Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 15 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee596b_spring_2016/

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May 22nd, 2016

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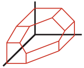
$= f(A_r) + 2f(C) + f(B_r)$ $\quad = f(A_r) + f(C) + f(B_r)$ $\quad = f(A \cap B)$



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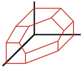


Cumulative Outstanding Reading

- Read chapters 2 and 3 from Fujishige's book.
- Read chapter 1 from Fujishige's book.

Announcements, Assignments, and Reminders

- Homework 4, available at our assignment dropbox (https://canvas.uw.edu/courses/1039754/assignments), due (electronically) Wednesday (5/25) at 11:55pm.
- Homework 3, available at our assignment dropbox (https://canvas.uw.edu/courses/1039754/assignments), due (electronically) Monday (5/2) at 11:55pm.
- Homework 2, available at our assignment dropbox (https://canvas.uw.edu/courses/1039754/assignments), due (electronically) Monday (4/18) at 11:55pm.
- Homework 1, available at our assignment dropbox (https://canvas.uw.edu/courses/1039754/assignments), due (electronically) Friday (4/8) at 11:55pm.
- Weekly Office Hours: Mondays, 3:30-4:30, or by skype or google hangout (set up meeting via our our discussion board (https://canvas.uw.edu/courses/1039754/discussion_topics)).

Class Road Map - IT-I

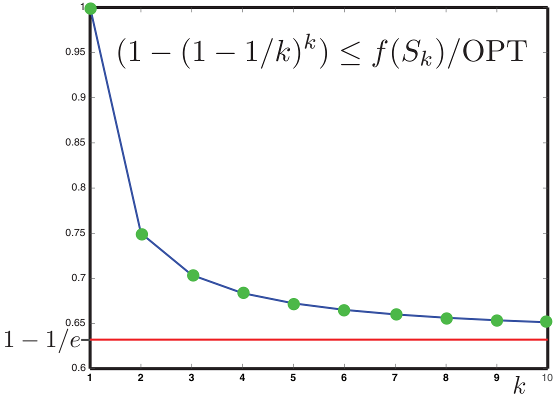
- L1(3/28): Motivation, Applications, & Basic Definitions
- L2(3/30): Machine Learning Apps (diversity, complexity, parameter, learning target, surrogate).
- L3(4/4): Info theory exs, more apps, definitions, graph/combinatorial examples, matrix rank example, visualization
- L4(4/6): Graph and Combinatorial Examples, matrix rank, Venn diagrams, examples of proofs of submodularity, some useful properties
- L5(4/11): Examples & Properties, Other Defs., Independence
- L6(4/13): Independence, Matroids, Matroid Examples, matroid rank is submodular
- L7(4/18): Matroid Rank, More on Partition Matroid, System of Distinct Reps, Transversals, Transversal Matroid,
- L8(4/20): Transversals, Matroid and representation, Dual Matroids,
- L9(4/25): Dual Matroids, Properties, Combinatorial Geometries, Matroid and Greedy
- L10(4/27): Matroid and Greedy, Polyhedra, Matroid Polytopes,
- L11(5/2): From Matroids to Polymatroids, Polymatroids
- L12(5/4): Polymatroids, Polymatroids and Greedy
- L13(5/9): Polymatroids and Greedy; Possible Polytopes; Extreme Points; Polymatroids, Greedy, and Cardinality Constrained Maximization
- L14(5/11): Cardinality Constrained Maximization; Curvature; Submodular Max w. Other Constraints
- L15(5/16): Submodular Max w. Other Constraints, Most Violated $\leq$, Matroids cont., Closure/Sat, Closure/Sat, Fund. Circuit/Dep
- L16(5/18):
- L17(5/23):
- L18(5/25):
- L19(6/1):
- L20(6/6): Final Presentations maximization.

Finals Week: June 6th-10th, 2016.

The Greedy Algorithm: $1 - 1/e$ intuition.

- At step $i < k$, greedy chooses v_i to maximize $f(v|S_i)$.
- Let S^* be optimal solution (of size k) and OPT $= f(S^*)$. By submodularity, we will show:

$$\exists v \in V \setminus S_i : f(v|S_i) = f(S_i + v|S_i) \geq \frac{1}{k}(\text{OPT} - f(S_i)) \qquad (15.1)$$



Equation (**??**) will show that Equation (**??**) $\Rightarrow$:

$$\begin{aligned} &\text{OPT} - f(S_{i+1}) \\ &\qquad \leq (1-1/k)(\text{OPT} - f(S_i)) \\ &\Rightarrow \text{OPT} - f(S_k) \\ &\qquad \leq (1-1/k)^k \text{OPT} \\ &\qquad \leq 1/e\text{OPT} \\ &\Rightarrow \text{OPT}(1-1/e) \leq f(S_k) \end{aligned}$$

Priority Queue

- Use a priority queue Q as a data structure: operations include:
  - Insert an item (v, α) into queue, with $v \in V$ and $\alpha \in \mathbb{R}$.
$$\text{INSERT}(Q, (v, \alpha)) \tag{15.14}$$
  - Pop the item (v, α) with maximum value α off the queue.
$$(v, \alpha) \leftarrow \text{POP}(Q) \tag{15.15}$$
  - Query the value of the max item in the queue
$$\text{MAX}(Q) \in \mathbb{R} \tag{15.16}$$
- On next slide, we call a popped item "fresh" if the value (v, α) popped has the correct value $\alpha = f(v|S_i)$. Use extra "bit" to store this info
- If a popped item is fresh, it must be the maximum — this can happen if, at given iteration, v was first popped and neither fresh nor maximum so placed back in the queue, and it then percolates back to the top at which point it is fresh — thereby avoid extra queue check.

Minoux's Accelerated Greedy Algorithm Submodular Max

Algorithm 3: Minoux's Accelerated Greedy Algorithm

```
1  Set S_0 ← ∅ ; i ← 0 ; Initialize priority queue Q ;
2  for v ∈ E do
3      INSERT(Q, f(v))
4  repeat
5      (v, α) ← POP(Q) ;
6      if α not "fresh" then
7          recompute α ← f(v|S_i)
8      if (popped α in line 5 was "fresh") OR (α ≥ MAX(Q)) then
9          Set S_{i+1} ← S_i ∪ {v} ;
10         i ← i + 1 ;
11     else
12         INSERT(Q, (v, α))
13 until i = |E|;
```

(Minimum) Submodular Set Cover

- Given polymatroid f, goal is to find a covering set of minimum cost:

$$S^* \in \operatorname*{argmin}_{S \subseteq V} |S| \text{ such that } f(S) \geq \alpha \tag{15.14}$$

where α is a "cover" requirement.

- Normally take $\alpha = f(V)$ but defining $f'(A) = \min\{f(A), \alpha\}$ we can take any α. Hence, we have equivalent formulation:

$$S^* \in \operatorname*{argmin}_{S \subseteq V} |S| \text{ such that } f'(S) \geq f'(V) \tag{15.15}$$

- Note that this immediately generalizes standard set cover, in which case $f(A)$ is the cardinality of the union of sets indexed by A.
- Greedy Algorithm: Pick the first chain item S_i chosen by aforementioned greedy algorithm such that $f(S_i) \geq \alpha$ and output that as solution.

(Minimum) Submodular Set Cover: Approximation Analysis

- For integer valued f, this greedy algorithm an $O(\log(\max_{s \in V} f(\{s\})))$ approximation. Let S^* be optimal, and S^{G} be greedy solution, then

$$|S^{\mathsf{G}}| \leq |S^*| H(\max_{s \in V} f(\{s\})) = |S^*| O(\log_e(\max_{s \in V} f(\{s\}))) \tag{15.14}$$

where H is the harmonic function, i.e., $H(d) = \sum_{i=1}^{d} (1/i)$.

- If f is not integral value, then bounds we get are of the form:

$$|S^{\mathsf{G}}| \leq |S^*| \left(1 + \log_e \frac{f(V)}{f(V) - f(S_{T-1})}\right) \tag{15.15}$$

wehre S_T is the final greedy solution that occurs at step T.

- Set cover is hard to approximate with a factor better than $(1 - \epsilon) \log \alpha$, where α is the desired cover constraint.

Curvature of a Submodular function

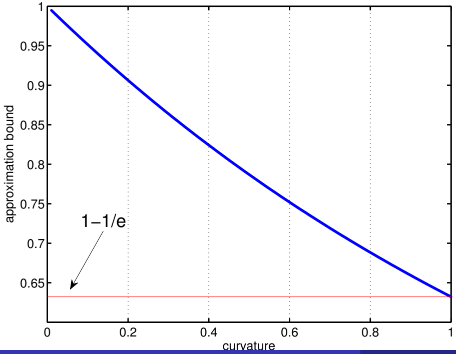
- By submodularity, total curvature can be computed in either form:

$$c \triangleq 1 - \min_{S, j \notin S : f(j|\emptyset) \neq 0} \frac{f(j|S)}{f(j|\emptyset)} = 1 - \min_{j : f(j|\emptyset) \neq 0} \frac{f(j|V \setminus \{j\})}{f(j|\emptyset)} \quad (15.17)$$

- Note: Matroid rank is either modular $c = 0$ or maximally curved $c = 1$ — hence, matroid rank can have only the extreme points of curvature, namely 0 or 1.
- Polymatroid functions are, in this sense, more nuanced, in that they allow non-extreme curvature, with $c \in [0, 1]$.
- It will be remembered the notion of "partial dependence" within polymatroid functions.

Curvature and approximation

- Curvature limitation can help the greedy algorithm in terms of approximation bounds.
- Conforti & Cornuéjols showed that greedy gives a $1/(1+c)$ approximation to $\max\{f(S) : S \in \mathcal{I}\}$ when f has total curvature c.
- Hence, greedy subject to matroid constraint is a $\max(1/(1+c), 1/2)$ approximation algorithm, and if $c < 1$ then it is better than 1/2 (e.g., with $c = 1/4$ then we have a 0.8 algorithm).
- For k-uniform matroid (i.e., k-cardinality constraints), then approximation factor becomes $\frac{1}{c}(1 - e^{-c})$



Greedy over multiple matroids

- Obvious heuristic is to use the greedy step but always stay feasible.
- I.e., Starting with $S_0 = \emptyset$, we repeat the following greedy step

$$S_{i+1} = S_i \cup \left\{ \underset{v \in V \setminus S_i \,:\, S_i + v \in \bigcap_{i=1}^p \mathcal{I}_i}{\operatorname{argmax}} f(S_i \cup \{v\}) \right\} \tag{15.17}$$

- That is, we keep choosing next whatever feasible element looks best.
- This algorithm is simple and also has a guarantee

Theorem 15.2.2

Given a polymatroid function f, and set of matroids $\{M_j = (E, \mathcal{I}_j)\}_{j=1}^p$, the above greedy algorithm returns sets S_i such that for each i we have $f(S_i) \geq \frac{1}{p+1} \max_{|S| \leq i, S \in \bigcap_{i=1}^p \mathcal{I}_i} f(S)$, assuming such sets exists.

- For one matroid, we have a 1/2 approximation.
- Very easy algorithm, Minoux trick still possible, while addresses multiple matroid constraints — but the bound is not that good when there are many matroids.

Greedy over multiple matroids: Generalized Bipartite Matching

- Generalized bipartite matching (i.e., max bipartite matching with submodular costs on the edges). Use two partition matroids (as mentioned earlier in class)

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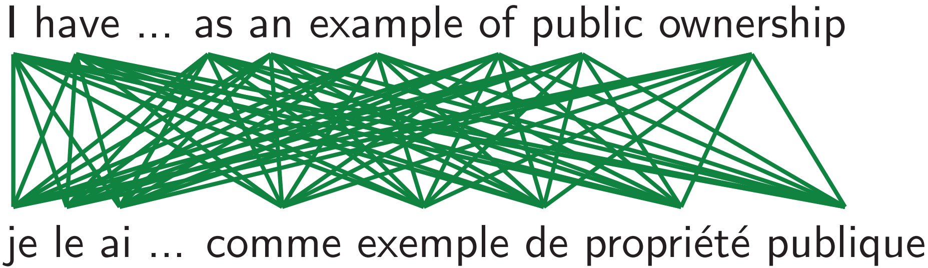
- Generalized bipartite matching (i.e., max bipartite matching with submodular costs on the edges). Use two partition matroids (as mentioned earlier in class)
- Useful in natural language processing: Ex. Computer language translation, find an alignment between two language strings.
- Consider bipartite graph $G = (E, F, V)$ where E and F are the left/right set of nodes, respectively, and V is the set of edges.
- E corresponds to, say, an English language sentence and F corresponds to a French language sentence — goal is to form a matching (an alignment) between the two.

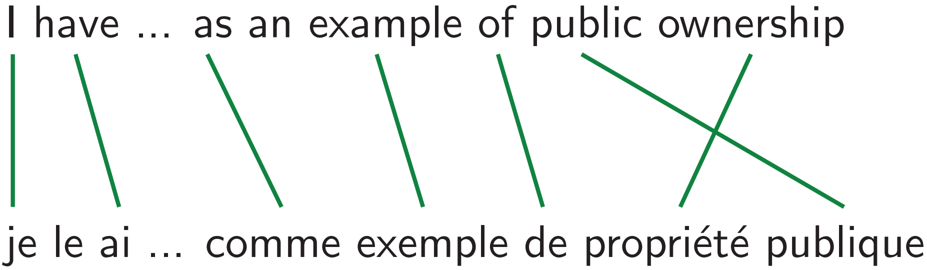
Greedy over > 1 matroids: Multiple Language Alignment

- Consider English string and French string, set up as a bipartite graph.



Greedy over > 1 matroids: Multiple Language Alignment

- One possible alignment, a matching, with score as sum of edge weights.



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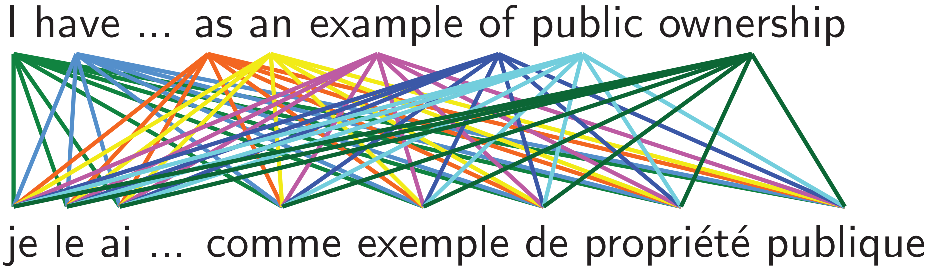
- Edges incident to English words constitute an edge partition



- The two edge partitions can be used to set up two 1-partition matroids on the edges.
- For each matroid, a set of edges is independent only if the set intersects each partition block no more than one time.

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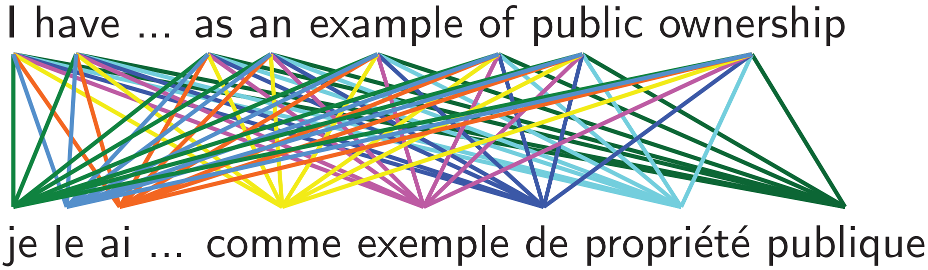
- Edges incident to French words constitute an edge partition



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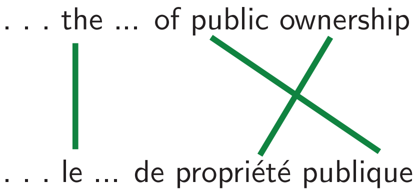
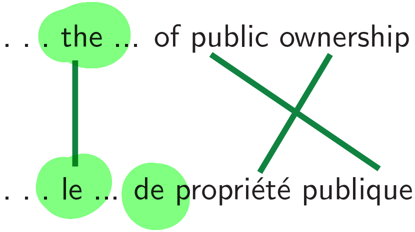
Fertility at most 1





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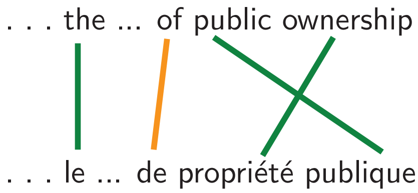
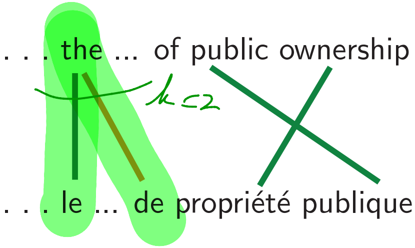
Fertility at most 2





Greedy over > 1 matroids: Multiple Language Alignment

- Generalizing further, each block of edges in each partition matroid can have its own "fertility" limit:
  $\mathcal{I} = \{X \subseteq V : |X \cap V_i| \leq k_i$ for all $i = 1, \ldots, \ell\}$.

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- Maximizing submodular function subject to multiple matroid constraints addresses this problem.

Greedy over multiple matroids: Submodular Welfare

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- Goal of submodular welfare: Partition the goods $E = E_1 \cup E_2 \cup \cdots \cup E_n$ into n blocks in order to maximize the submodular social welfare, measured as:

$$\text{submodular-social-welfare}(E_1, E_2, \ldots, E_n) = \sum_{i=1}^{n} g_i(E_i). \tag{15.1}$$

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- We can solve this via submodular maximization subject to multiple matroid independence constraints as we next describe ...

Submodular Welfare: Submodular Max over matroid partition

- Create new ground set E' as disjoint union of n copies of the ground set. I.e.,

$$E' = \underbrace{E \uplus E \uplus \cdots \uplus E}_{n\times} \tag{15.2}$$

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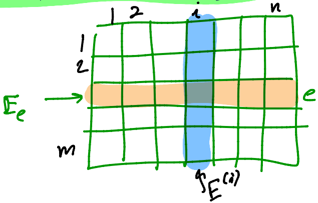

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- Hence, $\{E_e\}_{e \in E}$ is a partition of E', each block of the partition for one of the original elements in E.

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- Hence, $\{E_e\}_{e \in E}$ is a partition of E', each block of the partition for one of the original elements in E.
- Create a 1-partition matroid $\mathcal{M} = (E', \mathcal{I})$ where

$$\mathcal{I} = \left\{ S \subseteq E' : \forall e \in E, |S \cap E_e| \le 1 \right\} \tag{15.3}$$

Submodular Welfare: Submodular Max over matroid partition

- Hence, S is independent in matroid $\mathcal{M} = (E', I)$ if S uses each original element no more than once.

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- Create submodular function $f' : 2^{E'} \to \mathbb{R}_+$ with $f'(S) = \sum_{i=1}^n g_i(S \cap E^{(i)})$.

$S \subseteq E'$

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- Create submodular function $f' : 2^{E'} \to \mathbb{R}_+$ with $f'(S) = \sum_{i=1}^{n} g_i(S \cap E^{(i)})$.
- Submodular welfare maximization becomes matroid constrained submodular max $\max \{f'(S) : S \in \mathcal{I}\}$, so greedy algorithm gives a $1/2$ approximation.

Submodular Social Welfare

- Have $n = 6$ people (who don't like to share) and $|E| = m = 7$ pieces of sushi. E.g., $e \in E$ might be $e =$ "salmon roll".

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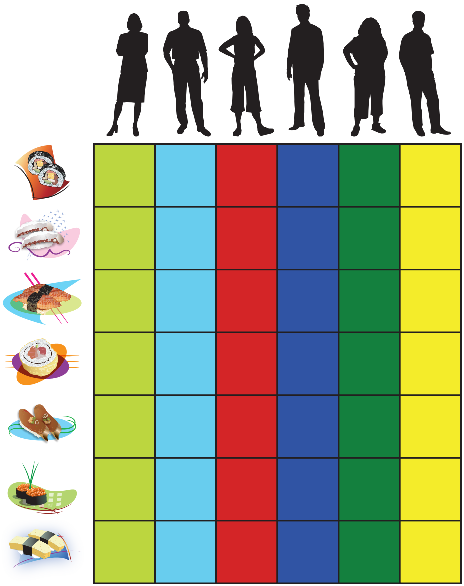

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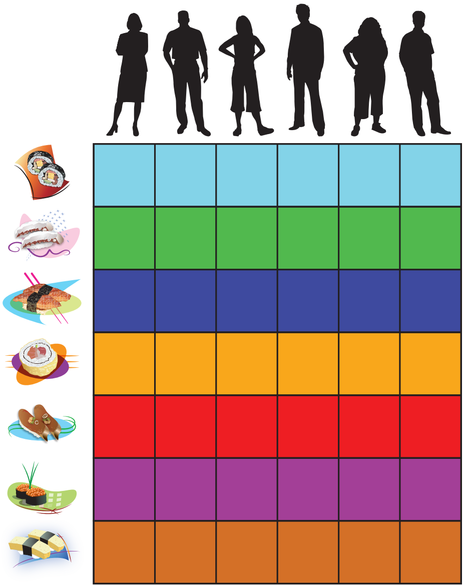
Submodular Social Welfare



- Have $n = 6$ people (who don't like to share) and $|E| = m = 7$ pieces of sushi. E.g., $e \in E$ might be $e =$ "salmon roll".
- Goal: distribute sushi to people to maximize social welfare.
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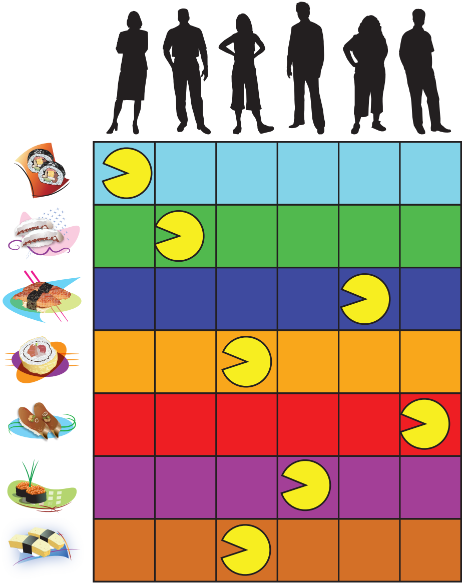

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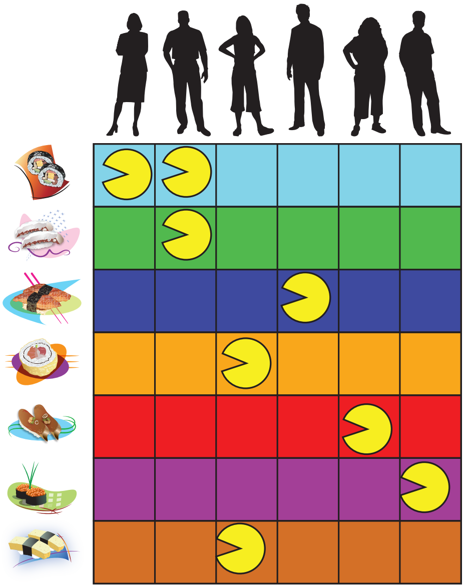

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- Important: A knapsack constraint yields an independence system (down closed) but it is not a matroid!
- $c(e)$ may be seen as the cost of item e and if $c(e) = 1$ for all e, then we recover the cardinality constraint we saw earlier.

Monotone Submodular over Knapsack Constraint

- Greedy can be seen as choosing the best gain: Starting with $S_0 = \emptyset$, we repeat the following greedy step

$$S_{i+1} = S_i \cup \left\{ \underset{v \in V \setminus S_i}{\operatorname{argmax}} \Big(f(S_i \cup \{v\}) - f(S_i) \Big) \right\} \tag{15.4}$$

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- Core idea in knapsack case: Greedy can be extended to choose next whatever looks cost-normalized best, i.e., Starting some initial set S_0, we repeat the following cost-normalized greedy step

$$S_{i+1} = S_i \cup \left\{ \operatorname*{argmax}_{v \in V \setminus S_i} \frac{f(S_i \cup \{v\}) - f(S_i)}{c(v)} \right\} \tag{15.5}$$

  which we repeat until $c(S_{i+1}) > b$ and then take S_i as the solution.

A Knapsack Constraint

- There are a number of ways of getting approximation bounds using this strategy.
- If we run the normalized greedy procedure starting with $S_0 = \emptyset$, and compare the solution found with the max of the singletons $\max_{v \in V} f(\{v\})$, choosing the max, then we get a $(1 - e^{-1/2}) \approx 0.39$ approximation, in $O(n^2)$ time (Minoux trick also possible for further speed)
- Partial enumeration: On the other hand, we can get a $(1 - e^{-1}) \approx 0.63$ approximation in $O(n^5)$ time if we run the above procedure starting from all sets of cardinality three (so restart for all S_0 such that $|S_0| = 3$), and compare that with the best singleton and pairwise solution.
- Extending something similar to this to d simultaneous knapsack constraints is possible as well.

Local Search Algorithms

From J. Vondrak

- Local search involves switching up to t elements, as long as it provides a (non-trivial) improvement; can iterate in several phases. Some examples follow:
- 1/3 approximation to unconstrained non-monotone maximization [Feige, Mirrokni, Vondrak, 2007]
- $1/(k+2+\frac{1}{k}+\delta_t)$ approximation for non-monotone maximization subject to k matroids [Lee, Mirrokni, Nagarajan, Sviridenko, 2009]
- $1/(k+\delta_t)$ approximation for monotone submodular maximization subject to $k \geq 2$ matroids [Lee, Sviridenko, Vondrak, 2010].

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$$f(\tilde{S}) \geq \alpha \cdot f(S^{opt})$$

$$\alpha(n) \geq 0$$

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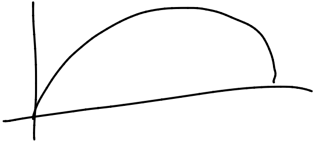

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- Thus, any approximation algorithm must be for unipolar submodular functions. E.g., non-negative but otherwise arbitrary submodular functions.
- We may get a $(\frac{1}{3} - \frac{\epsilon}{n})$ approximation for maximizing non-monotone non-negative submodular functions, with most $O(\frac{1}{\epsilon}n^3 \log n)$ function calls using approximate local maxima.

Submodularity and local optima

- Given any submodular function f, a set $S \subseteq V$ is a local maximum of f if $f(S - v) \leq f(S)$ for all $v \in S$ and $f(S + v) \leq f(S)$ for all $v \in V \setminus S$ (i.e., local in a Hamming ball of radius 1).

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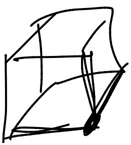

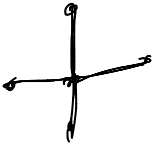

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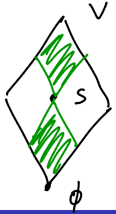

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- This is the approach that yields the $(\frac{1}{3} - \frac{\epsilon}{n})$ approximation algorithm.

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Algorithm 7: Randomized Linear-time non-monotone submodular max

Set $L \leftarrow \emptyset$; $U \leftarrow V$ /* Lower L, upper U. Invariant: $L \subseteq U$ */ ;
Order elements of $V = (v_1, v_2, \ldots, v_n)$ arbitrarily ;
for $i \leftarrow 0 \ldots |V|$ **do**
  $a \leftarrow [f(v_i|L)]_+$; $b \leftarrow [-f(U|U \setminus \{v_i\})]_+$;
  if $a = b = 0$ **then** $p \leftarrow 1/2$;
  ;
  else $p \leftarrow a/(a+b)$;
  ;
  if *Flip of coin with* $\Pr($*heads*$) = p$ *draws heads* **then**
    $L \leftarrow L \cup \{v_i\}$;
  Otherwise /* if the coin drew tails, an event with prob. $1 - p$ */
    $U \leftarrow U \setminus \{v\}$
return L *(which is the same as* U *at this point)*

$L = U$

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- It may be possible to choose the random order smartly to get better results in practice.

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- Often the computational costs of the algorithms are prohibitive (e.g., exponential in k) with large constants, so these algorithms might not scale.
- On the other hand, these algorithms offer deep and interesting intuition into submodular functions, beyond what we have covered here.

Some results on submodular maximization

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- More general constraints are possible too, as we see on the next table (for references, see Jan Vondrak's publications `http://theory.stanford.edu/~jvondrak/`).

Submodular Max Summary - 2012: From J. Vondrak

Monotone Maximization

Constraint	Approximation	Hardness	Technique
$\lvert S\rvert \leq k$	$1 - 1/e$	$1 - 1/e$	greedy
matroid	$1 - 1/e$	$1 - 1/e$	multilinear ext.
$O(1)$ knapsacks	$1 - 1/e$	$1 - 1/e$	multilinear ext.
k matroids	$k + \epsilon$	$k/\log k$	local search
k matroids and $O(1)$ knapsacks	$O(k)$	$k/\log k$	multilinear ext.

Nonmonotone Maximization

Constraint	Approximation	Hardness	Technique
Unconstrained	$1/2$	$1/2$	combinatorial
matroid	$1/e$	0.48	multilinear ext.
$O(1)$ knapsacks	$1/e$	0.49	multilinear ext.
k matroids	$k + O(1)$	$k/\log k$	local search
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- A paper by Chekuri, Vondrak, and Zenklusen (2011) make some progress on this front using multilinear extensions.

Multilinear extension

Definition 15.3.2

For a set function $f : 2^V \to \mathbb{R}$, define its multilinear extension $F : [0,1]^V \to \mathbb{R}$ by

$$F(x) = \sum_{S \subseteq V} f(S) \prod_{i \in S} x_i \prod_{j \in V \setminus S} (1 - x_j) \tag{15.6}$$

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$x = 1_A \quad F(1_A) = f(A)$

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- While this is defined for any set function, we have:

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- Note that $F(x) = Ef(\hat{x})$ where $\hat{x}$ is a random binary vector over $\{0,1\}^V$ with elements independent w. probability x_i for $\hat{x}_i$.
- While this is defined for any set function, we have:

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- Another connection between submodularity and convexity/concavity
- but note, unlike the Lovász extension, this function is neither.

Submodular Max and polyhedral approaches

- Basic idea: Given a set of constraints $\mathcal{I}$, we form a polytope $P_{\mathcal{I}}$ such that $\{\mathbf{1}_I : I \in \mathcal{I}\} \subseteq P_{\mathcal{I}}$
- We find $\max_{x \in P_{\mathcal{I}}} F(x)$ where $F(x)$ is the multi-linear extension of f, to find a fractional solution x^*
- We then round x^* to a point on the hypercube, thus giving us a solution to the discrete problem.

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- Also, Vondrak showed that this scheme achieves the $\frac{1}{c}(1 - e^{-c})$ curvature based bound for any matroid, which matches the bound we had earlier for uniform matroids with standard greedy.
- In general, one needs to do Monte-Carlo methods to estimate the multilinear extension (so further approximations would apply).

Review from lecture 11

The next slide comes from lecture 11.

A polymatroid function's polyhedron is a polymatroid.

Theorem 15.4.1

Let f be a polymatroid function defined on subsets of E. For any $x \in \mathbb{R}_+^E$, and any P_f^+-basis $y^x \in \mathbb{R}_+^E$ of x, the component sum of y^x is

$$y^x(E) = \text{rank}(x) = \max\left(y(E) : y \leq x, y \in P_f^+\right)$$
$$= \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \quad (15.10)$$

As a consequence, P_f^+ is a polymatroid, since r.h.s. is constant w.r.t. y^x.

Taking $E \setminus B = \text{supp}(x)$ (so elements B are all zeros in x), and for $b \notin B$ we make $x(b)$ is big enough, the r.h.s. min has solution $A^* = B$. We recover submodular function from the polymatroid polyhedron via the following:

$$\text{rank}\left(\frac{1}{\epsilon}\mathbf{1}_{E \setminus B}\right) = f(B) = \max\left\{y(B) : y \in P_f^+\right\} \quad (15.11)$$

In fact, we will ultimately see a number of important consequences of this theorem (other than just that P_f^+ is a polymatroid)

Review from lecture 12

The next slide comes from lecture 12.

Matroid instance of Theorem ??

- Considering Theorem **??**, the matroid case is now a special case, where we have that:

Corollary 15.4.2

We have that:

$$\max\{y(E) : y \in P_{\textit{ind. set}}(M), y \le x\} = \min\{r_M(A) + x(E \setminus A) : A \subseteq E\} \tag{15.30}$$

where r_M is the matroid rank function of some matroid.

Most violated inequality problem in matroid polytope case

- Consider

$$P_r^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r_M(A), \forall A \subseteq E\} \tag{15.7}$$

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- Hence, there must be a set of $\mathcal{W} \subseteq 2^V$, each member of which corresponds to a violated inequality, i.e., equations of the form $x(A) > r_M(A)$ for $A \in \mathcal{W}$.

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- The most violated inequality when x is considered w.r.t. P_r^+ corresponds to the set A that maximizes $x(A) - r_M(A)$, i.e., the most violated inequality is valuated as:

$$\max \left\{ x(A) - r_M(A) : A \in \mathcal{W} \right\} = \max \left\{ x(A) - r_M(A) : A \subseteq E \right\} \tag{15.8}$$

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- Since x is modular and $x(E \setminus A) = x(E) - x(A)$, we can express this via a min as in;:

$$\min \{r_M(A) + x(E \setminus A) : A \subseteq E\} \tag{15.9}$$

Most violated inequality/polymatroid membership/SFM

- Consider

$$P_f^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\} \tag{15.10}$$

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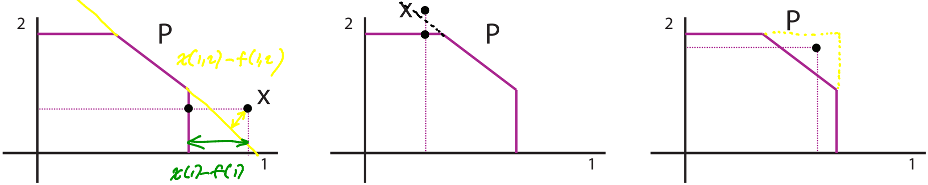

Left: $\mathcal{W} = \{\{1\}\}$ Center: $\mathcal{W} = \{\{2\}\}$ Right: $\mathcal{W} = \{\{1, 2\}\}$

{1,2} {1,2}

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$$\min\{f(A) + x(E \setminus A) : A \subseteq E\} \quad (15.12)$$

$f(A) + x(E \setminus A)$

$= f(A) - x(A) + \text{const.}$

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- We will ultimatley answer how general this form of SFM is.

Review from Lecture 6

The following three slides are review from lecture 6.

Matroids, other definitions using matroid rank $r : 2^V \to \mathbb{Z}_+$

Definition 15.5.3 (closed/flat/subspace)

A subset $A \subseteq E$ is closed (equivalently, a flat or a subspace) of matroid M if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

Definition: A hyperplane is a flat of rank $r(M) - 1$.

Definition 15.5.4 (closure)

Given $A \subseteq E$, the closure (or span) of A, is defined by
$\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}$.

Therefore, a closed set A has $\text{span}(A) = A$.

Definition 15.5.5 (circuit)

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise-minimal dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

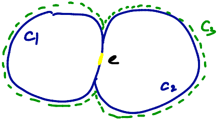
Matroids by circuits

A set is independent if and only if it contains no circuit. Therefore, it is not surprising that circuits can also characterize a matroid.

Theorem 15.5.3 (Matroid by circuits)

Let E be a set and $\mathcal{C}$ be a collection of subsets of E that satisfy the following three properties:

1. *(C1): $\emptyset \notin \mathcal{C}$*
2. *(C2): if $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.*
3. *(C3): if $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq C_2$, and $e \in C_1 \cap C_2$, then there exists a $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.*



Matroids by circuits

Several circuit definitions for matroids.

Theorem 15.5.3 (Matroid by circuits)

Let E be a set and $\mathcal{C}$ be a collection of nonempty subsets of E, such that no two sets in $\mathcal{C}$ are contained in each other. Then the following are equivalent.

1. *$\mathcal{C}$ is the collection of circuits of a matroid;*
2. *if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$;*
3. *if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, and $y \in C \setminus C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$ containing y;*

Again, think about this for a moment in terms of linear spaces and matrices, and spanning trees.

Fundamental circuits in matroids

Lemma 15.5.1

Let $I \in \mathcal{I}(M)$, and $e \in E$, then $I \cup \{e\}$ contains at most one circuit in M.

Proof.

$\square$

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- Suppose, to the contrary, that there are two distinct circuits C_1, C_2 such that $C_1 \cup C_2 \subseteq I \cup \{e\}$.

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In general, let $C(I, e)$ be the unique circuit associated with $I \cup \{e\}$ (commonly called the fundamental circuit in M w.r.t. I and e).

Matroids: The Fundamental Circuit

- Define $C(I, e)$ be the unique circuit associated with $I \cup \{e\}$ (the fundamental circuit in M w.r.t. I and e, if it exists).

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- If $e \notin \text{span}(I)$, then $C(I, e) = \emptyset$, since no circuit is created in this case.

Union of matroid bases of a set

Lemma 15.5.2 any set.

Let $\mathcal{B}(C)$ be the set of bases of C. Then, given matroid $\mathcal{M} = (E, \mathcal{I})$, and any loop-free (i.e., no dependent singleton elements) set $C \subseteq E$, we have:

$$\bigcup_{B \in \mathcal{B}(C)} B = C. \tag{15.13}$$

Bases of a set C constitute a cover.

$\forall B \in \mathcal{B}(C), \quad B \subseteq \text{span}(C)$

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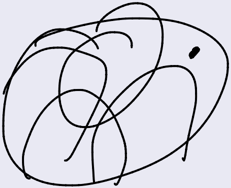
- Define $C' \triangleq \bigcup_{B \in \mathcal{B}(C)}$, and suppose $\exists c \in C$ such that $c \notin C'$.



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$$\bigcup_{B \in \mathcal{B}(C)} B = C. \tag{15.13}$$

Proof.

- Define $C' \triangleq \bigcup_{B \in \mathcal{B}(C)}$, and suppose $\exists c \in C$ such that $c \notin C'$.
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- Then choose $c' \in C(B, c)$ with $c' \neq c$.
- Then $B + c - c'$ is independent size $|B|$ subset of C and hence spans C, and thus is a c-containing member of $\mathcal{B}(C)$, contradicting $c \notin C'$.

□

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- Recall, for a given $x \in P_f$, we have defined this tight family as

$$\mathcal{D}(x) = \{A : A \subseteq E, x(A) = f(A)\} \tag{15.14}$$

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$g(\cdot): f(\cdot) - x(\cdot)$

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- The zero-valued minimizers of f' are thus closed under union and intersection.
- In fact, this is true for all minimizers of a submodular function as stated in the next theorem.

Minimizers of a Submodular Function form a lattice

Theorem 15.6.1

For arbitrary submodular f, the minimizers are closed under union and intersection. That is, let $\mathcal{M} = \operatorname{argmin}_{X \subseteq E} f(X)$ be the set of minimizers of f. Let $A, B \in \mathcal{M}$. Then $A \cup B \in \mathcal{M}$ and $A \cap B \in \mathcal{M}$.

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Proof.

Since A and B are minimizers, we have $f(A) = f(B) \leq f(A \cap B)$ and $f(A) = f(B) \leq f(A \cup B)$.

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Thus, the minimizers of a submodular function form a lattice, and there is a maximal and a minimal minimizer of every submodular function.

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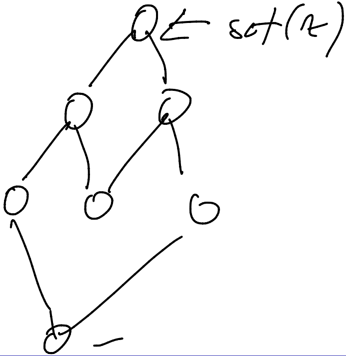
$$\mathrm{cl}(x) \stackrel{\text{def}}{=} \mathrm{sat}(x) \stackrel{\text{def}}{=} \bigcup \{A : A \in \mathcal{D}(x)\} \tag{15.18}$$



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- First, we see how sat generalizes matroid closure.

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- Consider matroid $(E, \mathcal{I}) = (E, r)$, some $I \in \mathcal{I}$. Then $\mathbf{1}_I \in P_r$ and

$$\mathcal{D}(\mathbf{1}_I) = \{A : \mathbf{1}_I(A) = r(A)\} \tag{15.21}$$

$$1_I(A) = \sum_{a \in A} 1_I(e) = |A \cap I|$$

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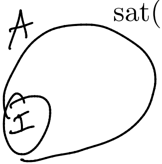

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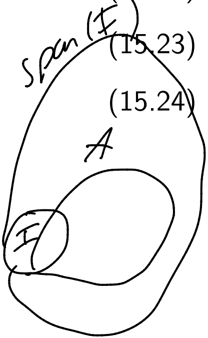

$I \cap A \in \mathcal{I}$

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- Notice that $\mathbf{1}_I(A) = |I \cap A| \leq |I|$.

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- We formalize this next.

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Lemma 15.7.1 (Matroid $\mathrm{sat} : \mathbb{R}_+^E \to 2^E$ is the same as closure.)

For $I \in \mathcal{I}$, we have $\mathrm{sat}(\mathbf{1}_I) = \mathrm{span}(I)$ (15.25)

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- Therefore, $\text{sat}(\mathbf{1}_I) \supseteq \text{span}(I)$.

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- Now, consider $b \in \mathrm{sat}(\mathbf{1}_I) \setminus I$.

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- Now $r(A) = |A \cap I| \leq |I| = r(I)$.

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The sat function = Polymatroid Closure

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- Now, consider $b \in \text{sat}(\mathbf{1}_I) \setminus I$.
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  In which case, we also get $\mathrm{sat}(\mathbf{1}_C) = \mathrm{span}(C)$ (in general, could define $\mathrm{sat}(y) = \mathrm{sat}(\text{P-basis}(y))$).

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Exercise: is $\mathrm{span}(C) = \mathrm{sat}(\mathbf{1}_C)$? Prove or disprove it.

The sat function, span, and submodular function minimization

- Thus, for a matroid, $\mathrm{sat}(\mathbf{1}_I)$ is exactly the closure (or span) of I in the matroid. I.e., for matroid (E, r), we have $\mathrm{span}(I) = \mathrm{sat}(\mathbf{1}_B)$.

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- Submodular function minimization can solve "span" queries in a matroid or "sat" queries in a polymatroid.

sat, as tight polymatroidal elements

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- We next show more formally that these are the same.

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- On the other hand, for any $e \in \text{sat}(x)$ defined as in Eq. (15.35), since e is itself a member of a tight set, there is a set $A \ni e$ such that $x(A) = f(A)$, giving

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- Therefore, the two definitions of sat are identical.

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since any $B \subseteq E$ such that $e \notin B$ does not change in a $\mathbf{1}_e$ adjustment, meaning $(x + \alpha \mathbf{1}_e)(B) = x(B)$.

- Again, this is identical to:

$$\max \{\alpha : x(A) + \alpha \leq f(A), \forall A \supseteq \{e\}\} \tag{15.41}$$

or

$$\max \{\alpha : \alpha \leq f(A) - x(A), \forall A \supseteq \{e\}\} \tag{15.42}$$

Saturation Capacity

- The max is achieved when

$$\alpha = \hat{c}(x;e) \stackrel{\text{def}}{=} \min\{f(A) - x(A), \forall A \supseteq \{e\}\} \tag{15.43}$$

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- Note that any α with $0 \leq \alpha \leq \hat{c}(x;e)$ we have $x + \alpha \mathbf{1}_e \in P_f$.
- We also see that computing $\hat{c}(x;e)$ is a form of submodular function minimization.

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- Given $x \in P_f$, and $e \in \text{sat}(x)$, define

$$\mathcal{D}(x, e) = \{A : e \in A \subseteq E, x(A) = f(A)\} \tag{15.46}$$

$$= \mathcal{D}(x) \cap \{A : A \subseteq E, e \in A\} \tag{15.47}$$

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- Therefore, we can define a unique minimal element of $\mathcal{D}(x, e)$ denoted as follows:

$$\text{dep}(x, e) = \begin{cases} \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases} \tag{15.48}$$

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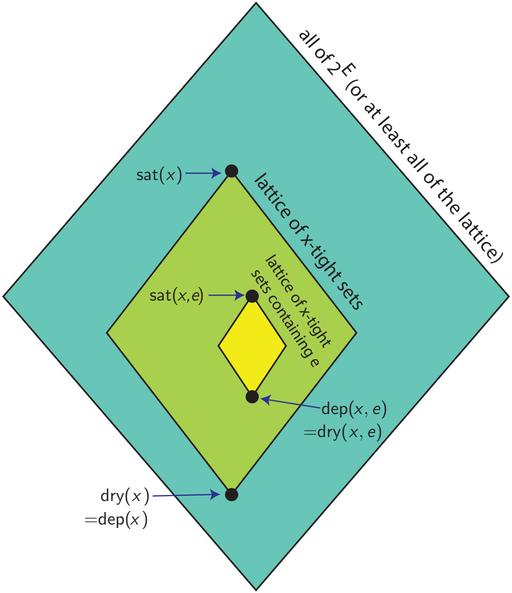
- I.e., $\text{dep}(x, e)$ is the minimal element in $\mathcal{D}(x)$ that contains e (the minimal x-tight set containing e).

dep and sat in a lattice

- Given some $x \in P_f$,
- The picture on the right summarizes the relationships between the lattices and sublattices.
- Note, $\bigcap_e \mathrm{dep}(x, e) = \mathrm{dep}(x)$.



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- Note that dry need not be the empty set. Exercise: give example.

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- Now, given $x \in P_f$, and $e \in \text{sat}(x)$, recall distributive sub-lattice of e-containing tight sets $\mathcal{D}(x, e) = \{A : e \in A, x(A) = f(A)\}$

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- But actually, $\text{dry}(x, e) = \text{dep}(x, e)$, so we have derived another expression for $\text{dep}(x, e)$ in Eq. (15.50).

Dependence Function and Fundamental Matroid Circuit

- Now, let $(E, \mathcal{I}) = (E, r)$ be a matroid, and let $I \in \mathcal{I}$ giving $\mathbf{1}_I \in P_r$. We have $\text{sat}(\mathbf{1}_I) = \text{span}(I) = \text{closure}(I)$.

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- By SFM lattice, $\exists$ a unique minimal $A \ni e$ with $|I \cap A| = r(A)$.
- Thus, $\text{dep}(\mathbf{1}_I, e)$ must be a circuit since if it included more than a circuit, it would not be minimal in this sense.

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- Therefore, when $e \in \text{sat}(\mathbf{1}_I) \setminus I$, then $\text{dep}(\mathbf{1}_I, e) = C(I, e)$ where $C(I, e)$ is the unique circuit contained in $I + e$ in a matroid (the fundamental circuit of e and I that we encountered before).

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- We are thus free to take subsets of I as A, all of which must contain e, but all of which have rank equal to size.
- Also note: in general for $x \in P_f$ and $e \in \text{sat}(x)$, we have $\text{dep}(x, e)$ is tight by definition.

Summary of sat, and dep

- For $x \in P_f$, $\text{sat}(x)$ (span, closure) is the maximal saturated (x-tight) set w.r.t. x. I.e., $\text{sat}(x) = \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\}$. That is,

$$\text{cl}(x) \stackrel{\text{def}}{=} \text{sat}(x) \triangleq \bigcup \{A : A \in \mathcal{D}(x)\} \tag{15.54}$$

$$= \bigcup \{A : A \subseteq E, x(A) = f(A)\} \tag{15.55}$$

$$= \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\} \tag{15.56}$$

- For $e \in \text{sat}(x)$, we have $\text{dep}(x, e) \subseteq \text{sat}(x)$ (fundamental circuit) is the minimal (common) saturated (x-tight) set w.r.t. x containing e. I.e.,

$$\text{dep}(x, e) = \begin{cases} \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases}$$

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- But, analogous to the circuit case, is there an exchange property for $\text{dep}(x, e)$ in the form of vector movement restriction?
- We might expect the vector $\text{dep}(x, e)$ property to take the form: a positive move in the e-direction stays within P_f^+ only if we simultaneously take a negative move in one of the $\text{dep}(x, e)$ directions.

Dependence Function and exchange in 2D

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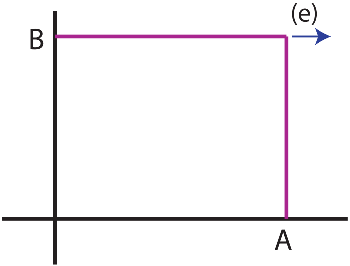
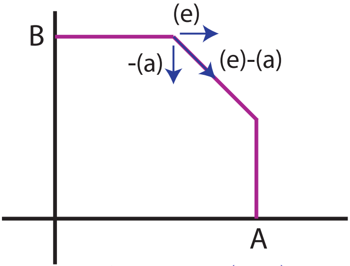
- Viewable in 2D, we have for $A, B \subseteq E$, $A \cap B = \emptyset$:



Left: $A \cap \text{dep}(x, e) = \emptyset$, and we can't move further in (e) direction, and moving in any negative $a \in A$ direction doesn't change that. Notice no dependence between (e) and any element in A.



Right: $A \subseteq \text{dep}(x, e)$, and we can't move further in the (e) direction, but we can move further in (e) direction by moving in some $a \in A$ negative direction. Notice dependence between (e) and elements in A.

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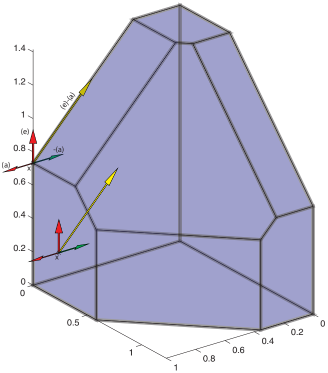

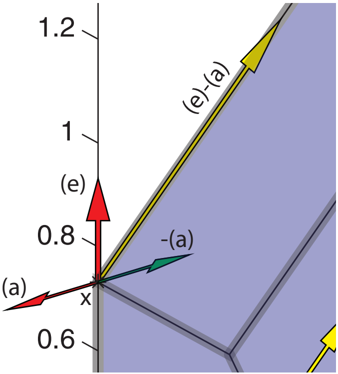

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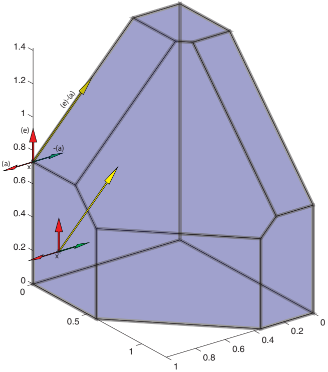

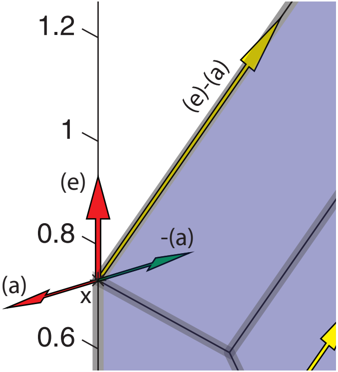

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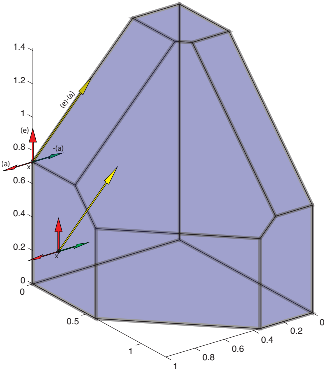

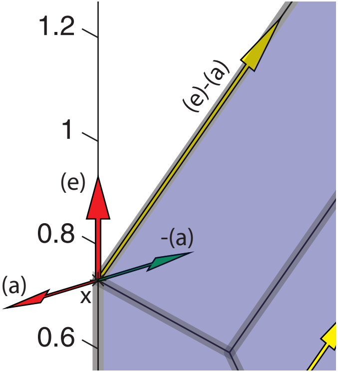

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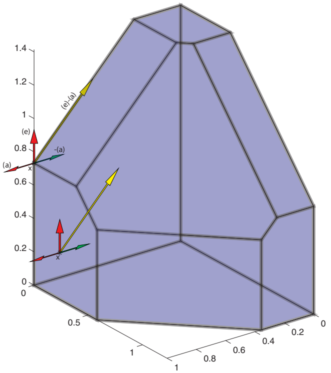

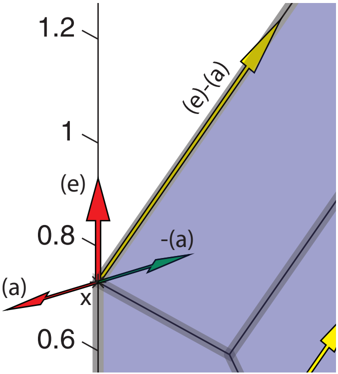

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- We next show this formally ...

dep and exchange derived

- The derivation for $\text{dep}(x, e)$ involves turning a strict inequality into a non-strict one with a strict explicit slack variable α:

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- Also, if $e' \in A$ but $e \notin A$, then $x(A) + \alpha(\mathbf{1}_e(A) - \mathbf{1}_{e'}(A)) = x(A) - \alpha \leq f(A)$ since $x \in P_f$.

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- thus, we get the same in the above if we remove the constraint $A \not\ni e', e \in A$, that is we get

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- Compare with original, the minimal element of $\mathcal{D}(x,e)$, with $e \in \operatorname{sat}(x)$:

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$$\text{dep}(x,e) = \begin{cases} \bigcap\{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases}$$

$$= \{e' : \exists \alpha > 0, \text{ s.t. } x + \alpha(\mathbf{1}_e - \mathbf{1}_{e'}) \in P_f\}$$