Submodular Functions, Optimization, and Applications to Machine Learning
— Spring Quarter, Lecture 14 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee596b_spring_2016/

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\[
f(A) + f(B) \geq f(A \cup B) + f(A \cap B)
\]

Cumulative Outstanding Reading

- Read chapters 2 and 3 from Fujishige's book.
- Read chapter 1 from Fujishige's book.
Logistics

Announcements, Assignments, and Reminders

- Homework 3, available at our assignment dropbox (https://canvas.uw.edu/courses/1039754/assignments), due (electronically) Monday (5/2) at 11:55pm.
- Homework 1, available at our assignment dropbox (https://canvas.uw.edu/courses/1039754/assignments), due (electronically) Friday (4/8) at 11:55pm.
- Weekly Office Hours: Mondays, 3:30-4:30, or by skype or google hangout (set up meeting via our our discussion board (https://canvas.uw.edu/courses/1039754/discussion_topics)).

Class Road Map - IT-I

- L1(3/28): Motivation, Applications, & Basic Definitions
- L3(4/4): Info theory exs, more apps, definitions, graph/combinatorial examples, matrix rank example, visualization
- L4(4/6): Graph and Combinatorial Examples, matrix rank, Venn diagrams, examples of proofs of submodularity, some useful properties
- L5(4/11): Examples & Properties, Other Defs., Independence
- L6(4/13): Independence, Matroids, Matroid Examples, matroid rank is submodular
- L8(4/20): Transversals, Matroid and representation, Dual Matroids
- L10(4/27): Matroid and Greedy, Polyhedra, Matroid Polytopes
- L11(5/2): From Matroids to Polymatroids, Polymatroids
- L12(5/4): Polymatroids, Polymatroids and Greedy
- L13(5/9): Polymatroids and Greedy; Possible Polytopes; Extreme Points; Polymatroids, Greedy, and Cardinality Constrained Maximization
- L14(5/11): Cardinality Constrained Maximization; Curvature; Submodular Max w. Other Constraints
- L15(5/16):
- L16(5/18):
- L17(5/23):
- L18(5/25):
- L19(6/1):

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Polymatroidal polyhedron and greedy

- Thus, restating the above results into a single complete theorem, we have a result very similar to what we saw for matroids (i.e., Theorem ??)

**Theorem 14.2.1**

If \( f : 2^E \to \mathbb{R}_+ \) is given, and \( P \) is a polytope in \( \mathbb{R}_+^E \) of the form
\[
P = \{ x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E \},
\]
then the greedy solution to the problem \( \max(\omega x : x \in P) \) is \( \forall \omega \) optimum iff \( f \) is monotone non-decreasing submodular (i.e., iff \( P \) is a polymatroid).

Multiple Polytopes associated with arbitrary \( f \)

- Given an arbitrary submodular function \( f : 2^V \to \mathbb{R} \) (not necessarily a polymatroid function, so it need not be positive, monotone, etc.).
- If \( f(\emptyset) \neq 0 \), can set \( f'(A) = f(A) - f(\emptyset) \) without destroying submodularity. This does not change any minima, (i.e., \( \text{argmin}_A f(A) = \text{argmin}_A f'(A) \)) so assume all functions are normalized \( f(\emptyset) = 0 \).

Note that due to constraint \( x(\emptyset) \leq f(\emptyset) \), we must have \( f(\emptyset) \geq 0 \) since if not (i.e., if \( f(\emptyset) < 0 \)), then \( P_f^+ \) doesn’t exist.
Another form of normalization can do is:

\[
f'(A) = \begin{cases} f(A) & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset \end{cases}
\]  
(14.1)

This preserves submodularity due to \( f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \), and if \( A \cap B = \emptyset \) then r.h.s. only gets smaller when \( f(\emptyset) \geq 0 \).

- We can define several polytopes:
\[
P_f = \{ x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E \}
\]  
(14.2)
\[
P_f^+ = P_f \cap \{ x \in \mathbb{R}^E : x \geq 0 \}
\]  
(14.3)
\[
B_f = P_f \cap \{ x \in \mathbb{R}^E : x(E) = f(E) \}
\]  
(14.4)

- \( P_f \) is what is sometimes called the extended polytope (sometimes is sometimes called the extended polytope (sometimes...
Multiple Polytopes in 2D associated with $f$

\[ P_f^+ = P_f \cap \{ x \in \mathbb{R}^E : x \geq 0 \} \]  
\[ P_f = \{ x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E \} \]  
\[ B_f = P_f \cap \{ x \in \mathbb{R}^E : x(E) = f(E) \} \]

A polymatroid function’s polyhedron is a polymatroid.

**Theorem 14.2.1**

Let $f$ be a submodular function defined on subsets of $E$. For any $x \in \mathbb{R}^E$, we have:

\[ \text{rank}(x) = \max (y(E) : y \leq x, y \in P_f) = \min (x(A) + f(E \setminus A) : A \subseteq E) \]  

Essentially the same theorem as Theorem ??, but note $P_f$ rather than $P_f^+$. Taking $x = 0$ we get:

**Corollary 14.2.2**

Let $f$ be a submodular function defined on subsets of $E$. We have:

\[ \text{rank}(0) = \max (y(E) : y \leq 0, y \in P_f) = \min (f(A) : A \subseteq E) \]
Logistics

Review

Polymatroid with labeled edge lengths

- Recall
  \[ f(e|A) = f(A+e) - f(A) \]

- Notice how
  submodularity,
  \[ f(e|B) \leq f(e|A) \text{ for } A \subseteq B, \]
  defines the shape of the polytope.

- In fact, we have
  strictness here
  \[ f(e|B) < f(e|A) \text{ for } A \subset B. \]

- Also, consider how the
  greedy algorithm
  proceeds along the edges
  of the polytope.

Intuition: why greedy works with polymatroids

- Given \( w \), the goal is
  to find
  \[ x = (x(e_1), x(e_2)) \]
  that maximizes
  \[ x^\top w = x(e_1)w(e_1) + x(e_2)w(e_2). \]

- If \( w(e_2) > w(e_1) \) the
  upper extreme point
  indicated maximizes
  \( x^\top w \) over \( x \in P_f^+ \).

- If \( w(e_2) < w(e_1) \) the
  lower extreme point
  indicated maximizes
  \( x^\top w \) over \( x \in P_f^+ \).
### The Greedy Algorithm: $1 - 1/e$ intuition.

- At step $i < k$, greedy chooses $v_i$ to maximize $f(v|S_i)$.
- Let $S^*$ be optimal solution (of size $k$) and $\text{OPT} = f(S^*)$. By submodularity, we will show:

\[ \exists v \in V \setminus S_i : f(v|S_i) = f(S_i + v|S_i) \geq \frac{1}{k} (\text{OPT} - f(S_i)) \quad (14.1) \]

Equation (14.1) shows that $f(S_1)$ of size 1 is at least $1/e$ of the optimal.

- Equation (14.10) will show that Equation (14.1) implies:

\[ \text{OPT} - f(S_{i+1}) \leq (1 - 1/k)(\text{OPT} - f(S_i)) \]

\[ \Rightarrow \text{OPT} - f(S_k) \leq (1 - 1/k)^k \text{OPT} \]

\[ \Rightarrow \text{OPT}(1 - 1/e) \leq f(S_k) \]

### Cardinality Constrained Polymatroid Max Theorem

**Theorem 14.3.1 (Nemhauser et al. 1978)**

Given non-negative monotone submodular function $f : 2^V \rightarrow \mathbb{R}_+$, define $\{S_i\}_{i \geq 0}$ to be the chain formed by the greedy algorithm (Eqn. (??)). Then for all $k, \ell \in \mathbb{Z}_{++}$, we have:

\[ f(S_\ell) \geq (1 - e^{-\ell/k}) \max_{S : |S| \leq k} f(S) \quad (14.2) \]

and in particular, for $\ell = k$, we have:

\[ f(S_k) \geq (1 - 1/e) \max_{S : |S| \leq k} f(S). \]

- $k$ is size of optimal set, i.e., $\text{OPT} = f(S^*)$ with $|S^*| = k$.
- $\ell$ is size of set we are choosing (i.e., we choose $S_\ell$ from greedy chain).
- Bound is how well does $S_\ell$ (of size $\ell$) do relative to $S^*$, the optimal set of size $k$.
- Intuitively, bound should get worse when $\ell < k$ and get better when $\ell > k$. 
Proof of Theorem 14.3.1.

- Fix \( \ell \) (number of items greedy will chose) and \( k \) (size of optimal set to compare against).
- Set \( S^* \in \arg\max \{ f(S) : |S| \leq k \} \)
- w.l.o.g. assume \( |S^*| = k \).
- Order \( S^* = (v_1^*, v_2^*, \ldots, v_k^*) \) arbitrarily.
- Let \( S_i = (v_1, v_2, \ldots, v_i) \) be the greedy order chain chosen by the algorithm, for \( i \in \{1, 2, \ldots, \ell\} \).
- Then the following inequalities (on the next slide) follow:

\[ f(S^*) \leq f(S^* \cup S_i) = f(S_i) + f(S^*|S_i) \]  \hfill (14.3)

\[ = f(S_i) + \sum_{j=1}^{k} f(v_j^*|S_i \cup \{v_1^*, v_2^*, \ldots, v_{j-1}^*\}) \]  \hfill (14.4)

\[ \leq f(S_i) + \sum_{v \in S^*} f(v|S_i) \]  \hfill (14.5)

\[ \leq f(S_i) + \sum_{v \in S^*} f(v|S_i) = f(S_i) + \sum_{v \in S^*} f(S_{i+1}|S_i) \]  \hfill (14.6)

\[ = f(S_i) + kf(S_{i+1}|S_i) \]  \hfill (14.7)

Therefore, we have Equation 14.1, i.e.,:

\[ f(S^*) - f(S_i) \leq kf(S_{i+1}|S_i) = k(f(S_{i+1}) - f(S_i)) \]  \hfill (14.8)

... proof of Theorem 14.3.1 cont.
Cardinality Constrained Polymatroid Max Theorem

... proof of Theorem 14.3.1 cont.

- Define gap \( \delta_i \triangleq f(S^*) - f(S_i) \), so \( \delta_i - \delta_{i+1} = f(S_{i+1}) - f(S_{i}) \), giving
  \[
  \delta_i \leq k(\delta_i - \delta_{i+1}) \tag{14.9}
  \]
  or
  \[
  \delta_{i+1} \leq (1 - \frac{1}{k})\delta_i \tag{14.10}
  \]

- The relationship between \( \delta_0 \) and \( \delta_\ell \) is then
  \[
  \delta_\ell \leq (1 - \frac{1}{k})^\ell \delta_0 \tag{14.11}
  \]

- Now, \( \delta_0 = f(S^*) - f(\emptyset) \leq f(S^*) \) since \( f \geq 0 \).

- Also, by variational bound \( 1 - x \leq e^{-x} \) for \( x \in \mathbb{R} \), we have
  \[
  \delta_\ell \leq (1 - \frac{1}{k})^\ell \delta_0 \leq e^{-\ell/k} f(S^*) \tag{14.12}
  \]

With \( \ell = k \), when picking \( k \) items, greedy gets \( (1 - 1/e) \approx 0.6321 \) bound. This means that if \( S_k \) is greedy solution of size \( k \), and \( S^* \) is an optimal solution of size \( k \), \( f(S_k) \geq (1 - 1/e) f(S^*) \approx 0.6321 f(S^*) \).

What if we want to guarantee a solution no worse than \( 0.95 f(S^*) \) where \(|S^*| = k\)? Set \( 0.95 = (1 - e^{-\ell/k}) \), which gives 
\[
\ell = \lceil -k \ln(1 - 0.95) \rceil = 4k. \quad \text{And} \quad \lceil -\ln(1 - 0.999) \rceil = 7.
\]

So solution, in the worst case, quickly gets very good. Typical/practical case is much better.
Greedy running time

- Greedy computes a new maximum $n = |V|$ times, and each maximum computation requires $O(n)$ comparisons, leading to $O(n^2)$ computation for greedy.
- This is the best we can do for arbitrary functions, but $O(n^2)$ is not practical to some.
- Greedy can be made much faster in practice by a simple strategy made possible, once again, via the use of submodularity.
- This is called Minoux’s 1977 Accelerated Greedy strategy (and has been rediscovered a few times, e.g., “Lazy greedy”), and runs much faster while still producing same answer.
- We describe it next:

Minoux’s Accelerated Greedy for Submodular Functions

- At stage $i$ in the algorithm, we have a set of gains $f(v|S_i)$ for all $v \notin S_i$. Store these values $\alpha_v \leftarrow f(v|S_i)$ in sorted priority queue.
- Priority queue, $O(1)$ to find max, $O(\log n)$ to insert in right place.
- Once we choose a max $v$, then set $S_{i+1} \leftarrow S_i + v$.
- For $v \notin S_{i+1}$ we have $f(v|S_{i+1}) \leq f(v|S_i)$ by submodularity.
- Therefore, if we find a $v'$ such that $f(v'|S_{i+1}) \geq \alpha_v$ for all $v \neq v'$, then since

$$f(v'|S_{i+1}) \geq \alpha_v = f(v|S_i) \geq f(v|S_{i+1})$$

(14.14)

we have the true max, and we need not re-evaluate gains of other elements again.
- Strategy is: find the $\arg\max_{v' \in V \setminus S_{i+1}} \alpha_{v'}$, and then compute the real $f(v'|S_{i+1})$. If it is greater than all other $\alpha_v$'s then that’s the next greedy step. Otherwise, replace $\alpha_{v'}$ with its real value, resort $(O(\log n))$, and repeat.
Minoux’s Accelerated Greedy for Submodular Functions

- Minoux’s algorithm is exact, in that it has the same guarantees as does the $O(n^2)$ greedy Algorithm 2 (this means it will return either the same answers, or answers that have the $1 - 1/e$ guarantee).
- In practice: Minoux’s trick has enormous speedups ($\approx 700 \times$) over the standard greedy procedure due to reduced function evaluations and use of good data structures (priority queue).
- When choosing a of size $k$, naïve greedy algorithm is $O(nk)$ but accelerated variant at the very best does $O(n + k)$, so this limits the speedup.
- Algorithm has been rediscovered (I think) independently (CELF - cost-effective lazy forward selection, Leskovec et al., 2007)
- Can be used for “big data” sets (e.g., social networks, selecting blogs of greatest influence, document summarization, etc.).
Minoux’s Accelerated Greedy Algorithm Submodular Max

Algorithm 2: Minoux’s Accelerated Greedy Algorithm

1. Set $S_0 \leftarrow \emptyset$; $i \leftarrow 0$; Initialize priority queue $Q$
2. for $v \in E$ do
   3. INSERT$(Q, f(v))$
4. repeat
   5. $(v, \alpha) \leftarrow \text{POP}(Q)$;
   6. if $\alpha$ not “fresh” then
      7. recompute $\alpha \leftarrow f(v | S_i)$
   8. if (popped $\alpha$ in line 5 was “fresh”) OR ($\alpha \geq \text{MAX}(Q)$) then
      9. Set $S_{i+1} \leftarrow S_i \cup \{v\}$;
     10. $i \leftarrow i + 1$
   else
     12. INSERT$(Q, (v, \alpha))$
5. until $i = |E|$

Minimum Submodular Cover

- Given polymatroid $f$, goal is to find a covering set of minimum cost:
  \[
  S^* \in \arg\min_{S \subseteq V} |S| \text{ such that } f(S) \geq \alpha \tag{14.18}
  \]
  where $\alpha$ is a “cover” requirement.
- Normally take $\alpha = f(V)$ but defining $f'(A) = \min \{f(A), \alpha\}$ we can take any $\alpha$. Hence, we have equivalent formulation:
  \[
  S^* \in \arg\min_{S \subseteq V} |S| \text{ such that } f'(S) \geq f'(V) \tag{14.19}
  \]
- Note that this immediately generalizes standard set cover, in which case $f(A)$ is the cardinality of the union of sets indexed by $A$.
- Algorithm: Pick the first $S_i$ chosen by aforementioned greedy algorithm such that $f(S_i) \geq \alpha$.
- For integer valued $f$, this greedy algorithm an $O(\log(\max_{s \in V} f(\{s\})))$ approximation. Set cover is hard to approximate with a factor better than $(1 - \epsilon) \log \alpha$, where $\alpha$ is the desired cover constraint.
Summary: Monotone Submodular Maximization

- Only makes sense when there is a constraint.
- We discussed cardinality constraint.
- Generalizes the max \( k \)-cover problem, and also similar to the set cover problem.
- Simple greedy algorithm gets \( 1 - e^{-\ell/k} \) approximation, where \( k \) is size of optimal set we compare against, and \( \ell \) is size of set greedy algorithm chooses.
- Submodular cover: min. \(|S|\) s.t. \( f(S) \geq \alpha\).
- Minoux’s accelerated greedy trick.

The Greedy Algorithm: \( 1 - 1/e \) intuition.

- At step \( i < k \), greedy chooses \( v_i \) to maximize \( f(v_i|S_i) \).
- Let \( S^* \) be optimal solution (of size \( k \)) and \( \OPT = f(S^*) \). By submodularity, we will show:

\[
\exists v \in V \setminus S_i : f(v|S_i) = f(S_i + v|S_i) \geq \frac{1}{k}(\OPT - f(S_i)) \quad (14.1)
\]

Equation (14.10) will show that Equation (14.1)

\[
(1 - (1 - 1/k)^k) \leq f(S_k)/\OPT
\]

\Rightarrow:

\[
\OPT - f(S_{i+1}) \\
\leq (1 - 1/k)(\OPT - f(S_i)) \\
\Rightarrow \OPT - f(S_k) \\
\leq (1 - 1/k)^k \OPT \\
\leq 1/e \OPT \\
\Rightarrow \OPT (1 - 1/e) \leq f(S_k)
\]
Randomized greedy

- How can we produce a randomized greedy strategy, one where each greedy sweep produces a set that, on average, has a $1 - 1/e$ guarantee?
- Suppose the following holds:

$$E[f(a_{i+1}|A_i)] \geq \frac{f(OPT) - f(A_i)}{k}$$

(14.20)

where $A_i = (a_1, a_2, \ldots, a_i)$ are the first $i$ elements chosen by the strategy.
- See problem 5, homework 4.

Curvature of a Submodular function

- For any submodular function, we have $f(j|S) \leq f(j|\emptyset)$ so that $f(j|S)/f(j|\emptyset) \leq 1$ whenever $f(j|\emptyset) \neq 0$.
- For $f : 2^V \to \mathbb{R}_+$ (non-negative) functions, we also have $f(j|S)/f(j|\emptyset) \geq 0$ — and $= 0$ whenever $j$ is "spanned" by $S$.
- The total curvature of a submodular function is defined as follows:

$$c \triangleq 1 - \min_{S,j \notin S: f(j|\emptyset) \neq 0} \frac{f(j|S)}{f(j|\emptyset)} = 1 - \min_{f(j) \neq 0} \frac{f(j|V \setminus j)}{f(j)}$$

(14.21)

- $c \in [0, 1]$. When $c = 0$, $f(j|S) = f(j|\emptyset)$ for all $S, j$, a sufficient condition for modularity, and we saw in Theorem ?? that greedy is optimal for max weight indep. set of a matroid.
- For $f$ with curvature $c$, then $\forall A \subseteq V, \forall v \notin a, \forall c' \geq c$:

$$f(A + v) - f(A) \geq (1 - c')f(v)$$

(14.22)

- When $c = 1$ then submodular function is “maximally curved”, i.e., there exists is a subset that fully spans some other element.
- Matroid rank functions with some dependence is infinitely curved.
Curvature of a Submodular function

- By submodularity, total curvature can be computed in either form:

\[
c \triangleq 1 - \min_{S,j \not\in S : f(j|\emptyset) \neq 0} \frac{f(j|S)}{f(j|\emptyset)} = 1 - \min_{j : f(j|\emptyset) \neq 0} \frac{f(j|V \setminus \{j\})}{f(j|\emptyset)} \quad (14.23)
\]

- Note: Matroid rank is either modular \( c = 0 \) or infinitely curved \( c = 1 \) — hence, matroid rank can have only the extreme points of curvature, namely 0 or 1.
- Polymatroid functions are, in this sense, more nuanced, in that they allow non-extreme curvature, with \( c \in [0, 1] \).
- It will be remembered the notion of “partial dependence” within polymatroid functions.

Curvature for \( f(S) = \sqrt{|S|} \)

- Curvature of \( f(S) = \sqrt{|S|} \) as function of \( |V| = n \)

- \( f(S) = \sqrt{|S|} \) with \( |V| = n \) has curvature \( 1 - (\sqrt{n} - \sqrt{n-1}) \).
- Approximation gets worse with bigger ground set.
- Functions of the form \( f(S) = \sqrt{m(S)} \) where \( m : V \to \mathbb{R}_+ \), approximation worse with \( n \) if \( \min_{i,j} |m(i) - m(j)| \) has a fixed lower bound with increasing \( n \).
Curvature and approximation

- Curvature limitation can help the greedy algorithm in terms of approximation bounds.
- Conforti & Cornuéjols showed that greedy gives a $1/(1 + c)$ approximation to $\max \{ f(S) : S \in \mathcal{I} \}$ when $f$ has total curvature $c$.
- Hence, greedy subject to matroid constraint is a $\max(1/(1 + c), 1/2)$ approximation algorithm, and if $c < 1$ then it is better than $1/2$ (e.g., with $c = 1/4$ then we have a 0.8 algorithm).

For $k$-uniform matroid (i.e., $k$-cardinality constraints), then approximation factor becomes $\frac{1}{c}(1 - e^{-c})$.

Generalizations

- Consider a $k$-uniform matroid $\mathcal{M} = (V, \mathcal{I})$ where $\mathcal{I} = \{ S \subseteq V : |S| \leq k \}$, and consider problem $\max \{ f(A) : A \in \mathcal{I} \}$.
- Hence, the greedy algorithm is $1 - 1/e$ optimal for maximizing polymatroidal $f$ subject to a $k$-uniform matroid constraint.
- Might be useful to allow an arbitrary matroid (e.g., partition matroid $\mathcal{I} = \{ X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell \}$, or a transversal, etc).
- Knapsack constraint: if each item $v \in V$ has a cost $c(v)$, we may ask for $c(S) \leq b$ where $b$ is a budget, in units of costs. Q: Is $\mathcal{I} = \{ I : c(I) \leq b \}$ the independent sets of a matroid?
- We may wish to maximize $f$ subject to multiple matroid constraints. I.e., $S \in \mathcal{I}_1, S \in \mathcal{I}_2, \ldots, S \in \mathcal{I}_p$ where $\mathcal{I}_i$ are independent sets of the $i^{th}$ matroid.
- Combinations of the above (e.g., knapsack & multiple matroid constraints).
Greedy over multiple matroids

- Obvious heuristic is to use the greedy step but always stay feasible.
- I.e., Starting with $S_0 = \emptyset$, we repeat the following greedy step

$$S_{i+1} = S_i \cup \left\{ \arg\max_{v \in V \setminus S_i : S_i + v \in \bigcap_{i=1}^p I_i} f(S_i \cup \{v\}) \right\}$$ \hspace{1cm} (14.24)

- That is, we keep choosing next whatever feasible element looks best.
- This algorithm is simple and also has a guarantee

**Theorem 14.5.1**

*Given a polymatroid function $f$, and set of matroids $\{M_j = (E, I_j)\}_{j=1}^p$, the above greedy algorithm returns sets $S_i$ such that for each $i$ we have $f(S_i) \geq \frac{1}{p+1} \max_{|S| \leq i, S \in \bigcap_{i=1}^p I_i} f(S)$, assuming such sets exists.*

- For one matroid, we have a 1/2 approximation.
- Very easy algorithm, Minoux trick still possible, while addresses multiple matroid constraints — but the bound is not that good when there are many matroids.

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Matroid Intersection and Bipartite Matching

- Why might we want to do matroid intersection?
- Consider bipartite graph $G = (V, F, E)$. Define two partition matroids $M_V = (E, I_V)$, and $M_F = (E, I_F)$.
- Independence in each matroid corresponds to:
  1. $I \in I_V$ if $|I \cap (V, f)| \leq 1$ for all $f \in F$,
  2. and $I \in I_F$ if $|I \cap (v, F)| \leq 1$ for all $v \in V$.

Therefore, a matching in $G$ is simultaneously independent in both $M_V$ and $M_F$ and finding the maximum matching is finding the maximum cardinality set independent in both matroids.
- In bipartite graph case, therefore, can be solved in polynomial time.
Let $G_1 = (V_1, E)$ and $G_2 = (V_2, E)$ be two graphs on an isomorphic set of edges (let’s just give them same names $E$).

Consider two cycle matroids associated with these graphs $M_1 = (E, I_1)$ and $M_2 = (E, I_2)$. They might be very different (e.g., an edge might be between two distinct nodes in $G_1$ but the same edge is a loop in multi-graph $G_2$.)

We may wish to find the maximum size edge-induced subgraph that is still forest in both graphs (i.e., adding any edges will create a circuit in either $M_1$, $M_2$, or both).

This is again a matroid intersection problem.

Definition: a Hamiltonian cycle is a cycle that passes through each node exactly once.

Given directed graph $G$, goal is to find such a Hamiltonian cycle.

From $G$ with $n$ nodes, create $G'$ with $n + 1$ nodes by duplicating (w.l.o.g.) a particular node $v_1 \in V(G)$ to $v_1^+, v_1^-$, and have all outgoing edges from $v_1$ come instead from $v_1^-$ and all edges incoming to $v_1$ go instead to $v_1^+$.

Let $M_1$ be the cycle matroid on $G'$.

Let $M_2$ be the partition matroid having as independent sets those that have no more than one edge leaving any node — i.e., $I \in \mathcal{I}(M_2)$ if $|I \cap \delta^-(v)| \leq 1$ for all $v \in V(G')$.

Let $M_3$ be the partition matroid having as independent sets those that have no more than one edge entering any node — i.e., $I \in \mathcal{I}(M_3)$ if $|I \cap \delta^+(v)| \leq 1$ for all $v \in V(G')$.

Then a Hamiltonian cycle exists iff there is an $n$-element intersection of $M_1$, $M_2$, and $M_3$. Recall, the traveling salesperson problem (TSP) is the problem to, given a directed graph, start at a node, visit all cities, and return to the starting point. Optimal solution should be as small as possible.
Greedy over multiple matroids: Generalized Bipartite Matching

- Generalized bipartite matching (i.e., max bipartite matching with submodular costs on the edges). Use two partition matroids (as mentioned earlier in class)
- Useful in natural language processing: Ex. Computer language translation, find an alignment between two language strings.
- Consider bipartite graph $G = (E, F, V)$ where $E$ and $F$ are the left/right set of nodes, respectively, and $V$ is the set of edges.
- $E$ corresponds to, say, an English language sentence and $F$ corresponds to a French language sentence — goal is to form a matching (an alignment) between the two.

Greedy over $> 1$ matroids: Multiple Language Alignment

- Consider English string and French string, set up as a bipartite graph.

I have ... as an example of public ownership

je le ai ... comme exemple de propriété publique
Greedy over > 1 matroids: Multiple Language Alignment

- One possible alignment, a matching, with score as sum of edge weights.

I have ... as an example of public ownership

je le ai ... comme exemple de propriété publique

- Edges incident to English words constitute an edge partition

I have ... as an example of public ownership

je le ai ... comme exemple de propriété publique

- The two edge partitions can be used to set up two 1-partition matroids on the edges.

- For each matroid, a set of edges is independent only if the set intersects each partition block no more than one time.
Greedy over $> 1$ matroids: Multiple Language Alignment

- Edges incident to French words constitute an edge partition

I have ... as an example of public ownership

je le ai ... comme exemple de propriété publique

- The two edge partitions can be used to set up two 1-partition matroids on the edges.
- For each matroid, a set of edges is independent only if the set intersects each partition block no more than one time.

Typical to use bipartite matching to find an alignment between the two language strings.

As we saw, this is equivalent to two 1-partition matroids and a non-negative modular cost function on the edges.

We can generalize this using a polymatroid cost function on the edges, and two $k$-partition matroids, allowing for “fertility” in the models:

Fertility at most 1

. . . the ... of public ownership

. . . le ... de propriété publique
Typical to use bipartite matching to find an alignment between the two language strings.

As we saw, this is equivalent to two 1-partition matroids and a non-negative modular cost function on the edges.

We can generalize this using a polymatroid cost function on the edges, and two $k$-partition matroids, allowing for “fertility” in the models:

Fertility at most 2

Generalizing further, each block of edges in each partition matroid can have its own “fertility” limit:

$$\mathcal{I} = \{X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell\}.$$
Greedy over multiple matroids: Submodular Welfare

- Submodular Welfare Maximization: Consider $E$ a set of $m$ goods to be distributed/partitioned among $n$ people (“players”).
- Each player has a submodular “valuation” function, $g_i : 2^E \to \mathbb{R}_+$ that measures how “desirable” or “valuable” a given subset $A \subseteq E$ of goods are to that player.
- Assumption: No good can be shared between multiple players, each good must be allocated to a single player.
- Goal of submodular welfare: Partition the goods $E = E_1 \cup E_2 \cup \cdots \cup E_n$ into $n$ blocks in order to maximize the submodular social welfare, measured as:

$$\text{submodular-social-welfare}(E_1, E_2, \ldots, E_n) = \sum_{i=1}^{n} g_i(E_i). \quad (14.25)$$

- We can solve this via submodular maximization subject to multiple matroid independence constraints as we next describe...

Submodular Welfare: Submodular Max over matroid partition

- Create new ground set $E'$ as disjoint union of $n$ copies of the ground set. I.e.,

$$E' = E \uplus E \uplus \cdots \uplus E$$

(14.26)

- Let $E^{(i)} \subset E'$ be the $i^{th}$ block of $E'$.
- For any $e \in E$, the corresponding element in $E^{(i)}$ is called $(e, i) \in E^{(i)}$ (each original element is tagged by integer).
- For $e \in E$, define $E_e = \{(e', i) \in E' : e' = e\}$.
- Hence, $\{E_e\}_{e \in E}$ is a partition of $E'$, each block of the partition for one of the original elements in $E$.
- Create a 1-partition matroid $\mathcal{M} = (E', \mathcal{I})$ where

$$\mathcal{I} = \{ S \subseteq E' : \forall e \in E, |S \cap E_e| \leq 1 \} \quad (14.27)$$
Hence, \( S \) is independent in matroid \( \mathcal{M} = (E', I) \) if \( S \) uses each original element no more than once.

Create submodular function \( f' : 2^{E'} \rightarrow \mathbb{R}_+ \) with 
\[
f'(S) = \sum_{i=1}^{n} g_i(S \cap E^{(i)}).
\]

Submodular welfare maximization becomes matroid constrained submodular max \( \max \{ f'(S) : S \in I \} \), so greedy algorithm gives a 1/2 approximation.

---

Submodular Social Welfare

- Have \( n = 6 \) people (who don’t like to share) and \( |E| = m = 7 \) pieces of sushi. E.g., \( e \in E \) might be \( e = "salmon roll" \).
- Goal: distribute sushi to people to maximize social welfare.
- Ground set disjoint union 
  \[
  E \uplus E \uplus E \uplus E \uplus E \uplus E.
  \]
- Partition matroid partitions: 
  \[
  E_{e_1} \uplus E_{e_2} \uplus E_{e_3} \uplus E_{e_4} \uplus E_{e_5} \uplus E_{e_6} \uplus E_{e_7}.
  \]
- independent allocation
- non-independent allocation
Have \( n = 6 \) people (who don’t like to share) and \( |E| = m = 7 \) pieces of sushi. E.g., \( e \in E \) might be \( e = \text{"salmon roll"} \).

Goal: distribute sushi to people to maximize social welfare.

Ground set disjoint union
\[
E \uplus E \uplus E \uplus E \uplus E \uplus E \uplus E.
\]

Partition matroid partitions:
\[
E_{e_1} \cup E_{e_2} \cup E_{e_3} \cup E_{e_4} \cup E_{e_5} \cup E_{e_6} \cup E_{e_7}.
\]

independent allocation
non-independent allocation
Have $n = 6$ people (who don’t like to share) and $|E| = m = 7$ pieces of sushi. E.g., $e \in E$ might be $e = \text{"salmon roll"}$. 

Goal: distribute sushi to people to maximize social welfare.

Ground set disjoint union $E \cup E \cup E \cup E \cup E \cup E$.

Partition matroid partitions: $E_{e_1} \cup E_{e_2} \cup E_{e_3} \cup E_{e_4} \cup E_{e_5} \cup E_{e_6} \cup E_{e_7}$.

- independent allocation
- non-independent allocation
Monotone Submodular over Knapsack Constraint

- The constraint $|A| \leq k$ is a simple cardinality constraint.
- Consider a non-negative integral modular function $c : E \rightarrow \mathbb{Z}_+$.  
- A knapsack constraint would be of the form $c(A) \leq b$ where $B$ is some integer budget that must not be exceeded. That is $
\max \{ f(A) : A \subseteq V, c(A) \leq b \}$. 
- Important: A knapsack constraint yields an independence system (down closed) but it is not a matroid!
- $c(e)$ may be seen as the cost of item $e$ and if $c(e) = 1$ for all $e$, then we recover the cardinality constraint we saw earlier.

Greedy can be seen as choosing the best gain: Starting with $S_0 = \emptyset$, we repeat the following greedy step

$$S_{i+1} = S_i \cup \left\{ \arg \max_{v \in V \setminus S_i} \left( f(S_i \cup \{v\}) - f(S_i) \right) \right\} \quad (14.28)$$

the gain is $f(\{v\}|S_i) = f(S_i + v) - f(S_i)$, so greedy just chooses next the currently unselected element with greatest gain.

Core idea in knapsack case: Greedy can be extended to choose next whatever looks cost-normalized best, i.e., Starting some initial set $S_0$, we repeat the following cost-normalized greedy step

$$S_{i+1} = S_i \cup \left\{ \arg \max_{v \in V \setminus S_i} \frac{f(S_i \cup \{v\}) - f(S_i)}{c(v)} \right\} \quad (14.29)$$

which we repeat until $c(S_{i+1}) > b$ and then take $S_i$ as the solution.
A Knapsack Constraint

- There are a number of ways of getting approximation bounds using this strategy.
- If we run the normalized greedy procedure starting with $S_0 = \emptyset$, and compare the solution found with the max of the singletons $\max_{v \in V} f(\{v\})$, choosing the max, then we get a $(1 - e^{-1/2}) \approx 0.39$ approximation, in $O(n^2)$ time (Minoux trick also possible for further speed).
- Partial enumeration: On the other hand, we can get a $(1 - e^{-1}) \approx 0.63$ approximation in $O(n^5)$ time if we run the above procedure starting from all sets of cardinality three (so restart for all $S_0$ such that $|S_0| = 3$), and compare that with the best singleton and pairwise solution.
- Extending something similar to this to $d$ simultaneous knapsack constraints is possible as well.

Local Search Algorithms

From J. Vondrak

- Local search involves switching up to $t$ elements, as long as it provides a (non-trivial) improvement; can iterate in several phases. Some examples follow:
  - $1/3$ approximation to unconstrained non-monotone maximization [Feige, Mirrokni, Vondrak, 2007]
  - $1/(k + 2 + \frac{1}{k} + \delta_t)$ approximation for non-monotone maximization subject to $k$ matroids [Lee, Mirrokni, Nagarajan, Sviridenko, 2009]
  - $1/(k + \delta_t)$ approximation for monotone submodular maximization subject to $k \geq 2$ matroids [Lee, Sviridenko, Vondrak, 2010].
What About Non-monotone

- Alternatively, we may wish to maximize non-monotone submodular functions. This includes of course graph cuts, and this problem is APX-hard, so maximizing non-monotone functions, even unconstrainedly, is hard.
- If \( f \) is an arbitrary submodular function (so neither polymatroidal, nor necessarily positive or negative), then verifying if the maximum of \( f \) is positive or negative is already NP-hard.
- Therefore, submodular function max in such case is inapproximable unless P=NP (since any such procedure would give us the sign of the max).
- Thus, any approximation algorithm must be for unipolar submodular functions. E.g., non-negative but otherwise arbitrary submodular functions.
- We may get a \( \left( 1 - \frac{1}{3} \right) \) approximation for maximizing non-monotone non-negative submodular functions, with most \( O\left( \frac{1}{\varepsilon} n^3 \log n \right) \) function calls using approximate local maxima.

Submodularity and local optima

- Given any submodular function \( f \), a set \( S \subseteq V \) is a local maximum of \( f \) if \( f(S-v) \leq f(S) \) for all \( v \in S \) and \( f(S+v) \leq f(S) \) for all \( v \in V \setminus S \) (i.e., local in a Hamming ball of radius 1).
- The following interesting result is true for any submodular function:

**Lemma 14.5.2**

*Given a submodular function \( f \), if \( S \) is a local maximum of \( f \), and \( I \subseteq S \) or \( I \supseteq S \), then \( f(I) \leq f(S) \).*

- Idea of proof: Given \( v_1, v_2 \in S \), suppose \( f(S-v_1) \leq f(S) \) and \( f(S-v_2) \leq f(S) \). Submodularity requires \( f(S-v_1) + f(S-v_2) \geq f(S) + f(S-v_1-v_2) \) which would be impossible unless \( f(S-v_1-v_2) \leq f(S) \).
- Similarly, given \( v_1, v_2 \notin S \), and \( f(S+v_1) \leq f(S) \) and \( f(S+v_2) \leq f(S) \). Submodularity requires \( f(S+v_1) + f(S+v_2) \geq f(S) + f(S+v_1+v_2) \) which requires \( f(S+v_1+v_2) \leq f(S) \).
Submodularity and local optima

- Given any submodular function $f$, a set $S \subseteq V$ is a local maximum of $f$ if $f(S - v) \leq f(S)$ for all $v \in S$ and $f(S + v) \leq f(S)$ for all $v \in V \setminus S$ (i.e., local in a Hamming ball of radius 1).

- The following interesting result is true for any submodular function:

**Lemma 14.5.2**

Given a submodular function $f$, if $S$ is a local maximum of $f$, and $I \subseteq S$ or $I \supseteq S$, then $f(I) \leq f(S)$.

- In other words, once we have identified a local maximum, the two intervals in the Boolean lattice $[\emptyset, S]$ and $[S, V]$ can be ruled out as a possible improvement over $S$.

- Finding a local maximum is already hard (PLS-complete), but it is possible to find an approximate local maximum relatively efficiently.

- This is the approach that yields the $(\frac{1}{3} - \frac{\epsilon}{n})$ approximation algorithm.

Linear time algorithm unconstrained non-monotone max

- Tight randomized tight $1/2$ approximation algorithm for unconstrained non-monotone non-negative submodular maximization.

- Buchbinder, Feldman, Naor, Schwartz 2012. Recall $[a]_+ = \max(a, 0)$.

**Algorithm 3:** Randomized Linear-time non-monotone submodular max

1. Set $L \leftarrow \emptyset$; $U \leftarrow V$ /* Lower $L$, upper $U$. Invariant: $L \subseteq U$ */
2. Order elements of $V = (v_1, v_2, \ldots, v_n)$ arbitrarily;
3. for $i = 0 \ldots |V|$ do
   4. $a \leftarrow [f(v_i|L)]_+$; $b \leftarrow [-f(U|U \setminus \{v_i\})]_+$;
   5. if $a = b = 0$ then $p \leftarrow 1/2$;
   6. else $p \leftarrow a/(a + b)$;
   7. if Flip of coin with $\Pr(\text{heads}) = p$ draws heads then
      8. $L \leftarrow L \cup \{v_i\}$;
   9. Otherwise /* if the coin drew tails, an event with prob. $1 - p$ */
      10. $U \leftarrow U \setminus \{v\}$
11. return $L$ (which is the same as $U$ at this point)
Each “sweep” of the algorithm is $O(n)$.

Running the algorithm $1 \times$ (with an arbitrary variable order) results in a 1/3 approximation.

The 1/2 guarantee is in expected value (the expected solution has the 1/2 guarantee).

In practice, run it multiple times, each with a different random permutation of the elements, and then take the cumulative best.

It may be possible to choose the random order smartly to get better results in practice.

More general still: multiple constraints different types

In the past several years, there has been a plethora of papers on maximizing both monotone and non-monotone submodular functions under various combinations of one or more knapsack and/or matroid constraints.

The approximation quality is usually some function of the number of matroids, and is often not a function of the number of knapsacks.

Often the computational costs of the algorithms are prohibitive (e.g., exponential in $k$) with large constants, so these algorithms might not scale.

On the other hand, these algorithms offer deep and interesting intuition into submodular functions, beyond what we have covered here.
Some results on submodular maximization

- As we’ve seen, we can get $1 - 1/e$ for non-negative monotone submodular (polymatroid) functions with greedy algorithm under cardinality constraints, and this is tight.
- For general matroid, greedy reduces to $1/2$ approximation (as we’ve seen).
- We can recover $1 - 1/e$ approximation using the continuous greedy algorithm on the multilinear extension and then using pipage rounding to re-integerize the solution (see J. Vondrak’s publications).
- More general constraints are possible too, as we see on the next table (for references, see Jan Vondrak’s publications http://theory.stanford.edu/~jvondrak/).

Submodular Max Summary - 2012: From J. Vondrak

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Approximation</th>
<th>Hardness</th>
<th>Technique</th>
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<tbody>
<tr>
<td>$</td>
<td>S</td>
<td>\leq k$ matroid</td>
<td>$1 - 1/e$</td>
</tr>
<tr>
<td>$O(1)$ knapsacks</td>
<td>$1 - 1/e$</td>
<td>$1 - 1/e$</td>
<td>multilinear ext.</td>
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<tr>
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<td>$k + \epsilon$</td>
<td>$k/\log k$</td>
<td>local search</td>
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<tr>
<td>$k$ matroids and $O(1)$ knapsacks</td>
<td>$O(k)$</td>
<td>$k/\log k$</td>
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<tr>
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</table>
We’ve spent much time discussing SFM and the polymatroidal polytope, and in general polyhedral approaches for SFM. Most of the approaches for submodular max have not used such an approach, probably due to the difficulty in computing the “concave extension” of a submodular function (the convex extension is easy, namely the Lovász extension).

A paper by Chekuri, Vondrak, and Zenklusen (2011) make some progress on this front using multilinear extensions.

**Multilinear extension**

**Definition 14.5.3**

For a set function \( f : 2^V \rightarrow \mathbb{R} \), define its multilinear extension \( F : [0, 1]^V \rightarrow \mathbb{R} \) by

\[
F(x) = \sum_{S \subseteq V} f(S) \prod_{i \in S} x_i \prod_{j \in V \setminus S} (1 - x_j)
\]

Note that \( F(x) = E f(\hat{x}) \) where \( \hat{x} \) is a random binary vector over \( \{0, 1\}^V \) with elements independent w. probability \( x_i \) for \( \hat{x}_i \).

While this is defined for any set function, we have:

**Lemma 14.5.4**

Let \( F : [0, 1]^V \rightarrow \mathbb{R} \) be multilinear extension of set function \( f : 2^V \rightarrow \mathbb{R} \), then

- If \( f \) is monotone non-decreasing, then \( \frac{\partial F}{\partial x_i} \geq 0 \) for all \( i \in V, x \in [0, 1]^V \).
- If \( f \) is submodular, then \( \frac{\partial^2 F}{\partial x_i \partial x_j} \leq 0 \) for all \( i, j \in V, x \in [0, 1]^V \).
Multilinear extension

- Moreover, we have

**Lemma 14.5.5**

Let $F : [0, 1]^V \to \mathbb{R}$ be multilinear extension of set function $f : 2^V \to \mathbb{R}$, then

- If $f$ is monotone non-decreasing, then $F$ is non-decreasing along any line of direction $d \in \mathbb{R}^E$ with $d \geq 0$

- If $f$ is submodular, then $F$ is concave along any line of direction $d \geq 0$, and is convex along any line of direction $1_v - 1_w$ for any $v, w \in V$.

- Another connection between submodularity and convexity/concavity
- but note, unlike the Lovász extension, this function is neither.

Submodular Max and polyhedral approaches

- Basic idea: Given a set of constraints $\mathcal{I}$, we form a polytope $P_{\mathcal{I}}$ such that $\{1_I : I \in \mathcal{I}\} \subseteq P_{\mathcal{I}}$

- We find $\max_{x \in P_{\mathcal{I}}} F(x)$ where $F(x)$ is the multi-linear extension of $f$, to find a fractional solution $x^*$

- We then round $x^*$ to a point on the hypercube, thus giving us a solution to the discrete problem.
In the recent paper by Chekuri, Vondrak, and Zenklusen, they show:

1) constant factor approximation algorithm for \( \max \{ F(x) : x \in P \} \) for any down-monotone solvable polytope \( P \) and \( F \) multilinear extension of any non-negative submodular function.

2) A randomized rounding (pipage rounding) scheme to obtain an integer solution

3) An optimal \((1 - 1/e)\) instance of their rounding scheme that can be used for a variety of interesting independence systems, including \( O(1) \) knapsacks, \( k \) matroids and \( O(1) \) knapsacks, a \( k \)-matchoid and \( \ell \) sparse packing integer programs, and unsplittable flow in paths and trees.

Also, Vondrak showed that this scheme achieves the \( \frac{1}{c} (1 - e^{-c}) \) curvature based bound for any matroid, which matches the bound we had earlier for uniform matroids with standard greedy.

In practice, one needs to do Monte-Carlo methods to estimate the multilinear extension (so further approximations would apply).