Submodular Functions, Optimization, and Applications to Machine Learning
— Spring Quarter, Lecture 7 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee596b_spring_2016/

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Apr 20th, 2016

\[ f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \]

\[ = f(A) + 2f(C) + f(B) \]

\[ = f(A) + f(C) + f(B) \]

\[ = f(A \cap B) \]
Cumulative Outstanding Reading

- Read chapters 2 and 3 from Fujishige’s book.
- Read chapter 1 from Fujishige’s book.
Announcements, Assignments, and Reminders


- Homework 1, available at our assignment dropbox (https://canvas.uw.edu/courses/1039754/assignments), due (electronically) Friday (4/8) at 11:55pm.

- Weekly Office Hours: Mondays, 3:30-4:30, or by skype or google hangout (set up meeting via our discussion board (https://canvas.uw.edu/courses/1039754/discussion_topics)).
Class Road Map - IT-I

L1(3/28): Motivation, Applications, & Basic Definitions
L3(4/4): Info theory exs, more apps, definitions, graph/combinatorial examples, matrix rank example, visualization
L4(4/6): Graph and Combinatorial Examples, matrix rank, Venn diagrams, examples of proofs of submodularity, some useful properties
L5(4/11): Examples & Properties, OtherDefs., Independence
L6(4/13): Independence, Matroids, Matroid Examples, matroid rank is submodular
L8(4/20):
L9(4/25):
L10(4/27):
L11(5/2):
L12(5/4):
L13(5/9):
L14(5/11):
L15(5/16):
L16(5/18):
L17(5/23):
L18(5/25):
L19(6/1):
L20(6/6): Final Presentations maximization.

Finals Week: June 6th-10th, 2016.
Independent set definition of a matroid is perhaps most natural. Note, if \( J \in \mathcal{I} \), then \( J \) is said to be an independent set.

**Definition 7.2.3 (Matroid)**

A set system \((E, \mathcal{I})\) is a **Matroid** if

1. \( \emptyset \in \mathcal{I} \)
2. \( \forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I} \)
3. \( \forall I, J \in \mathcal{I}, \text{ with } |I| = |J| + 1, \text{ then there exists } x \in I \setminus J \text{ such that } J \cup \{x\} \in \mathcal{I}. \)

Why is (I1) is not redundant given (I2)? Because without (I1) could have a non-matroid where \( \mathcal{I} = \{\} \).
Matroids - important property

Proposition 7.2.3

In a matroid $M = (E, I)$, for any $U \subseteq E(M)$, any two bases of $U$ have the same size.

- In matrix terms, given a set of vectors $U$, all sets of independent vectors that span the space spanned by $U$ have the same size.
- In fact, under (I1),(I2), this condition is equivalent to (I3). Exercise: show the following is equivalent to the above.

Definition 7.2.4 (Matroid)

A set system $(V, I)$ is a Matroid if

(I1') $\emptyset \in I$ (emptyset containing)

(I2') $\forall I \in I, J \subseteq I \Rightarrow J \in I$ (down-closed or subclusive)

(I3') $\forall X \subseteq V$, and $I_1, I_2 \in \text{maxInd}(X)$, we have $|I_1| = |I_2|$ (all maximally independent subsets of $X$ have the same size).
Partition Matroid

- Let $V$ be our ground set.
- Let $V = V_1 \cup V_2 \cup \cdots \cup V_\ell$ be a partition of $V$ into $\ell$ blocks (i.e., disjoint sets). Define a set of subsets of $V$ as

$$\mathcal{I} = \{ X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell \}. \quad (7.5)$$

where $k_1, \ldots, k_\ell$ are fixed parameters, $k_i \geq 0$. Then $M = (V, \mathcal{I})$ is a matroid.

- Note that a $k$-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$.

- Parameters associated with a partition matroid: $\ell$ and $k_1, k_2, \ldots, k_\ell$ although often the $k_i$’s are all the same.

- We’ll show that property (I3’) in Def ?? holds. First note, for any $X \subseteq V$, $|X| = \sum_{i=1}^{\ell} |X \cap V_i|$ since $\{V_1, V_2, \ldots, V_\ell\}$ is a partition.

- If $X, Y \in \mathcal{I}$ with $|Y| > |X|$, then there must be at least one $i$ with $|Y \cap V_i| > |X \cap V_i|$. Therefore, adding one element $e \in V_i \cap (Y \setminus X)$ to $X$ won’t break independence.
The rank function \( r : 2^E \rightarrow \mathbb{Z}_+ \) of a matroid is submodular, that is,
\[
r(A) + r(B) \geq r(A \cup B) + r(A \cap B)
\]

**Proof.**

1. Let \( X \in \mathcal{I} \) be an inclusionwise maximal set with \( X \subseteq A \cap B \).
2. Let \( Y \in \mathcal{I} \) be inclusionwise maximal set with \( X \subseteq Y \subseteq A \cup B \).
3. Since \( M \) is a matroid, we know that \( r(A \cap B) = r(X) = |X| \), and \( r(A \cup B) = r(Y) = |Y| \). Also, for any \( U \in \mathcal{I} \), \( r(A) \geq |A \cap U| \).
4. Then we have (since \( X \subseteq A \cap B \), \( X \subseteq Y \), and \( Y \subseteq A \cup B \)),
\[
r(A) + r(B) \geq |Y \cap A| + |Y \cap B| \tag{7.5}
\]
\[
= |Y \cap (A \cap B)| + |Y \cap (A \cup B)| \tag{7.6}
\]
\[
\geq |X| + |Y| = r(A \cap B) + r(A \cup B) \tag{7.7}
\]
A matroid is defined from its rank function

**Theorem 7.2.3 (Matroid from rank)**

Let $E$ be a set and let $r : 2^E \rightarrow \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with $r$ being its rank function if and only if for all $A, B \subseteq E$:

1. **(R1)** $\forall A \subseteq E \quad 0 \leq r(A) \leq |A|$ (non-negative cardinality bounded)
2. **(R2)** $r(A) \leq r(B)$ whenever $A \subseteq B \subseteq E$ (monotone non-decreasing)
3. **(R3)** $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ for all $A, B \subseteq E$ (submodular)

- So submodularity and non-negative monotone non-decreasing, and unit increase is necessary and sufficient to define the matroid.
- Can name matroid as $(E, r)$, $E$ is ground set, $r$ is rank function.
- Given above, unit increment (if $r(A) = k$, then either $r(A \cup \{v\}) = k$ or $r(A \cup \{v\}) = k + 1$) holds.
- From above, $r(\emptyset) = 0$. Let $v \notin A$, then by monotonicity and submodularity, $r(A) \leq r(A \cup \{v\}) \leq r(A) + r(\{v\})$ which gives only two possible values to $r(A \cup \{v\})$. 

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Proof of Theorem ?? (matroid from rank).

- Given a matroid $M = (E, I)$, we see its rank function as defined in Eq. ?? satisfies (R1), (R2), and, as we saw in Lemma ??, (R3) too.
Proof of Theorem ?? (matroid from rank).

- Given a matroid $M = (E, \mathcal{I})$, we see its rank function as defined in Eq. ?? satisfies (R1), (R2), and, as we saw in Lemma ??, (R3) too.

- Next, assume we have (R1), (R2), and (R3). Define $\mathcal{I} = \{X \subseteq E : r(X) = |X|\}$. We will show that $(E, \mathcal{I})$ is a matroid.

...
Matroids from rank

Proof of Theorem ?? (matroid from rank).

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- First, $\emptyset \in \mathcal{I}$.
Proof of Theorem ?? (matroid from rank).

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- Next, assume we have (R1), (R2), and (R3). Define
  \[ I = \{ X \subseteq E : r(X) = |X| \}. \]
  We will show that $(E, I)$ is a matroid.
- First, $\emptyset \in I$.
- Also, if $Y \in I$ and $X \subseteq Y$ then by submodularity,
Proof of Theorem ?? (matroid from rank).

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- Next, assume we have (R1), (R2), and (R3). Define $\mathcal{I} = \{X \subseteq E : r(X) = |X|\}$. We will show that $(E, \mathcal{I})$ is a matroid.

- First, $\emptyset \in \mathcal{I}$.

- Also, if $Y \in \mathcal{I}$ and $X \subseteq Y$ then by submodularity,

$$r(X) \geq r(Y) - r(Y \setminus X) \quad \text{(7.1)}$$
Proof of Theorem ?? (matroid from rank).

- Given a matroid $M = (E, \mathcal{I})$, we see its rank function as defined in Eq. ?? satisfies (R1), (R2), and, as we saw in Lemma ??, (R3) too.
- Next, assume we have (R1), (R2), and (R3). Define $\mathcal{I} = \{X \subseteq E : r(X) = |X|\}$. We will show that $(E, \mathcal{I})$ is a matroid.
- First, $\emptyset \in \mathcal{I}$.
- Also, if $Y \in \mathcal{I}$ and $X \subseteq Y$ then by submodularity,

$$r(X) \geq r(Y) - r(Y \setminus X) + r(\emptyset) \tag{7.1}$$

\[\text{...}\]
Proof of Theorem ?? (matroid from rank).

- Given a matroid $M = (E, \mathcal{I})$, we see its rank function as defined in Eq. ?? satisfies (R1), (R2), and, as we saw in Lemma ??, (R3) too.
- Next, assume we have (R1), (R2), and (R3). Define $\mathcal{I} = \{ X \subseteq E : r(X) = |X| \}$. We will show that $(E, \mathcal{I})$ is a matroid.
- First, $\emptyset \in \mathcal{I}$.
- Also, if $Y \in \mathcal{I}$ and $X \subseteq Y$ then by submodularity,

$$r(X) \geq r(Y) - r(Y \setminus X) + r(\emptyset) \geq |Y| - |Y \setminus X|$$  \hspace{1cm} (7.1)

$$r(Y \setminus X) \leq |Y \setminus X|$$  \hspace{1cm} (7.2)
Proof of Theorem ?? (matroid from rank).

- Given a matroid $M = (E, \mathcal{I})$, we see its rank function as defined in Eq. ?? satisfies (R1), (R2), and, as we saw in Lemma ??, (R3) too.

- Next, assume we have (R1), (R2), and (R3). Define $\mathcal{I} = \{ X \subseteq E : r(X) = |X| \}$. We will show that $(E, \mathcal{I})$ is a matroid.

- First, $\emptyset \in \mathcal{I}$.

- Also, if $Y \in \mathcal{I}$ and $X \subseteq Y$ then by submodularity,

\[
r(X) \geq r(Y) - r(Y \setminus X) + r(\emptyset)
\geq |Y| - |Y \setminus X|
= |X|
\]  

(7.1)  

(7.2)  

(7.3)
Proof of Theorem 7.1 (matroid from rank).

- Given a matroid $M = (E, I)$, we see its rank function as defined in Eq. 7.1 satisfies (R1), (R2), and, as we saw in Lemma 7.1, (R3) too.

- Next, assume we have (R1), (R2), and (R3). Define 
  \[ I = \{ X \subseteq E : r(X) = |X| \}. \]
  We will show that $(E, I)$ is a matroid.

- First, $\emptyset \in I$.

- Also, if $Y \in I$ and $X \subseteq Y$ then by submodularity,
  \[
  r(X) \geq r(Y) - r(Y \setminus X) + r(\emptyset) \\
  \geq |Y| - |Y \setminus X| \\
  = |X|
  \]

  implying $r(X) = |X|$, and thus $X \in I$. ...
Proof of Theorem ?? (matroid from rank) cont.

Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).
Proof of Theorem ?? (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).

- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A| < |A + b|$. Then...
Proof of Theorem ?? (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).

- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \not\in \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A| < |A + b|$. Then

$$r(B) \leq r(A \cup B) \quad (7.4)$$
Proof of Theorem ?? (matroid from rank) cont.

Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).

Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A| < |A + b|$. Then

$$r(B) \leq r(A \cup B)$$

$$\leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A)$$

$$r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) \geq r(C \cup D)$$

$$+ r(C \cup D)$$

Giving a contradiction since $B \notin \mathcal{I}$.
Matroids from rank

Proof of Theorem ?? (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).

- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A| < |A + b|$. Then

\[
\begin{align*}
    r(B) &\leq r(A \cup B) \\
    &\leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) \\
    &= r(A \cup (B \setminus \{b_1\}))
\end{align*}
\]  

(7.4) (7.5) (7.6)
Proof of Theorem ?? (matroid from rank) cont.

Let \( A, B \in \mathcal{I} \), with \( |A| < |B| \), so \( r(A) = |A| < r(B) = |B| \). Let \( B \setminus A = \{b_1, b_2, \ldots, b_k\} \) (note \( k \leq |B| \)).

Suppose, to the contrary, that \( \forall b \in B \setminus A \), \( A + b \notin \mathcal{I} \), which means for all such \( b \), \( r(A + b) = r(A) = |A| < |A + b| \). Then

\[
\begin{align*}
r(B) &\leq r(A \cup B) \\
&\leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) \\
&= r(A \cup (B \setminus \{b_1\})) \\
&\leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A)
\end{align*}
\]  

(7.4)  
(7.5)  
(7.6)  
(7.7)
Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).

Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A| < |A + b|$. Then

$$r(B) \leq r(A \cup B) \leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) = r(A \cup (B \setminus \{b_1\})) \leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) = r(A \cup (B \setminus \{b_1, b_2\}))$$

(7.4) (7.5) (7.6) (7.7) (7.8)
Proof of Theorem ?? (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).

- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A| < |A + b|$. Then

$$r(B) \leq r(A \cup B)$$

$$\leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A)$$

$$= r(A \cup (B \setminus \{b_1\}))$$

$$\leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A)$$

$$= r(A \cup (B \setminus \{b_1, b_2\}))$$

$$\leq \ldots \leq r(A) = |A| < |B|$$
Proof of Theorem ?? (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).

- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A| < |A + b|$. Then

\[
    r(B) \leq r(A \cup B) \\
    \leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) \\
    = r(A \cup (B \setminus \{b_1\})) \\
    \leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) \\
    = r(A \cup (B \setminus \{b_1, b_2\})) \\
    \leq \ldots \leq r(A) = |A| < |B|
\]

(7.4) \hspace{1cm} (7.5) \hspace{1cm} (7.6) \hspace{1cm} (7.7) \hspace{1cm} (7.8) \hspace{1cm} (7.9)

giving a contradiction since $B \in \mathcal{I}$. 

\[\square\]
Another way of using function $r$ to define a matroid.

**Theorem 7.3.1 (Matroid from rank II)**

Let $E$ be a finite set and let $r : 2^E \rightarrow \mathbb{Z}_+^+$ be a function. Then $r(\cdot)$ defines a matroid with $r$ being its rank function if and only if for all $A \subseteq E$, and $x, y \in E$:

\begin{align*}
(R1') \quad & r(\emptyset) = 0; \\
(R2') \quad & r(X) \leq r(X \cup \{y\}) \leq r(X) + 1; \\
(R3') \quad & \text{If } r(X \cup \{x\}) = r(X \cup \{y\}) = r(X), \text{ then } r(X \cup \{x,y\}) = r(X). \\
\end{align*}
Theorem 7.3.2 (Matroid by submodular functions)

Let \( f : 2^E \rightarrow \mathbb{Z} \) be an integer-valued monotone non-decreasing submodular function. Define a set of sets as follows:

\[
\mathcal{C}(f) = \left\{ C \subseteq E : C \text{ is non-empty, is inclusionwise-minimal, and has } f(C) < |C| \right\}
\]

Then \( \mathcal{C}(f) \) is the collection of circuits of a matroid on \( E \).

Inclusionwise-minimal in this case means that if \( C \in \mathcal{C}(f) \), then there exists no \( C' \subseteq C \) with \( C' \in \mathcal{C}(f) \) (i.e., \( C' \subseteq C \) would either be empty or have \( f(C') \geq |C'| \)). Also, recall inclusionwise-minimal in Definition ??, the definition of a circuit.
Summarizing: Many ways to define a Matroid

Summarizing what we’ve so far seen, we saw that it is possible to uniquely define a matroid based on any of:

- Independence (define the independent sets).
Summarizing: Many ways to define a Matroid

Summarizing what we’ve so far seen, we saw that it is possible to uniquely define a matroid based on any of:

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- Base axioms (exchangeability)
Summarizing: Many ways to define a Matroid

Summarizing what we’ve so far seen, we saw that it is possible to uniquely define a matroid based on any of:

- Independence (define the independent sets).
- Base axioms (exchangeability)
- Circuit axioms
Summarizing what we’ve so far seen, we saw that it is possible to uniquely define a matroid based on any of:

- Independence (define the independent sets).
- Base axioms (exchangeability)
- Circuit axioms
- Closure axioms (we didn’t see this, but it is possible)
Summarizing: Many ways to define a Matroid

Summarizing what we’ve so far seen, we saw that it is possible to uniquely define a matroid based on any of:

- Independence (define the independent sets).
- Base axioms (exchangeability)
- Circuit axioms
- Closure axioms (we didn’t see this, but it is possible)
- Rank axioms (normalized, monotone, cardinality bounded, submodular)
Summarizing what we’ve so far seen, we saw that it is possible to uniquely define a matroid based on any of:

- Independence (define the independent sets).
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- Closure axioms (we didn’t see this, but it is possible)
- Rank axioms (normalized, monotone, cardinality bounded, submodular)
- Matroids by submodular functions.
Maximization problems for matroids

- Given a matroid $M = (E, I)$ and a modular cost function $c : E \rightarrow \mathbb{R}$, the task is to find an $X \in I$ such that $c(X) = \sum_{x \in X} c(x)$ is maximum.
- This seems remarkably similar to the max spanning tree problem.
Minimization problems for matroids

- Given a matroid $M = (E, \mathcal{I})$ and a modular cost function $c : E \to \mathbb{R}$, the task is to find a basis $B \in \mathcal{B}$ such that $c(B)$ is minimized.

- This sounds like a set cover problem (find the minimum cost covering set of sets).
What is the partition matroid’s rank function?

\[ r(A) = \sum_{i=1}^{\infty} \min(\vert A \setminus V_i \vert, k_i) \] (7.11)

which we also immediately see is submodular using properties we spoke about last week. That is:

- \( \vert A \setminus V_i \vert \) is submodular (in fact modular) in \( A \)
- \( \min(\text{submodular}(A), k_i) \) is submodular in \( A \) since \( \vert A \setminus V_i \vert \) is monotone.
- Sums of submodular functions are submodular.

\( r(A) \) is also non-negative integral monotone non-decreasing, so it defines a matroid (the partition matroid).
What is the partition matroid’s rank function?

A partition matroids rank function:

\[
    r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i) \tag{7.11}
\]

which we also immediately see is submodular using properties we spoke about last week. That is:
What is the partition matroid’s rank function?

A partition matroids rank function:

\[ r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i) \]  

(7.11)

which we also immediately see is submodular using properties we spoke about last week. That is:

1. \(|A \cap V_i|\) is submodular (in fact modular) in \(A\)

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\]  

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1. \(|A \cap V_i|\) is submodular (in fact modular) in \(A\)
2. \(\min(\text{submodular}(A), k_i)\) is submodular in \(A\) since \(|A \cap V_i|\) is monotone.
What is the partition matroid’s rank function?

A partition matroids rank function:

\[ r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i) \]  

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1. \(|A \cap V_i|\) is submodular (in fact modular) in \(A\)
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3. sums of submodular functions are submodular.
What is the partition matroid’s rank function?

A partition matroids rank function:

\[
r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i)
\]  

which we also immediately see is submodular using properties we spoke about last week. That is:

1. \(|A \cap V_i|\) is submodular (in fact modular) in \(A\)
2. \(\min(\text{submodular}(A), k_i)\) is submodular in \(A\) since \(|A \cap V_i|\) is monotone.
3. Sums of submodular functions are submodular.

\(r(A)\) is also non-negative integral monotone non-decreasing, so it defines a matroid (the partition matroid).
From 2-partition matroid rank to truncated matroid rank

Example: 2-partition matroid rank function: Given natural numbers $a, b \in \mathbb{Z}^+$ with $a < b$, and any set $R \subseteq V$ with $|R| = b$. 
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Create two-block partition $V = (R, \overline{R})$, where $\overline{R} = V \setminus R$ so $|\overline{R}| = |V| - b$. Gives 2-partition matroid rank function as follows:

$$r(A) = \min(|A \cap R|, a) + \min(|A \cap \overline{R}|, |\overline{R}|)$$  \hspace{1cm} (7.12)

$$= \min(|A \cap R|, a) + |A \cap \overline{R}|$$  \hspace{1cm} (7.13)

$$= \min(|A \cap \overline{R}| + |A \cap R|, |A \cap \overline{R}| + a)$$  \hspace{1cm} (7.14)

$$= \min(|A|, |A \cap \overline{R}| + a)$$  \hspace{1cm} (7.15)

$$k_R = a \quad k_{\overline{R}} = |R|$$
From 2-partition matroid rank to truncated matroid rank

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(7.15)

- Figure showing partition blocks and partition matroid limits.

Since $|\bar{R}| = |V| - b$

the limit on $\bar{R}$ is vacuous.

$a < |R| = b$
Example: 2-partition matroid rank function: Given natural numbers \( a, b \in \mathbb{Z}_+ \) with \( a < b \), and any set \( R \subseteq V \) with \( |R| = b \).

Create two-block partition \( V = (R, \bar{R}) \), where \( \bar{R} = V \setminus R \) so \( |\bar{R}| = |V| - b \). Gives 2-partition matroid rank function as follows:

\[
\begin{align*}
    r(A) &= \min(|A \cap R|, a) + \min(|A \cap \bar{R}|, |\bar{R}|) \\
    &= \min(|A \cap R|, a) + |A \cap \bar{R}| \\
    &= \min(|A \cap \bar{R}| + |A \cap R|, |A \cap \bar{R}| + a) \\
    &= \min(|A|, |A \cap \bar{R}| + a)
\end{align*}
\]

Figure showing partition blocks and partition matroid limits.

Since \( |\bar{R}| = |V| - b \)

the limit on \( \bar{R} \) is vacuous.

\[ a < |R| = b \]
Define truncated matroid rank function. Start with 2-partition matroid rank \( r(A) = \min(|A \cap R|, a) + \min(|A \cap \bar{R}|, |\bar{R}|), a < b \). Define:

\[
f_R(A) = \min \left\{ r(A), b \right\}
\]

(7.16)

\[
= \min \left\{ \min(|A|, |A \cap \bar{R}| + a), b \right\}
\]

(7.17)

\[
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Defines a matroid \( M = (V, f_R) = (V, \mathcal{I}) \) (Goemans et. al.) with 
\[
\mathcal{I} = \{ I \subseteq V : |I| \leq b \text{ and } |I \cap R| \leq a \},
\]

(7.19)
**Truncated Matroid Rank Function**

- Define **truncated matroid rank** function. Start with 2-partition matroid rank \( r(A) = \min(|A \cap R|, a) + \min(|A \cap \bar{R}|, |\bar{R}|) \), \( a < b \). Define:

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Useful for showing hardness of constrained submodular minimization.

Consider sets \( B \subseteq V \) with \( |B| = b \).
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Truncated Matroid Rank Function

- **Define truncated matroid rank** function. Start with 2-partition matroid rank $r(A) = \min(|A \cap R|, a) + \min(|A \cap \bar{R}|, |\bar{R}|)$, $a < b$. Define:

  \[ f_R(A) = \min \left\{ r(A), b \right\} \tag{7.16} \]

  \[ = \min \left\{ \min(|A|, |A \cap \bar{R}| + a), b \right\} \tag{7.17} \]

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  Useful for showing hardness of constrained submodular minimization.

Consider sets $B \subseteq V$ with $|B| = b$.

- For $R$, we have $f_R(R) = \min(b, a, b) = a < b$.
- For any $B$ with $|B \cap R| \leq a$, $f_R(B) = b$.
- For any $B$ with $|B \cap R| = \ell$, with $a \leq \ell \leq b$, $f_R(B) = a + b - \ell$. 

Truncated Matroid Rank Function

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- Useful for showing hardness of constrained submodular minimization.

Consider sets \( B \subseteq V \) with \( |B| = b \).

- For \( R \), we have \( f_R(R) = \min(b, a, b) = a < b \).
- For any \( B \) with \( |B \cap R| \leq a \), \( f_R(B) = b \).
- For any \( B \) with \( |B \cap R| = \ell \), with \( a \leq \ell \leq b \), \( f_R(B) = a + b - \ell \).
- \( R \), the set with minimum valuation amongst size-\( b \) sets, is hidden within an exponentially larger set of size-\( b \) sets with larger valuation.
A partition matroid can be viewed using a bipartite graph.

Letting $V$ denote the ground set, and $V_1, V_2, \ldots$ the partition, the graph is $G = (V, I, E)$ where $V$ is the ground set, $I$ is a set of “indices”, and $E$ is the set of edges.

$I = (I_1, I_2, \ldots, I_\ell)$ is a set of $k = \sum_{i=1}^\ell k_i$ nodes, grouped into $\ell$ clusters, where there are $k_i$ nodes in the $i^{\text{th}}$ group $I_i$.

$(v, i) \in E(G)$ iff $v \in V_j$ and $i \in I_j$. $|I_i| = k_i$. 
Example where $\ell = 5$, $(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3)$.

Recall, $\mathcal{V}$ is the neighbor function in a bipartite graph, the neighbors of $X$ is defined as $\{v \in \mathcal{V}(G) : (X, \{v\}) \notin E\}$, and recall that $|\mathcal{V}(X)|$ is submodular.

Here, for $X \subseteq \mathcal{V}$, we have $\mathcal{V}(X) = \{i \in \mathcal{I} : (v, i) \in E(G) \text{ and } v \in X\}$.

For such a constructed bipartite graph, the rank function of a partition matroid is $r(X) = \max$ involving $X$. 
Example where $\ell = 5$, 
$$(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3).$$

Recall, $\Gamma : 2^V \rightarrow \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of $X$ is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.
Example where $\ell = 5$, 

$$(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3).$$

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For such a constructed bipartite graph, the rank function of a partition matroid is $r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i) =$ the maximum matching involving $X$. 
Laminar Family and Laminar Matroid

- We can define a matroid with structures richer than just partitions.
Laminar Family and Laminar Matroid

- We can define a matroid with structures richer than just partitions.

- A set system \((V, \mathcal{F})\) is called a **laminar** family if for any two sets \(A, B \in \mathcal{F}\), at least one of the three sets \(A \cap B\), \(A \setminus B\), or \(B \setminus A\) is empty.
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A set system \((V, \mathcal{F})\) is called a \textit{laminar} family if for any two sets \(A, B \in \mathcal{F}\), at least one of the three sets \(A \cap B, A \setminus B, \text{ or } B \setminus A\) is empty.

Family is laminar \(\exists\) no two \underline{properly} intersecting members: \(\forall A, B \in \mathcal{F}\), either \(A, B\) disjoint \((A \cap B = \emptyset)\) or comparable \((A \subseteq B \text{ or } B \subseteq A)\).
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- Suppose we have a laminar family \(\mathcal{F}\) of subsets of \(V\) and an integer \(k_A\) for every set \(A \in \mathcal{F}\).
Laminar Family and Laminar Matroid

- We can define a matroid with structures richer than just partitions.
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![Laminar Family Diagram]

- Family is laminar \(\exists\) no two properly intersecting members: \(\forall A, B \in F\), either \(A, B\) disjoint \((A \cap B = \emptyset)\) or comparable \((A \subseteq B \text{ or } B \subseteq A)\).
- Suppose we have a laminar family \(F\) of subsets of \(V\) and an integer \(k_A\) for every set \(A \in F\). Then \((V, \mathcal{I})\) defines a matroid where

\[
\mathcal{I} = \{ I \subseteq E : |I \cap A| \leq k_A \text{ for all } A \in F \}\]  

\((7.20)\)
Laminar Family and Laminar Matroid

- We can define a matroid with structures richer than just partitions.
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![Image of laminar family](image.png)

- Family is laminar \(\exists\) no two \textit{properly} intersecting members: \(\forall A, B \in \mathcal{F}\), either \(A, B\) disjoint \((A \cap B = \emptyset)\) or comparable \((A \subseteq B\) or \(B \subseteq A\)).
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- Exercise: what is the rank function here?
Let \((V, \mathcal{V})\) be a set system (i.e., \(\mathcal{V} = (V_i : i \in I)\) where \(\emptyset \subset V_i \subseteq V\) for all \(i\), and \(I\) is an index set. Hence, \(|I| = |\mathcal{V}|\).
System of Representatives

- Let \((V, \mathcal{V})\) be a set system (i.e., \(\mathcal{V} = (V_i : i \in I)\) where \(\emptyset \subset V_i \subseteq V\) for all \(i\)), and \(I\) is an index set. Hence, \(|I| = |\mathcal{V}|\).

- Here, the sets \(V_i \in \mathcal{V}\) are like “groups” and any \(v \in V\) with \(v \in V_i\) is a member of group \(i\). Groups need not be disjoint (e.g., interest groups of individuals).
Let \((V, \mathcal{V})\) be a set system (i.e., \(\mathcal{V} = (V_i : i \in I)\) where \(\emptyset \subset V_i \subset V\) for all \(i\)), and \(I\) is an index set. Hence, \(|I| = |\mathcal{V}|\).

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A family \((v_i : i \in I)\) with \(v_i \in V\) is said to be a system of representatives of \(\mathcal{V}\) if \(\exists\) a bijection \(\pi : I \rightarrow I\) such that \(v_i \in V_{\pi(i)}\).

\[\forall i \neq j, v_i = v_j \quad v_i, v_j \in V\]
Let \((V, \mathcal{V})\) be a set system (i.e., \(\mathcal{V} = (V_i : i \in I)\) where \(\emptyset \subset V_i \subseteq V\) for all \(i\)), and \(I\) is an index set. Hence, \(|I| = |\mathcal{V}|\).

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\(v_i\) is the representative of set (or group) \(V_{\pi(i)}\), meaning the \(i^{th}\) representative is meant to represent set \(V_{\pi(i)}\).
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- A family \((v_i : i \in I)\) with \(v_i \in V\) is said to be a system of representatives of \(\mathcal{V}\) if there exists a bijection \(\pi : I \to I\) such that \(v_i \in V_{\pi(i)}\).
- \(v_i\) is the representative of set (or group) \(V_{\pi(i)}\), meaning the \(i^{th}\) representative is meant to represent set \(V_{\pi(i)}\).
- Example: Consider the house of representatives, \(v_i = “Jim McDermott”\), while \(i = “King County, WA-7”\).
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- Example: Consider the house of representatives, \(v_i = “Jim McDermott”\), while \(i = “King County, WA-7”\).
- In a system of representatives, there is no requirement for the representatives to be distinct. I.e., we could have some \(v_1 \in V_1 \cap V_2\), where \(v_1\) represents both \(V_1\) and \(V_2\).
Let \( (V, \mathcal{V}) \) be a set system (i.e., \( \mathcal{V} = (V_i : i \in I) \) where \( \emptyset \subset V_i \subseteq V \) for all \( i \)), and \( I \) is an index set. Hence, \(|I| = |\mathcal{V}| \).

Here, the sets \( V_i \in \mathcal{V} \) are like “groups” and any \( v \in V \) with \( v \in V_i \) is a member of group \( i \). Groups need not be disjoint (e.g., interest groups of individuals).

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We can view this as a bipartite graph.
We can view this as a bipartite graph. The groups of $V$ are marked by color tags on the left, and also via right neighbors in the graph.

Here, $\ell = 6$ groups, with $\mathcal{V} = (V_1, V_2, \ldots, V_6)$:

$$\mathcal{V} = (\{e, f, h\}, \{d, e, g\}, \{b, c, e, h\}, \{a, b, h\}, \{a\}, \{a\}).$$

A system of representatives would make sure that there is a representative for each color group. For example, the representatives ( \{a, c, d, f, h\} ) are shown as colors on the left.

Here, the set of representatives is not distinct. Why?

In fact, due to the red and pink group, a distinct group of representatives is impossible (since there is only one common choice to represent both color groups).
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A system of representatives would make sure that there is a representative for each color group. For example,

- The representatives ($\{a, c, d, f, h\}$) are shown as colors on the left.
- Here, the set of representatives is not distinct. Why? In fact, due to the red and pink group, a distinct group of representatives is impossible (since there is only one common choice to represent both color groups).
Let \((V, \mathcal{V})\) be a set system (i.e., \(\mathcal{V} = (V_i : i \in I)\) where \(V_i \subseteq V\) for all \(i\)), and \(I\) is an index set. Hence, \(|I| = |\mathcal{V}|\).
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A family \((v_i : i \in I)\) with \(v_i \in V\) is said to be a system of distinct representatives of \(\mathcal{V}\) if \(\exists\) a bijection \(\pi : I \leftrightarrow I\) such that \(v_i \in V_{\pi(i)}\) and \(v_i \neq v_j\) for all \(i \neq j\).
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A family \((v_i : i \in I)\) with \(v_i \in V\) is said to be a system of **distinct representatives** of \(\mathcal{V}\) if \(\exists\) a bijection \(\pi : I \leftrightarrow I\) such that \(v_i \in V_{\pi(i)}\) and \(v_i \neq v_j\) for all \(i \neq j\).

In a system of distinct representatives, there is a requirement for the representatives to be distinct. We can re-state (and rename) this as a:
System of Distinct Representatives

- Let \((V, \mathcal{V})\) be a set system (i.e., \(\mathcal{V} = (V_k : i \in I)\) where \(V_i \subseteq V\) for all \(i\)), and \(I\) is an index set. Hence, \(|I| = |\mathcal{V}|\).
- A family \((v_i : i \in I)\) with \(v_i \in V\) is said to be a system of distinct representatives of \(\mathcal{V}\) if \(\exists\) a bijection \(\pi : I \leftrightarrow I\) such that \(v_i \in V_{\pi(i)}\) and \(v_i \neq v_j\) for all \(i \neq j\).
- In a system of distinct representatives, there is a requirement for the representatives to be distinct. We can re-state (and rename) this as a:

**Definition 7.5.1 (transversal)**

Given a set system \((V, \mathcal{V})\) and index set \(I\) for \(\mathcal{V}\) as defined above, a set \(T \subseteq V\) is a transversal of \(\mathcal{V}\) if there is a bijection \(\pi : T \leftrightarrow I\) such that

\[
x \in V_{\pi(x)} \text{ for all } x \in T
\] (7.21)
Let \((V, \mathcal{V})\) be a set system (i.e., \(\mathcal{V} = (V_k : i \in I)\) where \(V_i \subseteq V\) for all \(i\)), and \(I\) is an index set. Hence, \(|I| = |\mathcal{V}|\).

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(7.21)

Note that due to \(\pi : T \leftrightarrow I\) being a bijection, all of \(I\) and \(T\) are “covered” (so this makes things distinct automatically).
A set $T' \subseteq V$ is a **partial transversal** if $T'$ is a transversal of some subfamily $\mathcal{V}' = (V_i : i \in I')$ where $I' \subseteq I$. 

Therefore, for any transversal $T$, any subset $T' \subseteq T$ is a partial transversal.

Thus, transversals are down closed (subclusive).
A set $T' \subseteq V$ is a **partial transversal** if $T'$ is a transversal of some subfamily $\mathcal{V}' = (V_i : i \in I')$ where $I' \subseteq I$.

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Thus, transversals are down closed (subclusive).
When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
- Given a set system \((V, \mathcal{V})\) with \(\mathcal{V} = (V_i : i \in I)\), and \(V_i \subseteq V\) for all \(i\).
  
  Then, for any \(J \subseteq I\), let 
  
  \[ V(J) = \bigcup_{j \in J} V_j \]
  
  so \(|V(J)| : 2^I \rightarrow \mathbb{Z}_+\) is the set cover func. (we know is submodular).

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We have

**Theorem 7.6.1 (Hall’s theorem)**

*Given a set system \((V, \mathcal{V})\), the family of subsets \(\mathcal{V} = (V_i : i \in I)\) has a transversal \((v_i : i \in I)\) iff for all \(J \subseteq I\)*

\[
\bigcap \left\{ |V(J)| \geq |J| \right\}
\]

(7.23)
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- Hall’s theorem \((\forall J \subseteq I, |V(J)| \geq |J|)\) as a bipartite graph.

\[
\begin{array}{c}
V \\
\downarrow \\
I
\end{array}
\]

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When do transversals exist?

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![Bipartite Graph](image)
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- Moreover, we have

**Theorem 7.6.2 (Rado’s theorem (1942))**

If \(M = (V, r)\) is a matroid on \(V\) with rank function \(r\), then the family of subsets \((V_i : i \in I)\) of \(V\) has a transversal \((v_i : i \in I)\) that is independent in \(M\) iff for all \(J \subseteq I\)

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r(V(J)) \geq |J|
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\[
r(V(J)) \geq |J|
\]

(7.24)

- Note, a transversal \(T\) independent in \(M\) means that \(r(T) = |T|\).
More general conditions for existence of transversals

**Theorem 7.6.3 (Polymatroid transversal theorem)**

If $\mathcal{V} = (V_i : i \in I)$ is a finite family of non-empty subsets of $V$, and $f : 2^V \to \mathbb{Z}_+$ is a non-negative, integral, monotone non-decreasing, and submodular function, then $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that

$$f(\bigcup_{i \in J} \{v_i\}) \geq |J| \text{ for all } J \subseteq I \tag{7.25}$$

if and only if

$$f(V(J)) \geq |J| \text{ for all } J \subseteq I \tag{7.26}$$
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Theorem 7.6.3 (Polymatroid transversal theorem)

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\[
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(7.26)

Given Theorem 7.6.3, we immediately get Theorem 7.6.1 by taking \( f(S) = |S| \) for \( S \subseteq V \). In which case, Eq. 7.25 requires the system of representatives to be distinct.
Theorem 7.6.3 (Polymatroid transversal theorem)

If \( V = (V_i : i \in I) \) is a finite family of non-empty subsets of \( V \), and \( f : 2^V \to \mathbb{Z}_+ \) is a non-negative, integral, monotone non-decreasing, and submodular function, then \( V \) has a system of representatives \( (v_i : i \in I) \) such that

\[
|A| = c \quad \text{for all } A \subseteq I
\]

If and only if

\[
f \left( \bigcup_{i \in J} \{v_i\} \right) \geq |J| \quad \text{for all } J \subseteq I
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\[
f (V(J)) \geq |J| \quad \text{for all } J \subseteq I
\]

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Given Theorem 7.6.3, we immediately get Theorem 7.6.1 by taking \( f(S) = |S| \) for \( S \subseteq V \).

We get Theorem 7.6.2 by taking \( f(S) = r(S) \) for \( S \subseteq V \), the rank function of the matroid. where, Eq. 7.25 insists the system of representatives is independent in \( M \), and hence also distinct.
Note the condition in Theorem 7.6.3 is $f(V(J)) \geq |J|$ for all $J \subseteq I$, where $f : 2^V \to \mathbb{Z}_+$ is non-negative, integral, monotone non-decreasing and submodular, and $V(J) = \bigcup_{j \in J} V_j$ with $V_i \subseteq V$. 

Proposition 7.6.4: $g$ as given above is submodular. Hence, the condition for existence can be solved by (a special case of) submodular function minimization, or vice versa!
Submodular Composition with Set-to-Set functions

- Note the condition in Theorem 7.6.3 is $f(V(J)) \geq |J|$ for all $J \subseteq I$, where $f : 2^V \rightarrow \mathbb{Z}_+$ is non-negative, integral, monotone non-decreasing and submodular, and $V(J) = \bigcup_{j \in J} V_j$ with $V_i \subseteq V$.
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Define $g : 2^I \rightarrow \mathbb{Z}$ with $g(J) = f(V(J)) - |J|$, then the condition for the existence of a system of representatives, with quality Equation 7.25, becomes:

$$\min_{J \subseteq I} g(J) \geq 0$$

(7.27)
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Submodular Composition with Set-to-Set functions

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**Proposition 7.6.4**

$g$ as given above is submodular.
Submodular Composition with Set-to-Set functions

- Note the condition in Theorem 7.6.3 is \( f(V(J)) \geq |J| \) for all \( J \subseteq I \), where \( f : 2^V \to \mathbb{Z}_+ \) is non-negative, integral, monotone non-decreasing and submodular, and \( V(J) = \bigcup_{j \in J} V_j \) with \( V_i \subseteq V \).
- Note \( V(\cdot) : 2^I \to 2^V \) is a set-to-set function, composable with a submodular function.
- Define \( g : 2^I \to \mathbb{Z} \) with \( g(J) = f(V(J)) - |J| \), then the condition for the existence of a system of representatives, with quality Equation 7.25, becomes:

\[
\min_{J \subseteq I} g(J) \geq 0 \tag{7.27}
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- What kind of function is \( g \)?

**Proposition 7.6.4**

*\( g \) as given above is submodular.*

- Hence, the condition for existence can be solved by (a special case of) submodular function minimization, or vice versa!
first part proof of Theorem 7.6.3.

- Suppose \( \mathcal{V} \) has a system of representatives \((v_i : i \in I)\) such that Eq. 7.25 (i.e., \( f(\bigcup_{i \in J} \{v_i\}) \geq |J| \) for all \( J \subseteq I \)) is true.
More general conditions for existence of transversals

first part proof of Theorem 7.6.3.

- Suppose $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that Eq. 7.25 (i.e., $f(\bigcup_{i \in J} \{v_i\}) \geq |J|$ for all $J \subseteq I$) is true.

- Then since $f$ is monotone, and since $V(J) \supseteq \bigcup_{i \in J} \{v_i\}$ when $(v_i : i \in I)$ is a system of representatives, then Eq. 7.26 (i.e., $f(V(J)) \geq |J|$ for all $J \subseteq I$) immediately follows.

$$f(V(J)) \geq f(\bigcup_{y \in J} \{v_y\}) \geq |J|$$
### Lemma 7.6.5 (contraction lemma)

Suppose Eq. 7.26 \( (f(V(J)) \geq |J|, \forall J \subseteq I) \) is true for \( V = (V_i : i \in I) \), and there exists an \( i \) such that \( |V_i| \geq 2 \) (w.l.o.g., say \( i = 1 \)). Then there exists \( \bar{v} \in V_1 \) such that the family of subsets \( (V_1 \setminus \{ \bar{v} \}, V_2, \ldots, V_{|I|}) \) also satisfies Eq 7.26.

### Proof.

- When Eq. 7.26 holds, this means that for any subsets \( J \in \{J_1, J_2\} \), we have that, for \( J \in \{J_1, J_2\} \), consider index \( J \cup \{1\} \)
  
  \[
  f(V(J \cup \{1\})) \geq |J \cup \{1\}| \tag{7.28}
  \]
  
  and hence
  
  \[
  f(V_1 \cup V(J_1)) \geq |J_1| + 1 \tag{7.29}
  \]
  
  \[
  f(V_1 \cup V(J_2)) \geq |J_2| + 1 \tag{7.30}
  \]
  
  \[\ldots\]
Lemma 7.6.5 (contraction lemma)

Suppose Eq. 7.26 ($f(V(J)) \geq |J|$, $\forall J \subseteq I$) is true for $V = (V_i : i \in I)$, and there exists an $i$ such that $|V_i| \geq 2$ (w.l.o.g., say $i = 1$). Then there exists $\bar{v} \in V_1$ such that the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq. 7.26.

Proof.

- Suppose, to the contrary, the consequent is false. Then we may take any $\bar{v}_1, \bar{v}_2 \in V_1$ as two distinct elements in $V_1$ . . .

  ...
### Lemma 7.6.5 (contraction lemma)

Suppose Eq. 7.26 \((f(V(J)) \geq |J|, \forall J \subseteq I)\) is true for \(V = (V_i : i \in I)\), and there exists an \(i\) such that \(|V_i| \geq 2\) (w.l.o.g., say \(i = 1\)). Then there exists \(\bar{v} \in V_1\) such that the family of subsets \((V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})\) also satisfies Eq 7.26.

### Proof.

- Suppose, to the contrary, the consequent is false. Then we may take any \(\bar{v}_1, \bar{v}_2 \in V_1\) as two distinct elements in \(V_1\)...
- ...and there must exist subsets \(J_1, J_2\) of \(I \setminus \{1\}\) such that

\[
\begin{align*}
    f((V_1 \setminus \{\bar{v}_1\}) \cup V(J_1)) &< |J_1| + 1, \quad (7.31) \\
    f((V_1 \setminus \{\bar{v}_2\}) \cup V(J_2)) &< |J_2| + 1, \quad (7.32)
\end{align*}
\]

(note that either one or both of \(J_1, J_2\) could be empty). ...
Lemma 7.6.5 (contraction lemma)

Suppose Eq. 7.26 \( f(V(J)) \geq |J|, \forall J \subseteq I \) is true for \( V = (V_i : i \in I) \), and there exists an \( i \) such that \( |V_i| \geq 2 \) (w.l.o.g., say \( i = 1 \)). Then there exists \( \bar{v} \in V_1 \) such that the family of subsets \( (V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|}) \) also satisfies Eq 7.26.

Proof.

- Taking \( X = (V_1 \setminus \{\bar{v}_1\}) \cup V(J_1) \) and \( Y = (V_1 \setminus \{\bar{v}_2\}) \cup V(J_2) \), we have \( f(X) \leq |J_1|, f(Y) \leq |J_2| \), and that:

\[
X \cup Y = V_1 \cup V(J_1 \cup J_2), \tag{7.33}
\]

\[
X \cap Y \supseteq V(J_1 \cap J_2), \tag{7.34}
\]

and

\[
|J_1| + |J_2| \geq f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \tag{7.35}
\]

...
More general conditions for existence of transversals

Lemma 7.6.5 (contraction lemma)

Suppose Eq. 7.26 ($f(V(J)) \geq |J|, \forall J \subseteq I$) is true for $V = (V_i : i \in I)$, and there exists an $i$ such that $|V_i| \geq 2$ (w.l.o.g., say $i = 1$). Then there exists $\bar{v} \in V_1$ such that the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 7.26.

Proof.

- since $f$ submodular monotone non-decreasing, & Eqs 7.33-7.35,

$$|J_1| + |J_2| \geq f(V_1 \cup V(J_1 \cup J_2)) + f(V(J_1 \cap J_2)) \quad (7.36)$$

...
More general conditions for existence of transversals

Lemma 7.6.5 (contraction lemma)

Suppose Eq. 7.26 ($f(V(J)) \geq |J|, \forall J \subseteq I$) is true for $\mathcal{V} = (V_i : i \in I)$, and there exists an $i$ such that $|V_i| \geq 2$ (w.l.o.g., say $i = 1$). Then there exists $v \in V_1$ such that the family of subsets $(V_1 \setminus \{v\}, V_2, \ldots, V_{|I|})$ also satisfies Eq. 7.26.

Proof.

- since $f$ submodular monotone non-decreasing, & Eqs 7.33-7.35,

\[ |J_1| + |J_2| \geq f(V_1 \cup V(J_1 \cup J_2)) + f(V(J_1 \cap J_2)) \]  \hspace{1cm} (7.36)

- Since $\mathcal{V}$ satisfies Eq. 7.26, $1 \notin J_1 \cup J_2$, & Eqs 7.29-7.30, this gives

\[ |J_1| + |J_2| \geq |J_1 \cup J_2| + 1 + |J_1 \cap J_2| \]  \hspace{1cm} (7.37)

which is a contradiction since cardinality is modular.
More general conditions for existence of transversals

**Theorem 7.6.3 (Polymatroid transversal theorem)**

If \( \mathcal{V} = (V_i : i \in I) \) is a finite family of non-empty subsets of \( V \), and \( f : 2^V \to \mathbb{Z}_+ \) is a non-negative, integral, monotone non-decreasing, and submodular function, then \( \mathcal{V} \) has a system of representatives \((v_i : i \in I)\) such that

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f(\bigcup_{i \in J} \{v_i\}) \geq |J| \text{ for all } J \subseteq I \tag{7.25}
\]

if and only if

\[
f(V(J)) \geq |J| \text{ for all } J \subseteq I \tag{7.26}
\]

- Given Theorem 7.6.3, we immediately get Theorem 7.6.1 by taking \( f(S) = |S| \) for \( S \subseteq V \).
- We get Theorem 7.6.2 by taking \( f(S) = r(S) \) for \( S \subseteq V \), the rank function of the matroid.
converse proof of Theorem 7.6.3.

- Conversely, suppose Eq. 7.26 is true.
converse proof of Theorem 7.6.3.

- Conversely, suppose Eq. 7.26 is true.
- If each $V_i$ is a singleton set, then the result follows immediately.

\[ \nu(\overline{U}) = \bigvee_{i \in U} \exists_{z \in V_i} \]
**converse proof of Theorem 7.6.3.**

- Conversely, suppose Eq. 7.26 is true.
- If each $V_i$ is a singleton set, then the result follows immediately.
- W.l.o.g., let $|V_1| \geq 2$, then by Lemma 7.6.5, the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_1^{|I|})$ also satisfies Eq 7.26 for the right $\bar{v}$. 

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converse proof of Theorem 7.6.3.

- Conversely, suppose Eq. 7.26 is true.
- If each $V_i$ is a singleton set, then the result follows immediately.
- W.l.o.g., let $|V_1| \geq 2$, then by Lemma 7.6.5, the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 7.26 for the right $\bar{v}$.
- We can continue to reduce the family, deleting elements from $V_i$ for some $i$ while $|V_i| \geq 2$, until we arrive at a family of singleton sets.
converse proof of Theorem 7.6.3.

- Conversely, suppose Eq. 7.26 is true.
- If each $V_i$ is a singleton set, then the result follows immediately.
- W.l.o.g., let $|V_1| \geq 2$, then by Lemma 7.6.5, the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 7.26 for the right $\bar{v}$.
- We can continue to reduce the family, deleting elements from $V_i$ for some $i$ while $|V_i| \geq 2$, until we arrive at a family of singleton sets.
- This family will be the required system of representatives.
converse proof of Theorem 7.6.3.

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- This family will be the required system of representatives.

This theorem can be used to produce a variety of other results quite easily, and shows how submodularity is the key ingredient in its truth.
Transversal Matroid

Transversals, themselves, define a matroid.

**Theorem 7.7.1**

*If \( \mathcal{V} \) is a family of finite subsets of a ground set \( V \), then the collection of partial transversals of \( \mathcal{V} \) is the set of independent sets of a matroid \( M = (V, \mathcal{V}) \) on \( V \).*
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- This means that the transversals of \( \mathcal{V} \) are the bases of matroid \( M \).
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*If $\mathcal{V}$ is a family of finite subsets of a ground set $V$, then the collection of partial transversals of $\mathcal{V}$ is the set of independent sets of a matroid $M = (V, \mathcal{V})$ on $V$.***

- This means that the transversals of $\mathcal{V}$ are the bases of matroid $M$.
- Therefore, all maximal partial transversals of $\mathcal{V}$ have the same cardinality!
Transversals and Bipartite Matchings

- Transversals correspond exactly to matchings in bipartite graphs.

Given a set system \((V, \mathcal{I})\) with \(V = \{V_i : i \in \mathcal{I}\}\), we can define a bipartite graph \(G = (V, \mathcal{I}, E)\) associated with \(V\) that has edge set \(\{(v, i) : v \in V, i \in \mathcal{I}, v \in V_i\}\). A matching in this graph is a set of edges no two of which have a common endpoint. In fact, we easily have:

**Lemma 7.7.2** As a subset \(T \subseteq V\) is a partial transversal of \(V\) iff there is a matching in \((V, \mathcal{I}, E)\) in which every edge has one endpoint in \(T\) matched into \(\mathcal{I}\) (\(T\) matched into \(\mathcal{I}\)).
Transversals and Bipartite Matchings

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Transversals and Bipartite Matchings

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- Given a set system $(V, \mathcal{V})$, with $\mathcal{V} = (V_i : i \in I)$, we can define a bipartite graph $G = (V, I, E)$ associated with $\mathcal{V}$ that has edge set $\{(v, i) : v \in V, i \in I, v \in V_i\}$.
- A matching in this graph is a set of edges no two of which that have a common endpoint. In fact, we easily have:

**Lemma 7.7.2**

As a subset $T \subseteq V$ is a partial transversal of $V$ iff there is a matching in $(V, I, E)$ in which every edge has one endpoint in $T$ ($T$ matched into $I$).
Transversals and Bipartite Matchings

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Are arbitrary matchings matroids?
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Consider the following graph (left), and two max-matchings (two right instances)
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\[ \{AC\} \text{ is a maximum matching, as is } \{AD, BC\}, \text{ but they are not the same size.} \]
Example where $\ell = 5$,

$$(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3).$$

Recall, $\Gamma : 2^V \rightarrow \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of $X$ is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.

Here, for $X \subseteq V$, we have $\Gamma(X) = \{i \in I : (v, i) \in E(G) \text{ and } v \in X\}$.

For such a constructed bipartite graph, the rank function of a partition matroid is $r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i) =$ the maximum matching involving $X$. 

Recall the partition matroid rank function. Note, \( k_i = |I_i| \) in the bipartite graph representation, and since a matroid, w.l.o.g., \( |V_i| \geq k_i \) (also, recall, \( V(J) = \bigcup_{j \in J} V_j \)).
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Start with partition matroid rank function in the subsequent equations.

\[
r(A) = \sum_{i \in \{1, \ldots, \ell\}} \min(|A \cap V_i|, k_i)
\]

(7.38)
Morphing Partition Matroid Rank

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- Start with partition matroid rank function in the subsequent equations.

$$r(A) = \sum_{i \in \{1, \ldots, \ell\}} \min(|A \cap V_i|, k_i)$$  \hspace{1cm} (7.38)

$$= \sum_{i=1}^{\ell} \min(|A \cap V(I_i)|, |I_i|)$$  \hspace{1cm} (7.39)
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= \sum_{i=1}^{\ell} \min_{J_i \in \{\emptyset, I_i\}} \left( |A \cap V(I_i)| \text{ if } J_i \neq \emptyset \right) \left( 0 \text{ if } J_i = \emptyset \right) + |I_i \setminus J_i| \tag{7.40}
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\[
\begin{align*}
r(A) &= \sum_{i \in \{1, \ldots, \ell\}} \min(|A \cap V_i|, k_i) \\
&= \sum_{i=1}^{\ell} \min(|A \cap V(I_i)|, |I_i|) \\
&= \sum_{i \in \{1, \ldots, \ell\}} \min_{\emptyset \neq J_i \subseteq I_i} \left( \begin{cases} 
|A \cap V(I_i)| & \text{if } J_i \neq \emptyset \\
0 & \text{if } J_i = \emptyset 
\end{cases} \right) + |I_i \setminus J_i| \\
&= \sum_{i \in \{1, \ldots, \ell\}} \min_{J_i \subseteq I_i} \left( \begin{cases} 
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Recall the partition matroid rank function. Note, \( k_i = |I_i| \) in the bipartite graph representation, and since a matroid, w.l.o.g., \( |V_i| \geq k_i \) (also, recall, \( V(J) = \cup_{j \in J} V_j \)).

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0 & \text{if } J_i = \emptyset
\end{array} \right) + |I_i \setminus J_i| \tag{7.41}
\]

\[
= \sum_{i \in \{1, \ldots, \ell\}} \min_{J_i \subseteq I_i} (|V(J_i) \cap A| + |I_i \setminus J_i|) \tag{7.42}
\]
Continuing,

\[
    r(A) = \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} (|V(J_i) \cap V(I_i) \cap A| - |I_i \cap J_i| + |I_i|) \quad (7.43)
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(7.46)

In fact, this bottom (more general) expression is the expression for the rank of a transversal matroid.
Partial Transversals Are Independent Sets in a Matroid

In fact, we have

**Theorem 7.7.3**

Let \((V, \mathcal{V})\) where \(\mathcal{V} = (V_1, V_2, \ldots, V_\ell)\) be a subset system. Let \(I = \{1, \ldots, \ell\}\). Let \(\mathcal{I}\) be the set of partial transversals of \(\mathcal{V}\). Then \((V, \mathcal{I})\) is a matroid.

**Proof.**
In fact, we have

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**Proof.**

- We note that \(\emptyset \in \mathcal{I}\) since the empty set is a transversal of the empty subfamily of \(\mathcal{V}\), thus (I1’) holds.
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- We already saw that if \(T\) is a partial transversal of \(\mathcal{V}\), and if \(T' \subseteq T\), then \(T'\) is also a partial transversal. So (I2') holds.
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**Proof.**

- We note that \(\emptyset \in I\) since the empty set is a transversal of the empty subfamily of \(\mathcal{V}\), thus (I1’) holds.
- We already saw that if \(T\) is a partial transversal of \(\mathcal{V}\), and if \(T' \subseteq T\), then \(T'\) is also a partial transversal. So (I2’) holds.
- Suppose that \(T_1\) and \(T_2\) are partial transversals of \(\mathcal{V}\) such that \(|T_1| < |T_2|\). Exercise: show that (I3’) holds.
Transversal Matroid Rank

- Transversal matroid has rank

\[ r(A) = \min_{J \subseteq I} (|V(J) \cap A| - |J| + |I|) \] (7.47)

Therefore, this function is submodular. Note that it is a minimum over a set of modular functions. Is this true in general?

Exercise:
Transversal Matroid Rank

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- Therefore, this function is submodular.

- Note that it is a minimum over a set of modular functions. Is this true in general? Exercise:
A circuit in a matroids is well defined, a subset $A \subseteq E$ is **circuit** if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).
Matroid loops

- A circuit in a matroids is well defined, a subset $A \subseteq E$ is circuit if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

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Such an $\{a\}$ is called a loop.
Matroid loops

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- There is no reason in a matroid such an \( A \) could not consist of a single element.
- Such an \( \{a\} \) is called a loop.
- In a matric (i.e., linear) matroid, the only such loop is the value 0, as all non-zero vectors have rank 1. The 0 can appear \( > 1 \) time with different indices, as can a self loop in a graph appear on different nodes.
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- Such an \( \{a\} \) is called a loop.
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- Note, we also say that two elements \( s, t \) are said to be parallel if \( \{s, t\} \) is a circuit.
Definition 7.8.1 (Matroid isomorphism)

Two matroids $M_1$ and $M_2$ respectively on ground sets $V_1$ and $V_2$ are isomorphic if there is a bijection $\pi : V_1 \rightarrow V_2$ which preserves independence (equivalently, rank, circuits, and so on).
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- Let $\mathbb{F}$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $\mathbb{F}$, such as a Galois field $\text{GF}(p)$ where $p$ is prime (such as $\text{GF}(2)$).

  Succinctly: A field is a set with $+$, $\times$, closure, associativity, commutativity, and additive and multiplicative identities and inverses.
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- We can more generally define matroids on a field.
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- Succinctly: A field is a set with $+$, $\cdot$, closure, associativity, commutativity, and additive and multiplicative identities and inverses.
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Definition 7.8.2 (linear matroids on a field)

Let $X$ be an $n \times m$ matrix and $E = \{1, \ldots, m\}$, where $X_{ij} \in \mathbb{F}$ for some field, and let $\mathcal{I}$ be the set of subsets of $E$ such that the columns of $X$ are linearly independent over $\mathbb{F}$. 
Definition 7.8.1 (Matroid isomorphism)

Two matroids $M_1$ and $M_2$ respectively on ground sets $V_1$ and $V_2$ are isomorphic if there is a bijection $\pi : V_1 \rightarrow V_2$ which preserves independence (equivalently, rank, circuits, and so on).

- Let $F$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $F$, such as a Galois field $GF(p)$ where $p$ is prime (such as $GF(2)$)).
  Succinctly: A field is a set with $+$, $\times$, closure, associativity, commutativity, and additive and multiplicative identities and inverses.
- We can more generally define matroids on a field.

Definition 7.8.3 (representable (as a linear matroid))

Any matroid isomorphic to a linear matroid on a field is called representable over $F$. 

Prof. Jeff Bilmes
Piff and Welsh in 1970, and Adkin in 1972 proved an important theorem about representability of transversal matroids.
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In particular:

**Theorem 7.8.4**

Transversal matroids are representable over all finite fields of sufficiently large cardinality, and are representable over any infinite field.
Converse: Representability of Transversal Matroids

The converse is not true, however.

Example 7.8.5

Let \( V = \{1, 2, 3, 4, 5, 6\} \) be a ground set and let \( M = (V, \mathcal{I}) \) be a set system where \( \mathcal{I} \) is all subsets of \( V \) of cardinality \( \leq 2 \) except for the pairs \( \{1, 2\}, \{3, 4\}, \{5, 6\} \).
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Example 7.8.5

Let $V = \{1, 2, 3, 4, 5, 6\}$ be a ground set and let $M = (V, \mathcal{I})$ be a set system where $\mathcal{I}$ is all subsets of $V$ of cardinality $\leq 2$ except for the pairs $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$.

- It can be shown that this is a matroid and is representable.
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**Example 7.8.5**

Let $V = \{1, 2, 3, 4, 5, 6\}$ be a ground set and let $M = (V, \mathcal{I})$ be a set system where $\mathcal{I}$ is all subsets of $V$ of cardinality $\leq 2$ except for the pairs $\{1, 2\}, \{3, 4\}, \{5, 6\}$.

- It can be shown that this is a matroid and is representable.
- However, this matroid is not isomorphic to any transversal matroid.
Definition 7.9.3 (closed/flat/subspace)

A subset $A \subseteq E$ is closed (equivalently, a flat or a subspace) of matroid $M$ if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

Definition: A hyperplane is a flat of rank $r(M) - 1$.

Definition 7.9.4 (closure)

Given $A \subseteq E$, the closure (or span) of $A$, is defined by

$$\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}.$$ 

Therefore, a closed set $A$ has $\text{span}(A) = A$.

Definition 7.9.5 (circuit)

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise-minimal dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).
We have the following definitions:

Definition 7.9.1 (spanning set of a set) Given a matroid $M = (V, I)$, a set $Y \subseteq V$, the set $X \subseteq Y$ such that $r(X) = r(Y)$ is called a spanning set of $Y$.

Definition 7.9.2 (spanning set of a matroid) Given a matroid $M = (V, I)$, a set $A \subseteq V$ such that $r(A) = r(V)$ is called a spanning set of the matroid.

A base of a matroid is a minimal spanning set (and it is independent) but supersets of a base are also spanning. $V$ is always trivially spanning.

Consider the terminology: "spanning tree in a graph", comes from spanning in a matroid sense.
Spanning Sets

- We have the following definitions:

**Definition 7.9.1 (spanning set of a set)**

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- Consider the terminology: “spanning tree in a graph”, comes from spanning in a matroid sense.
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- Given a matroid $M = (V, \mathcal{I})$, a dual matroid $M^* = (V, \mathcal{I}^*)$ can be defined on the same ground set $V$, but using a very different set of independent sets $\mathcal{I}^*$. 

\[ I^* = \{ A \subseteq V : V \cap A \text{ is a spanning set of } M \} \]  

\[ I^* = \{ A \subseteq V : \text{rank}_M(V \cap A) = \text{rank}_M(V) \} \]  

In other words, a set $A \subseteq V$ is independent in the dual $M^*$ (i.e., $A \in \mathcal{I}^*$) if its complement is spanning in $M$ (residual $V \cap A$ must contain a base in $M$). 

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