Submodular Functions, Optimization, and Applications to Machine Learning
— Spring Quarter, Lecture 16 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee596b_spring_2016/

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Cumulative Outstanding Reading

- Read chapters 2 and 3 from Fujishige’s book.
- Read chapter 1 from Fujishige’s book.
Announcements, Assignments, and Reminders


- Homework 3, available at our assignment dropbox (https://canvas.uw.edu/courses/1039754/assignments), due (electronically) Monday (5/2) at 11:55pm.


- Homework 1, available at our assignment dropbox (https://canvas.uw.edu/courses/1039754/assignments), due (electronically) Friday (4/8) at 11:55pm.

- Weekly Office Hours: Mondays, 3:30-4:30, or by skype or google hangout (set up meeting via our discussion board (https://canvas.uw.edu/courses/1039754/discussion_topics)).
Class Road Map - IT-I

L1(3/28): Motivation, Applications, & Basic Definitions
L3(4/4): Info theory exs, more apps, definitions, graph/combinatorial examples, matrix rank example, visualization
L4(4/6): Graph and Combinatorial Examples, matrix rank, Venn diagrams, examples of proofs of submodularity, some useful properties
L5(4/11): Examples & Properties, OtherDefs., Independence
L6(4/13): Independence, Matroids, Matroid Examples, matroid rank is submodular
L7(4/18): Matroid Rank, More on Partition Matroid, System of Distinct Reps, Transversals, Transversal Matroid,
L8(4/20): Transversals, Matroid and representation, Dual Matroids,
L10(4/27): Matroid and Greedy, Polyhedra, Matroid Polytopes,
L11(5/2): From Matroids to Polymatroids, Polymatroids
L12(5/4): Polymatroids, Polymatroids and Greedy
L13(5/9): Polymatroids and Greedy; Possible Polytopes; Extreme Points; Polymatroids, Greedy, and Cardinality Constrained Maximization
L14(5/11): Cardinality Constrained Maximization; Curvature; Submodular Max w. Other Constraints
L15(5/16): Submodular Max w. Other Constraints, Most Violated \( \leq \), Matroids cont., Closure/Sat,
L16(5/18): Closure/Sat, Fund. Circuit/Dep, Min-Norm Point and SFM, Min-Norm Point Algorithm, Proof that min-norm gives optimal.
L17(5/23):
L18(5/25):
L19(6/1):
L20(6/6): Final Presentations maximization.

Finals Week: June 6th-10th, 2016.
Consider

\[ P^+_r = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r_M(A), \forall A \subseteq E \} \]  \hspace{1cm} (16.7)

Suppose we have any \( x \in \mathbb{R}_+^E \) such that \( x \not\in P^+_r \).

Hence, there must be a set of \( \mathcal{W} \subseteq 2^V \), each member of which corresponds to a 
violated inequality, i.e., equations of the form \( x(A) > r_M(A) \) for \( A \in \mathcal{W} \).

The most violated inequality when \( x \) is considered w.r.t. \( P^+_r \) corresponds to the set \( A \) that maximizes \( x(A) - r_M(A) \), i.e., the most violated inequality is valuated as:

\[ \max \{ x(A) - r_M(A) : A \in \mathcal{W} \} = \max \{ x(A) - r_M(A) : A \subseteq E \} \]  \hspace{1cm} (16.8)

Since \( x \) is modular and \( x(E \setminus A) = x(E) - x(A) \), we can express this via a
min as in;:

\[ \min \{ r_M(A) + x(E \setminus A) : A \subseteq E \} \]  \hspace{1cm} (16.9)
Consider

\[ P_f^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E \} \]

(16.7)

Suppose we have any \( x \in \mathbb{R}^E_+ \) such that \( x \notin P_f^+ \).

Hence, there must be a set of \( \mathcal{W} \subseteq 2^V \), each member of which corresponds to a violated inequality, i.e., equations of the form \( x(A) > r_M(A) \) for \( A \in \mathcal{W} \).
The most violated inequality when $x$ is considered w.r.t. $P_f^+$ corresponds to the set $A$ that maximizes $x(A) - f(A)$, i.e., the most violated inequality is valuated as:

$$\max \{x(A) - f(A) : A \in \mathcal{W}\} = \max \{x(A) - f(A) : A \subseteq E\} \quad (16.7)$$

Since $x$ is modular and $x(E \setminus A) = x(E) - x(A)$, we can express this via a min as in;:

$$\min \{f(A) + x(E \setminus A) : A \subseteq E\} \quad (16.8)$$

More importantly, $\min \{f(A) + x(E \setminus A) : A \subseteq E\}$ is a form of submodular function minimization, namely $\min \{f(A) - x(A) : A \subseteq E\}$ for a submodular $f$ and $x \in \mathbb{R}_E^+$, consisting of a difference of polymatroid and modular function (so $f - x$ is no longer necessarily monotone, nor positive).

We will ultimately answer how general this form of SFM is.
Lemma 16.2.5

Let $I \in \mathcal{I}(M)$, and $e \in E$, then $I \cup \{e\}$ contains at most one circuit in $M$.

Proof.

- Suppose, to the contrary, that there are two distinct circuits $C_1, C_2$ such that $C_1 \cup C_2 \subseteq I \cup \{e\}$.
- Then $e \in C_1 \cap C_2$, and by (C2), there is a circuit $C_3$ of $M$ s.t. $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\} \subseteq I$.
- This contradicts the independence of $I$.

In general, let $C(I,e)$ be the unique circuit associated with $I \cup \{e\}$ (commonly called the fundamental circuit in $M$ w.r.t. $I$ and $e$).
Matroids: The Fundamental Circuit

- Define $C(I, e)$ be the unique circuit associated with $I \cup \{e\}$ (the fundamental circuit in $M$ w.r.t. $I$ and $e$, if it exists).
- If $e \in \text{span}(I) \setminus I$, then $C(I, e)$ is well defined ($I + e$ creates one circuit).
- If $e \in I$, then $I + e = I$ doesn’t create a circuit. In such cases, $C(I, e)$ is not really defined.
- In such cases, we define $C(I, e) = \{e\}$, and we will soon see why.
- If $e \notin \text{span}(I)$, then $C(I, e) = \emptyset$, since no circuit is created in this case.
The sat function $=$ Polymatroid Closure

- Thus, in a matroid, closure (span) of a set $A$ are all items that $A$ spans (eq. that depend on $A$).
- We wish to generalize closure to polymatroids.
- Consider $x \in P_f$ for polymatroid function $f$.
- Again, recall, tight sets are closed under union and intersection, and therefore form a distributive lattice.
- That is, we saw in Lecture 7 that for any $A, B \in \mathcal{D}(x)$, we have that $A \cup B \in \mathcal{D}(x)$ and $A \cap B \in \mathcal{D}(x)$, which can constitute a join and meet.
- Recall, for a given $x \in P_f$, we have defined this tight family as

$$\mathcal{D}(x) = \{ A : A \subseteq E, x(A) = f(A) \}$$

(16.8)
Minimizers of a Submodular Function form a lattice

**Theorem 16.2.6**

For arbitrary submodular $f$, the minimizers are closed under union and intersection. That is, let $\mathcal{M} = \arg\min_{X \subseteq E} f(X)$ be the set of minimizers of $f$. Let $A, B \in \mathcal{M}$. Then $A \cup B \in \mathcal{M}$ and $A \cap B \in \mathcal{M}$.

**Proof.**

Since $A$ and $B$ are minimizers, we have $f(A) = f(B) \leq f(A \cap B)$ and $f(A) = f(B) \leq f(A \cup B)$. By submodularity, we have

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \quad (16.10)$$

Hence, we must have $f(A) = f(B) = f(A \cup B) = f(A \cap B)$.

Thus, the minimizers of a submodular function form a lattice, and there is a maximal and a minimal minimizer of every submodular function.
The \textit{sat} function $\equiv$ Polymatroid Closure

- Matroid closure is generalized by the unique maximal element in $\mathcal{D}(x)$, also called the polymatroid closure or \textit{sat} (\textit{saturation function}).

- For some $x \in P_f$, we have defined:

\begin{align*}
\text{cl}(x) \overset{\text{def}}{=} \text{sat}(x) & \overset{\text{def}}{=} \bigcup \{ A : A \in \mathcal{D}(x) \} \\
& = \bigcup \{ A : A \subseteq E, x(A) = f(A) \} \\
& = \{ e : e \in E, \forall \alpha > 0, x + \alpha 1_e \notin P_f \}
\end{align*}

\hspace{1cm} (16.10) \hspace{1cm} (16.11) \hspace{1cm} (16.12)

- Hence, \textit{sat}(x) is the maximal (zero-valued) minimizer of the submodular function $f_x(A) \triangleq f(A) - x(A)$.

- Eq. (??) says that \textit{sat} consists of any point $x$ that is $P_f$ saturated (any additional positive movement, in that dimension, leaves $P_f$). We’ll revisit this in a few slides.

- First, we see how \textit{sat} generalizes matroid closure.
The $\text{sat}$ function $= \text{Polymatroid Closure}$

**Lemma 16.3.1 (Matroid $\text{sat} : \mathbb{R}_+^E \rightarrow 2^E$ is the same as closure.)**

For $I \in \mathcal{I}$, we have $\text{sat}(1_I) = \text{span}(I)$  \hfill (16.1)
The \textit{sat} function $= \text{Polymatroid Closure}$

\textbf{Lemma 16.3.1} (Matroid \text{sat} : \mathbb{R}_+^E \to 2^E \text{ is the same as closure.})

For $I \in \mathcal{I}$, we have $\text{sat}(1_I) = \text{span}(I)$ \hfill (16.1)

\textbf{Proof.}

- For $1_I(I) = |I| = r(I)$, so $I \in \mathcal{D}(1_I)$ and $I \subseteq \text{sat}(1_I)$. Also, $I \subseteq \text{span}(I)$. ...
The sat function $= \text{Polymatroid Closure}$

**Lemma 16.3.1** (Matroid sat : $\mathbb{R}_+^E \rightarrow 2^E$ is the same as closure.)

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**Proof.**

- For $1_I(I) = |I| = r(I)$, so $I \in \mathcal{D}(1_I)$ and $I \subseteq \text{sat}(1_I)$. Also, $I \subseteq \text{span}(I)$.
- Consider some $b \in \text{span}(I) \setminus I$.
The \( \text{sat} \) function $= \text{Polymatroid Closure}$

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For $I \in \mathcal{I}$, we have $\text{sat}(1_I) = \text{span}(I)$ (16.1)

**Proof.**

- For $1_I(I) = |I| = r(I)$, so $I \in \mathcal{D}(1_I)$ and $I \subseteq \text{sat}(1_I)$. Also, $I \subseteq \text{span}(I)$.
- Consider some $b \in \text{span}(I) \setminus I$.
- Then $I \cup \{b\} \in \mathcal{D}(1_I)$ since $1_I(I \cup \{b\}) = |I| = r(I \cup \{b\}) = r(I)$. 

...
The \textit{sat} function $= \text{Polymatroid Closure}$

\begin{lemma} \label{lemma:matroid_closure}
(Matroid sat : $\mathbb{R}^E_+ \to 2^E$ is the same as closure.)

For $I \in \mathcal{I}$, we have $\text{sat}(\mathbf{1}_I) = \text{span}(I)$ \hfill (16.1)

\end{lemma}

\begin{proof}
\begin{itemize}
    \item For $\mathbf{1}_I(I) = |I| = r(I)$, so $I \in \mathcal{D}(\mathbf{1}_I)$ and $I \subseteq \text{sat}(\mathbf{1}_I)$. Also, $I \subseteq \text{span}(I)$.
    \item Consider some $b \in \text{span}(I) \setminus I$.
    \item Then $I \cup \{b\} \in \mathcal{D}(\mathbf{1}_I)$ since $\mathbf{1}_I(I \cup \{b\}) = |I| = r(I \cup \{b\}) = r(I)$.
    \item Thus, $b \in \text{sat}(\mathbf{1}_I)$.
\end{itemize}

\end{proof}
The \textit{sat} function $= \text{Polymatroid Closure}$

\textbf{Lemma 16.3.1 (Matroid sat : $\mathbb{R}^E_+ \rightarrow 2^E$ is the same as closure.)}

\[ \text{For } I \in \mathcal{I}, \text{ we have } \text{sat}(1_I) = \text{span}(I) \]  
\hspace{1cm} (16.1)

\textbf{Proof.}

- For $1_I(I) = |I| = r(I)$, so $I \in \mathcal{D}(1_I)$ and $I \subseteq \text{sat}(1_I)$. Also, $I \subseteq \text{span}(I)$.
- Consider some $b \in \text{span}(I) \setminus I$.
- Then $I \cup \{b\} \in \mathcal{D}(1_I)$ since $1_I(I \cup \{b\}) = |I| = r(I) = r(I)$.
- Thus, $b \in \text{sat}(1_I)$.
- Therefore, $\text{sat}(1_I) \supseteq \text{span}(I)$. 

...
The sat function = Polymatroid Closure

... proof continued.

Now, consider \( b \in \text{sat}(\mathbf{1}_I) \setminus I \).
The \textit{sat} function $= \text{Polymatroid Closure}$

... proof continued.

- Now, consider $b \in \text{sat}(1_I) \setminus I$.
- Choose any $A \in \mathcal{D}(1_I)$ with $b \in A$, thus $b \in A \setminus I$. 

\[ A \setminus I \]

\[ I \]

\[ \Rightarrow \]

Since $b \in A \cap I$, we get $b \in \text{span}(I)$. 

Thus, $\text{sat}(1_I) \subseteq \text{span}(I)$. 

Hence $\text{sat}(1_I) = \text{span}(I)$.
The \textit{sat} function \(=\) Polymatroid Closure

... proof continued.

- Now, consider \(b \in \text{sat}(1_I) \setminus I\).
- Choose any \(A \in \mathcal{D}(1_I)\) with \(b \in A\), thus \(b \in A \setminus I\).
- Then \(1_I(A) = |A \cap I| = r(A) \geq r(A \cap I)\).
The \textit{sat} function = Polymatroid Closure

... proof continued.

- Now, consider $b \in \text{sat}(1_I) \setminus I$.
- Choose any $A \in \mathcal{D}(1_I)$ with $b \in A$, thus $b \in A \setminus I$.
- Then $1_I(A) = |A \cap I| = r(A)$.
- Now $r(A) = |A \cap I| \leq |I| = r(I)$. 


The \textit{sat} function $= \text{Polymatroid Closure}$

... proof continued.

- Now, consider $b \in \text{sat}(\mathbf{1}_I) \setminus I$.
- Choose any $A \in \mathcal{D}(\mathbf{1}_I)$ with $b \in A$, thus $b \in A \setminus I$.
- Then $\mathbf{1}_I(A) = |A \cap I| = r(A) = r(A \cap I)$
- Now $r(A) = |A \cap I| \leq |I| = r(I)$.
- Also, $r(A \cap I) = |A \cap I|$ since $A \cap I \in \mathcal{I}$.
The sat function = Polymatroid Closure

... proof continued.

- Now, consider $b \in \text{sat}(1_I) \setminus I$.
- Choose any $A \in \mathcal{D}(1_I)$ with $b \in A$, thus $b \in A \setminus I$.
- Then $1_I(A) = |A \cap I| = r(A)$.
- Now $r(A) = |A \cap I| \leq |I| = r(I)$.
- Also, $r(A \cap I) = |A \cap I|$ since $A \cap I \in \mathcal{I}$.
- Hence, $r(A \cap I) = r(A) = r((A \cap I) \cup (A \setminus I))$ meaning $(A \setminus I) \subseteq \text{span}(A \cap I) \subseteq \text{span}(I)$. 
The \textit{sat} function $= \text{Polymatroid Closure}$

...proof continued.

- Now, consider $b \in \text{sat}(1_I) \setminus I$.
- Choose any $A \in \mathcal{D}(1_I)$ with $b \in A$, thus $b \in A \setminus I$.
- Then $1_I(A) = |A \cap I| = r(A)$.
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- Since $b \in A \setminus I$, we get $b \in \text{span}(I)$.
The sat function = Polymatroid Closure

... proof continued.

- Now, consider $b \in \text{sat}(1_I) \setminus I$.
- Choose any $A \in \mathcal{D}(1_I)$ with $b \in A$, thus $b \in A \setminus I$.
- Then $1_I(A) = |A \cap I| = r(A)$.
- Now $r(A) = |A \cap I| \leq |I| = r(I)$.
- Also, $r(A \cap I) = |A \cap I|$ since $A \cap I \in \mathcal{I}$.
- Hence, $r(A \cap I) = r(A) = r((A \cap I) \cup (A \setminus I))$ meaning $(A \setminus I) \subseteq \text{span}(A \cap I) \subseteq \text{span}(I)$.
- Since $b \in A \setminus I$, we get $b \in \text{span}(I)$.
- Thus, $\text{sat}(1_I) \subseteq \text{span}(I)$. 
The sat function $= \text{Polymatroid Closure}$

... proof continued.

- Now, consider $b \in \text{sat}(1_I) \setminus I$.
- Choose any $A \in D(1_I)$ with $b \in A$, thus $b \in A \setminus I$.
- Then $1_I(A) = |A \cap I| = r(A)$.
- Now $r(A) = |A \cap I| \leq |I| = r(I)$.
- Also, $r(A \cap I) = |A \cap I|$ since $A \cap I \in \mathcal{I}$.
- Hence, $r(A \cap I) = r(A) = r((A \cap I) \cup (A \setminus I))$ meaning $(A \setminus I) \subseteq \text{span}(A \cap I) \subseteq \text{span}(I)$.
- Since $b \in A \setminus I$, we get $b \in \text{span}(I)$.
- Thus, $\text{sat}(1_I) \subseteq \text{span}(I)$.
- Hence $\text{sat}(1_I) = \text{span}(I)$.
The \textit{sat} function $= \text{Polymatroid Closure}$

- Now, consider a matroid $(E, r)$ and some $C \subseteq E$ with $C \notin \mathcal{I}$, and consider $1_C$. 

\textbf{Exercise:} is $\text{span}(C) = \text{sat}(1_C)$? Prove or disprove it.
Now, consider a matroid \((E, r)\) and some \(C \subseteq E\) with \(C \notin \mathcal{I}\), and consider \(1_C\). Is \(1_C \in P_r\)?
The \textit{sat} function $\Rightarrow$ Polymatroid Closure

- Now, consider a matroid $(E, r)$ and some $C \subseteq E$ with $C \notin \mathcal{I}$, and consider $1_C$. Is $1_C \in P_r$? No, it might not be a vertex, or even a member, of $P_r$. 
The \textit{sat} function $\equiv$ Polymatroid Closure

- Now, consider a matroid $(E, r)$ and some $C \subseteq E$ with $C \notin \mathcal{I}$, and consider $1_C$. Is $1_C \in P_r$? No, it might not be a vertex, or even a member, of $P_r$.

- $\text{span}(\cdot)$ operates on more than just independent sets, so $\text{span}(C)$ is perfectly sensible.
The \textit{sat} function $\equiv$ Polymatroid Closure

- Now, consider a matroid $(E, r)$ and some $C \subseteq E$ with $C \notin \mathcal{I}$, and consider $\mathbf{1}_C$. Is $\mathbf{1}_C \in P_r$? No, it might not be a vertex, or even a member, of $P_r$.
- $\text{span}(\cdot)$ operates on more than just independent sets, so $\text{span}(C)$ is perfectly sensible.
- \textbf{Note} $\text{span}(C) = \text{span}(B)$ where $\mathcal{I} \ni B \in \mathcal{B}(C)$ is a base of $C$. 

Exercise: is $\text{span}(C) = \text{sat}(\mathbf{1}_C)$? Prove or disprove it.
The sat function \(=\) Polymatroid Closure

- Now, consider a matroid \((E, r)\) and some \(C \subseteq E\) with \(C \notin \mathcal{I}\), and consider \(1_C\). Is \(1_C \in P_r\)? No, it might not be a vertex, or even a member, of \(P_r\).
- \(\text{span}()\) operates on more than just independent sets, so \(\text{span}(C)\) is perfectly sensible.
- Note \(\text{span}(C) = \text{span}(B)\) where \(\mathcal{I} \ni B \in \mathcal{B}(C)\) is a base of \(C\).
- Then we have \(1_B \leq 1_C \leq 1_{\text{span}(C)}\), and that \(1_B \in P_r\). We can then make the definition:
  \[
  \text{sat}(1_C) \triangleq \text{sat}(1_B) \quad \text{for} \quad B \in \mathcal{B}(C)
  \]
  \[
  \text{sat}(y) = \text{sat}(\text{P-basis}(y))
  \]

In which case, we also get \(\text{sat}(1_C) = \text{span}(C)\) (in general, could define \(\text{sat}(y) = \text{sat}(\text{P-basis}(y))\)).
The \textit{sat} function $\equiv$ Polymatroid Closure

- Now, consider a matroid $(E, r)$ and some $C \subseteq E$ with $C \notin \mathcal{I}$, and consider $1_C$. Is $1_C \in P_r$? No, it might not be a vertex, or even a member, of $P_r$.
- $\text{span}(\cdot)$ operates on more than just independent sets, so $\text{span}(C)$ is perfectly sensible.
- Note $\text{span}(C) = \text{span}(B)$ where $\mathcal{I} \ni B \in \mathcal{B}(C)$ is a base of $C$.
- Then we have $1_B \leq 1_C \leq 1_{\text{span}(C)}$, and that $1_B \in P_r$. We can then make the definition:

$$\text{sat}(1_C) \triangleq \text{sat}(1_B) \text{ for } B \in \mathcal{B}(C)$$

(16.2)

In which case, we also get $\text{sat}(1_C) = \text{span}(C)$ (in general, could define $\text{sat}(y) = \text{sat}(P\text{-basis}(y))$).
- However, consider the following form

$$\text{sat}(1_C) = \bigcup \{A : A \subseteq E, |A \cap C| = r(A)\}$$

(16.3)
The sat function $\equiv$ Polymatroid Closure

- Now, consider a matroid $(E, r)$ and some $C \subseteq E$ with $C \notin \mathcal{I}$, and consider $1_C$. Is $1_C \in P_r$? No, it might not be a vertex, or even a member, of $P_r$.

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  $$\text{sat}(1_C) \triangleq \text{sat}(1_B) \text{ for } B \in \mathcal{B}(C)$$  \hspace{1cm} (16.2)

  In which case, we also get $\text{sat}(1_C) = \text{span}(C)$ (in general, could define $\text{sat}(y) = \text{sat}(P\text{-basis}(y))$).

- However, consider the following form

  $$\text{sat}(1_C) = \bigcup \{A : A \subseteq E, |A \cap C| = r(A)\}$$  \hspace{1cm} (16.3)

  **Exercise:** is $\text{span}(C) = \text{sat}(1_C)$? Prove or disprove it.
The sat function, span, and submodular function minimization

- Thus, for a matroid, $\text{sat}(1_I)$ is exactly the closure (or span) of $I$ in the matroid. I.e., for matroid $(E, r)$, we have $\text{span}(I) = \text{sat}(1_B)$. 
The \textit{sat} function, span, and submodular function minimization

Thus, for a matroid, \textit{sat}(1_I) is exactly the closure (or span) of \textit{I} in the matroid. I.e., for matroid \((E, r)\), we have \text{span}(I) = \text{sat}(1_B).

Recall, for \(x \in P_f\) and polymatroidal \(f\), \textit{sat}(x) is the maximal (by inclusion) minimizer of \(f(A) - x(A)\), and thus in a matroid, \text{span}(I) is the maximal minimizer of the submodular function formed by \(r(A) - 1_I(A)\).
The $\text{sat}$ function, span, and submodular function minimization

- Thus, for a matroid, $\text{sat}(1_I)$ is exactly the closure (or span) of $I$ in the matroid. I.e., for matroid $(E, r)$, we have $\text{span}(I) = \text{sat}(1_B)$.

- Recall, for $x \in P_f$ and polymatroidal $f$, $\text{sat}(x)$ is the maximal (by inclusion) minimizer of $f(A) - x(A)$, and thus in a matroid, $\text{span}(I)$ is the maximal minimizer of the submodular function formed by $r(A) - 1_I(A)$.

- Submodular function minimization can solve “span” queries in a matroid or “sat” queries in a polymatroid.
We are given an \( x \in P_f^+ \) for submodular function \( f \).
We are given an $x \in P_f^+$ for submodular function $f$.

Recall that for such an $x$, $\text{sat}(x)$ is defined as

$$\text{sat}(x) = \bigcup \{ A : x(A) = f(A) \}$$  

(16.4)
We are given an $x \in P_f^+$ for submodular function $f$.

Recall that for such an $x$, $\text{sat}(x)$ is defined as

$$\text{sat}(x) = \bigcup \{ A : x(A) = f(A) \}$$

(16.4)

We also have stated that $\text{sat}(x)$ can be defined as:

$$\text{sat}(x) = \left\{ e : \forall \alpha > 0, x + \alpha e \notin P_f^+ \right\}$$

(16.5)
We are given an $x \in P_f^+$ for submodular function $f$.

Recall that for such an $x$, $\text{sat}(x)$ is defined as

$$\text{sat}(x) = \bigcup \{ A : x(A) = f(A) \} \quad (16.4)$$

We also have stated that $\text{sat}(x)$ can be defined as:

$$\text{sat}(x) = \left\{ e : \forall \alpha > 0, x + \alpha 1_e \notin P_f^+ \right\} \quad (16.5)$$

We next show more formally that these are the same.
Let's start with one definition and derive the other.

\[
sat(x)
\]
sat, as tight polymatroidal elements

- Lets start with one definition and derive the other.

\[
\text{sat}(x) \overset{\text{def}}{=} \left\{ e : \forall \alpha > 0, x + \alpha 1_e \notin P_f^+ \right\} \quad \text{(16.6)}
\]
Let's start with one definition and derive the other.

\[
\text{sat}(x) \overset{\text{def}}{=} \left\{ e : \forall \alpha > 0, \ x + \alpha 1_e \notin P_f^+ \right\} = \left\{ e : \forall \alpha > 0, \exists A \text{ s.t. } (x + \alpha 1_e)(A) > f(A) \right\}
\]
sat, as tight polymatroidal elements

Let's start with one definition and derive the other.

\[
\text{sat}(x) \overset{\text{def}}{=} \left\{ e : \forall \alpha > 0, x + \alpha 1_e \notin P_f^+ \right\} \tag{16.6}
\]

\[
= \left\{ e : \forall \alpha > 0, \exists A \text{ s.t. } (x + \alpha 1_e)(A) > f(A) \right\} \tag{16.7}
\]

\[
= \left\{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } (x + \alpha 1_e)(A) > f(A) \right\} \tag{16.8}
\]
sat, as tight polymatroidal elements

- Let's start with one definition and derive the other.
  \[ \text{sat}(x) \overset{\text{def}}{=} \left\{ e : \forall \alpha > 0, x + \alpha 1_e \notin P^+_f \right\} \]  
  \[ = \left\{ e : \forall \alpha > 0, \exists A \text{ s.t. } (x + \alpha 1_e)(A) > f(A) \right\} \]  
  \[ = \left\{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } (x + \alpha 1_e)(A) > f(A) \right\} \]  

- This last bit follows since \( 1_e(A) = 1 \iff e \in A. \)
sat, as tight polymatroidal elements

- Let's start with one definition and derive the other.

\[
sat(x) \overset{\text{def}}{=} \left\{ e : \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f^+ \right\}
\]

(16.6)

\[
= \left\{ e : \forall \alpha > 0, \exists A \text{ s.t. } (x + \alpha \mathbf{1}_e)(A) > f(A) \right\}
\]

(16.7)

\[
= \left\{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } (x + \alpha \mathbf{1}_e)(A) > f(A) \right\}
\]

(16.8)

- This last bit follows since \( \mathbf{1}_e(A) = 1 \iff e \in A \). Continuing, we get

\[
sat(x) = \left\{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } x(A) + \alpha > f(A) \right\}
\]

(16.9)
Let's start with one definition and derive the other.

\[
sat(x) \overset{\text{def}}{=} \left\{ e : \forall \alpha > 0, x + \alpha 1_e \notin P^+_f \right\}
\]

(16.6)

\[
= \left\{ e : \forall \alpha > 0, \exists A \text{ s.t. } (x + \alpha 1_e)(A) > f(A) \right\}
\]

(16.7)

\[
= \left\{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } (x + \alpha 1_e)(A) > f(A) \right\}
\]

(16.8)

This last bit follows since \(1_e(A) = 1 \iff e \in A\). Continuing, we get

\[
sat(x) = \left\{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } x(A) + \alpha > f(A) \right\}
\]

(16.9)

given that \(x \in P^+_f\), meaning \(x(A) \leq f(A)\) for all \(A\), we must have

\[
sat(x)
\]
sat, as tight polymatroidal elements

- Lets start with one definition and derive the other.

\[ \text{sat}(x) \overset{\text{def}}{=} \left\{ e : \forall \alpha > 0, x + \alpha 1_e \notin P_f^+ \right\} \quad (16.6) \]

\[ = \left\{ e : \forall \alpha > 0, \exists A \text{ s.t. } (x + \alpha 1_e)(A) > f(A) \right\} \quad (16.7) \]

\[ = \left\{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } (x + \alpha 1_e)(A) > f(A) \right\} \quad (16.8) \]

- this last bit follows since \(1_e(A) = 1 \iff e \in A\). Continuing, we get

\[ \text{sat}(x) = \left\{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } x(A) + \alpha > f(A) \right\} \quad (16.9) \]

- given that \(x \in P_f^+\), meaning \(x(A) \leq f(A)\) for all \(A\), we must have

\[ \text{sat}(x) = \left\{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } x(A) = f(A) \right\} \quad (16.10) \]
sat, as tight polymatroidal elements

- Let's start with one definition and derive the other.

\[
\text{sat}(x) \overset{\text{def}}{=} \{ e : \forall \alpha > 0, x + \alpha 1_e \notin P_f^+ \} \quad (16.6)
\]

\[
= \{ e : \forall \alpha > 0, \exists A \text{ s.t. } (x + \alpha 1_e)(A) > f(A) \} \quad (16.7)
\]

\[
= \{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } (x + \alpha 1_e)(A) > f(A) \} \quad (16.8)
\]

- This last bit follows since \(1_e(A) = 1 \iff e \in A\). Continuing, we get

\[
\text{sat}(x) = \{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } x(A) + \alpha > f(A) \} \quad (16.9)
\]

- Given that \(x \in P_f^+\), meaning \(x(A) \leq f(A)\) for all \(A\), we must have

\[
\text{sat}(x) = \{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } x(A) = f(A) \} \quad (16.10)
\]

\[
= \{ e : \exists A \ni e \text{ s.t. } x(A) = f(A) \} \quad (16.11)
\]
sat, as tight polymatroidal elements

- Lets start with one definition and derive the other.
  \[
  \text{sat}(x) \overset{\text{def}}{=} \left\{ e : \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f^+ \right\} 
  \]
  \[
  = \left\{ e : \forall \alpha > 0, \exists A \text{ s.t. } (x + \alpha \mathbf{1}_e)(A) > f(A) \right\} 
  \]
  \[
  = \left\{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } (x + \alpha \mathbf{1}_e)(A) > f(A) \right\} 
  \]

- this last bit follows since \( \mathbf{1}_e(A) = 1 \iff e \in A \). Continuing, we get
  \[
  \text{sat}(x) = \left\{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } x(A) + \alpha > f(A) \right\} 
  \]

- given that \( x \in P_f^+ \), meaning \( x(A) \leq f(A) \) for all \( A \), we must have
  \[
  \text{sat}(x) = \left\{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } x(A) = f(A) \right\} 
  \]
  \[
  = \left\{ e : \exists A \ni e \text{ s.t. } x(A) = f(A) \right\} 
  \]

- So now, if \( A \) is any set such that \( x(A) = f(A) \), then we clearly have
sat, as tight polymatroidal elements

- Lets start with one definition and derive the other.

\[
sat(x) \overset{\text{def}}{=} \left\{ e : \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f^+ \right\}
\]

\[
= \left\{ e : \forall \alpha > 0, \exists A \text{ s.t. } (x + \alpha \mathbf{1}_e)(A) > f(A) \right\}
\] (16.7)

\[
= \left\{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } (x + \alpha \mathbf{1}_e)(A) > f(A) \right\}
\] (16.8)

- this last bit follows since \( \mathbf{1}_e(A) = 1 \iff e \in A \). Continuing, we get

\[
sat(x) = \left\{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } x(A) + \alpha > f(A) \right\}
\] (16.9)

- given that \( x \in P_f^+ \), meaning \( x(A) \leq f(A) \) for all \( A \), we must have

\[
sat(x) = \left\{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } x(A) = f(A) \right\}
\]

\[
= \left\{ e : \exists A \ni e \text{ s.t. } x(A) = f(A) \right\}
\] (16.11)

- So now, if \( A \) is any set such that \( x(A) = f(A) \), then we clearly have

\[
\forall e \in A, e \in sat(x),
\] (16.12)
sat, as tight polymatroidal elements

- Let's start with one definition and derive the other.

\[
sat(x) \overset{\text{def}}{=} \{ e : \forall \alpha > 0, x + \alpha 1_e \notin P_f^+ \} \quad (16.6)
\]

\[
= \{ e : \forall \alpha > 0, \exists A \text{ s.t. } (x + \alpha 1_e)(A) > f(A) \} \quad (16.7)
\]

\[
= \{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } (x + \alpha 1_e)(A) > f(A) \} \quad (16.8)
\]

- This last bit follows since \(1_e(A) = 1 \iff e \in A\). Continuing, we get

\[
sat(x) = \{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } x(A) + \alpha > f(A) \} \quad (16.9)
\]

- Given that \(x \in P_f^+\), meaning \(x(A) \leq f(A)\) for all \(A\), we must have

\[
sat(x) = \{ e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } x(A) = f(A) \} \quad (16.10)
\]

\[
= \{ e : \exists A \ni e \text{ s.t. } x(A) = f(A) \} \quad (16.11)
\]

- So now, if \(A\) is any set such that \(x(A) = f(A)\), then we clearly have

\[
\forall e \in A, e \in sat(x), \text{ and therefore that } sat(x) \supseteq A \quad (16.12)
\]
...and therefore, with sat as defined in Eq. (??),

\[ \text{sat}(x) \supseteq \bigcup \{ A : x(A) = f(A) \} \]  \hspace{1cm} (16.13)
... and therefore, with \( sat \) as defined in Eq. (??),

\[
sat(x) \supseteq \bigcup \{ A : x(A) = f(A) \} \quad (16.13)
\]

On the other hand, for any \( e \in sat(x) \) defined as in Eq. (16.11), since \( e \) is itself a member of a tight set, there is a set \( A \ni e \) such that \( x(A) = f(A) \), giving

\[
sat(x) \subseteq \bigcup \{ A : x(A) = f(A) \} \quad (16.14)
\]
...and therefore, with $\text{sat}$ as defined in Eq. (??),

$$\text{sat}(x) \supseteq \bigcup \{ A : x(A) = f(A) \}$$  \hspace{1cm} (16.13)

On the other hand, for any $e \in \text{sat}(x)$ defined as in Eq. (16.11), since $e$ is itself a member of a tight set, there is a set $A \ni e$ such that $x(A) = f(A)$, giving

$$\text{sat}(x) \subseteq \bigcup \{ A : x(A) = f(A) \}$$  \hspace{1cm} (16.14)

Therefore, the two definitions of $\text{sat}$ are identical.
Another useful concept is saturation capacity which we develop next.

\[
\begin{align*}
&\max \left\{ x \in \mathbb{R}^n, x + 1 e \right\} \\
&\max \left\{ x \in \mathbb{R}^n, x + 1 e \right\} \cdot (A) \\
&\max \left\{ x \in \mathbb{R}^n, x + 1 e \right\} \cdot (A) \\
&\max \left\{ x \in \mathbb{R}^n, x + 1 e \right\} \cdot (A)
\end{align*}
\]
Saturation Capacity

- Another useful concept is saturation capacity which we develop next.
- For $x \in P_f$, and $e \in E$, consider finding

$$\max \left\{ \alpha : \alpha \in \mathbb{R}, x + \alpha 1_e \in P_f \right\} \quad (16.15)$$
Another useful concept is saturation capacity which we develop next.

For $x \in P_f$, and $e \in E$, consider finding

$$\max \{ \alpha : \alpha \in \mathbb{R}, x + \alpha 1_e \in P_f \} \quad (16.15)$$

This is identical to:

$$\max \{ \alpha : (x + \alpha 1_e)(A) \leq f(A), \forall A \supseteq \{e\} \} \quad (16.16)$$

since any $B \subseteq E$ such that $e \notin B$ does not change in a $1_e$ adjustment, meaning $(x + \alpha 1_e)(B) = x(B)$.  

---

**Saturation Capacity**

\[ \hat{c}(x, e) \]
Saturation Capacity

- Another useful concept is saturation capacity which we develop next.
- For $x \in P_f$, and $e \in E$, consider finding

  $$\max \{ \alpha : \alpha \in \mathbb{R}, x + \alpha 1_e \in P_f \}$$  \hspace{1cm} (16.15)

- This is identical to:

  $$\max \{ \alpha : (x + \alpha 1_e)(A) \leq f(A), \forall A \supseteq \{e\} \}$$  \hspace{1cm} (16.16)

  since any $B \subseteq E$ such that $e \notin B$ does not change in a $1_e$ adjustment, meaning $(x + \alpha 1_e)(B) = x(B)$.

- Again, this is identical to:

  $$\max \{ \alpha : x(A) + \alpha \leq f(A), \forall A \supseteq \{e\} \}$$  \hspace{1cm} (16.17)
Saturation Capacity

Another useful concept is saturation capacity which we develop next.

For $x \in P_f$, and $e \in E$, consider finding

$$\max \{\alpha : \alpha \in \mathbb{R}, x + \alpha 1_e \in P_f\} \quad (16.15)$$

This is identical to:

$$\max \{\alpha : (x + \alpha 1_e)(A) \leq f(A), \forall A \supseteq \{e\}\} \quad (16.16)$$

since any $B \subseteq E$ such that $e \notin B$ does not change in a $1_e$ adjustment, meaning $(x + \alpha 1_e)(B) = x(B)$.

Again, this is identical to:

$$\max \{\alpha : x(A) + \alpha \leq f(A), \forall A \supseteq \{e\}\} \quad (16.17)$$

or

$$\max \{\alpha : \alpha \leq f(A) - x(A), \forall A \supseteq \{e\}\} \quad (16.18)$$
Saturation Capacity

- The max is achieved when

\[ \alpha = \hat{c}(x; e) \overset{\text{def}}{=} \min \{ f(A) - x(A), \forall A \supseteq \{e\} \} \quad (16.19) \]
Saturation Capacity

- The max is achieved when

\[ \alpha = \hat{c}(x; e) \overset{\text{def}}{=} \min \{ f(A) - x(A), \forall A \supseteq \{e\} \} \quad (16.19) \]

- \( \hat{c}(x; e) \) is known as the saturation capacity associated with \( x \in P_f \) and \( e \).
Saturation Capacity

- The max is achieved when

\[ \alpha = \hat{c}(x; e) \overset{\text{def}}{=} \min \{ f(A) - x(A), \forall A \supseteq \{ e \} \} \quad (16.19) \]

- \( \hat{c}(x; e) \) is known as the saturation capacity associated with \( x \in P_f \) and \( e \).

- Thus we have for \( x \in P_f \),

\[ \hat{c}(x; e) \overset{\text{def}}{=} \min \{ f(A) - x(A), \forall A \supseteq e \} \]

\[ = \max \{ \alpha : \alpha \in \mathbb{R}, x + \alpha 1_e \in P_f \} \quad (16.20) \]

\[ = \max \{ \alpha : \alpha \in \mathbb{R}, x + \alpha 1_e \in P_f \} \quad (16.21) \]
Saturation Capacity

- The max is achieved when

\[
\alpha = \hat{c}(x; e) \overset{\text{def}}{=} \min \{ f(A) - x(A), \forall A \supseteq \{e\} \} \quad (16.19)
\]

- \(\hat{c}(x; e)\) is known as the saturation capacity associated with \(x \in P_f\) and \(e\).

- Thus we have for \(x \in P_f\),

\[
\hat{c}(x; e) \overset{\text{def}}{=} \min \{ f(A) - x(A), \forall A \supseteq e \} = \max \{ \alpha : \alpha \in \mathbb{R}, x + \alpha \mathbf{1}_e \in P_f \} \quad (16.20)
\]

- We immediately see that for \(e \in E \setminus \text{sat}(x)\), we have that \(\hat{c}(x; e) > 0\).
Saturation Capacity

- The max is achieved when
  \[ \alpha = \hat{c}(x; e) \overset{\text{def}}{=} \min \{ f(A) - x(A), \forall A \supseteq \{e\} \} \quad (16.19) \]

- \( \hat{c}(x; e) \) is known as the saturation capacity associated with \( x \in P_f \) and \( e \).

- Thus we have for \( x \in P_f \),
  \[ \hat{c}(x; e) \overset{\text{def}}{=} \min \{ f(A) - x(A), \forall A \ni e \} \]
  \[ = \max \{ \alpha : \alpha \in \mathbb{R}, x + \alpha \cdot 1_e \in P_f \} \quad (16.20) \]

- We immediately see that for \( e \in E \setminus \text{sat}(x) \), we have that \( \hat{c}(x; e) > 0 \).
- Also, for \( e \in \text{sat}(x) \), we have that \( \hat{c}(x; e) = 0 \).
Saturation Capacity

- The max is achieved when
  \[ \alpha = \hat{c}(x; e) \overset{\text{def}}{=} \min \{ f(A) - x(A), \forall A \supseteq \{e\} \} \]  \hspace{1cm} (16.19)

- \( \hat{c}(x; e) \) is known as the saturation capacity associated with \( x \in P_f \) and \( e \).

- Thus we have for \( x \in P_f \),
  \[ \hat{c}(x; e) \overset{\text{def}}{=} \min \{ f(A) - x(A), \forall A \ni e \} \]
  \[ \hspace{1cm} = \max \{ \alpha : \alpha \in \mathbb{R}, x + \alpha 1_e \in P_f \} \]  \hspace{1cm} (16.20)

- We immediately see that for \( e \in E \setminus \text{sat}(x) \), we have that \( \hat{c}(x; e) > 0 \).
- Also, for \( e \in \text{sat}(x) \), we have that \( \hat{c}(x; e) = 0 \).
- Note that any \( \alpha \) with \( 0 \leq \alpha \leq \hat{c}(x; e) \) we have \( x + \alpha 1_e \in P_f \).
Saturation Capacity

- The max is achieved when
  \[ \alpha = \hat{c}(x; e) \overset{\text{def}}{=} \min \{ f(A) - x(A), \forall A \supseteq \{e\} \} \quad (16.19) \]

- \( \hat{c}(x; e) \) is known as the saturation capacity associated with \( x \in P_f \) and \( e \).

- Thus we have for \( x \in P_f \),
  \[ \hat{c}(x; e) \overset{\text{def}}{=} \min \{ f(A) - x(A), \forall A \ni e \} \]
  \[ = \max \{ \alpha : \alpha \in \mathbb{R}, x + \alpha 1_e \in P_f \} \quad (16.20) \]

- We immediately see that for \( e \in E \setminus \text{sat}(x) \), we have that \( \hat{c}(x; e) > 0 \).
- Also, for \( e \in \text{sat}(x) \), we have that \( \hat{c}(x; e) = 0 \).
- Note that any \( \alpha \) with \( 0 \leq \alpha \leq \hat{c}(x; e) \) we have \( x + \alpha 1_e \in P_f \).
- We also see that computing \( \hat{c}(x; e) \) is a form of submodular function minimization.
Dependence Function

- Tight sets can be restricted to contain a particular element.

\[
\begin{align*}
T(x, \epsilon) &= T(x) \setminus \{A \in \mathcal{E} : A \not\subseteq \mathcal{E}, x(A) = f(A)\} \\
\text{if } \epsilon \in \text{sat}(x) \\
\text{else }
\end{align*}
\]
Dependence Function

- Tight sets can be restricted to contain a particular element.
- Given $x \in P_f$, and $e \in \text{sat}(x)$, define

$$D(x, e) = \{ A : e \in A \subseteq E, x(A) = f(A) \}$$

$$= D(x) \cap \{ A : A \subseteq E, e \in A \}$$

Thus, $D(x, e) \subseteq D(x)$, and $D(x, e)$ is a sublattice of $D(x)$.

Therefore, we can define a unique minimal element of $D(x, e)$ denoted as follows:

$$\text{dep}(x, e) = \begin{cases} T \{ A : e \in A \subseteq E, x(A) = f(A) \} & \text{if } e \in \text{sat}(x) \\ \text{otherwise} \end{cases}$$

I.e., $\text{dep}(x, e)$ is the minimal $x$-tight set containing $e$ (the minimal $x$-tight set containing $e$).
Dependence Function

- Tight sets can be restricted to contain a particular element.
- Given $x \in P_f$, and $e \in \text{sat}(x)$, define

\[
D(x, e) = \{ A : e \in A \subseteq E, x(A) = f(A) \} \tag{16.22}
\]

\[
= D(x) \cap \{ A : A \subseteq E, e \in A \} \tag{16.23}
\]

- Thus, $D(x, e) \subseteq D(x)$, and $D(x, e)$ is a sublattice of $D(x)$. 
Dependence Function

- Tight sets can be restricted to contain a particular element.
- Given $x \in P_f$, and $e \in \text{sat}(x)$, define

$$\mathcal{D}(x, e) = \{A : e \in A \subseteq E, x(A) = f(A)\}$$

$$= \mathcal{D}(x) \cap \{A : A \subseteq E, e \in A\}$$

(16.22)

(16.23)

- Thus, $\mathcal{D}(x, e) \subseteq \mathcal{D}(x)$, and $\mathcal{D}(x, e)$ is a sublattice of $\mathcal{D}(x)$.
- Therefore, we can define a unique minimal element of $\mathcal{D}(x, e)$ denoted as follows:

$$\text{dep}(x, e) = \begin{cases} \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases}$$

(16.24)
Dependence Function

- Tight sets can be restricted to contain a particular element.
- Given \( x \in P_f \), and \( e \in \text{sat}(x) \), define

\[
D(x, e) = \{ A : e \in A \subseteq E, x(A) = f(A) \} \tag{16.22}
\]

\[
= D(x) \cap \{ A : A \subseteq E, e \in A \} \tag{16.23}
\]

- Thus, \( D(x, e) \subseteq D(x) \), and \( D(x, e) \) is a sublattice of \( D(x) \).
- Therefore, we can define a unique minimal element of \( D(x, e) \) denoted as follows:

\[
\text{dep}(x, e) = \begin{cases} 
\bigcap \{ A : e \in A \subseteq E, x(A) = f(A) \} & \text{if } e \in \text{sat}(x) \\
\emptyset & \text{else}
\end{cases} \tag{16.24}
\]

- I.e., \( \text{dep}(x, e) \) is the minimal element in \( D(x) \) that contains \( e \) (the minimal \( x \)-tight set containing \( e \)).
**dep and sat in a lattice**

- Given some \( x \in P_f \),
- The picture on the right summarizes the relationships between the lattices and sublattices.
- Note, \( \bigcap_e \text{dep}(x, e) = \text{dep}(x) \).
**Proof that min-norm gives optimal dep and sat in a lattice**

- **Given** $x \in P_f$, recall distributive lattice of tight sets
  
  \[ \mathcal{D}(x) = \{ A : x(A) = f(A) \} \]
dep and sat in a lattice

- Given $x \in P_f$, recall distributive lattice of tight sets
  \[ \mathcal{D}(x) = \{ A : x(A) = f(A) \} \]
- We had that $\text{sat}(x) = \bigcup \{ A : A \in \mathcal{D}(x) \}$ is the “1” element of this lattice.
**dep and sat in a lattice**

- Given $x \in P_f$, recall distributive lattice of tight sets $\mathcal{D}(x) = \{ A : x(A) = f(A) \}$
- We had that $\text{sat}(x) = \bigcup \{ A : A \in \mathcal{D}(x) \}$ is the “1” element of this lattice.
- Consider the “0” element of $\mathcal{D}(x)$, i.e., $\text{dry}(x) \overset{\text{def}}{=} \bigcap \{ A : A \in \mathcal{D}(x) \}$
**dep and sat in a lattice**

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**dep and sat in a lattice**

- Given $x \in P_f$, recall distributive lattice of tight sets
  \[ D(x) = \{ A : x(A) = f(A) \} \]
- We had that $\text{sat}(x) = \bigcup \{ A : A \in D(x) \}$ is the “1” element of this lattice.
- Consider the “0” element of $D(x)$, i.e., $\text{dry}(x) \overset{\text{def}}{=} \bigcap \{ A : A \in D(x) \}$
- We can see $\text{dry}(x)$ as the elements that are necessary for tightness.
- That is, we can equivalently define $\text{dry}(x)$ as
  \[ \text{dry}(x) = \{ e' : x(A) < f(A), \forall A \not\ni e' \} \] (16.25)
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  This can be read as, for any \( e' \in \text{dry}(x) \), any set that does not contain \( e' \) is not tight for \( x \) (any set \( A \) that is missing any element of \( \text{dry}(x) \) is not tight).
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Perhaps, then, a better name for \( \text{dry} \) is \( \text{ntight}(x) \), for the necessary for tightness (but we'll actually use neither name).
dep and sat in a lattice

- Given $x \in P_f$, recall distributive lattice of tight sets $\mathcal{D}(x) = \{A : x(A) = f(A)\}$
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  This can be read as, for any $e' \in \text{dry}(x)$, any set that does not contain $e'$ is not tight for $x$ (any set $A$ that is missing any element of $\text{dry}(x)$ is not tight).
- Perhaps, then, a better name for $\text{dry}$ is $\text{ntight}(x)$, for the necessary for tightness (but we’ll actually use neither name).
- Note that $\text{dry}$ need not be the empty set. Exercise: give example.
An alternate expression for $\text{dep} = \text{dry}$

Now, given $x \in P_f$, and $e \in \text{sat}(x)$, recall distributive sub-lattice of $e$-containing tight sets $D(x, e) = \{ A : e \in A, x(A) = f(A) \}$
An alternate expression for $\text{dep} = \text{dry}$

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- We can define the “1” element of this sub-lattice as $\text{sat}(x, e) \overset{\text{def}}{=} \bigcup \{ A : A \in \mathcal{D}(x, e) \} = \text{sat}(x) \quad e \in \text{sat}(x)$. 


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- We can define the “1” element of this sub-lattice as \( \text{sat}(x, e) \stackrel{\text{def}}{=} \bigcup \{ A : A \in \mathcal{D}(x, e) \} \).
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- That is, we can view $\text{dry}(x, e)$ as

  \[
  \text{dry}(x, e) = \{ e' : x(A) < f(A), \forall A \not\ni e', e \in A \} \quad (16.26)
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An alternate expression for \( \text{dep} = \text{dry} \)

- Now, given \( x \in P_f \), and \( e \in \text{sat}(x) \), recall distributive sub-lattice of \( e \)-containing tight sets \( D(x, e) = \{ A : e \in A, x(A) = f(A) \} \)
- We can define the “1” element of this sub-lattice as \( \text{sat}(x, e) \overset{\text{def}}{=} \bigcup \{ A : A \in D(x, e) \} \).
- Analogously, we can define the “0” element of this sub-lattice as \( \text{dry}(x, e) \overset{\text{def}}{=} \bigcap \{ A : A \in D(x, e) \} \).
- We can see \( \text{dry}(x, e) \) as the elements that are necessary for \( e \)-containing tightness, with \( e \in \text{sat}(x) \).
- That is, we can view \( \text{dry}(x, e) \) as
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  \text{dry}(x, e) = \{ e' : x(A) < f(A), \forall A \not\ni e', e \in A \} \tag{16.26}
  \]
- This can be read as, for any \( e' \in \text{dry}(x, e) \), any \( e \)-containing set that does not contain \( e' \) is not tight for \( x \).
An alternate expression for $\text{dep} = \text{dry}$

- Now, given $x \in P_f$, and $e \in \text{sat}(x)$, recall distributive sub-lattice of $e$-containing tight sets $\mathcal{D}(x, e) = \{A : e \in A, x(A) = f(A)\}$
- We can define the “1” element of this sub-lattice as $\text{sat}(x, e) \overset{\text{def}}{=} \bigcup \{A : A \in \mathcal{D}(x, e)\}$.
- Analogously, we can define the “0” element of this sub-lattice as $\text{dry}(x, e) \overset{\text{def}}{=} \bigcap \{A : A \in \mathcal{D}(x, e)\}$.
- We can see $\text{dry}(x, e)$ as the elements that are necessary for $e$-containing tightness, with $e \in \text{sat}(x)$.
- That is, we can view $\text{dry}(x, e)$ as
  \[
  \text{dry}(x, e) = \{e' : x(A) < f(A), \forall A \not\supseteq e', e \in A\} \tag{16.26}
  \]
- This can be read as, for any $e' \in \text{dry}(x, e)$, any $e$-containing set that does not contain $e'$ is not tight for $x$.
- But actually, $\text{dry}(x, e) = \text{dep}(x, e)$, so we have derived another expression for $\text{dep}(x, e)$ in Eq. (16.26).
Now, let \((E, \mathcal{I}) = (E, r)\) be a matroid, and let \(I \in \mathcal{I}\) giving \(1_I \in P_r\). We have \(\text{sat}(1_I) = \text{span}(I) = \text{closure}(I)\).
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Given \(e \in \text{sat}(1_I) \setminus I\) and then consider an \(A \ni e\) with \(|I \cap A| = r(A)|.\)
Dependence Function and Fundamental Matroid Circuit

- Now, let \((E, \mathcal{I}) = (E, r)\) be a matroid, and let \(I \in \mathcal{I}\) giving \(1_I \in P_r\). We have \(\text{sat}(1_I) = \text{span}(I) = \text{closure}(I)\).
- Given \(e \in \text{sat}(1_I) \setminus I\) and then consider an \(A \ni e\) with \(|I \cap A| = r(A)|\).
- Then \(I \cap A\) serves as a base for \(A\) (i.e., \(I \cap A\) spans \(A\)) and any such \(A\) contains a circuit (i.e., we can add \(e \in A \setminus I\) to \(I \cap A\) w/o increasing rank).
Now, let \((E, \mathcal{I}) = (E, r)\) be a matroid, and let \(I \in \mathcal{I}\) giving \(1_I \in P_r\). We have \(\text{sat}(1_I) = \text{span}(I) = \text{closure}(I)\).

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Given \(e \in \text{sat}(1_I) \setminus I\), and consider \(\text{dep}(1_I, e)\), with

\[
\text{dep}(1_I, e) = \bigcap \{A : e \in A \subseteq E, 1_I(A) = r(A)\} \quad (16.27)
\]

\[
= \bigcap \{A : e \in A \subseteq E, |I \cap A| = r(A)\} \quad (16.28)
\]

\[
= \bigcap \{A : e \in A \subseteq E, r(A) - |I \cap A| = 0\} \quad (16.29)
\]
Now, let \((E, \mathcal{I}) = (E, r)\) be a matroid, and let \(I \in \mathcal{I}\) giving \(1_I \in P_r\). We have \(\text{sat}(1_I) = \text{span}(I) = \text{closure}(I)\).

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By SFM lattice, \(\exists\) a unique minimal \(A \ni e\) with \(|I \cap A| = r(A)|\).
Now, let \((E, \mathcal{I}) = (E, r)\) be a matroid, and let \(I \in \mathcal{I}\) giving \(1_I \in P_r\). We have \(\text{sat}(1_I) = \text{span}(I) = \text{closure}(I)\).

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\]

By SFM lattice, \(\exists\) a unique minimal \(A \ni e\) with \(|I \cap A| = r(A)\).

Thus, \(\text{dep}(1_I, e)\) must be a circuit since if it included more than a circuit, it would not be minimal in this sense.
Therefore, when $e \in \text{sat}(1_I) \setminus I$, then $\text{dep}(1_I, e) = C(I, e)$ where $C(I, e)$ is the unique circuit contained in $I + e$ in a matroid (the fundamental circuit of $e$ and $I$ that we encountered before).
Dependence Function and Fundamental Matroid Circuit

Therefore, when $e \in \text{sat}(1_I) \setminus I$, then $\text{dep}(1_I, e) = C(I, e)$ where $C(I, e)$ is the unique circuit contained in $I + e$ in a matroid (the fundamental circuit of $e$ and $I$ that we encountered before).

Now, if $e \in \text{sat}(1_I) \cap I$ with $I \in \mathcal{I}$, we said that $C(I, e)$ was undefined (since no circuit is created in this case) and so we defined it as $C(I, e) = \{e\}$.
**Dependence Function and Fundamental Matroid Circuit**

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- In this case, for such an $e$, we have $\text{dep}(1_I, e) = \{e\}$ since all such sets $A \ni e$ with $|I \cap A| = r(A)$ contain $e$, but in this case no cycle is created, i.e., $|I \cap A| \geq |I \cap \{e\}| = r(e) = 1$. 
Therefore, when \( e \in \text{sat}(1_I) \setminus I \), then \( \text{dep}(1_I, e) = C(I, e) \) where \( C(I, e) \) is the unique circuit contained in \( I + e \) in a matroid (the fundamental circuit of \( e \) and \( I \) that we encountered before).

Now, if \( e \in \text{sat}(1_I) \cap I \) with \( I \in \mathcal{I} \), we said that \( C(I, e) \) was undefined (since no circuit is created in this case) and so we defined it as \( C(I, e) = \{e\} \).

In this case, for such an \( e \), we have \( \text{dep}(1_I, e) = \{e\} \) since all such sets \( A \supset e \) with \( |I \cap A| = r(A) \) contain \( e \), but in this case no cycle is created, i.e., \( |I \cap A| \geq |I \cap \{e\}| = r(e) = 1 \).

We are thus free to take subsets of \( I \) as \( A \), all of which must contain \( e \), but all of which have rank equal to size, and min size is 1.
Dependence Function and Fundamental Matroid Circuit

- Therefore, when \( e \in \text{sat}(1_I) \setminus I \), then \( \text{dep}(1_I, e) = C(I, e) \) where \( C(I, e) \) is the unique circuit contained in \( I + e \) in a matroid (the fundamental circuit of \( e \) and \( I \) that we encountered before).

- Now, if \( e \in \text{sat}(1_I) \cap I \) with \( I \in \mathcal{I} \), we said that \( C(I, e) \) was undefined (since no circuit is created in this case) and so we defined it as \( C(I, e) = \{e\} \).

- In this case, for such an \( e \), we have \( \text{dep}(1_I, e) = \{e\} \) since all such sets \( A \ni e \) with \( |I \cap A| = r(A) \) contain \( e \), but in this case no cycle is created, i.e., \( |I \cap A| \geq |I \cap \{e\}| = r(e) = 1 \).

- We are thus free to take subsets of \( I \) as \( A \), all of which must contain \( e \), but all of which have rank equal to size, and min size is 1.

- Also note: in general for \( x \in P_f \) and \( e \in \text{sat}(x) \), we have \( \text{dep}(x, e) \) is tight by definition.

\[ x(\text{dep}(x, e)) = f(\text{dep}(x, e)) \]
Summary of sat, and dep

For $x \in P_f$, $\text{sat}(x)$ (span, closure) is the maximal saturated ($x$-tight) set w.r.t. $x$. I.e., $\text{sat}(x) = \{e : e \in E, \forall \alpha > 0, x + \alpha 1_e \notin P_f \}$. That is,

$$\text{cl}(x) \overset{\text{def}}{=} \text{sat}(x) \triangleq \bigcup \{A : A \in \mathcal{D}(x)\}$$

$$= \bigcup \{A : A \subseteq E, x(A) = f(A)\}$$

$$= \{e : e \in E, \forall \alpha > 0, x + \alpha 1_e \notin P_f\}$$

For $e \in \text{sat}(x)$, we have $\text{dep}(x, e) \subseteq \text{sat}(x)$ (fundamental circuit) is the minimal (common) saturated ($x$-tight) set w.r.t. $x$ containing $e$. I.e.,

$$\text{dep}(x, e) = \begin{cases} \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases}$$

$$= \{e' : \exists \alpha > 0, \text{ s.t. } x + \alpha (1_e - 1_{e'}) \in P_f\}$$
Dependence Function and exchange

For \( e \in \text{span}(I) \setminus I \), we have that \( I + e \notin \mathcal{I} \). This is a set addition restriction property.
Dependence Function and exchange

- For $e \in \text{span}(I) \setminus I$, we have that $I + e \notin \mathcal{I}$. This is a set addition restriction property.
- Analogously, for $e \in \text{sat}(x)$, any $x + \alpha e \notin P_f$ for $\alpha > 0$. This is a vector increase restriction property.
Dependence Function and exchange

- For $e \in \text{span}(I) \setminus I$, we have that $I + e \notin \mathcal{I}$. This is a set addition restriction property.

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- Recall, we have $C(I, e) \setminus e' \in \mathcal{I}$ for $e' \in C(I, e)$. I.e., $C(I, e)$ consists of elements that when removed recover independence.

$e$ exchange with any $e' \in C(I, e)$
## Dependence Function and exchange

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- Recall, we have $C(I, e) \setminus e' \in \mathcal{I}$ for $e' \in C(I, e)$. I.e., $C(I, e)$ consists of elements that when removed recover independence.

- In other words, for $e \in \text{span}(I) \setminus I$, we have that

$$C(I, e) = \{ a \in E : I + e - a \in \mathcal{I} \}$$

(16.34)
**Dependence Function and exchange**

- For $e \in \text{span}(I) \setminus I$, we have that $I + e \notin \mathcal{I}$. This is a set addition restriction property.
- Analogously, for $e \in \text{sat}(x)$, any $x + \alpha 1_e \notin P_f$ for $\alpha > 0$. This is a vector increase restriction property.
- Recall, we have $C(I, e) \setminus e' \in \mathcal{I}$ for $e' \in C(I, e)$. I.e., $C(I, e)$ consists of elements that when removed recover independence.
- In other words, for $e \in \text{span}(I) \setminus I$, we have that
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- I.e., an addition of $e$ to $I$ stays within $\mathcal{I}$ only if we simultaneously remove one of the elements of $C(I, e)$.
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- Recall, we have $C(I, e) \setminus e' \in \mathcal{I}$ for $e' \in C(I, e)$. I.e., $C(I, e)$ consists of elements that when removed recover independence.

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- But, analogous to the circuit case, is there an exchange property for $\text{dep}(x, e)$ in the form of vector movement restriction?
Dependence Function and exchange

- For $e \in \text{span}(I) \setminus I$, we have that $I + e \notin \mathcal{I}$. This is a set addition restriction property.
- Analogously, for $e \in \text{sat}(x)$, any $x + \alpha \mathbf{1}_e \notin P_f$ for $\alpha > 0$. This is a vector increase restriction property.
- Recall, we have $C(I, e) \setminus e' \in \mathcal{I}$ for $e' \in C(I, e)$. I.e., $C(I, e)$ consists of elements that when removed recover independence.
- In other words, for $e \in \text{span}(I) \setminus I$, we have that
  \[ C(I, e) = \{ a \in E : I + e - a \in \mathcal{I} \} \] (16.34)
  
  I.e., an addition of $e$ to $I$ stays within $\mathcal{I}$ only if we simultaneously remove one of the elements of $C(I, e)$.
- But, analogous to the circuit case, is there an exchange property for $\text{dep}(x, e)$ in the form of vector movement restriction?
- We might expect the vector $\text{dep}(x, e)$ property to take the form: a positive move in the $e$-direction stays within $P_f^+$ only if we simultaneously take a negative move in one of the $\text{dep}(x, e)$ directions.
 Closure/Sat  Fund. Circuit/Dep  Min-Norm Point and SFM  Min-Norm Point Algorithm  Proof that min-norm gives optimal

Dependence Function and exchange in 2D

- $\text{dep}(x, e)$ is set of neg. directions we must move if we want to move in pos. $e$ direction, starting at $x$ and staying within $P_f$. 
Dependence Function and exchange in 2D

- $\text{dep}(x, e)$ is set of neg. directions we must move if we want to move in pos. $e$ direction, starting at $x$ and staying within $P_f$.

- Viewable in 2D, we have for $A, B \subseteq E$, $A \cap B = \emptyset$:

Left: $e \in B$ and $A \cap \text{dep}(x, e) = \emptyset$, and we can’t move further in $(e)$ direction, and moving in any negative $a \in A$ direction doesn’t change that. **No dependence** between $(e)$ and any element in $A$.

Right: $A \subseteq \text{dep}(x, e)$. We can’t move further in the $(e)$ direction, but we can move further in $(e)$ direction by moving in some negative $a \in A$ direction. **Dependence** between $(e)$ and elements in $A$. 
\[ A_1 \cup A_2 \subseteq V \]

\[ f' : \mathbb{R}^{+,\geq 1,2,3} \rightarrow \mathbb{R} \]

\[ f'(x) = f\left( \bigcup_{i \in X} A_i \right) \]

\[ \Rightarrow f(A_1 \cup A_2) = f(A_1) + f(B) \]
Dependence Function and exchange in 3D

- We can move neither in the (e) nor the (a) direction, but we can move in the (e) direction if we simultaneously move in the -(a) direction.
Dependence Function and exchange in 3D

- We can move neither in the (e) nor the (a) direction, but we can move in the (e) direction if we simultaneously move in the -(a) direction.
- In 3D, we have:
Dependence Function and exchange in 3D

- We can move neither in the (e) nor the (a) direction, but we can move in the (e) direction if we simultaneously move in the -(a) direction.
- In 3D, we have: \( e \in \text{dep}(y, a) \)

\[ \begin{align*}
0 & \quad 0.2 & \quad 0.4 & \quad 0.6 & \quad 0.8 & \quad 1 \\
0.5 & \quad 0.6 & \quad 0.7 & \quad 0.8 & \quad 0.9 & \quad 1 \\
-0.5 & \quad -0.6 & \quad -0.7 & \quad -0.8 & \quad -0.9 & \quad -1 \\
\end{align*} \]

\[ \begin{align*}
0 & \quad 0.2 & \quad 0.4 & \quad 0.6 & \quad 0.8 & \quad 1 \\
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\end{align*} \]

I.e., for \( e \in \text{sat}(x), \ a \in \text{sat}(x), \ a \in \text{dep}(x, e), \ e \notin \text{dep}(x, a), \)
Dependence Function and exchange in 3D

- We can move neither in the (e) nor the (a) direction, but we can move in the (e) direction if we simultaneously move in the -(a) direction.
- In 3D, we have:

\[ \text{dep}(x, e) = \{ a : a \in E, \exists \alpha > 0 : x + \alpha(1_e - 1_a) \in P_f \} \]  \hspace{1cm} (16.35)

I.e., for \( e \in \text{sat}(x), a \in \text{sat}(x), a \in \text{dep}(x, e), e \not\in \text{dep}(x, a) \), and
Dependence Function and exchange in 3D

- We can move neither in the (e) nor the (a) direction, but we can move in the (e) direction if we simultaneously move in the -(a) direction.
- In 3D, we have:

\[ \text{i.e., for } e \in \text{sat}(x), \ a \in \text{sat}(x), \ a \in \text{dep}(x, e), \ e \notin \text{dep}(x, a), \text{ and} \]
\[ \text{dep}(x, e) = \{ a : a \in E, \exists \alpha > 0 : x + \alpha (1_e - 1_a) \in P_f \} \quad (16.35) \]
- We next show this formally . . .
The derivation for \( \text{dep}(x, e) \) involves turning a strict inequality into a non-strict one with a strict explicit slack variable \( \alpha \):

\[
\text{dep}(x, e) = \text{ntight}(x, e) = e_0
\]

(16.36)
The derivation for \( \text{dep}(x, e) \) involves turning a strict inequality into a non-strict one with a strict explicit slack variable \( \alpha \):

\[
\text{dep}(x, e) = \text{ntight}(x, e) = \\
\{ e' : x(A) < f(A), \forall A \not= e', e \in A \}
\]  

(16.36) 

(16.37)
The derivation for \( \text{dep}(x, e) \) involves turning a strict inequality into a non-strict one with a strict explicit slack variable \( \alpha \):

\[
\text{dep}(x, e) = \text{ntight}(x, e) =
\]

\[
= \{ e' : x(A) < f(A), \forall A \not
\}
\]

\[
= \{ e' : \exists \alpha > 0, \text{ s.t. } \alpha \leq f(A) - x(A), \forall A \not
\}
\]
The derivation for $\text{dep}(x, e)$ involves turning a strict inequality into a non-strict one with a strict explicit slack variable $\alpha$:

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$$= \{ e' : \exists \alpha > 0, \text{ s.t. } \alpha 1_e(A) \leq f(A) - x(A), \forall A \not\ni e', e \in A \}$$
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= \{ e' : \exists \alpha > 0, \text{ s.t. } \alpha \mathbf{1}_e(A) \leq f(A) - x(A), \forall A \not\ni e', e \in A \} \\
= \{ e' : \exists \alpha > 0, \text{ s.t. } \alpha(\mathbf{1}_e(A) - \mathbf{1}_{e'}(A)) \leq f(A) - x(A), \forall A \not\ni e', e \in A \}
\]
The derivation for $\text{dep}(x, e)$ involves turning a strict inequality into a non-strict one with a strict explicit slack variable $\alpha$:

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$$= \{ e' : \exists \alpha > 0, \text{ s.t. } \alpha(1_e(A) - 1_{e'}(A)) \leq f(A) - x(A), \forall A \not\ni e', e \in A \}$$

$$= \{ e' : \exists \alpha > 0, \text{ s.t. } x(A) + \alpha(1_e(A) - 1_{e'}(A)) \leq f(A), \forall A \not\ni e', e \in A \}$$
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Now, $1_e(A) - 1_{e'}(A) = 0$ if either $\{e, e'\} \subseteq A$, or $\{e, e'\} \cap A = \emptyset$. 
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= \{ e' : \exists \alpha > 0, \text{ s.t. } x(A) + \alpha (1_e(A) - 1_{e'}(A)) \leq f(A), \forall A \not\ni e', e \in A \}
\]

Now, $1_e(A) - 1_{e'}(A) = 0$ if either $\{e, e'\} \subseteq A$, or $\{e, e'\} \cap A = \emptyset$.

Also, if $e' \in A$ but $e \not\in A$, then

\[
x(A) + \alpha (1_e(A) - 1_{e'}(A)) = x(A) - \alpha \leq f(A) \text{ since } x \in P_f.
\]
dep and exchange derived

thus, we get the same in the above if we remove the constraint $A \not\ni e', e \in A$, that is we get

$$\text{dep}(x, e) = \{e' : \exists \alpha > 0, \text{s.t. } x(A) + \alpha(1_e(A) - 1_{e'}(A)) \leq f(A), \forall A\}$$

(16.42)
dep and exchange derived

- thus, we get the same in the above if we remove the constraint
  \( A \not\ni e', e \in A \), that is we get

\[
\text{dep}(x, e) = \{ e' : \exists \alpha > 0, \text{ s.t. } x(A) + \alpha (1e(A) - 1e'(A)) \leq f(A), \forall A \}
\]

(16.42)

- This is then identical to

\[
\text{dep}(x, e) = \{ e' : \exists \alpha > 0, \text{ s.t. } x + \alpha (1e - 1e') \in P_f \}
\]

(16.43)
Thus, we get the same in the above if we remove the constraint $A \not
i e', e \in A$, that is we get

$$\text{dep}(x, e) = \{ e' : \exists \alpha > 0, \text{ s.t. } x(A) + \alpha(1_e(A) - 1_{e'}(A)) \leq f(A), \forall A \}$$  \hspace{1cm} (16.42)

This is then identical to

$$\text{dep}(x, e) = \{ e' : \exists \alpha > 0, \text{ s.t. } x + \alpha(1_e - 1_{e'}) \in P_f \}$$  \hspace{1cm} (16.43)

Compare with original, the minimal element of $D(x, e)$, with $e \in \text{sat}(x)$:

$$\text{dep}(x, e) = \begin{cases} \bigcap \{ A : e \in A \subseteq E, x(A) = f(A) \} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases}$$  \hspace{1cm} (16.44)
Summary of Concepts

- Most violated inequality \( \max \{ x(A) - f(A) : A \subseteq E \} \)
Summary of Concepts

- Most violated inequality $\max \{x(A) - f(A) : A \subseteq E\}$
- Matroid by circuits, and the fundamental circuit $C(I, e) \subseteq I + e$. 
Summary of Concepts

- Most violated inequality $\max \{x(A) - f(A) : A \subseteq E\}$
- Matroid by circuits, and the fundamental circuit $C(I, e) \subseteq I + e$.
- Minimizers of submodular functions form a lattice.
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- $e$-containing tight sets
Summary of Concepts

- Most violated inequality: \( \max \{ x(A) - f(A) : A \subseteq E \} \)
- Matroid by circuits, and the fundamental circuit \( C'(I, e) \subseteq I + e \).
- Minimizers of submodular functions form a lattice.
- Minimal and maximal element of a lattice.
- \( x \)-tight sets, maximal and minimal tight set.
- sat function & Closure
- Saturation Capacity
- e-containing tight sets
- dep function & fundamental circuit of a matroid
Summary important definitions so far: tight, dep, & sat

- $x$-tight sets: For $x \in P_f$, $\mathcal{D}(x) \triangleq \{ A \subseteq E : x(A) = f(A) \}$.
Summary important definitions so far: tight, dep, & sat

- **$x$-tight sets**: For $x \in P_f$, $\mathcal{D}(x) \triangleq \{ A \subseteq E : x(A) = f(A) \}$.
- **Polymatroid closure/maximal $x$-tight set**: For $x \in P_f$,
  \[
  \text{sat}(x) \triangleq \bigcup\{ A : A \in \mathcal{D}(x) \} = \{ e : e \in E, \forall \alpha > 0, x + \alpha 1_e \notin P_f \}.
  \]
Summary important definitions so far: tight, dep, & sat

- **x-tight sets**: For $x \in P_f$, $\mathcal{D}(x) \triangleq \{A \subseteq E : x(A) = f(A)\}$.

- **Polymatroid closure/maximal x-tight set**: For $x \in P_f,$
  \[ \text{sat}(x) \triangleq \bigcup \{ A : A \in \mathcal{D}(x) \} = \{ e : e \in E, \forall \alpha > 0, x + \alpha 1_e \notin P_f \} \]

- **Saturation capacity**: for $x \in P_f$, $0 \leq \hat{c}(x; e) \triangleq \min \{ f(A) - x(A) | \forall A \ni e \} = \max \{ \alpha : \alpha \in \mathbb{R}, x + \alpha 1_e \in P_f \}$
Summary important definitions so far: tight, dep, & sat

- **$x$-tight sets**: For $x \in P_f$, $\mathcal{D}(x) \overset{\Delta}{=} \{ A \subseteq E : x(A) = f(A) \}$.

- **Polymatroid closure/maximal $x$-tight set**: For $x \in P_f$, 
  $\text{sat}(x) \overset{\Delta}{=} \bigcup \{ A : A \in \mathcal{D}(x) \} = \{ e : e \in E, \forall \alpha > 0, x + \alpha 1_e \notin P_f \}$.

- **Saturation capacity**: for $x \in P_f$, $0 \leq \hat{c}(x; e) \overset{\Delta}{=} \min \{ f(A) - x(A) | \forall A \ni e \} = \max \{ \alpha : \alpha \in \mathbb{R}, x + \alpha 1_e \in P_f \}$

- **Recall**: $\text{sat}(x) = \{ e : \hat{c}(x; e) = 0 \}$ and $E \setminus \text{sat}(x) = \{ e : \hat{c}(x; e) > 0 \}$. 
Summary important definitions so far: tight, dep, & sat

- **x-tight sets**: For $x \in P_f$, $\mathcal{D}(x) \triangleq \{ A \subseteq E : x(A) = f(A) \}$.

- **Polymatroid closure/maximal x-tight set**: For $x \in P_f$,
  
  $\text{sat}(x) \triangleq \bigcup \{ A : A \in \mathcal{D}(x) \} = \{ e : e \in E, \forall \alpha > 0, x + \alpha 1_e \notin P_f \}$.

- **Saturation capacity**: for $x \in P_f$, $0 \leq \hat{c}(x; e) \triangleq \min \{ f(A) - x(A) | \forall A \ni e \} = \max \{ \alpha : \alpha \in \mathbb{R}, x + \alpha 1_e \in P_f \}$

- **Recall**: $\text{sat}(x) = \{ e : \hat{c}(x; e) = 0 \}$ and $E \setminus \text{sat}(x) = \{ e : \hat{c}(x; e) > 0 \}$.

- **e-containing x-tight sets**: For $x \in P_f$,
  
  $\mathcal{D}(x, e) = \{ A : e \in A \subseteq E, x(A) = f(A) \} \subseteq \mathcal{D}(x)$.
Summary important definitions so far: tight, dep, & sat

- **x-tight sets**: For \( x \in P_f \), \( \mathcal{D}(x) \triangleq \{ A \subseteq E : x(A) = f(A) \} \).

- **Polymatroid closure/maximal x-tight set**: For \( x \in P_f \), 
  \[
  \text{sat}(x) \triangleq \bigcup \{ A : A \in \mathcal{D}(x) \} = \{ e : e \in E, \forall \alpha > 0, x + \alpha 1_e \notin P_f \}.
  \]

- **Saturation capacity**: for \( x \in P_f \), 
  \[
  0 \leq \hat{c}(x; e) \triangleq \min \{ f(A) - x(A) \mid \forall A \ni e \} = \max \{ \alpha : \alpha \in \mathbb{R}, x + \alpha 1_e \in P_f \}.
  \]

- Recall: \( \text{sat}(x) = \{ e : \hat{c}(x; e) = 0 \} \) and \( E \setminus \text{sat}(x) = \{ e : \hat{c}(x; e) > 0 \} \).

- **e-containing x-tight sets**: For \( x \in P_f \), 
  \[
  \mathcal{D}(x, e) = \{ A : e \in A \subseteq E, x(A) = f(A) \} \subseteq \mathcal{D}(x).
  \]

- **Minimal e-containing x-tight set/polymatroidal fundamental circuit/**:
  For \( x \in P_f \), 
  \[
  \text{dep}(x, e) = \begin{cases} 
  \bigcap \{ A : e \in A \subseteq E, x(A) = f(A) \} & \text{if } e \in \text{sat}(x) \\
  \emptyset & \text{else}
  \end{cases}
  = \{ e' : \exists \alpha > 0, \text{ s.t. } x + \alpha (1_e - 1_{e'}) \in P_f \}.
  \]
A polymatroid function’s polyhedron is a polymatroid.

**Theorem 16.5.1**

Let $f$ be a submodular function defined on subsets of $E$. For any $x \in \mathbb{R}^E$, we have:

$$\text{rank}(x) = \max \{ y(E) : y \leq x, y \in P_f \} = \min \{ x(A) + f(E \setminus A) : A \subseteq E \}$$

(16.1)

Essentially the same theorem as Theorem ??, but note $P_f$ rather than $P_f^+$. Taking $x = 0$ we get:

**Corollary 16.5.2**

Let $f$ be a submodular function defined on subsets of $E$. We have:

$$\text{rank}(0) = \max \{ y(E) : y \leq 0, y \in P_f \} = \min \{ f(A) : A \subseteq E \}$$

(16.2)
Restating what we saw before, we have:

$$\max \{ y(E) | y \in P_f, y \leq 0 \} = \min \{ f(X) | X \subseteq V \}$$

(16.45)
Restating what we saw before, we have:

\[
\max \{ y(E) | y \in P_f, y \leq 0 \} = \min \{ f(X) | X \subseteq V \} \tag{16.45}
\]

Consider the optimization:

\[
\begin{align*}
\text{minimize} & \quad \| x \|_2^2 \\
\text{subject to} & \quad x \in B_f
\end{align*}
\tag{16.46a}
\]

where \( B_f \) is the base polytope of submodular \( f \), and

\[
\| x \|_2^2 = \sum_{e \in E} x(e)^2
\]

is the squared 2-norm. Let \( x^* \) be the optimal solution.
Min-Norm Point: Definition

- Restating what we saw before, we have:

\[
\max \{ y(E) \mid y \in P_f, y \leq 0 \} = \min \{ f(X) \mid X \subseteq V \} \quad (16.45)
\]

- Consider the optimization:

\[
\begin{align*}
\text{minimize} & \quad \|x\|_2^2 \\
\text{subject to} & \quad x \in B_f
\end{align*}
\]

(16.46a)

(16.46b)

where \( B_f \) is the base polytope of submodular \( f \), and \( \|x\|_2^2 = \sum_{e \in E} x(e)^2 \) is the squared 2-norm. Let \( x^* \) be the optimal solution.

- Note, \( x^* \) is the unique optimal solution since we have a strictly convex objective over a set of convex constraints.
**Min-Norm Point: Definition**

- Restating what we saw before, we have:

\[
\max \{ y(E) | y \in P_f, y \leq 0 \} = \min \{ f(X) | X \subseteq V \} \tag{16.45}
\]

- Consider the optimization:

\[
\begin{align*}
\text{minimize} & \quad \|x\|_2^2 \\
\text{subject to} & \quad x \in B_f
\end{align*} \tag{16.46a, 16.46b}
\]

where \( B_f \) is the base polytope of submodular \( f \), and
\[
\|x\|_2^2 = \sum_{e \in E} x(e)^2
\]

is the squared 2-norm. Let \( x^* \) be the optimal solution.

- Note, \( x^* \) is the unique optimal solution since we have a strictly convex objective over a set of convex constraints.

- \( x^* \) is called the minimum norm point of the base polytope.
Min-Norm Point: Examples

$P_f$

$P_f$

$P_f$

$P_f$

$P_f$

$P_f$
Given optimal solution $x^*$ to the above, consider the quantities

\[ y^* = x^* \land 0 = (\min(x^*(e), 0) | e \in E) \] (16.47)

\[ A_- = \{ e : x^*(e) < 0 \} \] (16.48)

\[ A_0 = \{ e : x^*(e) \leq 0 \} \] (16.49)
Given optimal solution \( x^* \) to the above, consider the quantities

\[
y^* = x^* \land 0 = \min(x^*(e), 0) | e \in E
\]  

\[
A_- = \{ e : x^*(e) < 0 \}
\]

\[
A_0 = \{ e : x^*(e) \leq 0 \}
\]

Thus, we immediately have that:

\[
A_- \subseteq A_0
\]

and that

\[
x^*(A_-) = x^*(A_0) = y^*(A_-) = y^*(A_0)
\]
Given optimal solution $x^*$ to the above, consider the quantities

$$y^* = x^* \land 0 = \min(x^*(e), 0) | e \in E$$

(16.47)

$$A_- = \{ e : x^*(e) < 0 \}$$

(16.48)

$$A_0 = \{ e : x^*(e) \leq 0 \}$$

(16.49)

Thus, we immediately have that:

$$A_- \subseteq A_0$$

(16.50)

and that

$$x^*(A_-) = x^*(A_0) = y^*(A_-) = y^*(A_0)$$

(16.51)

It turns out, these quantities will solve the submodular function minimization problem, as we now show.
Min-Norm Point and Submodular Function Minimization

- Given optimal solution $x^*$ to the above, consider the quantities

  \[ y^* = x^* \wedge 0 = (\min(x^*(e), 0) | e \in E) \]  
  \[ A_- = \{ e : x^*(e) < 0 \} \]  
  \[ A_0 = \{ e : x^*(e) \leq 0 \} \]

- Thus, we immediately have that:

  \[ A_- \subseteq A_0 \]  

  and that

  \[ x^*(A_-) = x^*(A_0) = y^*(A_-) = y^*(A_0) \]

- It turns out, these quantities will solve the submodular function minimization problem, as we now show.

- The proof is nice since it uses the tools we’ve been recently developing.
The following three slides are review, and are from Lectures 13, and 16.
A polymatroid function’s polyhedron is a polymatroid.

**Theorem 16.6.1**

Let $f$ be a submodular function defined on subsets of $E$. For any $x \in \mathbb{R}^E$, we have:

$$
\text{rank}(x) = \max (y(E) : y \leq x, y \in P_f) = \min (x(A) + f(E \setminus A) : A \subseteq E)
$$

(16.1)

Essentially the same theorem as Theorem ??, but note $P_f$ rather than $P_f^+$. Taking $x = 0$ we get:

**Corollary 16.6.2**

Let $f$ be a submodular function defined on subsets of $E$. We have:

$$
\text{rank}(0) = \max (y(E) : y \leq 0, y \in P_f) = \min (f(A) : A \subseteq E)
$$

(16.2)
Restating what we saw before, we have:

$$\max \{ y(E) | y \in P_f, y \leq 0 \} = \min \{ f(X) | X \subseteq V \} \quad (16.45)$$

Consider the optimization:

\begin{align*}
\text{minimize} & \quad \| x \|^2_2 \\
\text{subject to} & \quad x \in B_f
\end{align*} \quad (16.46a, 16.46b)

where $B_f$ is the base polytope of submodular $f$, and

$$\| x \|^2_2 = \sum_{e \in E} x(e)^2$$

is the squared 2-norm. Let $x^*$ be the optimal solution.

Note, $x^*$ is the unique optimal solution since we have a strictly convex objective over a set of convex constraints.

$x^*$ is called the minimum norm point of the base polytope.
Given optimal solution $x^*$ to the above, consider the quantities

\[
y^* = x^* \land 0 = \min(x^*(e), 0) | e \in E \tag{16.47}
\]

\[
A_- = \{ e : x^*(e) < 0 \} \tag{16.48}
\]

\[
A_0 = \{ e : x^*(e) \leq 0 \} \tag{16.49}
\]

Thus, we immediately have that:

\[
A_- \subseteq A_0 \tag{16.50}
\]

and that

\[
x^*(A_-) = x^*(A_0) = y^*(A_-) = y^*(A_0) \tag{16.51}
\]

It turns out, these quantities will solve the submodular function minimization problem, as we now show.

The proof is nice since it uses the tools we’ve been recently developing.
Ex: 3D base $B_f$: permutahedron

Consider submodular function $f : 2^V \rightarrow \mathbb{R}$ with $|V| = 4$, and for $X \subseteq V$, concave $g$,

$$f(X) = g(|X|)$$

$$= \sum_{i=1}^{|X|} (4 - i + 1)$$

Then $B_f$ is a 3D polytope, and in this particular case gives us a permutahedron with 24 distinct extreme points, on the right (from wikipedia).
Modified max-min theorem

We have a variant of Theorem 12.5.2, the min-max theorem, namely that:
We have a variant of Theorem 12.5.2, the min-max theorem, namely that:

**Theorem 16.6.1 (Edmonds-1970)**

\[
\min \{ f(X) | X \subseteq E \} = \max \left\{ x^{-}(E) | x \in B_f \right\}
\]

where \( x^{-}(e) = \min \{ x(e), 0 \} \) for \( e \in E \).
Modified max-min theorem

- We have a variant of Theorem 12.5.2, the min-max theorem, namely that:

**Theorem 16.6.1 (Edmonds-1970)**

\[
\min \{ f(X) | X \subseteq E \} = \max \{ x^-(E) | x \in B_f \}
\]

*where* \( x^-(e) = \min \{ x(e), 0 \} \) *for* \( e \in E \).

**Proof.**

\[
\min \{ f(X) | X \subseteq E \} = \min_{w \in [0,1]^E} \tilde{f}(w) = \min_{w \in [0,1]^E} \max_{x \in P_f} w^T x
\]

\[
= \min_{w \in [0,1]^E} \max_{x \in B_f} w^T x
\]

\[
= \max_{x \in B_f} \min_{w \in [0,1]^E} w^T x
\]

\[
= \max_{x \in B_f} x^-(E)
\]
Alternate proof of modified max-min theorem

We start directly from Theorem 12.5.2.

\[
\max (y(E) : y \leq 0, y \in P_f) = \min (f(A) : A \subseteq E) \tag{16.62}
\]

Given \( y \in \mathbb{R}^E \), define \( y^- \in \mathbb{R}^E \) with \( y^-(e) = \min \{y(e), 0\} \) for \( e \in E \).

\[
\max (y(E) : y \leq 0, y \in P_f) = \max (y^-(E) : y \leq 0, y \in P_f) \tag{16.63}
\]

\[
= \max (y^-(E) : y \in P_f) \tag{16.64}
\]

\[
= \max (y^-(E) : y \in B_f) \tag{16.65}
\]

The first equality follows since \( y \leq 0 \). For the second equality, clearly l.h.s. \( \leq \) r.h.s. Also, l.h.s. \( \geq \) r.h.s. since the positive parts don’t matter.

\[
\max (y^-(E) : y \in P_f) = \max (y^-(E) : y(A) \leq f(A) \forall A) \tag{16.66}
\]

\[
= \max (y^-(E) : y^-(A) + y^+(A) \leq f(A) \forall A)
\]

The third equality follows since for any \( x \in P_f \) there exists a \( y \in B_f \) with \( x \leq y \) (follows from Theorem ??).
\[ \min \{ w^T x : x \in B_f \} \]

- Recall that the greedy algorithm solves, for \( w \in \mathbb{R}_+^E \)

\[
\max \{ w^T x \mid x \in P_f \} = \max \{ w^T x \mid x \in B_f \} \tag{16.67}
\]

since for all \( x \in P_f \), there exists \( y \geq x \) with \( y \in B_f \).
$\min \left\{ w^T x : x \in B_f \right\}$

- Recall that the greedy algorithm solves, for $w \in \mathbb{R}_+^E$
  
  $$\max \left\{ w^T x | x \in P_f \right\} = \max \left\{ w^T x | x \in B_f \right\}$$  

  (16.67)

  since for all $x \in P_f$, there exists $y \geq x$ with $y \in B_f$.

- For arbitrary $w \in \mathbb{R}^E$, greedy algorithm will also solve:

  $$\max \left\{ w^T x | x \in B_f \right\}$$  

  (16.68)
\[ \min \{ w^\top x : x \in B_f \} \]

- Recall that the greedy algorithm solves, for \( w \in \mathbb{R}^E \)

\[
\max \{ w^\top x | x \in P_f \} = \max \{ w^\top x | x \in B_f \} \tag{16.67}
\]

since for all \( x \in P_f \), there exists \( y \geq x \) with \( y \in B_f \).

- For arbitrary \( w \in \mathbb{R}^E \), greedy algorithm will also solve:

\[
\max \{ w^\top x | x \in B_f \} \tag{16.68}
\]

- Also, since

\[
\min \{ w^\top x | x \in B_f \} = - \max \{ -w^\top x | x \in B_f \} \tag{16.69}
\]

the greedy algorithm using ordering \( (e_1, e_2, \ldots, e_m) \) such that

\[
w(e_1) \leq w(e_2) \leq \cdots \leq w(e_m) \tag{16.70}
\]

will solve Equation (16.69).
\[ \max \{ w^T x \mid x \in B_f \} \text{ for arbitrary } w \in \mathbb{R}^E \]

Let \( f(A) \) be arbitrary submodular function, and \( f(A) = f'(A) - m(A) \) where \( f' \) is polymatroidal, and \( w \in \mathbb{R}^E \).

\[
\begin{align*}
\max \{ w^T x \mid x \in B_f \} & = \max \{ w^T x \mid x(A) \leq f(A) \forall A, x(E) = f(E) \} \\
& = \max \{ w^T x \mid x(A) \leq f'(A) - m(A) \forall A, x(E) = f'(E) - m(E) \} \\
& = \max \{ w^T x \mid x(A) + m(A) \leq f'(A) \forall A, x(E) + m(E) = f'(E) \} \\
& = \max \{ w^T x + w^T m \mid x(A) + m(A) \leq f'(A) \forall A, x(E) + m(E) = f'(E) \} - w^T m \\
& = \max \{ w^T y \mid y \in B_{f'} \} - w^T m \\
& = w^T y^* - w^T m = w^T (y^* - m)
\]

where \( y = x + m \), so that \( x^* = y^* - m \).

So \( y^* \) uses greedy algorithm with positive orthant \( B_{f'} \). To show, we use Theorem 12.4.1 in Lecture 12, but we don’t require \( y \geq 0 \), and don’t stop when \( w \) goes negative to ensure \( y^* \in B_{f'} \). Then when we subtract off \( m \) from \( y^* \), we get solution to the original problem.
Orthogonal $x$-containing hyperplane & convex/affine hulls

- Define $H(x)$ as the hyperplane that is orthogonal to the line from 0 to $x$, while also containing $x$, i.e.

$$H(x) \triangleq \left\{ y \in \mathbb{R}^V \mid x^T y = \|x\|_2^2 \right\}$$  \hfill (16.71)

Any set $\{y \in \mathbb{R}^V \mid x^T y = c\}$ is orthogonal to the line from 0 to $x$. To also contain $x$, we need $\|x\|_2 \|x\|_2 \cos 0 = c$ giving $c = \|x\|_2^2$. 

Orthogonal $x$-containing hyperplane & convex/affine hulls

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$$H(x) \triangleq \left\{ y \in \mathbb{R}^V \mid x^Ty = \|x\|_2^2 \right\}$$  \hspace{1cm} (16.71)

Any set $\{y \in \mathbb{R}^V \mid x^Ty = c\}$ is orthogonal to the line from 0 to $x$. To also contain $x$, we need $\|x\|_2 \|x\|_2 \cos 0 = c$ giving $c = \|x\|_2^2$.

- Given a set of points $P = \{p_1, p_2, \ldots, p_k\}$ with $p_i \in \mathbb{R}^V$, let $\text{conv } P$ be the convex hull of $P$, i.e.,

$$\text{conv } P \triangleq \left\{ \sum_{i=1}^{k} \lambda_i p_i : \sum_{i} \lambda_i = 1, \lambda_i \geq 0, i \in [k] \right\}.$$  \hspace{1cm} (16.72)
Orthogonal $x$-containing hyperplane & convex/affine hulls

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and for $Q = \{q_1, q_2, \ldots, q_k\}$, with $q_i \in \mathbb{R}^V$, let $\mathrm{aff} \ Q$ be the affine hull of $Q$, i.e.,

$$\mathrm{aff} \ Q \triangleq \left\{ \sum_{i=1}^{k} \lambda_i q_i : \sum_{i=1}^{k} \lambda_i = 1 \right\}. \quad (16.73)$$
Orthogonal $x$-containing hyperplane & convex/affine hulls

- Define $H(x)$ as the hyperplane that is orthogonal to the line from 0 to $x$, while also containing $x$, i.e.

$$H(x) \triangleq \left\{ y \in \mathbb{R}^V \mid x^\top y = \|x\|_2^2 \right\}$$ (16.71)

Any set $\{y \in \mathbb{R}^V \mid x^\top y = c\}$ is orthogonal to the line from 0 to $x$. To also contain $x$, we need $\|x\|_2 \|x\|_2 \cos 0 = c$ giving $c = \|x\|_2^2$.

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and for $Q = \{q_1, q_2, \ldots, q_k\}$, with $q_i \in \mathbb{R}^V$, let $\text{aff } Q$ be the affine hull of $Q$, i.e.,

$$\text{aff } Q \triangleq \left\{ \sum_{i=1}^k \lambda_i q_i : \sum_{i=1}^k \lambda_i = 1 \right\} \supseteq \text{conv } Q.$$ (16.73)
Notation

- The line between $x$ and $y$: given two points $x, y \in \mathbb{R}^V$, let $[x, y] \triangleq \{\lambda x + (1 - \lambda y) : \lambda \in [0, 1]\}$. Hence, $[x, y] = \text{conv} \{x, y\}$. 
Notation

- The line between \( x \) and \( y \): given two points \( x, y \in \mathbb{R}^V \), let 
  \[ [x, y] \triangleq \{ \lambda x + (1 - \lambda y) : \lambda \in [0, 1] \} \]. Hence, \([x, y] = \text{conv} \{x, y\}\).

- Note, if we wish to minimize the 2-norm of a vector \( \|x\|_2 \), we can equivalently minimize its square \( \|x\|_2^2 = \sum_i x_i^2 \), and vice verse.
Wolfe-1976 developed an algorithm to compute the minimum norm point of a polytope, specified as a set of vertices.
Fujishige-Wolfe Min-Norm Algorithm

- Wolfe-1976 developed an algorithm to compute the minimum norm point of a polytope, specified as a set of vertices.
- Fujishige-1984 “Submodular Systems and Related Topics” realized this algorithm can find the minimum norm point of $B_f$. 

Given set of points $P = \{p_1, \ldots, p_m\}$ where $p_i \in \mathbb{R}^n$ and the minimum norm point in convex hull of $P$:

$$\min_x \|x\|_2 \in \text{conv}(P) \quad (16.74)$$

Wolfe's algorithm is guaranteed terminating, and explicitly uses a representation of $x$ as a convex combination of points in $P$. Algorithm maintains a set of points $Q \subset P$, which is always assumed linearly independent.
Fujishige-Wolfe Min-Norm Algorithm

• Wolfe-1976 developed an algorithm to compute the minimum norm point of a polytope, specified as a set of vertices.
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- Seems to be (among) the fastest general purpose SFM algo.
- Given set of points $P = \{p_1, \cdots, p_m\}$ where $p_i \in \mathbb{R}^n$: find the minimum norm point in convex hull of $P$:

$$\min_{x \in \text{conv} \ P} \|x\|_2$$  \hspace{1cm} (16.74)
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(16.74)

- Wolfe’s algorithm is guaranteed terminating, and explicitly uses a representation of $x$ as a convex combination of points in $P$
- Algorithm maintains a set of points $Q \subseteq P$, which is always assuredly affinely independent.
Fujishige-Wolfe Min-Norm Algorithm

- When $Q$ are affinely independent, minimum norm point in the affine hull of $Q$ can easily be found, as a closed form solution for $\min_{x \in \text{aff } Q} \|x\|_2$ is available (see below).
Fujishige-Wolfe Min-Norm Algorithm

- When $Q$ are affinely independent, minimum norm point in the affine hull of $Q$ can easily be found, as a closed form solution for $\min_{x \in \text{aff } Q} \|x\|_2$ is available (see below).

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- Algorithm repeatedly produces min. norm point $x^*$ for selected set $Q$.
- If we find $w_i \geq 0, i = 1, \cdots, m$ for the minimum norm point, then $x^*$ also belongs to $\text{conv } Q$ and also a minimum norm point over $\text{conv } Q$. 
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- One of the most expensive parts of Wolfe’s algorithm is solving linear optimization problem over the polytope, doable by examining all the extreme points in the polytope.
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- One of the most expensive parts of Wolfe’s algorithm is solving linear optimization problem over the polytope, doable by examining all the extreme points in the polytope.
- If number of extreme points is exponential, hard to do in general.
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- When $Q$ are affinely independent, minimum norm point in the affine hull of $Q$ can easily be found, as a closed form solution for $\min_{x \in \text{aff } Q} \|x\|_2$ is available (see below).
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- If we find $w_i \geq 0$, $i = 1, \cdots, m$ for the minimum norm point, then $x^*$ also belongs to $\text{conv } Q$ and also a minimum norm point over $\text{conv } Q$.
- If $Q \subseteq P$ is suitably chosen, $x^*$ may even be the minimum norm point over $\text{conv } P$ solving the original problem.
- One of the most expensive parts of Wolfe’s algorithm is solving linear optimization problem over the polytope, doable by examining all the extreme points in the polytope.
- If number of extreme points is exponential, hard to do in general.
- Number of extreme points of submodular base polytope is exponentially large, but linear optimization over the base polytope $B_f$ doable $O(n \log n)$ time via Edmonds’s greedy algorithm.
Pseudocode of Fujishige-Wolfe Min-Norm (MN) algorithm

Input: \( P = \{p_1, \ldots, p_m\}, p_i \in \mathbb{R}^n, i = 1, \ldots, m. \)

Output: \( x^* \): the minimum-norm-point in \( \text{conv} \ P. \)

1. \( x^* \leftarrow p_{i^*} \) where \( p_{i^*} \in \arg\min_{p \in P} \|p\|_2 \)  
   /* or choose it arbitrarily */ ;

2. \( Q \leftarrow \{x^*\}; \)

3. while 1 do  
   /* major loop */

   if \( x^* = 0 \) or \( H(x^*) \) separates \( P \) from origin then
   | return : \( x^* \)
   
   else
   
   Choose \( \hat{x} \in P \) on the near (closer to 0) side of \( H(x^*) \);
   
   \[ Q = Q \cup \{\hat{x}\}; \]

   while 1 do  
   /* minor loop */

   \[ x_0 \leftarrow \min_{x \in \text{aff} Q} \|x\|_2; \]

   if \( x_0 \in \text{conv} \ Q \) then
   
   \[ x^* \leftarrow x_0; \]

   break;

   else

   \[ y \leftarrow \min_{x \in \text{conv} Q \cap [x^*, x_0]} \|x - x_0\|_2; \]

   Delete from \( Q \) points not on the face of \( \text{conv} \ Q \) where \( y \) lies;

   \[ x^* \leftarrow y; \]
Fujishige-Wolfe Min-Norm algorithm: Geometric Example

- It is advised that for the next set of slides, you have a print out of the previous MN algorithm available on display/paper somewhere.
Fujishige-Wolfe Min-Norm algorithm: Geometric Example

- It is advised that for the next set of slides, you have a print out of the previous MN algorithm available on display/paper somewhere.
- Algorithm maintains an invariant, namely that:

  \[ x^* \in \text{conv } Q \subseteq \text{conv } P, \quad (16.75) \]

must hold at every possible assignment of \( x^* \) (Lines 1, 11, and 16):

1. True after Line 1 since \( Q = \{x^*\} \),
2. True after Line 11 since \( x_0 \in \text{conv } Q \),
3. and true after Line 16 since \( y \in \text{conv } Q \) even after deleting points.
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  3. and true after Line 16 since \( y \in \text{conv} \ Q \) even after deleting points.
- Note also for any \( x^* \in \text{conv} \ Q \subseteq \text{conv} \ P \), we have
  \[ \min_{x \in \text{aff} \ Q} \|x\|_2 \leq \min_{x \in \text{conv} \ Q} \|x\|_2 \leq \|x^*\|_2 \]  
  \[ (16.76) \]
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- Note also for any $x^* \in \text{conv } Q \subseteq \text{conv } P,$ we have

  $$\min_{x \in \text{aff } Q} \|x\|_2 \leq \min_{x \in \text{conv } Q} \|x\|_2 \leq \|x^*\|_2$$  

  (16.76)

- Note, the input, $P,$ consists of $m$ points. In the case of the base polytope, $P = B_f$ could be exponential in $n = |V|.$
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- Note also for any \( x^* \in \text{conv } Q \subseteq \text{conv } P \), we have

\[ \min_{x \in \text{aff } Q} \|x\|_2 \leq \min_{x \in \text{conv } Q} \|x\|_2 \leq \|x^*\|_2 \quad (16.76) \]

- Note, the input, \( P \), consists of \( m \) points. In the case of the base polytope, \( P = B_f \) could be exponential in \( n = |V| \).
- There are six places that might be seemingly tricky or expensive: Line 4, Line 6, Line 9, Line 10, Line 14, and Line 15.
Fujishige-Wolfe Min-Norm algorithm: Geometric Example

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1. True after Line 1 since \( Q = \{x^*\} \),
2. True after Line 11 since \( x_0 \in \text{conv } Q \),
3. and true after Line 16 since \( y \in \text{conv } Q \) even after deleting points.

- Note also for any \( x^* \in \text{conv } Q \subseteq \text{conv } P \), we have

\[ \min_{x \in \text{aff } Q} \|x\|_2 \leq \min_{x \in \text{conv } Q} \|x\|_2 \leq \|x^*\|_2 \quad (16.76) \]

- Note, the input, \( P \), consists of \( m \) points. In the case of the base polytope, \( P = B_f \) could be exponential in \( n = |V| \).
- There are six places that might be seemingly tricky or expensive: Line 4, Line 6, Line 9, Line 10, Line 14, and Line 15.
- We will consider each in turn, but first we do a geometric example.
Fujishige-Wolfe Min-Norm algorithm: Geometric Example

Polytope, and circles concentric at 0.
The initial polytope consisting of the convex hull of three points $p_1, p_2, p_3$, and the origin 0.
$p_1$ is the extreme point closest to 0 and so we choose it first, although we can choose any arbitrary extreme point as the initial point. We set $x^* \leftarrow p_1$ in Line 1, and $Q \leftarrow \{p_1\}$ in Line 2. $H(x^*) = H(p_1)$ (green dashed line) is not a supporting hyperplane of $\text{conv}(P)$ in Line 4, so we move on to the else condition in Line 5.
We need to add some extreme point $\hat{x}$ on the “near” side of $H(p_1)$ in Line 6, we choose $\hat{x} = p_2$. In Line 7, we set $Q \leftarrow Q \cup \{p_2\}$, so $Q = \{p_1, p_2\}$. 
$x_0 = R$ is the min-norm point in $\text{aff}\{p_1, p_2\}$ computed in Line 9.
\(x_0 = R\) is the min-norm point in \(\text{aff} \{p_1, p_2\}\) computed in Line 9. Also, with \(Q = \{p_1, p_2\}\), since \(R \in \text{conv} Q\), we set \(x^* \leftarrow x_0 = R\) in Line 11.

Note, after Line 11, we still have \(x^* \in P\) and \(\|x^*\|_2 = \|x^*_{\text{new}}\|_2 < \|x^*_{\text{old}}\|_2\) strictly.
Fujishige-Wolfe Min-Norm algorithm: Geometric Example

\[ R = x_0 = x^* \]. We consider next \( H(R) = H(x^*) \) in Line 4. \( H(x^*) \) is not a supporting hyperplane of \( \text{conv } P \). So we choose \( p_3 \) on the “near” side of \( H(x^*) \) in Line 6. Add \( Q \leftarrow Q \cup \{p_3\} \) in Line 7. Now \( Q = P = \{p_1, p_2, p_3\} \).
Fujishige-Wolfe Min-Norm algorithm: Geometric Example

\[ R = x_0 = x^*. \] We consider next \( H(R) = H(x^*) \) in Line 4. \( H(x^*) \) is not a supporting hyperplane of \( \text{conv} \ P \). So we choose \( p_3 \) on the “near” side of \( H(x^*) \) in Line 6. Add \( Q \leftarrow Q \cup \{p_3\} \) in Line 7. Now \( Q = P = \{p_1, p_2, p_3\} \). The origin \( x_0 = 0 \) is the min-norm point in \( \text{aff} \ Q \) (Line 9), and it is not in the interior of \( \text{conv} \ Q \) (condition in Line 10 is false).
Fujishige-Wolfe Min-Norm algorithm: Geometric Example

\[ Q = P = \{ p_1, p_2, p_3 \} \]. Line 14: \( S = y = \min_{x \in \text{conv } Q \cap [x^*, x_0]} \| x - x_0 \|_2 \) where \( x_0 \) is 0 and \( x^* \) is \( R \) here. Thus, \( y \) lies on the boundary of \( \text{conv } Q \). Note, \( \| y \|_2 < \| x^* \|_2 \) since \( x^* \in \text{conv } Q \), \( \| x_0 \|_2 < \| x^* \|_2 \).
Fujishige-Wolfe Min-Norm algorithm: Geometric Example

\[ Q = P = \{p_1, p_2, p_3\} \]. Line 14: \( S = y = \min_{x \in \text{conv } Q \cap [x^*, x_0]} \|x - x_0\|_2 \) where \( x_0 \) is 0 and \( x^* \) is \( R \) here. Thus, \( y \) lies on the boundary of \( \text{conv } Q \).

Note, \( \|y\|_2 < \|x^*\|_2 \) since \( x^* \in \text{conv } Q \), \( \|x_0\|_2 < \|x^*\|_2 \). Line 15: Delete \( p_1 \) from \( Q \) since it is not on the face where \( S \) lies. \( Q = \{p_2, p_3\} \) after Line 15.

Note, we still have \( y = S \in \text{conv } Q \) for the updated \( Q \).
Q = P = \{p_1, p_2, p_3\}. Line 14: S = y = \min_{x \in \text{conv } Q \cap [x^*, x_0]} \|x - x_0\|_2 \quad \text{where } x_0 \text{ is 0 and } x^* \text{ is } R \text{ here. Thus, } y \text{ lies on the boundary of conv } Q. \quad \text{Note, } \|y\|_2 < \|x^*\|_2 \text{ since } x^* \in \text{conv } Q, \|x_0\|_2 < \|x^*\|_2. \quad \text{Line 15: Delete } p_1 \text{ from } Q \text{ since it is not on the face where } S \text{ lies. } Q = \{p_2, p_3\} \text{ after Line 15.} \quad \text{Note, we still have } y = S \in \text{conv } Q \text{ for the updated } Q. \quad \text{Line 16: } x^* \leftarrow y, \quad \text{hence we again have } \|x^*\|_2 = \|x^*\text{new}\|_2 < \|x^*\text{old}\|_2 \text{ strictly.}
Fujishige-Wolfe Min-Norm algorithm: Geometric Example

$Q = \{p_2, p_3\}$, and so $x_0 = T$ computed in Line 9 is the min-norm point in aff $Q$. We also have $x_0 \in \text{conv} \ Q$ in Line 10 so we assign $x^* \leftarrow x_0$ in Line 11 and break.
Fujishige-Wolfe Min-Norm algorithm: Geometric Example

$H(T)$ separates $P$ from the origin in Line 4, and therefore is a supporting hyperplane, and therefore $x^*$ is the min-norm point in $\text{conv } P$, so we return with $x^*$. 
**Condition for Min-Norm Point**

**Theorem 16.6.2**

\[ P = \{p_1, p_2, \ldots, p_m\}, \text{ } x^* \in \text{conv } P \text{ is the min. norm point in conv } P \text{ iff } \]

\[ p_i^T x^* \geq \|x^*\|_2^2 \quad \forall i = 1, \ldots, m. \]  \hspace{1cm} (16.77)

**Proof.**

- Assume \( x^* \) is the min-norm point, let \( y \in \text{conv } P \), and \( 0 \leq \theta \leq 1 \).
**Condition for Min-Norm Point**

**Theorem 16.6.2**

\[ P = \{p_1, p_2, \ldots, p_m\}, \ x^* \in \text{conv } P \text{ is the min. norm point in conv } P \text{ iff } \]
\[ p_i^T x^* \geq \|x^*\|_2^2 \quad \forall i = 1, \ldots, m. \]  \hspace{1cm} (16.77)

**Proof.**

- Assume \( x^* \) is the min-norm point, let \( y \in \text{conv } P \), and \( 0 \leq \theta \leq 1 \).

- Then \( z \triangleq x^* + \theta(y - x^*) = (1 - \theta)x^* + \theta y \in \text{conv } P \), and

\[
\|z\|_2^2 = \|x^* + \theta(y - x^*)\|_2^2 \\
= \|x^*\|_2^2 + 2\theta(x^T y - x^T x^*) + \theta^2 \|y - x^*\|_2^2 \]  \hspace{1cm} (16.78)

It is possible for \( \|z\|_2^2 < \|x^*\|_2^2 \) for small \( \theta \), unless \( x^* \mid y \mid x^* \mid \) for all \( y \in \text{conv } P \). Equation (16.77).

Conversely, given Eq (16.77), and given that \( y = \sum_i p_i \in \text{conv } P \),

\[
y \mid x^* = \sum_i p_i x^* = x^* \\
\Rightarrow \|z\|_2^2 > \|x^*\|_2^2 \text{ in Equation 16.79 for arbitrary } z \in \text{conv } P.
\]
Theorem 16.6.2

\[ P = \{p_1, p_2, \ldots, p_m\}, \ x^* \in \text{conv} P \text{ is the min. norm point in } \text{conv} P \text{ iff } \]

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(16.77)

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- Then \( z \triangleq x^* + \theta(y - x^*) = (1 - \theta)x^* + \theta y \in \text{conv} P \), and

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\|z\|^2_2 = \|x^* + \theta(y - x^*)\|^2_2 = \|x^*\|^2_2 + 2\theta(x^T y - x^T x^*) + \theta^2 \|y - x^*\|^2_2
\]  

(16.78)

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- It is possible for \( \|z\|^2_2 < \|x^*\|^2_2 \) for small \( \theta \), unless \( x^T y \geq x^T x^* \) for all \( y \in \text{conv} P \Rightarrow \) Equation (16.77).
Condition for Min-Norm Point

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\[ P = \{p_1, p_2, \ldots, p_m\}, \; x^* \in \text{conv} \; P \text{ is the min. norm point in } \text{conv} \; P \text{ iff} \]

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- Assume \( x^* \) is the min-norm point, let \( y \in \text{conv} \; P \), and \( 0 \leq \theta \leq 1 \).
- Then \( z \triangleq x^* + \theta(y - x^*) = (1 - \theta)x^* + \theta y \in \text{conv} \; P \), and
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  \]
  \[
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- It is possible for \( \|z\|^2_2 < \|x^*\|^2_2 \) for small \( \theta \), unless \( x^T y \geq x^T x^* \) for all \( y \in \text{conv} \; P \Rightarrow \) Equation (16.77).

- Conversely, given Eq (16.77), and given that \( y = \sum_i \lambda_i p_i \in \text{conv} \; P \),
  \[
  y^T x^* = \sum_i \lambda_i p_i^T x^* \geq \sum_i \lambda_i x^T x^* = x^T x^* \quad (16.80)
  \]
  implying that \( \|z\|^2_2 > \|x^*\|^2_2 \) in Equation 16.79 for arbitrary \( z \in \text{conv} \; P \).
The set $Q$ is always affinely independent.

**Lemma 16.6.3**

The set $Q$ in the MN Algorithm is always affinely independent.
The set \( Q \) is always affinely independent

**Lemma 16.6.3**

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**Proof.**

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- $Q$ is of course affinely independent when there is at most one point in it (e.g., after Line 2).
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- $Q$ is of course affinely independent when there is at most one point in it (e.g., after Line 2).
- After the initialization, it changes only by deletion of points, or adding a single point. Deletion does not change the independence.
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- Before adding $\hat{x}$ at Line 7, we know $x^*$ is the minimum norm point in $\text{aff } Q$ (since we break only at Line 11).
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- Before adding $\hat{x}$ at Line 7, we know $x^*$ is the minimum norm point in $\text{aff } Q$ (since we break only at Line 11).
- Therefore, $x^*$ is normal to $\text{aff } Q$, which implies $\text{aff } Q \subseteq H(x^*)$. 
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- Therefore, $x^*$ is normal to $\text{aff} \ Q$, which implies $\text{aff} \ Q \subseteq H(x^*)$.
- Since $\hat{x} \notin H(x^*)$ chosen at Line 6, we have $\hat{x} \notin \text{aff} \ Q$. 

□
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- Therefore, $x^*$ is normal to $\text{aff } Q$, which implies $\text{aff } Q \subseteq H(x^*)$.
- Since $\hat{x} \not\in H(x^*)$ chosen at Line 6, we have $\hat{x} \not\in \text{aff } Q$.
- ∴ update $Q \cup \{\hat{x}\}$ at Line 7 is affinely independent as long as $Q$ is.
The set $Q$ is always affinely independent

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- $Q$ is of course affinely independent when there is at most one point in it (e.g., after Line 2).
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- Before adding $\hat{x}$ at Line 7, we know $x^*$ is the minimum norm point in $\text{aff } Q$ (since we break only at Line 11).
- Therefore, $x^*$ is normal to $\text{aff } Q$, which implies $\text{aff } Q \subseteq H(x^*)$.
- Since $\hat{x} \notin H(x^*)$ chosen at Line 6, we have $\hat{x} \notin \text{aff } Q$.
- $\therefore$ update $Q \cup \{\hat{x}\}$ at Line 7 is affinely independent as long as $Q$ is.

Thus, by Lemma 16.6.3, we have for any $x \in \text{aff } Q$ such that $x = \sum_i w_i q_i$ with $\sum_i w_i = 1$, the weights $w_i$ are uniquely determined.
Minimum Norm in an affine set

- Line 9 of the algorithm requires $x_0 \leftarrow \min_{x \in \text{aff } Q} \|x\|_2$. 
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- When $Q$ is affinely independent, this is relatively easy.
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- When $Q$ is affinely independent, this is relatively easy.
- Let $Q$ also represent the $n \times k$ matrix with points as columns $q \in Q$. We get the following, solvable with matrix inversion/linear solver:

  \[
  \begin{aligned}
  \text{minimize} & \quad \|x\|_2^2 = w^T Q^T Q w \\
  \text{subject to} & \quad 1^T w = 1
  \end{aligned}
  \]  
  \tag{16.81}

  \[1^T w = 1 \tag{16.82}\]
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- Note, this also solves Line 10, since feasibility requires $\sum_i w_i = 1$, we need only check $w \geq 0$ to ensure $x_0 = \sum_i w_i q_i \in \text{conv } Q$. 

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- In fact, a feature of the algorithm (in Wolfe’s 1976 paper) is that we keep the convex coefficients $\{w_i\}_i$ where $x^* = \sum_i \lambda_i p_i$ of $x^*$ and from this vector. We also keep $v$ such that $x_0 = \sum_i v_i q_i$ for points $q_i \in Q$, from Line 9.
- Given $w$ and $v$, we can also easily solve Lines 14 and 15 (see “Step 3” on page 133 of Wolfe-1976, which also defines numerical tolerances).
Minimum Norm in an affine set

- Line 9 of the algorithm requires $x_0 \leftarrow \min_{x \in \text{aff } Q} \| x \|_2$.
- When $Q$ is affinely independent, this is relatively easy.
- Let $Q$ also represent the $n \times k$ matrix with points as columns $q \in Q$. We get the following, solvable with matrix inversion/linear solver:

$$\begin{align*}
\text{minimize} & \quad \| x \|_2^2 = w^T Q^T Q w \\
\text{subject to} & \quad 1^T w = 1
\end{align*}$$

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- Note, this also solves Line 10, since feasibility requires $\sum_i w_i = 1$, we need only check $w \geq 0$ to ensure $x_0 = \sum_i w_i q_i \in \text{conv } Q$.
- In fact, a feature of the algorithm (in Wolfe’s 1976 paper) is that we keep the convex coefficients $\{w_i\}_i$ where $x^* = \sum_i \lambda_i p_i$ of $x^*$ and from this vector. We also keep $v$ such that $x_0 = \sum_i v_i q_i$ for points $q_i \in Q$, from Line 9.

Given $w$ and $v$, we can also easily solve Lines 14 and 15 (see “Step 3” on page 133 of Wolfe-1976, which also defines numerical tolerances).

- We have yet to see how to efficiently solve Lines 4 and 6, however.
MN Algorithm finds the MN point in finite time.

**Theorem 16.6.4**

*The MN Algorithm finds the minimum norm point in $\text{conv } P$ after a finite number of iterations of the major loop.*

**Proof.**

- In minor loop, we always have $x^* \in \text{conv } Q$, since whenever $Q$ is modified, $x^*$ is updated as well (Line 16) such that the updated $x^*$ remains in new $\text{conv } Q$.
The MN Algorithm finds the minimum norm point in $\text{conv} \, P$ after a finite number of iterations of the major loop.

Proof.

1. In minor loop, we always have $x^* \in \text{conv} \, Q$, since whenever $Q$ is modified, $x^*$ is updated as well (Line 16) such that the updated $x^*$ remains in new $\text{conv} \, Q$.

2. Hence, every time $x^*$ is updated (in minor loop), its norm never increases i.e., before Line 11, $\|x_0\|_2 \leq \|x^*\|_2$ since $x^* \in \text{aff} \, Q$ and $x_0 = \min_{x \in \text{aff} \, Q} \|x\|_2$.
The MN Algorithm finds the minimum norm point in $\text{conv } P$ after a finite number of iterations of the major loop.

Proof.

- In minor loop, we always have $x^* \in \text{conv } Q$, since whenever $Q$ is modified, $x^*$ is updated as well (Line 16) such that the updated $x^*$ remains in new $\text{conv } Q$.

- Hence, every time $x^*$ is updated (in minor loop), its norm never increases i.e., before Line 11, $\|x_0\|_2 \leq \|x^*\|_2$ since $x^* \in \text{aff } Q$ and $x_0 = \min_{x \in \text{aff } Q} \|x\|_2$. Similarly, before Line 16, $\|y\|_2 \leq \|x^*\|_2$, since invariant $x^* \in \text{conv } Q$ but while $x_0 \in \text{aff } Q$, we have $x_0 \notin \text{conv } Q$, and $\|x_0\|_2 < \|x^*\|_2$. ...

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MN Algorithm finds the MN point in finite time.

...proof of Theorem 16.6.4 continued.

Moreover, there can be no more iterations within a minor loop than the dimension of $\text{conv } Q$ for the initial $Q$ given to the minor loop initially at Line 8 (dimension of $\text{conv } Q$ is $|Q| - 1$ since $Q$ is affinely independent).
MN Algorithm finds the MN point in finite time.

... proof of Theorem 16.6.4 continued.

Moreover, there can be no more iterations within a minor loop than the dimension of $\text{conv } Q$ for the initial $Q$ given to the minor loop initially at Line 8 (dimension of $\text{conv } Q$ is $|Q| - 1$ since $Q$ is affinely independent).

Each iteration of the minor loop removes at least one point from $Q$ in Line 15.
MN Algorithm finds the MN point in finite time.

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- Each iteration of the minor loop removes at least one point from $Q$ in Line 15.

- When $Q$ reduces to a singleton, the minor loop always terminates.
proof of Theorem 16.6.4 continued.

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- Each iteration of the minor loop removes at least one point from $Q$ in Line 15.

- When $Q$ reduces to a singleton, the minor loop always terminates.

- Thus, the minor loop terminates in finite number of iterations, at most dimension of $Q$. 
MN Algorithm finds the MN point in finite time.

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- Moreover, there can be no more iterations within a minor loop than the dimension of $\text{conv } Q$ for the initial $Q$ given to the minor loop initially at Line 8 (dimension of $\text{conv } Q$ is $|Q| - 1$ since $Q$ is affinely independent).
- Each iteration of the minor loop removes at least one point from $Q$ in Line 15.
- When $Q$ reduces to a singleton, the minor loop always terminates.
- Thus, the minor loop terminates in finite number of iterations, at most dimension of $Q$.
- In fact, total number of iterations of minor loop in entire algorithm is at most number of points in $P$ since we never add back in points to $Q$ that have been removed.

...
...proof of Theorem 16.6.4 continued.

Each time $Q$ is augmented with $\hat{x}$ at Line 7, followed by updating $x^*$ with $x_0$ at Line 11, (i.e., when the minor loop returns with only one iteration), $\|x^*\|_2$ strictly decreases from what it was before.
...proof of Theorem 16.6.4 continued.

- Each time $Q$ is augmented with $\hat{x}$ at Line 7, followed by updating $x^*$ with $x_0$ at Line 11, (i.e., when the minor loop returns with only one iteration), $\|x^*\|_2$ strictly decreases from what it was before.

- To see this, consider $x^* + \theta(\hat{x} - x^*)$ where $0 \leq \theta \leq 1$. Since both $\hat{x}, x^* \in \text{conv } Q$, we have $x^* + \theta(\hat{x} - x^*) \in \text{conv } Q$. 


... proof of Theorem 16.6.4 continued.

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- Therefore, we have $\|x^* + \theta(\hat{x} - x^*)\|_2 \geq \|x_0\|_2$, which implies

$$\|x^* + \theta(\hat{x} - x^*)\|_2^2 = \|x^*\|_2^2 + 2\theta (x^*)^\top \hat{x} - \|x^*\|_2^2 + \theta^2 \|\hat{x} - x^*\|_2^2 \geq \|x_0\|_2^2$$

(16.83)

$\hat{x}$ is on the same side of $H(x^*)$ as the origin, i.e. $(x^*)^\top \hat{x} < \|x^*\|_2^2$. 

...
MN Algorithm finds the MN point in finite time.

... proof of Theorem 16.6.4 continued.

Therefore, for sufficiently small $\theta$, specifically for

$$\theta < \frac{2 \left( \| x^* \|^2_2 - (x^*)^\top \hat{x} \right)}{\| \hat{x} - x^* \|^2_2}$$

(16.84)

we have that $\| x^* \|^2_2 > \| x_0 \|^2_2$. 
MN Algorithm finds the MN point in finite time.

... proof of Theorem 16.6.4 continued.

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  \[
  \theta < \frac{2 \left( \|x^*\|_2^2 - (x^*)^\top \hat{x} \right)}{\|\hat{x} - x^*\|_2^2}
  \]  
  (16.84)

  we have that $\|x^*\|_2^2 > \|x_0\|_2^2$.

- For a similar reason, we have $\|x^*\|_2$ strictly decreases each time $Q$ is updated at Line 7 and followed by updating $x^*$ with $y$ at Line 16.
MN Algorithm finds the MN point in finite time.

...proof of Theorem 16.6.4 continued.

- Therefore, for sufficiently small $\theta$, specifically for

$$\theta < \frac{2 \left( \|x^*\|_2^2 - (x^*)^\top \hat{x} \right)}{\|\hat{x} - x^*\|_2^2}$$

(16.84)

we have that $\|x^*\|_2^2 > \|x_0\|_2^2$.

- For a similar reason, we have $\|x^*\|_2$ strictly decreases each time $Q$ is updated at Line 7 and followed by updating $x^*$ with $y$ at Line 16.

- Therefore, in each iteration of major loop, $\|x^*\|_2$ strictly decreases, and the MN Algorithm must terminate and it can only do so when the optimal is found.
Line: 6: Finding $\hat{x} \in P$ on the near side of $H(x^*)$

- The “near” side means the side that contains the origin.
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Ideally, find \( \hat{x} \) such that the reduction of \( \|x^*\|_2 \) is maximized to reduce number of major iterations.

From Eqn. 16.83, reduction on norm is lower-bounded:

\[
\Delta = \|x^*\|_2^2 - \|x_0\|_2^2 \geq 2\theta \left( \|x^*\|_2^2 - (x^*)^\top \hat{x} \right) - \theta^2 \|\hat{x} - x^*\|_2^2 \equiv \Delta
\]

(16.85)
Line: 6: Finding $\hat{x} \in P$ on the near side of $H(x^*)$

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- Ideally, find $\hat{x}$ such that the reduction of $\|x^*\|^2_2$ is maximized to reduce number of major iterations.
- From Eqn. 16.83, reduction on norm is lower-bounded:

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(16.85)

- When $0 \leq \theta < \frac{2(\|x^*\|^2_2 - (x^*)^\top \hat{x})}{\|\hat{x} - x^*\|^2_2}$, we can get the maximal value of the lower bound, over $\theta$, as follows:

$$\max_{0 \leq \theta < \frac{2(\|x^*\|^2_2 - (x^*)^\top \hat{x})}{\|\hat{x} - x^*\|^2_2}} \Delta = \left(\frac{\|x^*\|^2_2 - (x^*)^\top \hat{x}}{\|\hat{x} - x^*\|^2_2}\right)^2$$

(16.86)
Line: 6: Finding $\hat{x} \in P$ on the near side of $H(x^*)$

- To maximize lower bound of norm reduction at each major iteration, want to find an $\hat{x}$ such that the above lower bound (Equation 16.86) is maximized.
To maximize lower bound of norm reduction at each major iteration, want to find an \( \hat{x} \) such that the above lower bound (Equation 16.86) is maximized.

That is, we want to find

\[
\hat{x} \in \arg\max_{x \in P} \left( \frac{\|x^*\|^2_2 - (x^*)^\top x}{\|x - x^*\|^2_2} \right)^2
\]

(16.87)

to ensure that a large norm reduction is assured.
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to ensure that a large norm reduction is assured.

This problem, however, is at least as hard as the MN problem itself as we have a quadratic term in the denominator.
Line: 6: Finding \( \hat{x} \in P \) on the near side of \( H(x^*) \)

- As a surrogate, we maximize numerator in Eqn. 16.87, i.e., find

\[
\hat{x} \in \arg\max_{x \in P} \|x^*\|_2^2 - (x^*)^T x = \arg\min_{x \in P} (x^*)^T x,
\]  

(16.88)
Line: 6: Finding $\hat{x} \in P$ on the near side of $H(x^*)$

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- Intuitively, by solving the above, we find $\hat{x}$ such that it has the largest distance to the hyperplane $H(x^*)$, and this is exactly the strategy used in the Wolfe-1976 algorithm.
This page discusses the min-norm point algorithm, particularly the proof that the minimum norm point gives optimal solutions. The algorithm's Line 6 focuses on finding a point \( \hat{x} \in P \) on the near side of \( H(x^*) \).

**As a surrogate, we maximize numerator in Eqn. 16.87, i.e., find**

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**Intuitively, by solving the above, we find \( \hat{x} \) such that it has the largest distance to the hyperplane \( H(x^*) \), and this is exactly the strategy used in the Wolfe-1976 algorithm.**

**Also, solution \( \hat{x} \) can be used to determine if hyperplane \( H(x^*) \) separates \( \text{conv} \ P \) from the origin (Line 4): if the point in \( P \) having greatest distance to \( H(x^*) \) is not on the side where origin lies, then \( H(x^*) \) separates \( \text{conv} \ P \) from the origin.**
**Line: 6: Finding \( \hat{x} \in P \) on the near side of \( H(x^*) \)**

- As a surrogate, we maximize numerator in Eqn. 16.87, i.e., find

  \[
  \hat{x} \in \text{argmax}_{x \in P} \|x^*\|_2^2 - (x^*)^\top x = \text{argmin}_{x \in P} (x^*)^\top x,
  \]

  \[
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  \]

- Intuitively, by solving the above, we find \( \hat{x} \) such that it has the largest distance to the hyperplane \( H(x^*) \), and this is exactly the strategy used in the Wolfe-1976 algorithm.

- Also, solution \( \hat{x} \) can be used to determine if hyperplane \( H(x^*) \) separates \( \text{conv } P \) from the origin (Line 4): if the point in \( P \) having greatest distance to \( H(x^*) \) is not on the side where origin lies, then \( H(x^*) \) separates \( \text{conv } P \) from the origin.

- Mathematically, we terminate the algorithm if

  \[
  (x^*)^\top \hat{x} \geq \|x^*\|_2^2,
  \]

  \[
  \text{(16.89)}
  \]

  where \( \hat{x} \) is the solution of Eq. 16.88.
Line: 6: Finding $\hat{x} \in P$ on the near side of $H(x^*)$

In practice, the above optimality test might never hold numerically. Hence, as suggested by Wolfe, we introduce a tolerance parameter $\epsilon > 0$, and terminates the algorithm if

$$
(x^*)^\top \hat{x} > \|x^*\|_2^2 - \epsilon \max_{x \in Q} \|x\|_2^2
$$

(16.90)
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- When $\text{conv } P$ is a submodular base polytope (i.e., $\text{conv } P = B_f$ for a submodular function $f$), then the problem in Eqn 16.88 can be solved efficiently by Edmonds’s greedy algorithm (even though there may be an exponential number of extreme points).
In practice, the above optimality test might never hold numerically. Hence, as suggested by Wolfe, we introduce a tolerance parameter $\epsilon > 0$, and terminates the algorithm if

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Hence, Edmonds's discovery is one of the main reasons that the MN algorithm is applicable to submodular function minimization.
SFM Summary (modified from S. Iwata’s slides)

General Submodular Function Minimization

- Wolfe (1976)/von Hohenbalken (1975)
- Bach (2012/13)
  - minimum norm point algorithm
  - Ellipsoid Method
- Edmonds (1965/1970)
- Bixby, Cunningham, Topkis (1984)
- Iwata, Fleischer, Fujishige (2000)
  - $O(n^5 \gamma \log M)$
  - $O(n^7 \gamma \log n)$
- Cunningham (1985)
  - $O(n^7 \gamma + n^8)$
- Iwata (2002)
  - Fully Combinatorial
- Iwata, Fleischer, Fujishige (2000)
- Schrijver (2000)
- Iwata (2003)
  - $O((n^4 \gamma + n^5) \log M)$
- Orlin (2007)
  - $O(n^5 \gamma + n^6)$
- Iwata, Orlin (2009)

...
The currently fastest strongly polynomial combinatorial algorithm for SFM achieves a running time of $O(n^5 T + n^6)$ (Orlin’09) where $T$ is the time for function evaluation, far from practical for large problem instances.
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  - Therefore, the complexity of each major iteration is
    \[ O(n^3 + n^{1+p}) \]
  - where each function oracle call requires $O(n^p)$ time.
**MN Algorithm Complexity**

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  - Therefore, the complexity of each major iteration is $O(n^3 + n^{1+p})$

where each function oracle call requires $O(n^p)$ time.

- Since the number of major iterations required is unknown, the complexity of MN is also unknown.
**MN Algorithm Empirical Complexity**

**Figure:** The number of major iteration for \( f(S) = -m_1(S) + 100 \cdot (w_1(N(S)))^\alpha \). The red lines are the linear interpolations of the worst case points, and the black lines are the linear interpolations of the average case points. From Lin&Bilmes 2014 (unpublished)
Min-Norm Point and SFM

Theorem 16.7.1

Let \( y^* \), \( A_- \), and \( A_0 \) be as given. Then \( y^* \) is a maximizer of the l.h.s. of Eqn. (??). Moreover, \( A_- \) is the unique minimal minimizer of \( f \) and \( A_0 \) is the unique maximal minimizer of \( f \).

Proof.

First note, since \( x^* \in B_f \), we have \( x^*(E) = f(E) \), meaning \( \text{sat}(x^*) = E \). Thus, we can consider any \( e \in E \) within \( \text{dep}(x^*, e) \).
Theorem 16.7.1

Let $y^*$, $A_-$, and $A_0$ be as given. Then $y^*$ is a maximizer of the l.h.s. of Eqn. (??). Moreover, $A_-$ is the unique minimal minimizer of $f$ and $A_0$ is the unique maximal minimizer of $f$.

Proof.

First note, since $x^* \in B_f$, we have $x^*(E) = f(E)$, meaning $\text{sat}(x^*) = E$. Thus, we can consider any $e \in E$ within $\text{dep}(x^*, e)$.

Consider any pair $(e, e')$ with $e' \in \text{dep}(x^*, e)$ and $e \in A_-$. Then $x^*(e) < 0$, and $\exists \alpha > 0$ s.t. $x^* + \alpha 1_e - \alpha 1_{e'} \in P_f$. 

...
**Theorem 16.7.1**

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- Consider any pair \((e, e')\) with \( e' \in \text{dep}(x^*, e) \) and \( e \in A_- \). Then \( x^*(e) < 0 \), and \( \exists \alpha > 0 \) s.t. \( x^* + \alpha 1_e - \alpha 1_{e'} \in P_f \).

- We have \( x^*(E) = f(E) \) and \( x^* \) is minimum in l2 sense. We have \( (x^* + \alpha 1_e - \alpha 1_{e'}) \in P_f \), and in fact

\[
(x^* + \alpha 1_e - \alpha 1_{e'})(E) = x^*(E) + \alpha - \alpha = f(E) \quad (16.91)
\]

so \( x^* + \alpha 1_e - \alpha 1_{e'} \in B_f \) also.

...
...proof of Thm. 16.7.1 cont.

Then

\[
(x^* + \alpha 1_e - \alpha 1_{e'}) (E) = x^* (E \setminus \{e, e'\}) + (x^*(e) + \alpha) + (x^*(e') - \alpha) = f(E).
\]

\[
x_{\text{new}}(e) \quad x_{\text{new}}(e')
\]
... proof of Thm. 16.7.1 cont.

- Then
  \[ (x^* + \alpha 1_e - \alpha 1_{e'}) (E) \]
  \[ = x^*(E \setminus \{e, e'\}) + (x^*(e) + \alpha) + (x^*(e') - \alpha) = f(E). \]
  \[ x_{\text{new}}^*(e) \quad x_{\text{new}}^*(e') \]

- Minimality of \( x^* \in B_f \) in \( l_2 \) sense requires that, with such an \( \alpha > 0 \),
  \[ (x^*(e))^2 + (x^*(e'))^2 < (x_{\text{new}}^*(e))^2 + (x_{\text{new}}^*(e'))^2 \]
... proof of Thm. 16.7.1 cont.

- Then \( (x^* + \alpha \mathbf{1}_e - \alpha \mathbf{1}_{e'}) (E) \)
  \[
  = x^* (E \setminus \{e, e'\}) + (x^*(e) + \alpha) + (x^*(e') - \alpha) = f(E).
  \]
  \[
  x^*_{\text{new}}(e) \quad \quad x^*_{\text{new}}(e')
  \]

- Minimality of \( x^* \in B_f \) in \( l^2 \) sense requires that, with such an \( \alpha > 0 \),
  \[
  \left( x^*(e) \right)^2 + \left( x^*(e') \right)^2 < \left( x^*_{\text{new}}(e) \right)^2 + \left( x^*_{\text{new}}(e') \right)^2
  \]

- Given that \( e \in A_- \), \( x^*(e) < 0 \). Thus, if \( x^*(e') > 0 \), we could have
  \[
  (x^*(e) + \alpha)^2 + (x^*(e') - \alpha)^2 < (x^*(e))^2 + (x^*(e'))^2
  \]
  contradicting the optimality of \( x^* \).
Min-Norm Point and SFM

... proof of Thm. 16.7.1 cont.

- Then \((x^* + \alpha 1_e - \alpha 1_{e'}) (E)\)
  \[= x^* (E \setminus \{e, e'\}) + (x^*(e) + \alpha) + (x^*(e') - \alpha) = f(E).\]

  \[\underbrace{x^*_{\text{new}} (e)}_{x^*_{\text{new}} (e')} \]

- Minimality of \(x^* \in B_f\) in \(l^2\) sense requires that, with such an \(\alpha > 0\),
  \[
  \left( x^*(e) \right)^2 + \left( x^*(e') \right)^2 < \left( x^*_{\text{new}} (e) \right)^2 + \left( x^*_{\text{new}} (e') \right)^2
  \]

- Given that \(e \in A_-, x^*(e) < 0\). Thus, if \(x^*(e') > 0\), we could have
  \[
  (x^*(e) + \alpha)^2 + (x^*(e') - \alpha)^2 < (x^*(e))^2 + (x^*(e'))^2,
  \]
  contradicting the optimality of \(x^*\).

- If \(x^*(e') = 0\), we would have \( (x^*(e) + \alpha)^2 + (\alpha)^2 < (x^*(e))^2 \), for any
  \(0 < \alpha < |x^*(e)|\) (Exercise:), again contradicting the optimality of \(x^*\).
Min-Norm Point and SFM

... proof of Thm. 16.7.1 cont.

- Then 
  \[
  (x^* + \alpha 1_e - \alpha 1_{e'}) (E) \\
  = x^*(E \setminus \{e, e'\}) + (x^*(e) + \alpha) + (x^*(e') - \alpha) = f(E).
  \]

- Minimality of \( x^* \in B_f \) in \( l_2 \) sense requires that, with such an \( \alpha > 0 \),
  \[
  (x^*(e))^2 + (x^*(e'))^2 > (x_{new}(e))^2 + (x_{new}(e'))^2
  \]

- Given that \( e \in A_-, x^*(e) < 0 \). Thus, if \( x^*(e') > 0 \), we could have
  \[
  (x^*(e) + \alpha)^2 + (x^*(e') - \alpha)^2 < (x^*(e))^2 + (x^*(e'))^2,
  \]
  contradicting the optimality of \( x^* \).

- If \( x^*(e') = 0 \), we would have
  \[
  (x^*(e) + \alpha)^2 + (\alpha)^2 < (x^*(e))^2,
  \]
  for any \( 0 < \alpha < |x^*(e)| \) (Exercise:), again contradicting the optimality of \( x^* \).

- Thus, we must have \( x^*(e') < 0 \) (strict negativity).
Thus, for a pair \((e, e')\) with \(e' \in \text{dep}(x^*, e)\) and \(e \in A_-\), we have \(x(e') < 0\) and hence \(e' \in A_-\).
... proof of Thm. 16.7.1 cont.

- Thus, for a pair \((e, e')\) with \(e' \in \text{dep}(x^*, e)\) and \(e \in A_\rightarrow\), we have \(x(e') < 0\) and hence \(e' \in A_\rightarrow\).
- Hence, \(\forall e \in A_\rightarrow\), we have \(\text{dep}(x^*, e) \subseteq A_\rightarrow\).
... proof of Thm. 16.7.1 cont.

Thus, for a pair \((e, e')\) with \(e' \in \text{dep}(x^*, e)\) and \(e \in A_-\), we have \(x(e') < 0\) and hence \(e' \in A_-\).

Hence, \(\forall e \in A_-\), we have \(\text{dep}(x^*, e) \subseteq A_-\).

A very similar argument can show that, \(\forall e \in A_0\), we have \(\text{dep}(x^*, e) \subseteq A_0\).
Therefore, we have $\bigcup_{e \in A_-} \dep(x^*, e) = A_-$ and $\bigcup_{e \in A_0} \dep(x^*, e) = A_0$.
Therefore, we have $\bigcup_{e \in A_-} \text{dep}(x^*, e) = A_-$ and $\bigcup_{e \in A_0} \text{dep}(x^*, e) = A_0$

i.e., $\{\text{dep}(x^*, e)\}_{e \in A_-}$ is cover for $A_-$, as is $\{\text{dep}(x^*, e)\}_{e \in A_0}$ for $A_0$. 

Proof that min-norm gives optimal
Min-Norm Point and SFM

... proof of Thm. 16.7.1 cont.

- Therefore, we have $\bigcup_{e \in A_-} \text{dep}(x^*, e) = A_-$ and $\bigcup_{e \in A_0} \text{dep}(x^*, e) = A_0$

- I.e., $\{\text{dep}(x^*, e)\}_{e \in A_-}$ is cover for $A_-$, as is $\{\text{dep}(x^*, e)\}_{e \in A_0}$ for $A_0$.

- $\text{dep}(x^*, e)$ is minimal tight set containing $e$, meaning $x^*(\text{dep}(x^*, e)) = f(\text{dep}(x^*, e))$, and since tight sets are closed under union, we have that $A_-$ and $A_0$ are also tight, meaning:
Therefore, we have $\bigcup_{e \in A_-} \text{dep}(x^*, e) = A_-$ and $\bigcup_{e \in A_0} \text{dep}(x^*, e) = A_0$.

- \textbf{I.e.}, $\{\text{dep}(x^*, e)\}_{e \in A_-}$ is cover for $A_-$, as is $\{\text{dep}(x^*, e)\}_{e \in A_0}$ for $A_0$.
- $\text{dep}(x^*, e)$ is minimal tight set containing $e$, meaning $x^*(\text{dep}(x^*, e)) = f(\text{dep}(x^*, e))$, and since tight sets are closed under union, we have that $A_-$ and $A_0$ are also tight, meaning:

$$x^*(A_-) = f(A_-)$$  \hspace{1cm} (16.92)
Min-Norm Point and SFM

... proof of Thm. 16.7.1 cont.

- Therefore, we have $\bigcup_{e \in A_-} \text{dep}(x^*, e) = A_-$ and $\bigcup_{e \in A_0} \text{dep}(x^*, e) = A_0$
- i.e., $\{\text{dep}(x^*, e)\}_{e \in A_-}$ is cover for $A_-$, as is $\{\text{dep}(x^*, e)\}_{e \in A_0}$ for $A_0$.
- $\text{dep}(x^*, e)$ is minimal tight set containing $e$, meaning $x^*(\text{deg}(x^*, e)) = f(\text{deg}(x^*, e))$, and since tight sets are closed under union, we have that $A_-$ and $A_0$ are also tight, meaning:

  $$x^*(A_-) = f(A_-)$$  \hspace{1cm} (16.92)

  $$x^*(A_0) = f(A_0)$$  \hspace{1cm} (16.93)
... proof of Thm. 16.7.1 cont.

- Therefore, we have $\bigcup_{e \in A_-} \text{dep}(x^*, e) = A_-$ and $\bigcup_{e \in A_0} \text{dep}(x^*, e) = A_0$

- i.e., $\{\text{dep}(x^*, e)\}_{e \in A_-}$ is cover for $A_-$, as is $\{\text{dep}(x^*, e)\}_{e \in A_0}$ for $A_0$.

- $\text{dep}(x^*, e)$ is minimal tight set containing $e$, meaning

\[ x^*(\text{dep}(x^*, e)) = f(\text{dep}(x^*, e)) \]

- and since tight sets are closed under union, we have that $A_-$ and $A_0$ are also tight, meaning:

\[
\begin{align*}
    x^*(A_-) &= f(A_-) & (16.92) \\
    x^*(A_0) &= f(A_0) & (16.93) \\
    x^*(A_-) &= x^*(A_0) = y^*(E) = y^*(A_0) + y^*(E \setminus A_0) & (16.94) \\
    & \quad \uparrow = 0
\end{align*}
\]
Min-Norm Point and SFM

... proof of Thm. 16.7.1 cont.

- Therefore, we have $\bigcup_{e \in A_-} \text{dep}(x^*, e) = A_-$ and $\bigcup_{e \in A_0} \text{dep}(x^*, e) = A_0$
- i.e., $\{\text{dep}(x^*, e)\}_{e \in A_-}$ is cover for $A_-$, as is $\{\text{dep}(x^*, e)\}_{e \in A_0}$ for $A_0$.
- $\text{dep}(x^*, e)$ is minimal tight set containing $e$, meaning $x^*(\text{dep}(x^*, e)) = f(\text{dep}(x^*, e))$, and since tight sets are closed under union, we have that $A_-$ and $A_0$ are also tight, meaning:

\[
x^*(A_-) = f(A_-) \quad (16.92)
\]
\[
x^*(A_0) = f(A_0) \quad (16.93)
\]
\[
x^*(A_-) = x^*(A_0) = y^*(E) = y^*(A_0) + \left(y^*(E \setminus A_0) = 0\right) \quad (16.94)
\]

and therefore, all together we have
Therefore, we have $\bigcup_{e \in A_-} \text{dep}(x^*, e) = A_-$ and $\bigcup_{e \in A_0} \text{dep}(x^*, e) = A_0$.

i.e., $\{\text{dep}(x^*, e)\}_{e \in A_-}$ is cover for $A_-$, as is $\{\text{dep}(x^*, e)\}_{e \in A_0}$ for $A_0$.

$\text{dep}(x^*, e)$ is minimal tight set containing $e$, meaning $x^*(\text{dep}(x^*, e)) = f(\text{dep}(x^*, e))$, and since tight sets are closed under union, we have that $A_-$ and $A_0$ are also tight, meaning:

\[
x^*(A_-) = f(A_-) \quad \text{(16.92)}
\]
\[
x^*(A_0) = f(A_0) \quad \text{(16.93)}
\]
\[
x^*(A_-) = x^*(A_0) = y^*(E) = y^*(A_0) + y^*(E \setminus A_0) = 0 \quad \text{(16.94)}
\]

and therefore, all together we have

\[
f(A_-) = f(A_0) = x^*(A_-) = x^*(A_0) = y^*(E) \quad \text{(16.95)}
\]
Min-Norm Point and SFM

... proof of Thm. 16.7.1 cont.

• Now, $y^*$ is feasible for the l.h.s. of Eqn. (??).
Now, $y^*$ is feasible for the l.h.s. of Eqn. (??). This follows since, we have $y^* = x^* \land 0 \leq 0$, and since $x^* \in B_f \subset P_f$, and $y^* \leq x^*$ and $P_f$ is down-closed, we have that $y^* \in P_f$. 
Min-Norm Point and SFM

... proof of Thm. 16.7.1 cont.

- Now, $y^*$ is feasible for the l.h.s. of Eqn. (??). This follows since, we have $y^* = x^* \land 0 \leq 0$, and since $x^* \in B_f \subset P_f$, and $y^* \leq x^*$ and $P_f$ is down-closed, we have that $y^* \in P_f$.

- Also, for any $y \in P_f$ with $y \leq 0$ and for any $X \subseteq E$, we have $y(E) \leq y(X) \leq f(X)$. 

...
Now, \( y^\ast \) is feasible for the l.h.s. of Eqn. (??). This follows since, we have \( y^\ast = x^\ast \land 0 \leq 0 \), and since \( x^\ast \in B_f \subset P_f \), and \( y^\ast \leq x^\ast \) and \( P_f \) is down-closed, we have that \( y^\ast \in P_f \).

Also, for any \( y \in P_f \) with \( y \leq 0 \) and for any \( X \subseteq E \), we have \( y(E) \leq y(X) \leq f(X) \).

Hence, we have found a feasible for l.h.s. of Eqn. (??), \( y^\ast \leq 0 \), \( y^\ast \in P_f \), so \( y^\ast(E) \leq f(X) \) for all \( X \).
Now, \( y^* \) is feasible for the l.h.s. of Eqn. (??). This follows since, we have \( y^* = x^* \land 0 \leq 0 \), and since \( x^* \in B_f \subset P_f \), and \( y^* \leq x^* \) and \( P_f \) is down-closed, we have that \( y^* \in P_f \).

Also, for any \( y \in P_f \) with \( y \leq 0 \) and for any \( X \subseteq E \), we have \( y(E) \leq y(X) \leq f(X) \).

Hence, we have found a feasible for l.h.s. of Eqn. (??), \( y^* \leq 0 \), \( y^* \in P_f \), so \( y^*(E) \leq f(X) \) for all \( X \).

So \( y^*(E) \leq \min \{ f(X) | X \subseteq V \} \).
Now, $y^*$ is feasible for the l.h.s. of Eqn. (??). This follows since, we have $y^* = x^* ∧ 0 ≤ 0$, and since $x^* ∈ B_f \subseteq P_f$, and $y^* ≤ x^*$ and $P_f$ is down-closed, we have that $y^* ∈ P_f$.

Also, for any $y ∈ P_f$ with $y ≤ 0$ and for any $X ⊆ E$, we have $y(E) ≤ y(X) ≤ f(X)$.

Hence, we have found a feasible for l.h.s. of Eqn. (??), $y^* ≤ 0$, $y^* ∈ P_f$, so $y^*(E) ≤ f(X)$ for all $X$.

So $y^*(E) ≤ \min \{f(X) | X ⊆ V\}$.

Considering Eqn. (16.92), we have found sets $A_-$ and $A_0$ with tightness in Eqn. (??), meaning $y^*(E') = f(A_-) = f(A_0)$. 

...
...proof of Thm. 16.7.1 cont.

Now, $y^*$ is feasible for the l.h.s. of Eqn. (??). This follows since, we have $y^* = x^* \land 0 \leq 0$, and since $x^* \in B_f \subset P_f$, and $y^* \leq x^*$ and $P_f$ is down-closed, we have that $y^* \in P_f$.

Also, for any $y \in P_f$ with $y \leq 0$ and for any $X \subseteq E$, we have $y(E) \leq y(X) \leq f(X)$.

Hence, we have found a feasible for l.h.s. of Eqn. (??), $y^* \leq 0$, $y^* \in P_f$, so $y^*(E) \leq f(X)$ for all $X$.

So $y^*(E) \leq \min \{ f(X) | X \subseteq V \}$.

Considering Eqn. (16.92), we have found sets $A_-$ and $A_0$ with tightness in Eqn. (??), meaning $y^*(E) = f(A_-) = f(A_0)$.

Hence, $y^*$ is a maximizer of l.h.s. of Eqn. (??), and $A_-$ and $A_0$ are minimizers of $f$. 

...
Now, for any $X \subset A_-$, we have

$$f(X) \geq x^*(X) > x^*(A_-) = f(A_-)$$  \hspace{1cm} (16.96)
... proof of Thm. 16.7.1 cont.

- Now, for any $X \subset A_-$, we have

$$f(X) \geq x^*(X) > x^*(A_-) = f(A_-) \quad (16.96)$$

- And for any $X \supset A_0$, we have

$$f(X) \geq x^*(X) > x^*(A_0) = f(A_0) \quad (16.97)$$
... proof of Thm. 16.7.1 cont.

- Now, for any $X \subset A_-$, we have

$$f(X) \geq x^*(X) > x^*(A_-) = f(A_-) \tag{16.96}$$

- And for any $X \supset A_0$, we have

$$f(X) \geq x^*(X) > x^*(A_0) = f(A_0) \tag{16.97}$$

- Hence, $A_-$ must be the unique minimal minimizer of $f$, and $A_0$ is the unique maximal minimizer of $f$. 
Min-Norm Point and SFM

- So, if we have a procedure to compute the min-norm point computation, we can solve SFM.
Min-Norm Point and SFM

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- Nice thing about previous proof is that it uses both expressions for dep for different purposes.
Min-Norm Point and SFM

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So, if we have a procedure to compute the min-norm point computation, we can solve SFM.

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An algorithm (by F. Wolfe) can find this min-norm point, essentially an active-set procedure for quadratic programming. It uses Edmonds’s greedy algorithm to make it efficient.
Min-Norm Point and SFM

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- An algorithm (by F. Wolfe) can find this min-norm point, essentially an active-set procedure for quadratic programming. It uses Edmonds’s greedy algorithm to make it efficient.
- This is currently the best practical algorithm for general purpose submodular function minimization.
So, if we have a procedure to compute the min-norm point computation, we can solve SFM.

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An algorithm (by F. Wolfe) can find this min-norm point, essentially an active-set procedure for quadratic programming. It uses Edmonds’s greedy algorithm to make it efficient.

This is currently the best practical algorithm for general purpose submodular function minimization.

But its underlying lower-bound complexity is unknown, although in practice its estimated empirical complexity runs anywhere from \( O(n^3) \) to \( O(n^{4.5}) \) or so (see Jegelka, Lin, Bilmes (NIPS 2011)).
Min-norm point and other minimizers of $f$

- Recall, that the set of minimizers of $f$ forms a lattice.
Recall, that the set of minimizers of $f$ forms a lattice.

In fact, with $x^*$ the min-norm point, and $A_-$ and $A_0$ as defined above, we have the following theorem:
Min-norm point and other minimizers of $f$

- Recall, that the set of minimizers of $f$ forms a lattice.
- In fact, with $x^*$ the min-norm point, and $A_-$ and $A_0$ as defined above, we have the following theorem:

**Theorem 16.7.2**

Let $A \subseteq E$ be any minimizer of submodular $f$, and let $x^*$ be the minimum-norm point. Then $A$ has the form:

$$A = A_- \cup \bigcup_{a \in A_m} \text{dep}(x^*, a)$$  \hspace{1cm} (16.98)

for some set $A_m \subseteq A_0 \setminus A_-$. 
Min-norm point and other minimizers of $f$

Proof of Thm. 16.7.2.

- If $A$ is a minimizer, then $A_- \subseteq A \subseteq A_0$, and $f(A) = y^*(E)$ is the minimum valuation of $f$. 
Min-norm point and other minimizers of $f$

**proof of Thm. 16.7.2.**

- **If** $A$ **is a minimizer,** then $A^- \subseteq A \subseteq A_0,$ **and** $f(A) = y^*(E)$ **is the minimum valuation of** $f.$

- **But** $x^* \in P_f,$ so $x^*(A) \leq f(A)$ **and** $f(A) = x^*(A^-) \leq x^*(A)$ **(or alternatively, just note that** $x^*(A_0 \setminus A) = 0).$
Min-norm point and other minimizers of $f$

Proof of Thm. 16.7.2.

- If $A$ is a minimizer, then $A_\bot \subseteq A \subseteq A_0$, and $f(A) = y^*(E)$ is the minimum valuation of $f$.
- But $x^* \in P_f$, so $x^*(A) \leq f(A)$ and $f(A) = x^*(A_\bot) \leq x^*(A)$ (or alternatively, just note that $x^*(A_0 \setminus A) = 0$).
- Hence, $x^*(A) = x^*(A_\bot) = f(A)$ so that $A$ is also a tight set for $x^*$. 
proof of Thm. 16.7.2.

- If \( A \) is a minimizer, then \( A_- \subseteq A \subseteq A_0 \), and \( f(A) = y^*(E) \) is the minimum valuation of \( f \).
- But \( x^* \in P_f \), so \( x^*(A) \leq f(A) \) and \( f(A) = x^*(A_-) \leq x^*(A) \) (or alternatively, just note that \( x^*(A_0 \setminus A) = 0 \)).
- Hence, \( x^*(A) = x^*(A_-) = f(A) \) so that \( A \) is also a tight set for \( x^* \).
- For any \( a \in A \), \( A \) is a tight set containing \( a \), and \( \text{dep}(x^*, a) \) is the minimal tight containing \( a \).
Min-norm point and other minimizers of $f$

**proof of Thm. 16.7.2.**

- If $A$ is a minimizer, then $A_\bot \subseteq A \subseteq A_0$, and $f(A) = y^*(E)$ is the minimum valuation of $f$.

- But $x^* \in P_f$, so $x^*(A) \leq f(A)$ and $f(A) = x^*(A_\bot) \leq x^*(A)$ (or alternatively, just note that $x^*(A_0 \setminus A) = 0$).

- Hence, $x^*(A) = x^*(A_\bot) = f(A)$ so that $A$ is also a tight set for $x^*$.

- For any $a \in A$, $A$ is a tight set containing $a$, and $\text{dep}(x^*, a)$ is the minimal tight containing $a$.

- Hence, for any $a \in A$, $\text{dep}(x^*, a) \subseteq A$. 

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proof of Thm. 16.7.2.

- If $A$ is a minimizer, then $A_\perp \subseteq A \subseteq A_0$, and $f(A) = y^*(E)$ is the minimum valuation of $f$.
- But $x^* \in P_f$, so $x^*(A) \leq f(A)$ and $f(A) = x^*(A_\perp) \leq x^*(A)$ (or alternatively, just note that $x^*(A_0 \setminus A) = 0$).
- Hence, $x^*(A) = x^*(A_\perp) = f(A)$ so that $A$ is also a tight set for $x^*$.
- For any $a \in A$, $A$ is a tight set containing $a$, and $\text{dep}(x^*, a)$ is the minimal tight containing $a$.
- Hence, for any $a \in A$, $\text{dep}(x^*, a) \subseteq A$.
- This means that $\bigcup_{a \in A} \text{dep}(x^*, a) = A$. 

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Min-norm point and other minimizers of $f$

**proof of Thm. 16.7.2.**

- If $A$ is a minimizer, then $A_- \subseteq A \subseteq A_0$, and $f(A) = y^*(E)$ is the minimum valuation of $f$.

- But $x^* \in P_f$, so $x^*(A) \leq f(A)$ and $f(A) = x^*(A_-) \leq x^*(A)$ (or alternatively, just note that $x^*(A_0 \setminus A) = 0$).

- Hence, $x^*(A) = x^*(A_-) = f(A)$ so that $A$ is also a tight set for $x^*$.

- For any $a \in A$, $A$ is a tight set containing $a$, and $\text{dep}(x^*, a)$ is the minimal tight containing $a$.

- Hence, for any $a \in A$, $\text{dep}(x^*, a) \subseteq A$.

- This means that $\bigcup_{a \in A} \text{dep}(x^*, a) = A$.

- Since $A_- \subseteq A \subseteq A_0$, then $\exists A_m \subseteq A \setminus A_-$ such that

$$A = \bigcup_{a \in A_-} \text{dep}(x^*, a) \cup \bigcup_{a \in A_m} \text{dep}(x^*, a) = A_- \cup \bigcup_{a \in A_m} \text{dep}(x^*, a)$$
On a unique minimizer \( f \)

- Note that if \( f(e|A) > 0, \forall A \subseteq E \) and \( e \in E \setminus A \), then we have \( A_- = A_0 \) (there is one unique minimizer).
On a unique minimizer $f$

- Note that if $f(e|A) > 0$, $\forall A \subseteq E$ and $e \in E \setminus A$, then we have $A_\neg = A_0$ (there is one unique minimizer).

- On the other hand, if $A_\neg = A_0$, it does not imply $f(e|A) > 0$ for all $A \subseteq E \setminus \{e\}$.
On a unique minimizer $f$

- Note that if $f(e|A) > 0$, $\forall A \subseteq E$ and $e \in E \setminus A$, then we have $A_- = A_0$ (there is one unique minimizer).
- On the other hand, if $A_- = A_0$, it does not imply $f(e|A) > 0$ for all $A \subseteq E \setminus \{e\}$.
- If $A_- = A_0$ then certainly $f(e|A_0) > 0$ for $e \in E \setminus A_0$ and $-f(e|A_0 \setminus \{e\}) > 0$ for all $e \in A_0$. 