct elements in V_1 ...
- ...and there must exist subsets J_1, J_2 of $I \setminus \{1\}$ such that

$$f((V_1 \setminus \{\bar{v}_1\}) \cup V(J_1)) < |J_1| + 1, \tag{7.31}$$

$$f((V_1 \setminus \{\bar{v}_2\}) \cup V(J_2)) < |J_2| + 1, \tag{7.32}$$

(note that either one or both of J_1, J_2 could be empty).

...

More general conditions for existence of transversals

Lemma 7.6.5 (contraction lemma)

Suppose Eq. 7.26 ($f(V(J)) \geq |J|, \forall J \subseteq I$) is true for $\mathcal{V} = (V_i : i \in I)$, and there exists an i such that $|V_i| \geq 2$ (w.l.o.g., say $i = 1$). Then there exists $\bar{v} \in V_1$ such that the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 7.26.

Proof.

- Taking $X = (V_1 \setminus \{\bar{v}_1\}) \cup V(J_1)$ and $Y = (V_1 \setminus \{\bar{v}_2\}) \cup V(J_2)$, we have $f(X) \leq |J_1|$, $f(Y) \leq |J_2|$, and that:

$$X \cup Y = V_1 \cup V(J_1 \cup J_2), \qquad (7.33)$$

$$X \cap Y \supseteq V(J_1 \cap J_2), \qquad (7.34)$$

and

$$|J_1| + |J_2| \geq f(X) + f(Y)$$
$$\geq f(X \cup Y) + f(X \cap Y) \qquad (7.35)$$

...

Matroid Rank | More on Partition Matroid | System of Distinct Reps | Transversals | Transversal Matroid | Matroid and representation | Dual Matroid

More general conditions for existence of transversals

Lemma 7.6.5 (contraction lemma)

Suppose Eq. 7.26 ($f(V(J)) \geq |J|, \forall J \subseteq I$) is true for $\mathcal{V} = (V_i : i \in I)$, and there exists an i such that $|V_i| \geq 2$ (w.l.o.g., say $i = 1$). Then there exists $\bar{v} \in V_1$ such that the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 7.26.

Proof.

- since f submodular monotone non-decreasing, & Eqs 7.33-7.35,

$$|J_1| + |J_2| \geq f(V_1 \cup V(J_1 \cup J_2)) + f(V(J_1 \cap J_2)) \qquad (7.36)$$

- Since $\mathcal{V}$ satisfies Eq. 7.26, $1 \notin J_1 \cup J_2$, & Eqs 7.29-7.30, this gives

$$|J_1| + |J_2| \geq |J_1 \cup J_2| + 1 + |J_1 \cap J_2| \qquad (7.37)$$

which is a contradiction since cardinality is modular.

...

More general conditions for existence of transversals

Theorem 7.6.3 (Polymatroid transversal theorem)

If $\mathcal{V} = (V_i : i \in I)$ is a finite family of non-empty subsets of V, and $f : 2^V \to \mathbb{Z}_+$ is a non-negative, integral, monotone non-decreasing, and submodular function, then $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that

$$f(\cup_{i \in J}\{v_i\}) \geq |J| \text{ for all } J \subseteq I \tag{7.25}$$

if and only if

$$f(V(J)) \geq |J| \text{ for all } J \subseteq I \tag{7.26}$$

- Given Theorem 7.6.3, we immediately get Theorem 7.6.1 by taking $f(S) = |S|$ for $S \subseteq V$. *In which case, Eq. 7.25 requires the system of representatives to be distinct.*
- We get Theorem 7.6.2 by taking $f(S) = r(S)$ for $S \subseteq V$, the rank function of the matroid. *where, Eq. 7.25 insists the system of representatives is independent in M, and hence also distinct.*

More general conditions for existence of transversals

converse proof of Theorem 7.6.3.

- Conversely, suppose Eq. 7.26 is true.
- If each V_i is a singleton set, then the result follows immediately.
- W.l.o.g., let $|V_1| \geq 2$, then by Lemma 7.6.5, the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 7.26 for the right $\bar{v}$.
- We can continue to reduce the family, deleting elements from V_i for some i while $|V_i| \geq 2$, until we arrive at a family of singleton sets.
- This family will be the required system of representatives.

□

This theorem can be used to produce a variety of other results quite easily, and shows how submodularity is the key ingredient in its truth.

Transversal Matroid

Transversals, themselves, define a matroid.

Theorem 7.7.1

If $\mathcal{V}$ is a family of finite subsets of a ground set V, then the collection of partial transversals of $\mathcal{V}$ is the set of independent sets of a matroid $M = (V, \mathcal{V})$ on V.

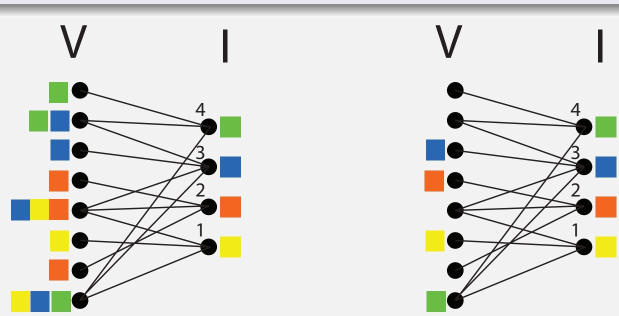
- This means that the transversals of $\mathcal{V}$ are the bases of matroid M.
- Therefore, all maximal partial transversals of $\mathcal{V}$ have the same cardinality!

Transversals and Bipartite Matchings

- Transversals correspond exactly to matchings in bipartite graphs.
- Given a set system $(V, \mathcal{V})$, with $\mathcal{V} = (V_i : i \in I)$, we can define a bipartite graph $G = (V, I, E)$ associated with $\mathcal{V}$ that has edge set $\{(v, i) : v \in V, i \in I, v \in V_i\}$.
- A matching in this graph is a set of edges no two of which that have a common endpoint. In fact, we easily have:

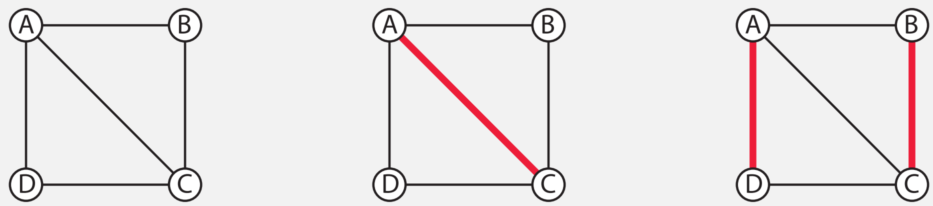
Lemma 7.7.2

A subset $T \subseteq V$ is a partial transversal of $\mathcal{V}$ iff there is a matching in (V, I, E) in which every edge has one endpoint in T (T matched into I).



Arbitrary Matchings and Matroids?

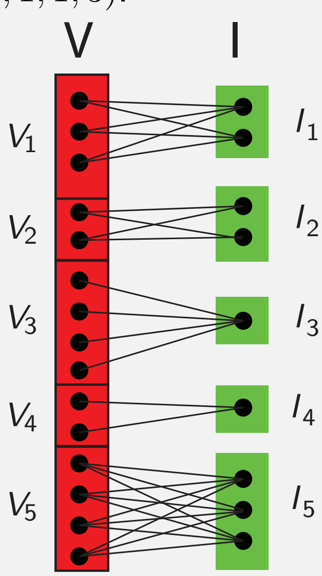
- Are arbitrary matchings matroids?
- Consider the following graph (left), and two max-matchings (two right instances)



- $\{AC\}$ is a maximum matching, as is $\{AD, BC\}$, but they are not the same size.

Partition Matroid, rank as matching

- Example where $\ell = 5$, $(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3)$.



- Recall, $\Gamma : 2^V \to \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of X is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.
- Here, for $X \subseteq V$, we have $\Gamma(X) = \{i \in I : (v, i) \in E(G) \text{ and } v \in X\}$.
- For such a constructed bipartite graph, the rank function of a partition matroid is $r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i) =$ the maximum matching involving X.

Morphing Partition Matroid Rank

- Recall the partition matroid rank function. Note, $k_i = |I_i|$ in the bipartite graph representation, and since a matroid, w.l.o.g., $|V_i| \geq k_i$ (also, recall, $V(J) = \cup_{j \in J} V_j$).
- Start with partition matroid rank function in the subsequent equations.

$$r(A) = \sum_{i \in \{1,\dots,\ell\}} \min(|A \cap V_i|, k_i) \quad (7.38)$$

$$= \sum_{i=1}^{\ell} \min(|A \cap V(I_i)|, |I_i|) \quad (7.39)$$

$$= \sum_{i \in \{1,\dots,\ell\}} \min_{J_i \in \{\emptyset, I_i\}} \left(\begin{Bmatrix} |A \cap V(I_i)| & \text{if } J_i \neq \emptyset \\ 0 & \text{if } J_i = \emptyset \end{Bmatrix} + |I_i \setminus J_i| \right) \quad (7.40)$$

$$= \sum_{i \in \{1,\dots,\ell\}} \min_{J_i \subseteq I_i} \left(\begin{Bmatrix} |A \cap V(I_i)| & \text{if } J_i \neq \emptyset \\ 0 & \text{if } J_i = \emptyset \end{Bmatrix} + |I_i \setminus J_i| \right) \quad (7.41)$$

$$= \sum_{i \in \{1,\dots,\ell\}} \min_{J_i \subseteq I_i} \left(|V(J_i) \cap A| + |I_i \setminus J_i| \right) \quad (7.42)$$

... Morphing Partition Matroid Rank

- Continuing,

$$r(A) = \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} \left(|V(J_i) \cap V(I_i) \cap A| - |I_i \cap J_i| + |I_i| \right) \quad (7.43)$$

$$= \min_{J \subseteq I} \left(\sum_{i=1}^{\ell} |V(J) \cap V(I_i) \cap A| - |I_i \cap J| + |I_i| \right) \quad (7.44)$$

$$= \min_{J \subseteq I} \left(|V(J) \cap V(I) \cap A| - |J| + |I| \right) \quad (7.45)$$

$$= \min_{J \subseteq I} \left(|V(J) \cap A| - |J| + |I| \right) \quad (7.46)$$

- In fact, this bottom (more general) expression is the expression for the rank of a transversal matroid.

Partial Transversals Are Independent Sets in a Matroid

In fact, we have

Theorem 7.7.3

Let $(V, \mathcal{V})$ where $\mathcal{V} = (V_1, V_2, \dots, V_\ell)$ be a subset system. Let $I = \{1, \dots, \ell\}$. Let $\mathcal{I}$ be the set of partial transversals of $\mathcal{V}$. Then $(V, \mathcal{I})$ is a matroid.

Proof.

- We note that $\emptyset \in \mathcal{I}$ since the empty set is a transversal of the empty subfamily of $\mathcal{V}$, thus (I1') holds.
- We already saw that if T is a partial transversal of $\mathcal{V}$, and if $T' \subseteq T$, then T' is also a partial transversal. So (I2') holds.
- Suppose that T_1 and T_2 are partial transversals of $\mathcal{V}$ such that $|T_1| < |T_2|$. Exercise: show that (I3') holds.

□

Transversal Matroid Rank

- Transversal matroid has rank

$$r(A) = \min_{J \subseteq I} \left(|V(J) \cap A| - |J| + |I| \right) \tag{7.47}$$

- Therefore, this function is submodular.
- Note that it is a minimum over a set of modular functions. Is this true in general? Exercise:

Matroid loops

- A circuit in a matroids is well defined, a subset $A \subseteq E$ is circuit if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).
- There is no reason in a matroid such an A could not consist of a single element.
- Such an $\{a\}$ is called a loop.
- In a matric (i.e., linear) matroid, the only such loop is the value $\mathbf{0}$, as all non-zero vectors have rank 1. The $\mathbf{0}$ can appear > 1 time with different indices, as can a self loop in a graph appear on different nodes.
- Note, we also say that two elements s, t are said to be parallel if $\{s, t\}$ is a circuit.

Representable

Definition 7.8.1 (Matroid isomorphism)

Two matroids M_1 and M_2 respectively on ground sets V_1 and V_2 are isomorphic if there is a bijection $\pi : V_1 \rightarrow V_2$ which preserves independence (equivalently, rank, circuits, and so on).

- Let $\mathbb{F}$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $\mathbb{F}$, such as a Galois field GF(p) where p is prime (such as GF(2)).
  Succinctly: A field is a set with $+$, $*$, closure, associativity, commutativity, and additive and multiplictaive identities and inverses.
- We can more generally define matroids on a field.

Definition 7.8.2 (linear matroids on a field)

Let $\mathbf{X}$ be an $n \times m$ matrix and $E = \{1, \ldots, m\}$, where $\mathbf{X}_{ij} \in \mathbb{F}$ for some field, and let $\mathcal{I}$ be the set of subsets of E such that the columns of $\mathbf{X}$ are linearly independent over $\mathbb{F}$.

Representable

Definition 7.8.1 (Matroid isomorphism)

Two matroids M_1 and M_2 respectively on ground sets V_1 and V_2 are isomorphic if there is a bijection $\pi : V_1 \to V_2$ which preserves independence (equivalently, rank, circuits, and so on).

- Let $\mathbb{F}$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $\mathbb{F}$, such as a Galois field GF(p) where p is prime (such as GF(2)).
  Succinctly: A field is a set with $+$, $*$, closure, associativity, commutativity, and additive and multiplictaive identities and inverses.
- We can more generally define matroids on a field.

Definition 7.8.3 (representable (as a linear matroid))

Any matroid isomorphic to a linear matroid on a field is called representable over $\mathbb{F}$

Representability of Transversal Matroids

- Piff and Welsh in 1970, and Adkin in 1972 proved an important theorem about representability of transversal matroids.
- In particular:

Theorem 7.8.4

Transversal matroids are representable over all finite fields of sufficiently large cardinality, and are representable over any infinite field.

Converse: Representability of Transversal Matroids

The converse is not true, however.

Example 7.8.5

Let $V = \{1, 2, 3, 4, 5, 6\}$ be a ground set and let $M = (V, \mathcal{I})$ be a set system where $\mathcal{I}$ is all subsets of V of cardinality ≤ 2 except for the pairs $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$.

- It can be shown that this is a matroid and is representable.
- However, this matroid is not isomorphic to any transversal matroid.

Matroids, other definitions using matroid rank $r : 2^V \to \mathbb{Z}_+$

Definition 7.9.3 (closed/flat/subspace)

A subset $A \subseteq E$ is closed (equivalently, a flat or a subspace) of matroid M if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

Definition: A hyperplane is a flat of rank $r(M) - 1$.

Definition 7.9.4 (closure)

Given $A \subseteq E$, the closure (or span) of A, is defined by $\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}$.

Therefore, a closed set A has $\text{span}(A) = A$.

Definition 7.9.5 (circuit)

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise-minimal dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

Spanning Sets

- We have the following definitions:

Definition 7.9.1 (spanning set of a set)

Given a matroid $\mathcal{M} = (V, \mathcal{I})$, and a set $Y \subseteq V$, then any set $X \subseteq Y$ such that $r(X) = r(Y)$ is called a spanning set of Y.

Definition 7.9.2 (spanning set of a matroid)

Given a matroid $\mathcal{M} = (V, \mathcal{I})$, any set $A \subseteq V$ such that $r(A) = r(V)$ is called a spanning set of the matroid.

- A base of a matroid is a minimal spanning set (and it is independent) but supersets of a base are also spanning.
- V is always trivially spanning.
- Consider the terminology: "spanning tree in a graph", comes from spanning in a matroid sense.

Dual of a Matroid

- Given a matroid $M = (V, \mathcal{I})$, a dual matroid $M^* = (V, \mathcal{I}^*)$ can be defined on the same ground set V, but using a very different set of independent sets $\mathcal{I}^*$.
- We define the set of sets $\mathcal{I}^*$ for M^* as follows:

$$\mathcal{I}^* = \{A \subseteq V : V \setminus A \text{ is a spanning set of } M\} \tag{7.48}$$

- That is, a set A is independent in the dual matroid M^* if removal of A from V does not decrease the rank in M:

$$\mathcal{I}^* = \{A \subseteq V : \text{rank}_M(V \setminus A) = \text{rank}_M(V)\} \tag{7.49}$$

- In other words, a set $A \subseteq V$ is independent in the dual M^* (i.e., $A \in \mathcal{I}^*$) if its complement is spanning in M (residual $V \setminus A$ must contain a base in M).
- Dual of the dual: Note, we have that $(M^*)^* = M$.