Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 7 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee596b_spring_2016/

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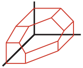
$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

$= f(A_r) + 2f(C) + f(B_r)$ $\quad = f(A_r) + f(C) + f(B_r)$ $\quad = f(A \cap B)$



Cumulative Outstanding Reading

- Read chapters 2 and 3 from Fujishige's book.
- Read chapter 1 from Fujishige's book.

Announcements, Assignments, and Reminders

- Homework 2, available at our assignment dropbox (https://canvas.uw.edu/courses/1039754/assignments), due (electronically) Monday (4/18) at 11:55pm.
- Homework 1, available at our assignment dropbox (https://canvas.uw.edu/courses/1039754/assignments), due (electronically) Friday (4/8) at 11:55pm.
- Weekly Office Hours: Mondays, 3:30-4:30, or by skype or google hangout (set up meeting via our our discussion board (https://canvas.uw.edu/courses/1039754/discussion_topics)).

Class Road Map - IT-I

- L1(3/28): Motivation, Applications, & Basic Definitions
- L2(3/30): Machine Learning Apps (diversity, complexity, parameter, learning target, surrogate).
- L3(4/4): Info theory exs, more apps, definitions, graph/combinatorial examples, matrix rank example, visualization
- L4(4/6): Graph and Combinatorial Examples, matrix rank, Venn diagrams, examples of proofs of submodularity, some useful properties
- L5(4/11): Examples & Properties, Other Defs., Independence
- L6(4/13): Independence, Matroids, Matroid Examples, matroid rank is submodular
- L7(4/18): Matroid Rank, More on Partition Matroid, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation, Dual Matroid
- L8(4/20):
- L9(4/25):
- L10(4/27):
- L11(5/2):
- L12(5/4):
- L13(5/9):
- L14(5/11):
- L15(5/16):
- L16(5/18):
- L17(5/23):
- L18(5/25):
- L19(6/1):
- L20(6/6): Final Presentations maximization.

Finals Week: June 6th-10th, 2016.

Matroid

Independent set definition of a matroid is perhaps most natural. Note, if $J \in \mathcal{I}$, then J is said to be an independent set.

Definition 7.2.3 (Matroid)

A set system $(E, \mathcal{I})$ is a Matroid if

(I1) $\emptyset \in \mathcal{I}$

(I2) $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$

(I3) $\forall I, J \in \mathcal{I}$, with $|I| = |J| + 1$, then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$.

Why is (I1) is not redundant given (I2)? Because without (I1) could have a non-matroid where $\mathcal{I} = \{\}$.

Matroids - important property

Proposition 7.2.3

In a matroid $M = (E, \mathcal{I})$, for any $U \subseteq E(M)$, any two bases of U have the same size.

- In matrix terms, given a set of vectors U, all sets of independent vectors that span the space spanned by U have the same size.
- In fact, under (I1),(I2), this condition is equivalent to (I3). Exercise: show the following is equivalent to the above.

Definition 7.2.4 (Matroid)

A set system $(V, \mathcal{I})$ is a Matroid if

(I1') $\emptyset \in \mathcal{I}$ (emptyset containing)

(I2') $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ (down-closed or subclusive)

(I3') $\forall X \subseteq V$, and $I_1, I_2 \in \text{maxInd}(X)$, we have $|I_1| = |I_2|$ (all maximally independent subsets of X have the same size).

Partition Matroid

- Let V be our ground set.
- Let $V = V_1 \cup V_2 \cup \dots \cup V_\ell$ be a partition of V into ℓ blocks (i.e., disjoint sets). Define a set of subsets of V as

$$\mathcal{I} = \{X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \dots, \ell\}. \tag{7.5}$$

  where $k_1, \dots, k_\ell$ are fixed parameters, $k_i \geq 0$. Then $M = (V, \mathcal{I})$ is a matroid.

  (limit)
- Note that a k-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$.
- Parameters associated with a partition matroid: ℓ and $k_1, k_2, \dots, k_\ell$ although often the k_i's are all the same.
- We'll show that property (I3') in Def **??** holds. First note, for any $X \subseteq V$, $|X| = \sum_{i=1}^{\ell} |X \cap V_i|$ since $\{V_1, V_2, \dots, V_\ell\}$ is a partition.
- If $X, Y \in \mathcal{I}$ with $|Y| > |X|$, then there must be at least one i with $|Y \cap V_i| > |X \cap V_i|$. Therefore, adding one element $e \in V_i \cap (Y \setminus X)$ to X won't break independence.

Matroids - rank function is submodular

Lemma 7.2.3

The rank function $r : 2^E \to \mathbb{Z}_+$ of a matroid is submodular, that is $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$

Proof.

1. Let $X \in \mathcal{I}$ be an inclusionwise maximal set with $X \subseteq A \cap B$
2. Let $Y \in \mathcal{I}$ be inclusionwise maximal set with $X \subseteq Y \subseteq A \cup B$.
3. Since M is a matroid, we know that $r(A \cap B) = r(X) = |X|$, and $r(A \cup B) = r(Y) = |Y|$. Also, for any $U \in \mathcal{I}$, $r(A) \geq |A \cap U|$.
4. Then we have (since $X \subseteq A \cap B$, $X \subseteq Y$, and $Y \subseteq A \cup B$),

$$
\begin{aligned}
r(A) + r(B) &\geq |Y \cap A| + |Y \cap B| && (7.5) \\
&= |Y \cap (A \cap B)| + |Y \cap (A \cup B)| && (7.6) \\
&\geq |X| + |Y| = r(A \cap B) + r(A \cup B) && (7.7)
\end{aligned}
$$

□

A matroid is defined from its rank function

Theorem 7.2.3 (Matroid from rank)

Let E be a set and let $r : 2^E \to \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with r being its rank function if and only if for all $A, B \subseteq E$:

(R1) $\forall A \subseteq E \;\; 0 \leq r(A) \leq |A|$ *(non-negative cardinality bounded)*

(R2) $r(A) \leq r(B)$ *whenever* $A \subseteq B \subseteq E$ *(monotone non-decreasing)*

(R3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ *for all* $A, B \subseteq E$ *(submodular)*

- So submodularity and non-negative monotone non-decreasing, and unit increase is necessary and sufficient to define the matroid.
- Can name matroid as (E, r), E is ground set, r is rank function.
- Given above, unit increment (if $r(A) = k$, then either $r(A \cup \{v\}) = k$ or $r(A \cup \{v\}) = k + 1$) holds.
- From above, $r(\emptyset) = 0$. Let $v \notin A$, then by monotonicity and submodularity, $r(A) \leq r(A \cup \{v\}) \leq r(A) + r(\{v\})$ which gives only two possible values to $r(A \cup \{v\})$.

Matroids from rank

7.2.3

Proof of Theorem ?? (matroid from rank).

- Given a matroid $M = (E, \mathcal{I})$, we see its rank function as defined in Eq. ?? satisfies (R1), (R2), and, as we saw in Lemma ??, (R3) too.

...

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- Next, assume we have (R1), (R2), and (R3). Define $\mathcal{I} = \{X \subseteq E : r(X) = |X|\}$. We will show that $(E, \mathcal{I})$ is a matroid.

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- First, $\emptyset \in \mathcal{I}$.
- Also, if $Y \in \mathcal{I}$ and $X \subseteq Y$ then by submodularity,

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Matroids from rank

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- First, $\emptyset \in \mathcal{I}$.
- Also, if $Y \in \mathcal{I}$ and $X \subseteq Y$ then by submodularity,

$$r(X) \geq r(Y) - r(Y \setminus X) \tag{7.1}$$

...

Matroids from rank

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- First, $\emptyset \in \mathcal{I}$.
- Also, if $Y \in \mathcal{I}$ and $X \subseteq Y$ then by submodularity,

$$r(X) \geq r(Y) - r(Y \setminus X) + r(\emptyset) \tag{7.1}$$

$$r(X) + r(Y \setminus X) \geq r(X \cup (Y \setminus X)) + r(\underbrace{X \cap (Y \setminus X)}_{\emptyset})$$

...

Matroids from rank

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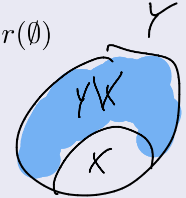
$$r(X) \geq r(Y) - r(Y \setminus X) + r(\emptyset) \tag{7.1}$$

$$\geq |Y| - |Y \setminus X| \tag{7.2}$$



$r(Y \setminus X) \leq |Y \setminus X|$

...

Matroids from rank

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$$r(X) \geq r(Y) - r(Y \setminus X) + r(\emptyset) \tag{7.1}$$

$$\geq |Y| - |Y \setminus X| \tag{7.2}$$

$$= |X| \tag{7.3}$$

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Matroids from rank

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- First, $\emptyset \in \mathcal{I}$.
- Also, if $Y \in \mathcal{I}$ and $X \subseteq Y$ then by submodularity,

$$r(X) \geq r(Y) - r(Y \setminus X) + r(\emptyset) \tag{7.1}$$
$$\geq |Y| - |Y \setminus X| \tag{7.2}$$
$$= |X| \tag{7.3}$$

implying $r(X) = |X|$, and thus $X \in \mathcal{I}$.

...

Matroids from rank

Proof of Theorem ?? (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \dots, b_k\}$ (note $k \leq |B|$).

$1 \leq$

$\square$

Matroids from rank

Proof of Theorem ?? (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).
- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such b, $r(A + b) = r(A) = |A| < |A + b|$. Then

$\square$

Matroids from rank

Proof of Theorem **??** (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).
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$$r(B) \leq r(A \cup B) \tag{7.4}$$

$\square$

Matroids from rank

Proof of Theorem ?? (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).
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$$r(B) \leq r(A \cup B) \tag{7.4}$$
$$\leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) \tag{7.5}$$

$$\underbrace{r(A \cup (B \setminus b_1))}_{C} + \underbrace{r(A \cup \{b_1\})}_{D} \geq r(C \cup D) + r(C \cap D)$$

□

Matroids from rank

Proof of Theorem **??** (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \dots, b_k\}$ (note $k \leq |B|$).
- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such b, $r(A + b) = r(A) = |A| < |A + b|$. Then

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$$\leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) \tag{7.5}$$
$$= r(A \cup (B \setminus \{b_1\})) \tag{7.6}$$

□

Matroids from rank

Proof of Theorem ?? (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).
- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such b, $r(A + b) = r(A) = |A| < |A + b|$. Then

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$$= r(A \cup (B \setminus \{b_1\}) \tag{7.6}$$

$$\leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) \tag{7.7}$$

□

Matroids from rank

Proof of Theorem **??** (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \dots, b_k\}$ (note $k \leq |B|$).
- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such b, $r(A + b) = r(A) = |A| < |A + b|$. Then

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$$\leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) \tag{7.7}$$
$$= r(A \cup (B \setminus \{b_1, b_2\})) \tag{7.8}$$

□

Matroids from rank

Proof of Theorem ?? (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).
- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such b, $r(A + b) = r(A) = |A| < |A + b|$. Then

$$
\begin{aligned}
r(B) &\leq r(A \cup B) && (7.4)\\
&\leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) && (7.5)\\
&= r(A \cup (B \setminus \{b_1\}) && (7.6)\\
&\leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) && (7.7)\\
&= r(A \cup (B \setminus \{b_1, b_2\})) && (7.8)\\
&\leq \ldots \leq r(A) = |A| < |B| && (7.9)
\end{aligned}
$$

□

Matroids from rank

Proof of Theorem ?? (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \dots, b_k\}$ (note $k \leq |B|$).
- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such b, $r(A+b) = r(A) = |A| < |A+b|$. Then

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&= r(A \cup (B \setminus \{b_1, b_2\})) && (7.8)\\
&\leq \ldots \leq r(A) = |A| < |B| && (7.9)
\end{aligned}
$$

giving a contradiction since $B \in \mathcal{I}$. $\square$

Matroids from rank II

Another way of using function r to define a matroid.

Theorem 7.3.1 (Matroid from rank II)

Let E be a finite set and let $r : 2^E \to \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with r being its rank function if and only if for all $\cancel{A}$ $X \subseteq E$, and $x, y \in E$:

(R1') $r(\emptyset) = 0$;

(R2') $r(X) \leq r(X \cup \{y\}) \leq r(X) + 1$;

(R3') *If* $r(X \cup \{x\}) = r(X \cup \{y\}) = r(X)$, *then* $r(X \cup \{x, y\}) = r(X)$.

Matroids by submodular functions

Theorem 7.3.2 (Matroid by submodular functions)

Let $f : 2^E \to \mathbb{Z}$ be a integer valued monotone non-decreasing submodular function. Define a set of sets as follows: $\mathcal{C}(f) = \{ \cancel{\emptyset}, \;) \quad) \quad \}$

$$\mathcal{C}(f) = \Big\{ C \subseteq E : C \text{ is non-empty, is inclusionwise-minimal, and has } f(C) < |C| \Big\} \tag{7.10}$$

Then $\mathcal{C}(f)$ is the collection of circuits of a matroid on E.

Inclusionwise-minimal in this case means that if $C \in \mathcal{C}(f)$, then there exists no $C' \subset C$ with $C' \in \mathcal{C}(f)$ (i.e., $C' \subset C$ would either be empty or have $f(C') \geq |C'|$). Also, recall inclusionwise-minimal in Definition **??**, the definition of a circuit.

Summarizing: Many ways to define a Matroid

Summarizing what we've so far seen, we saw that it is possible to uniquely define a matroid based on any of:

- Independence (define the independent sets).

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- Independence (define the independent sets).
- Base axioms (exchangeability)
- Circuit axioms

Summarizing: Many ways to define a Matroid

Summarizing what we've so far seen, we saw that it is possible to uniquely define a matroid based on any of:

- Independence (define the independent sets).
- Base axioms (exchangeability)
- Circuit axioms
- Closure axioms (we didn't see this, but it is possible)

Summarizing: Many ways to define a Matroid

Summarizing what we've so far seen, we saw that it is possible to uniquely define a matroid based on any of:

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- Base axioms (exchangeability)
- Circuit axioms
- Closure axioms (we didn't see this, but it is possible)
- Rank axioms (normalized, monotone, cardinality bounded, submodular)

non-neg integral

Summarizing: Many ways to define a Matroid

Summarizing what we've so far seen, we saw that it is possible to uniquely define a matroid based on any of:

- Independence (define the independent sets).
- Base axioms (exchangeability)
- Circuit axioms
- Closure axioms (we didn't see this, but it is possible)
- Rank axioms (normalized, monotone, cardinality bounded, submodular)
- Matroids by submodular functions.

integral

Maximization problems for matroids

- Given a matroid $M = (E, \mathcal{I})$ and a modular cost function $c : E \to \mathbb{R}$, the task is to find an $X \in \mathcal{I}$ such that $c(X) = \sum_{x \in X} c(x)$ is maximum.
- This seems remarkably similar to the max spanning tree problem.

Minimization problems for matroids

- Given a matroid $M = (E, \mathcal{I})$ and a modular cost function $c : E \to \mathbb{R}$, the task is to find a basis $B \in \mathcal{B}$ such that $c(B)$ is minimized.
- This sounds like a set cover problem (find the minimum cost covering set of sets).

Partition Matroid

- What is the partition matroid's rank function?

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- A partition matroids rank function:

$$r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i) \tag{7.11}$$

which we also immediately see is submodular using properties we spoke about last week. That is:

Partition Matroid

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1. $|A \cap V_i|$ is submodular (in fact modular) in A

Partition Matroid

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- $r(A)$ is also non-negative integral monotone non-decreasing, so it defines a matroid (the partition matroid).

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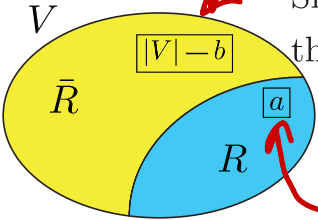

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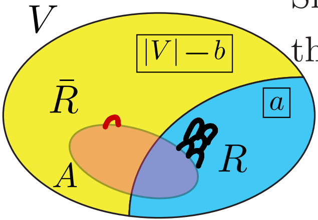

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- R, the set with minimum valuation amongst size-b sets, is hidden within an exponentially larger set of size-b sets with larger valuation.

Partition Matroid, rank as matching

- A partition matroid can be viewed using a bipartite graph.
- Letting V denote the ground set, and $V_1, V_2, \ldots$ the partition, the bipartite graph is $G = (V, I, E)$ where V is the ground set, I is a set of "indices", and E is the set of edges.
- $I = (I_1, I_2, \ldots, I_\ell)$ is a set of $k = \sum_{i=1}^{\ell} k_i$ nodes, grouped into ℓ clusters, where there are k_i nodes in the i^{th} group I_i.
- $(v, i) \in E(G)$ iff $v \in V_j$ and $i \in I_j$.

$|I_i| = k_i$

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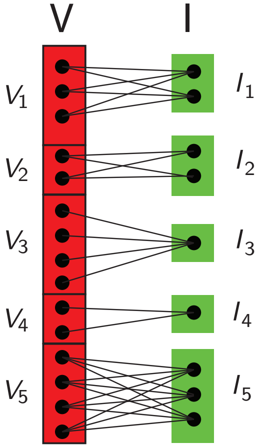
- Example where $\ell = 5$, $(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3)$.



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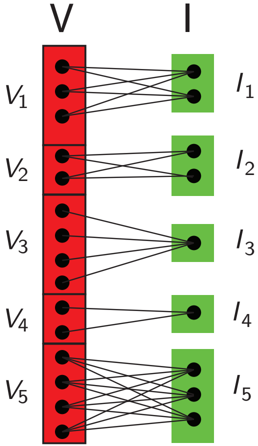

- Recall, $\Gamma : 2^V \to \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of X is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.

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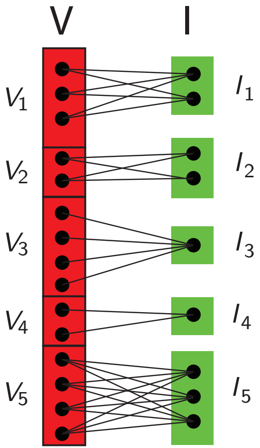

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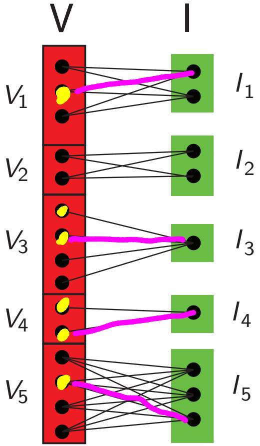

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- For such a constructed bipartite graph, the rank function of a partition matroid is $r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i) =$ the maximum matching involving X.

Laminar Family and Laminar Matroid

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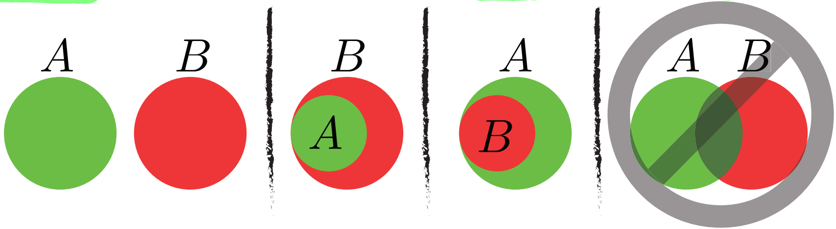

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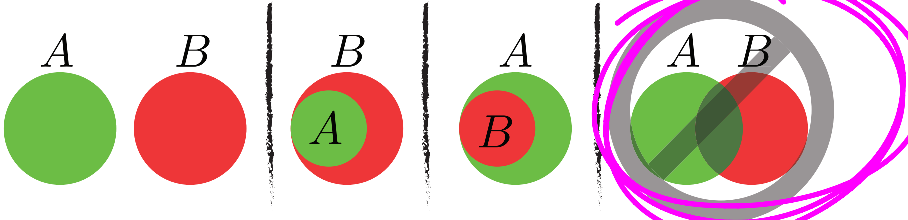

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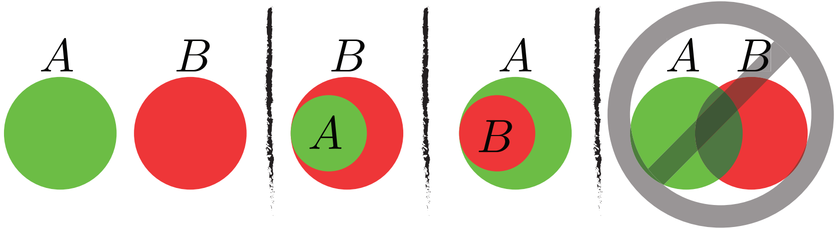

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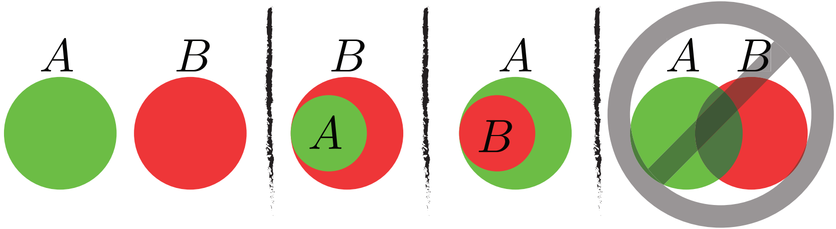

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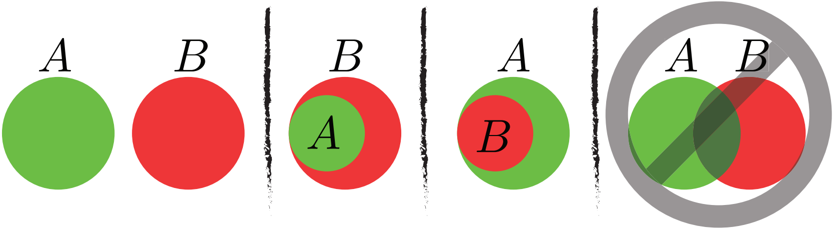

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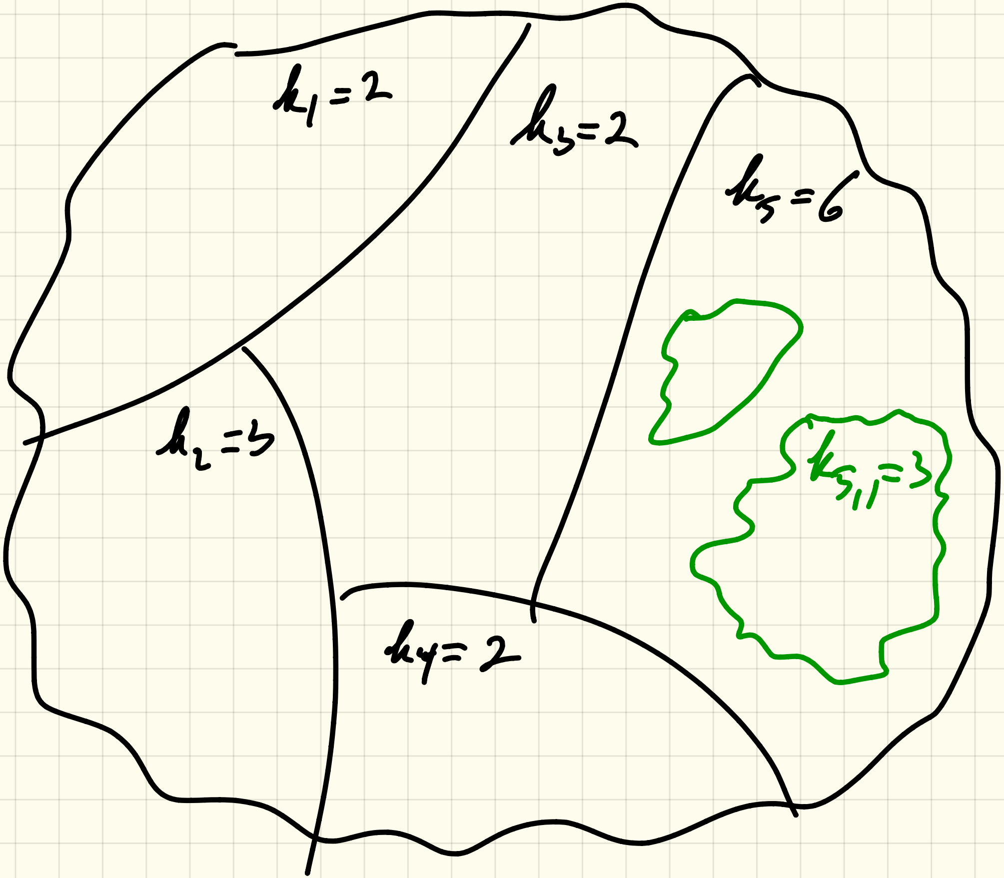
- Exercise: what is the rank function here?



$k_1 = 2$
$k_3 = 2$
$k_5 = 6$
$k_2 = 3$
$k_{5,1} = 3$
$k_4 = 2$

System of Representatives

- Let $(V, \mathcal{V})$ be a set system (i.e., $\mathcal{V} = (V_i : i \in I)$ where $\emptyset \subset V_i \subseteq V$ for all i), and I is an index set. Hence, $|I| = |\mathcal{V}|$.

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$v_i = v_j \quad i \neq j$

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- In a system of representatives, there is no requirement for the representatives to be distinct. I.e., we could have some $v_1 \in V_1 \cap V_2$, where v_1 represents both V_1 and V_2.

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- Here, the sets $V_i \in \mathcal{V}$ are like "groups" and any $v \in V$ with $v \in V_i$ is a member of group i. Groups need not be disjoint (e.g., interest groups of individuals).
- A family $(v_i : i \in I)$ with $v_i \in V$ is said to be a system of representatives of $\mathcal{V}$ if $\exists$ a bijection $\pi : I \to I$ such that $v_i \in V_{\pi(i)}$.
- v_i is the representative of set (or group) $V_{\pi(i)}$, meaning the i^{th} representative is meant to represent set $V_{\pi(i)}$.
- Example: Consider the house of representatives, $v_i =$ "Jim McDermott", while $i =$ "King County, WA-7".
- In a system of representatives, there is no requirement for the representatives to be distinct. I.e., we could have some $v_1 \in V_1 \cap V_2$, where v_1 represents both V_1 and V_2.
- We can view this as a bipartite graph.

System of Representatives

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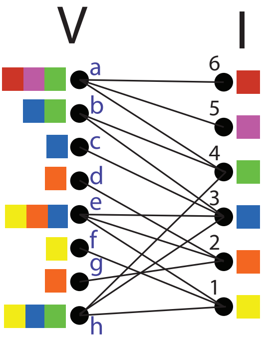
- Here, $\ell = 6$ groups, with $\mathcal{V} = (V_1, V_2, \dots, V_6)$ $= \Big(\{e, f, h\}, \{d, e, g\}, \{b, c, e, h\}, \{a, b, h\}, \{a\}, \{a\}\Big)$.



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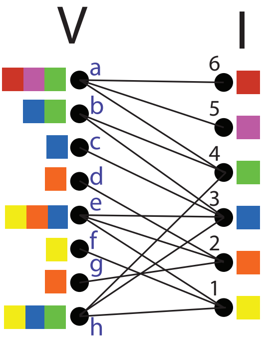

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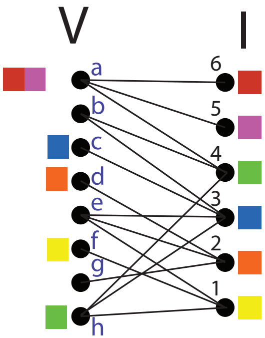

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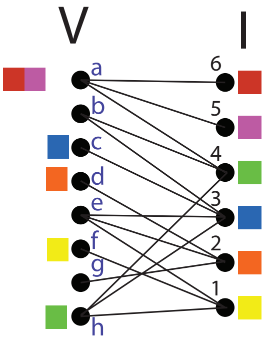

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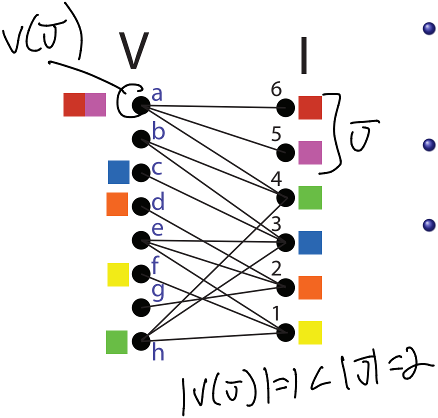

- A system of representatives would make sure that there is a representative for each color group. For example,
- The representatives $(\{a, c, d, f, h\})$ are shown as colors on the left.
- Here, the set of representatives is not distinct. Why? In fact, due to the red and pink group, a distinct group of representatives is impossible (since there is only one common choice to represent both color groups).

System of Distinct Representatives

- Let $(V, \mathcal{V})$ be a set system (i.e., $\mathcal{V} = (V_k : i \in I)$ where $V_i \subseteq V$ for all i), and I is an index set. Hence, $|I| = |\mathcal{V}|$.

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Definition 7.5.1 (transversal)

Given a set system $(V, \mathcal{V})$ and index set I for $\mathcal{V}$ as defined above, a set $T \subseteq V$ is a transversal of $\mathcal{V}$ if there is a bijection $\pi : T \leftrightarrow I$ such that

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- Note that due to $\pi : T \leftrightarrow I$ being a bijection, all of I and T are "covered" (so this makes things distinct automatically).

Transversals are Subclusive

- A set $T' \subseteq V$ is a partial transversal if T' is a transversal of some subfamily $\mathcal{V}' = (V_i : i \in I')$ where $I' \subseteq I$.

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- Thus, transversals are down closed (subclusive).

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- Given a set system $(V, \mathcal{V})$ with $\mathcal{V} = (V_i : i \in I)$, and $V_i \subseteq V$ for all i. Then, for any $J \subseteq I$, let

$J \subseteq I$ $V(J) \subseteq V$

$V: 2^I \to 2^V$

$$V(J) = \cup_{j \in J} V_j \tag{7.22}$$

so $|V(J)| : 2^I \to \mathbb{Z}_+$ is the set cover func. (we know is submodular).

$\shortparallel$
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Theorem 7.6.1 (Hall's theorem)

Given a set system $(V, \mathcal{V})$, the family of subsets $\mathcal{V} = (V_i : i \in I)$ has a transversal $(v_i : i \in I)$ iff for all $J \subseteq I$

$r(|V(J)| \geq |J|)$

$r(A) = |A|$

$$|V(J)| \geq |J| \tag{7.23}$$

Size of the set covered by the sets corresponding to the set of indices

$\geq$ size of the set of indices.

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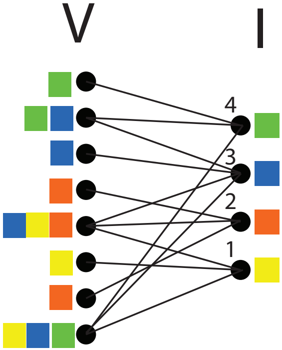
- Hall's theorem $(\forall J \subseteq I, |V(J)| \geq |J|)$ as a bipartite graph.



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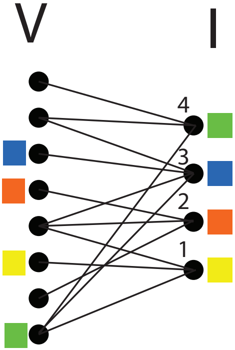

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Theorem 7.6.2 (Rado's theorem (1942))

If $M = (V, r)$ is a matroid on V with rank function r, then the family of subsets $(V_i : i \in I)$ of V has a transversal $(v_i : i \in I)$ that is independent in M iff for all $J \subseteq I$

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- Note, a transversal T independent in M means that $r(T) = |T|$.

More general conditions for existence of transversals

Theorem 7.6.3 (Polymatroid transversal theorem)

If $\mathcal{V} = (V_i : i \in I)$ is a finite family of non-empty subsets of V, and $f : 2^V \to \mathbb{Z}_+$ is a non-negative, integral, monotone non-decreasing, and submodular function, then $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that

$$f(\cup_{i \in J}\{v_i\}) \geq |J| \text{ for all } J \subseteq I \tag{7.25}$$

if and only if

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$|A| = c$

$r(A) \geq c$?

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- We get Theorem 7.6.2 by taking $f(S) = r(S)$ for $S \subseteq V$, the rank function of the matroid. *where, Eq. 7.25 insists the system of representatives is independent in M, and hence also distinct.*

Submodular Composition with Set-to-Set functions

- Note the condition in Theorem 7.6.3 is $f(V(J)) \geq |J|$ for all $J \subseteq I$, where $f : 2^V \to \mathbb{Z}_+$ is non-negative, integral, monotone non-decreasing and submodular, and $V(J) = \cup_{j \in J} V_j$ with $V_i \subseteq V$.

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- Hence, the condition for existence can be solved by (a special case of) submodular function minimization, or vice verse!

More general conditions for existence of transversals

first part proof of Theorem 7.6.3.

- Suppose $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that Eq. 7.25 (i.e., $f(\cup_{i \in J}\{v_i\}) \geq |J|$ for all $J \subseteq I$) is true.

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- Then since f is monotone, and since $V(J) \supseteq \cup_{i \in J}\{v_i\}$ when $(v_i : i \in I)$ is a system of representatives, then Eq. 7.26 (i.e., $f(V(J)) \geq |J|$ for all $J \subseteq I$) immediately follows.

...

$$f(V(J)) \geq f\Big(\bigcup_{j \in J} \{v_j\}\Big) \geq |J|$$

More general conditions for existence of transversals

Lemma 7.6.5 (contraction lemma)

Suppose Eq. 7.26 ($f(V(J)) \geq |J|, \forall J \subseteq I$) is true for $\mathcal{V} = (V_i : i \in I)$, and there exists an i such that $|V_i| \geq 2$ (w.l.o.g., say $i = 1$). Then there exists $\bar{v} \in V_1$ such that the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 7.26.

Proof.

- When Eq. 7.26 holds, this means that for any subsets $J_1, J_2 \subseteq I \setminus \{1\}$, we have that, for $J \in \{J_1, J_2\}$, consider index $J \cup \{1\}$

$$f(V(J \cup \{1\})) \geq |J \cup \{1\}| \tag{7.28}$$

and hence

$$f(V_1 \cup V(J_1)) \geq |J_1| + 1 \tag{7.29}$$

$$f(V_1 \cup V(J_2)) \geq |J_2| + 1 \tag{7.30}$$

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Proof.

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- ...and there must exist subsets J_1, J_2 of $I \setminus \{1\}$ such that

$$f((V_1 \setminus \{\bar{v}_1\}) \cup V(J_1)) < |J_1| + 1, \qquad (7.31)$$

$$f((V_1 \setminus \{\bar{v}_2\}) \cup V(J_2)) < |J_2| + 1, \qquad (7.32)$$

(note that either one or both of J_1, J_2 could be empty).

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Proof.

- Taking $X = (V_1 \setminus \{\bar{v}_1\}) \cup V(J_1)$ and $Y = (V_1 \setminus \{\bar{v}_2\}) \cup V(J_2)$, we have $f(X) \leq |J_1|$, $f(Y) \leq |J_2|$, and that:

$$X \cup Y = V_1 \cup V(J_1 \cup J_2), \tag{7.33}$$

$$X \cap Y \supseteq V(J_1 \cap J_2), \tag{7.34}$$

and

$$\begin{aligned} |J_1| + |J_2| &\geq f(X) + f(Y) \\ &\geq f(X \cup Y) + f(X \cap Y) \end{aligned} \tag{7.35}$$

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More general conditions for existence of transversals

Lemma 7.6.5 (contraction lemma)

Suppose Eq. 7.26 ($f(V(J)) \geq |J|, \forall J \subseteq I$) is true for $\mathcal{V} = (V_i : i \in I)$, and there exists an i such that $|V_i| \geq 2$ (w.l.o.g., say $i = 1$). Then there exists $\bar{v} \in V_1$ such that the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 7.26.

Proof.

- since f submodular monotone non-decreasing, & Eqs 7.33-7.35,

$$|J_1| + |J_2| \geq f(V_1 \cup V(J_1 \cup J_2)) + f(V(J_1 \cap J_2)) \tag{7.36}$$

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- Since $\mathcal{V}$ satisfies Eq. 7.26, $1 \notin J_1 \cup J_2$, & Eqs 7.29-7.30, this gives

$$|J_1| + |J_2| \geq |J_1 \cup J_2| + 1 + |J_1 \cap J_2| \tag{7.37}$$

which is a contradiction since cardinality is modular.

More general conditions for existence of transversals

Theorem 7.6.3 (Polymatroid transversal theorem)

If $\mathcal{V} = (V_i : i \in I)$ is a finite family of non-empty subsets of V, and $f : 2^V \to \mathbb{Z}_+$ is a non-negative, integral, monotone non-decreasing, and submodular function, then $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that

$$f(\cup_{i \in J}\{v_i\}) \geq |J| \text{ for all } J \subseteq I \tag{7.25}$$

if and only if

$$f(V(J)) \geq |J| \text{ for all } J \subseteq I \tag{7.26}$$

- Given Theorem 7.6.3, we immediately get Theorem 7.6.1 by taking $f(S) = |S|$ for $S \subseteq V$.
- We get Theorem 7.6.2 by taking $f(S) = r(S)$ for $S \subseteq V$, the rank function of the matroid.

More general conditions for existence of transversals

converse proof of Theorem 7.6.3.

- Conversely, suppose Eq. 7.26 is true.

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- Conversely, suppose Eq. 7.26 is true.
- If each V_i is a singleton set, then the result follows immediately.

$$V(\bar{v}) = \bigcup_{j \in \bar{v}} \{v_j\}$$

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- W.l.o.g., let $|V_1| \geq 2$, then by Lemma 7.6.5, the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 7.26 for the right $\bar{v}$.

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- We can continue to reduce the family, deleting elements from V_i for some i while $|V_i| \geq 2$, until we arrive at a family of singleton sets.

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This theorem can be used to produce a variety of other results quite easily, and shows how submodularity is the key ingredient in its truth.

Transversal Matroid

Transversals, themselves, define a matroid.

Theorem 7.7.1

If $\mathcal{V}$ is a family of finite subsets of a ground set V, then the collection of partial transversals of $\mathcal{V}$ is the set of independent sets of a matroid $M = (V, \mathcal{V})$ on V.

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- This means that the transversals of $\mathcal{V}$ are the bases of matroid M.
- Therefore, all maximal partial transversals of $\mathcal{V}$ have the same cardinality!

Transversals and Bipartite Matchings

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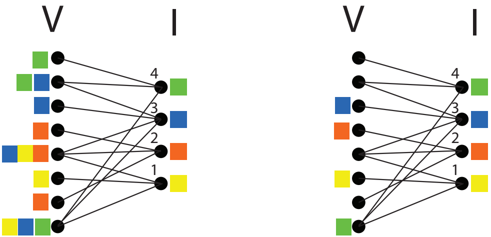
- A matching in this graph is a set of edges no two of which that have a common endpoint.



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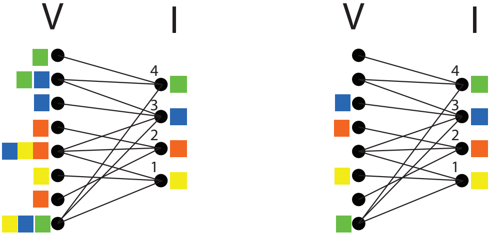

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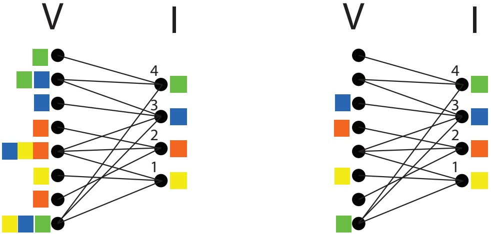
Lemma 7.7.2

A subset $T \subseteq V$ is a partial transversal of $\mathcal{V}$ iff there is a matching in (V, I, E) in which every edge has one endpoint in T (T matched into I).



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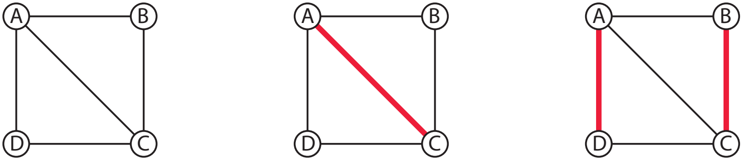

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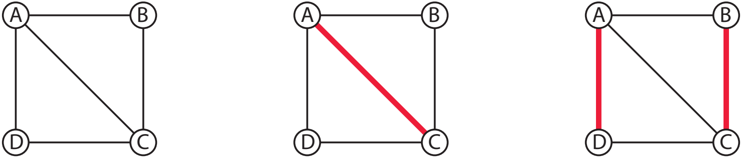
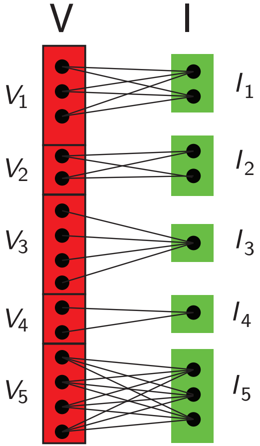


- $\{AC\}$ is a maximum matching, as is $\{AD, BC\}$, but they are not the same size.

Partition Matroid, rank as matching

- Example where $\ell = 5$, $(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3)$.



- Recall, $\Gamma : 2^V \to \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of X is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.
- Here, for $X \subseteq V$, we have $\Gamma(X) = \{i \in I : (v, i) \in E(G) \text{ and } v \in X\}$.
- For such a constructed bipartite graph, the rank function of a partition matroid is $r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i) =$ the maximum matching involving X.

Morphing Partition Matroid Rank

- Recall the partition matroid rank function. Note, $k_i = |I_i|$ in the bipartite graph representation, and since a matroid, w.l.o.g., $|V_i| \geq k_i$ (also, recall, $V(J) = \cup_{j \in J} V_j$).

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- Start with partition matroid rank function in the subsequent equations.

$$r(A) = \sum_{i \in \{1,\dots,\ell\}} \min(|A \cap V_i|, k_i) \tag{7.38}$$

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$$= \sum_{i \in \{1,\dots,\ell\}} \min_{J_i \subseteq I_i} \left(|V(J_i) \cap A| + |I_i \setminus J_i| \right) \tag{7.42}$$

... Morphing Partition Matroid Rank

- Continuing,

$$r(A) = \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} \left(|V(J_i) \cap V(I_i) \cap A| - |I_i \cap J_i| + |I_i| \right) \tag{7.43}$$

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- In fact, this bottom (more general) expression is the expression for the rank of a transversal matroid.

Partial Transversals Are Independent Sets in a Matroid

In fact, we have

Theorem 7.7.3

Let $(V, \mathcal{V})$ where $\mathcal{V} = (V_1, V_2, \ldots, V_\ell)$ be a subset system. Let $I = \{1, \ldots, \ell\}$. Let $\mathcal{I}$ be the set of partial transversals of $\mathcal{V}$. Then $(V, \mathcal{I})$ is a matroid.

Proof.

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- Suppose that T_1 and T_2 are partial transversals of $\mathcal{V}$ such that $|T_1| < |T_2|$. Exercise: show that (I3') holds.

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- Transversal matroid has rank

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- Therefore, this function is submodular.
- Note that it is a minimum over a set of modular functions. Is this true in general? Exercise:

Matroid loops

- A circuit in a matroids is well defined, a subset $A \subseteq E$ is circuit if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

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- Note, we also say that two elements s, t are said to be parallel if $\{s, t\}$ is a circuit.

Representable

Definition 7.8.1 (Matroid isomorphism)

Two matroids M_1 and M_2 respectively on ground sets V_1 and V_2 are isomorphic if there is a bijection $\pi : V_1 \to V_2$ which preserves independence (equivalently, rank, circuits, and so on).

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  Succinctly: A field is a set with $+$, $*$, closure, associativity, commutativity, and additive and multiplictaive identities and inverses.

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  Succinctly: A field is a set with $+$, $*$, closure, associativity, commutativity, and additive and multiplictaive identities and inverses.
- We can more generally define matroids on a field.

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Definition 7.8.2 (linear matroids on a field)

Let $\mathbf{X}$ be an $n \times m$ matrix and $E = \{1, \ldots, m\}$, where $\mathbf{X}_{ij} \in \mathbb{F}$ for some field, and let $\mathcal{I}$ be the set of subsets of E such that the columns of $\mathbf{X}$ are linearly independent over $\mathbb{F}$.

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- We can more generally define matroids on a field.

Definition 7.8.3 (representable (as a linear matroid))

Any matroid isomorphic to a linear matroid on a field is called representable over $\mathbb{F}$

Representability of Transversal Matroids

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- In particular:

Theorem 7.8.4

Transversal matroids are representable over all finite fields of sufficiently large cardinality, and are representable over any infinite field.

Converse: Representability of Transversal Matroids

The converse is not true, however.

Example 7.8.5

Let $V = \{1, 2, 3, 4, 5, 6\}$ be a ground set and let $M = (V, \mathcal{I})$ be a set system where $\mathcal{I}$ is all subsets of V of cardinality ≤ 2 except for the pairs $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$.

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- It can be shown that this is a matroid and is representable.
- However, this matroid is not isomorphic to any transversal matroid.

Matroids, other definitions using matroid rank $r : 2^V \to \mathbb{Z}_+$

Definition 7.9.3 (closed/flat/subspace)

A subset $A \subseteq E$ is closed (equivalently, a flat or a subspace) of matroid M if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

Definition: A hyperplane is a flat of rank $r(M) - 1$.

Definition 7.9.4 (closure)

Given $A \subseteq E$, the closure (or span) of A, is defined by $\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}$.

Therefore, a closed set A has $\text{span}(A) = A$.

Definition 7.9.5 (circuit)

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise-minimal dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

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- V is always trivially spanning.
- Consider the terminology: "spanning tree in a graph", comes from spanning in a matroid sense.

Dual of a Matroid

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- Dual of the dual: Note, we have that $(M^*)^* = M$.