Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 14 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee596b_spring_2016/

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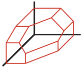
$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

$= f(A_r) + 2f(C) + f(B_r)$ $\quad = f(A_r) + f(C) + f(B_r)$ $\quad = f(A \cap B)$



Cumulative Outstanding Reading

- Read chapters 2 and 3 from Fujishige's book.
- Read chapter 1 from Fujishige's book.

Announcements, Assignments, and Reminders

- Homework 4, available now at our assignment dropbox (https://canvas.uw.edu/courses/1039754/assignments), due (electronically) Wednesday (5/25) at 11:55pm.
- Homework 3, available at our assignment dropbox (https://canvas.uw.edu/courses/1039754/assignments), due (electronically) Monday (5/2) at 11:55pm.
- Homework 2, available at our assignment dropbox (https://canvas.uw.edu/courses/1039754/assignments), due (electronically) Monday (4/18) at 11:55pm.
- Homework 1, available at our assignment dropbox (https://canvas.uw.edu/courses/1039754/assignments), due (electronically) Friday (4/8) at 11:55pm.
- Weekly Office Hours: Mondays, 3:30-4:30, or by skype or google hangout (set up meeting via our our discussion board (https://canvas.uw.edu/courses/1039754/discussion_topics)).

Class Road Map - IT-I

- L1(3/28): Motivation, Applications, & Basic Definitions
- L2(3/30): Machine Learning Apps (diversity, complexity, parameter, learning target, surrogate).
- L3(4/4): Info theory exs, more apps, definitions, graph/combinatorial examples, matrix rank example, visualization
- L4(4/6): Graph and Combinatorial Examples, matrix rank, Venn diagrams, examples of proofs of submodularity, some useful properties
- L5(4/11): Examples & Properties, Other Defs., Independence
- L6(4/13): Independence, Matroids, Matroid Examples, matroid rank is submodular
- L7(4/18): Matroid Rank, More on Partition Matroid, System of Distinct Reps, Transversals, Transversal Matroid,
- L8(4/20): Transversals, Matroid and representation, Dual Matroids,
- L9(4/25): Dual Matroids, Properties, Combinatorial Geometries, Matroid and Greedy
- L10(4/27): Matroid and Greedy, Polyhedra, Matroid Polytopes,
- L11(5/2): From Matroids to Polymatroids, Polymatroids
- L12(5/4): Polymatroids, Polymatroids and Greedy
- L13(5/9): Polymatroids and Greedy; Possible Polytopes; Extreme Points; Polymatroids, Greedy, and Cardinality Constrained Maximization
- L14(5/11): Cardinality Constrained Maximization; Curvature; Submodular Max w. Other Constraints
- L15(5/16):
- L16(5/18):
- L17(5/23):
- L18(5/25):
- L19(6/1):
- L20(6/6): Final Presentations maximization.

Finals Week: June 6th-10th, 2016.

Polymatroidal polyhedron and greedy

- Thus, restating the above results into a single complete theorem, we have a result very similar to what we saw for matroids (i.e., Theorem **??**)

Theorem 14.2.1

If $f : 2^E \to \mathbb{R}_+$ is given, and P is a polytope in $\mathbb{R}_+^E$ of the form $P = \left\{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\right\}$, then the greedy solution to the problem $\max(wx : x \in P)$ is $\forall w$ optimum iff f is monotone non-decreasing submodular (i.e., iff P is a polymatroid).

Multiple Polytopes associated with arbitrary f

- Given an arbitrary submodular function $f : 2^V \to R$ (not necessarily a polymatroid function, so it need not be positive, monotone, etc.).
- If $f(\emptyset) \neq 0$, can set $f'(A) = f(A) - f(\emptyset)$ without destroying submodularity. This does not change any minima, (i.e., $\operatorname{argmin}_A f(A) = \operatorname{argmin}_{A'} f'(A)$) so assume all functions are normalized $f(\emptyset) = 0$.
- We can define several polytopes:

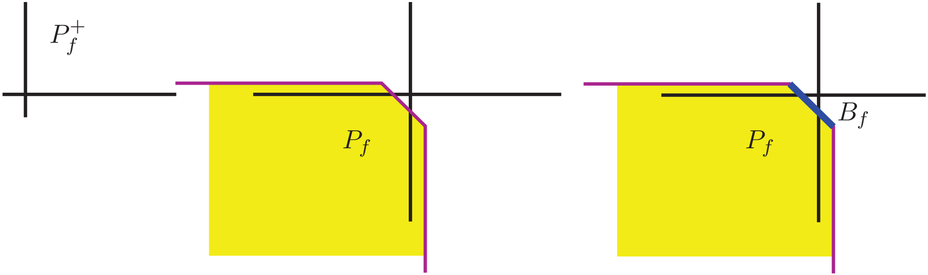
$$P_f = \left\{x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E\right\} \tag{14.1}$$

$$P_f^+ = P_f \cap \left\{x \in \mathbb{R}^E : x \geq 0\right\} \tag{14.2}$$

$$B_f = P_f \cap \left\{x \in \mathbb{R}^E : x(E) = f(E)\right\} \tag{14.3}$$

- P_f is what is sometimes called the extended polytope (sometimes notated as EP_f.
- P_f^+ is P_f restricted to the positive orthant.
- B_f is called the base polytope, analogous to the base in matroid.

Multiple Polytopes in 2D associated with f



$$P_f^+ = P_f \cap \left\{ x \in \mathbb{R}^E : x \geq 0 \right\} \tag{14.1}$$

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$$B_f = P_f \cap \left\{ x \in \mathbb{R}^E : x(E) = f(E) \right\} \tag{14.3}$$

A polymatroid function's polyhedron is a polymatroid.

Theorem 14.2.1

Let f be a submodular function defined on subsets of E. For any $x \in \mathbb{R}^E$, we have:

$$\textit{rank}(x) = \max\left(y(E) : y \leq x, y \in P_f\right) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{14.1}$$

Essentially the same theorem as Theorem **??**, but note P_f rather than P_f^+. Taking $x = 0$ we get:

Corollary 14.2.2

Let f be a submodular function defined on subsets of E. We have:

$$\textit{rank}(0) = \max\left(y(E) : y \leq 0, y \in P_f\right) = \min\left(f(A) : A \subseteq E\right) \tag{14.2}$$

Polymatroid extreme points

Theorem 14.2.1

For a given ordering $E = (e_1, \dots, e_m)$ of E and a given $E_i = (e_1, \dots, e_i)$ and x generated by E_i using the greedy procedure ($x(e_i) = f(e_i|E_{i-1})$), then x is an extreme point of P_f

Proof.

- We already saw that $x \in P_f$ (Theorem **??**).
- To show that x is an extreme point of P_f, note that it is the unique solution of the following system of equations

$$x(E_j) = f(E_j) \text{ for } 1 \leq j \leq i \leq m \tag{14.5}$$

$$x(e) = 0 \text{ for } e \in E \setminus E_i \tag{14.6}$$

There are $i \leq m$ equations and $i \leq m$ unknowns, and simple Gaussian elimination gives us back the x constructed via the Greedy algorithm!!

Polymatroid extreme points

- Moreover, we have (and will ultimately prove)

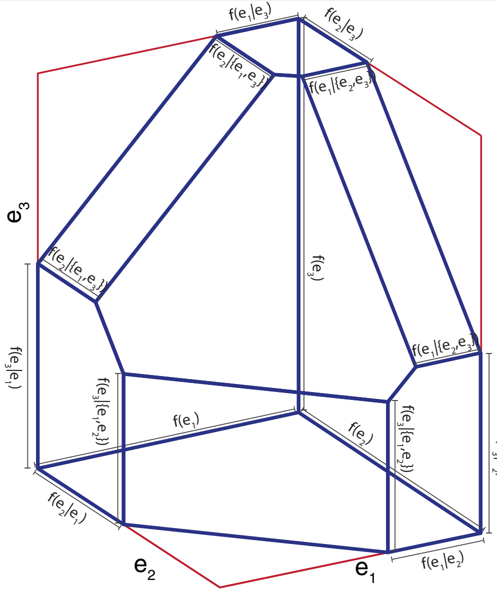
Corollary 14.2.2

If x is an extreme point of P_f and $B \subseteq E$ is given such that $\text{supp}(x) = \{e \in E : x(e) \neq 0\} \subseteq B \subseteq \cup(A : x(A) = f(A)) = \text{sat}(x)$, then x is generated using greedy by some ordering of B.

- Note, $\text{sat}(x) = \mathsf{cl}(x) = \cup(A : x(A) = f(A))$ is also called the closure of x (recall that sets A such that $x(A) = f(A)$ are called tight, and such sets are closed under union and intersection, as seen in Lecture 8, Theorem **??**)
- Thus, $\mathsf{cl}(x)$ is a tight set.
- Also, $\mathsf{supp}(x) = \{e \in E : x(e) \neq 0\}$ is called the support of x.
- For arbitrary x, $\mathsf{supp}(x)$ is not necessarily tight, but for an extreme point, $\mathsf{supp}(x)$ is.

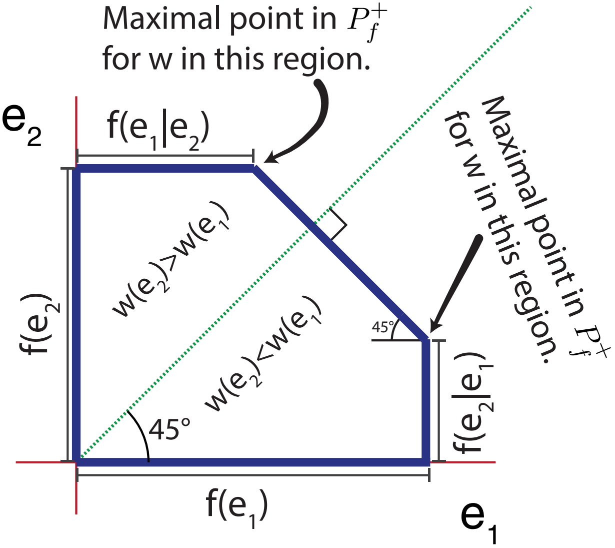
Polymatroid with labeled edge lengths

- Recall $f(e|A) = f(A+e) - f(A)$
- Notice how submodularity, $f(e|B) \le f(e|A)$ for $A \subseteq B$, defines the shape of the polytope.
- In fact, we have strictness here $f(e|B) < f(e|A)$ for $A \subset B$.
- Also, consider how the greedy algorithm proceeds along the edges of the polytope.



Intuition: why greedy works with polymatroids

- Given w, the goal is to find $x = (x(e_1), x(e_2))$ that maximizes $x^\intercal w = x(e_1)w(e_1) + x(e_2)w(e_2)$.
- If $w(e_2) > w(e_1)$ the upper extreme point indicated maximizes $x^\intercal w$ over $x \in P_f^+$.
- If $w(e_2) < w(e_1)$ the lower extreme point indicated maximizes $x^\intercal w$ over $x \in P_f^+$.



Maximal point in P_f^+ for w in this region.

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The Greedy Algorithm: $1 - 1/e$ intuition.

- At step $i < k$, greedy chooses v_i to maximize $f(v|S_i)$.

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- Let S^* be optimal solution (of size k) and $\text{OPT} = f(S^*)$. By submodularity, we will show:

$$\exists v \in V \setminus S_i : f(v|S_i) = f(S_i + v|S_i) \geq \frac{1}{k}(\text{OPT} - f(S_i)) \tag{14.1}$$

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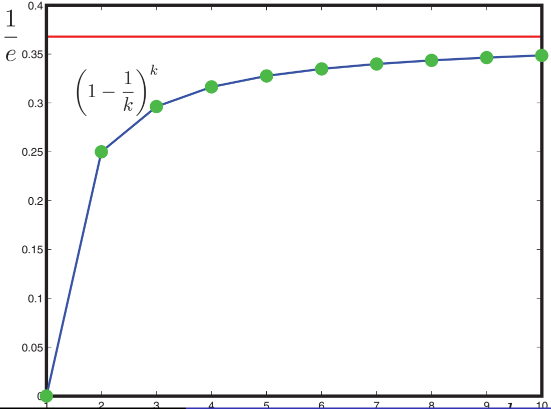

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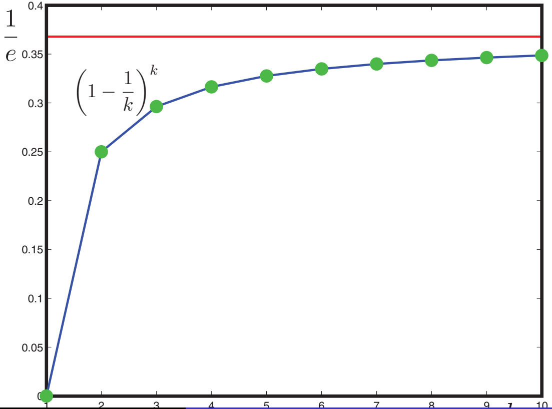

Equation (14.10) will show that Equation (14.1) $\Rightarrow$:

$$\text{OPT} - f(S_{i+1}) \leq (1 - 1/k)(\text{OPT} - f(S_i))$$

$$\Rightarrow \text{OPT} - f(S_k) \leq (1 - 1/k)^k \text{OPT} \leq 1/e\text{OPT}$$

$$\Rightarrow \text{OPT}(1 - 1/e) \leq f(S_k)$$

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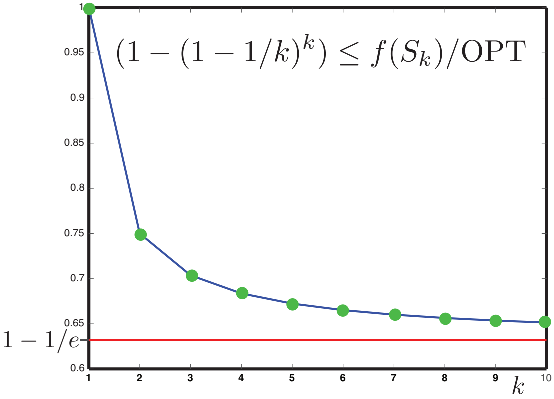

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Cardinality Constrained Polymatroid Max Theorem

Theorem 14.3.1 (Nemhauser et al. 1978)

Given non-negative monotone submodular function $f : 2^V \to \mathbb{R}_+$, *define* $\{S_i\}_{i\geq 0}$ *to be the chain formed by the greedy algorithm (Eqn.* (**??**)*). Then for all* $k, \ell \in \mathbb{Z}_{++}$, *we have:*

$$f(S_\ell) \geq (1 - e^{-\ell/k}) \max_{S:|S|\leq k} f(S) \tag{14.2}$$

and in particular, for $\ell = k$, *we have* $f(S_k) \geq (1 - 1/e) \max_{S:|S|\leq k} f(S)$.

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- Bound is how well does S_ℓ (of size ℓ) do relative to S^*, the optimal set of size k.
- Intuitively, bound should get worse when $\ell < k$ and get better when $\ell > k$.

Cardinality Constrained Polymatroid Max Theorem

Proof of Theorem 14.3.1.

. . .

Cardinality Constrained Polymatroid Max Theorem

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- Fix ℓ (number of items greedy will chose) and k (size of optimal set to compare against).

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- Order $S^* = (v_1^*, v_2^*, \dots, v_k^*)$ arbitrarily.

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- Let $S_i = (v_1, v_2, \dots, v_i)$ be the greedy order chain chosen by the algorithm, for $i \in \{1, 2, \dots, \ell\}$.
- Then the following inequalities (on the next slide) follow:

. . .

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 14.3.1 cont.

. . .

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 14.3.1 cont.

- For all $i < \ell$, we have

$$f(S^*)$$

. . .

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 14.3.1 cont.

- For all $i < \ell$, we have

$$f(S^*) \leq f(S^* \cup S_i)$$

. . .

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 14.3.1 cont.

- For all $i < \ell$, we have

$$f(S^*) \leq f(S^* \cup S_i) = f(S_i) + f(S^*|S_i) \tag{14.3}$$

. . .

Cardinality Constrained Polymatroid Max Theorem

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- For all $i < \ell$, we have

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$$= f(S_i) + \sum_{j=1}^{k} f(v_j^* | S_i \cup \{v_1^*, v_2^*, \dots, v_{j-1}^*\}) \tag{14.4}$$

. . .

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 14.3.1 cont.

- For all $i < \ell$, we have

$$
\begin{aligned}
f(S^*) &\leq f(S^* \cup S_i) = f(S_i) + f(S^*|S_i) && (14.3)\\
&= f(S_i) + \sum_{j=1}^{k} f(v_j^*|S_i \cup \{v_1^*, v_2^*, \ldots, v_{j-1}^*\}) && (14.4)\\
&\leq f(S_i) + \sum_{v \in S^*} f(v|S_i) && (14.5)
\end{aligned}
$$

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&\leq f(S_i) + \sum_{v \in S^*} f(v_{i+1}|S_i)
\end{aligned}
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Cardinality Constrained Polymatroid Max Theorem

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$$\leq f(S_i) + \sum_{v \in S^*} f(v_{i+1}|S_i) = f(S_i) + \sum_{v \in S^*} f(S_{i+1}|S_i) \tag{14.6}$$

$$= f(S_i) + kf(S_{i+1}|S_i) \tag{14.7}$$

. . .

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&\leq f(S_i) + \sum_{v \in S^*} f(v|S_i) && (14.5)\\
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&= f(S_i) + kf(S_{i+1}|S_i) && (14.7)
\end{aligned}$$

- Therefore, we have Equation 14.1, i.e.,:

$$f(S^*) - f(S_i) \leq kf(S_{i+1}|S_i) = k(f(S_{i+1}) - f(S_i)) \qquad (14.8)$$

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Cardinality Constrained Polymatroid Max Theorem

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- Now, $\delta_0 = f(S^*) - f(\emptyset) \leq f(S^*)$ since $f \geq 0$.
- Also, by variational bound $1 - x \leq e^{-x}$ for $x \in \mathbb{R}$, we have
$$\delta_\ell \leq (1 - \frac{1}{k})^\ell \delta_0 \leq e^{-\ell/k} f(S^*) \tag{14.12}$$

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- So solution, in the worst case, quickly gets very good. Typical/practical case is much better.

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- We describe it next:

Minoux's Accelerated Greedy for Submodular Functions

- At stage i in the algorithm, we have a set of gains $f(v|S_i)$ for all $v \notin S_i$. Store these values $\alpha_v \leftarrow f(v|S_i)$ in sorted priority queue.

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- Therefore, if we find a v' such that $f(v'|S_{i+1}) \geq \alpha_v$ for all $v \neq v'$, then since

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- Strategy is: find the $\operatorname{argmax}_{v' \in V \setminus S_{i+1}} \alpha_{v'}$, and then compute the real $f(v'|S_{i+1})$. If it is greater than all other α_v's then that's the next greedy step. Otherwise, replace $\alpha_{v'}$ with its real value, resort ($O(\log n)$), and repeat.

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- Algorithm has been rediscovered (I think) independently (CELF - cost-effective lazy forward selection, Leskovec et al., 2007)
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- If a popped item is fresh, it must be the maximum — this can happen if, at given iteration, v was first popped and neither fresh nor maximum so placed back in the queue, and it then percolates back to the top at which point it is fresh — thereby avoid extra queue check.

Minoux's Accelerated Greedy Algorithm Submodular Max

Algorithm 2: Minoux's Accelerated Greedy Algorithm

```
1  Set $S_0 \leftarrow \emptyset$ ; $i \leftarrow 0$ ; Initialize priority queue $Q$ ;
2  for $v \in E$ do
3      $\text{INSERT}(Q, f(v))$
4  repeat
5      $(v, \alpha) \leftarrow \text{POP}(Q)$ ;
6      if $\alpha$ not "fresh" then
7          recompute $\alpha \leftarrow f(v|S_i)$
8      if (popped $\alpha$ in line 5 was "fresh") OR ($\alpha \geq \text{MAX}(Q)$) then
9          Set $S_{i+1} \leftarrow S_i \cup \{v\}$ ;
10         $i \leftarrow i + 1$ ;
11     else
12         $\text{INSERT}(Q, (v, \alpha))$
13 until $i = |E|$;
```

Minimum Submodular Cover

- Given polymatroid f, goal is to find a covering set of minimum cost:

$$S^* \in \operatorname*{argmin}_{S \subseteq V} |S| \text{ such that } f(S) \geq \alpha \tag{14.18}$$

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- For integer valued f, this greedy algorithm an $O(\log(\max_{s \in V} f(\{s\})))$ approximation. Set cover is hard to approximate with a factor better than $(1 - \epsilon) \log \alpha$, where α is the desired cover constraint.

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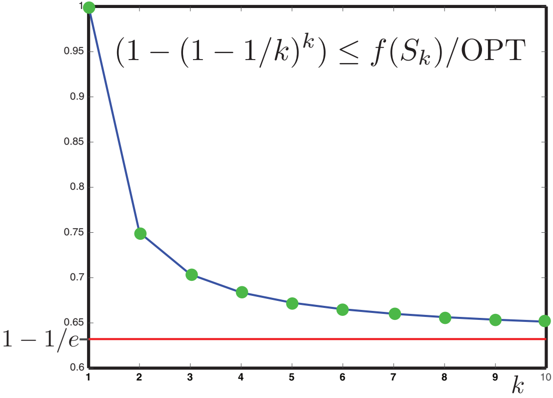
- Submodular cover: min. $|S|$ s.t. $f(S) \geq \alpha$.
- Minoux's accelerated greedy trick.

The Greedy Algorithm: $1 - 1/e$ intuition.

- At step $i < k$, greedy chooses v_i to maximize $f(v|S_i)$.
- Let S^* be optimal solution (of size k) and OPT $= f(S^*)$. By submodularity, we will show:

$$\exists v \in V \setminus S_i : f(v|S_i) = f(S_i + v|S_i) \geq \frac{1}{k}(\text{OPT} - f(S_i)) \tag{14.1}$$



Equation (14.10) will show that Equation (14.1) $\Rightarrow$:

$$\begin{aligned}\text{OPT} - f(S_{i+1}) &\\ &\leq (1-1/k)(\text{OPT} - f(S_i))\\ \Rightarrow \text{OPT} - f(S_k) &\\ &\leq (1-1/k)^k\text{OPT}\\ &\leq 1/e\text{OPT}\\ \Rightarrow \text{OPT}(1-1/e) &\leq f(S_k)\end{aligned}$$

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- Suppose the following holds:

$$E[f(a_{i+1}|A_i)] \geq \frac{f(OPT) - f(A_i)}{k} \tag{14.20}$$

where $A_i = (a_1, a_2, \ldots, a_i)$ are the first i elements chosen by the strategy.

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- How can we produce a randomized greedy strategy, one where each greedy sweep produces a set that, on average, has a $1 - 1/e$ guarantee?
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$$E[f(a_{i+1}|A_i)] \geq \frac{f(OPT) - f(A_i)}{k} \tag{14.20}$$

  where $A_i = (a_1, a_2, \ldots, a_i)$ are the first i elements chosen by the strategy.
- See problem 5, homework 4.

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- It will be remembered the notion of "partial dependence" within polymatroid functions.

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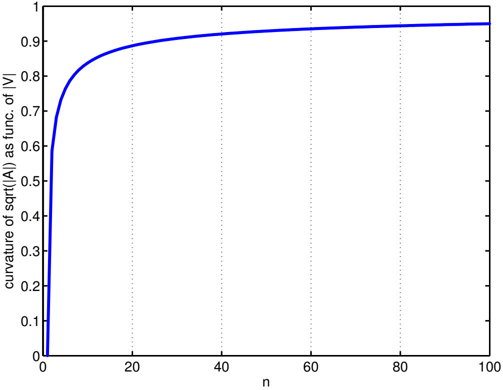

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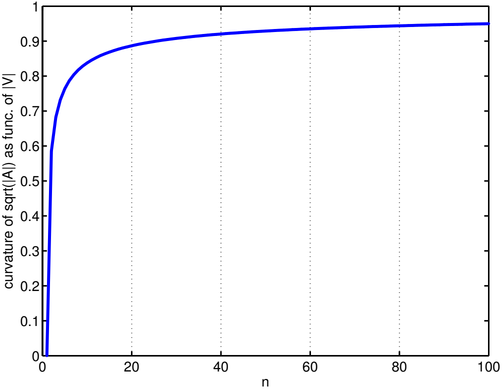

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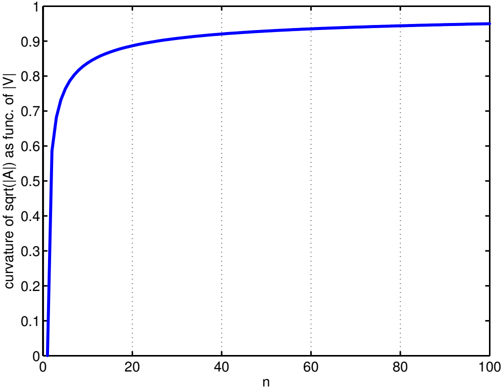

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- Functions of the form $f(S) = \sqrt{m(S)}$ where $m : V \to \mathbb{R}_+$, approximation worse with n if $\min_{i,j} |m(i) - m(j)|$ has a fixed lower bound with increasing n.

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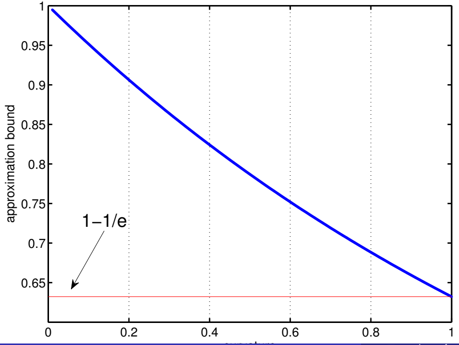
- For k-uniform matroid (i.e., k-cardinality constraints), then approximation factor becomes $\frac{1}{c}(1 - e^{-c})$



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- Combinations of the above (e.g., knapsack & multiple matroid constraints).

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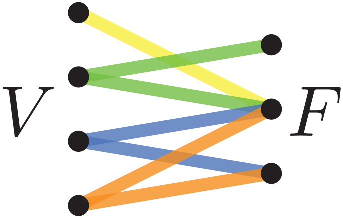

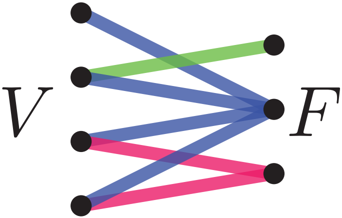

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- Consider bipartite graph $G = (V, F, E)$. Define two partition matroids $M_V = (E, \mathcal{I}_V)$, and $M_F = (E, \mathcal{I}_F)$.
- Independence in each matroid corresponds to:
  1. $I \in \mathcal{I}_V$ if $|I \cap (V, f)| \leq 1$ for all $f \in F$,
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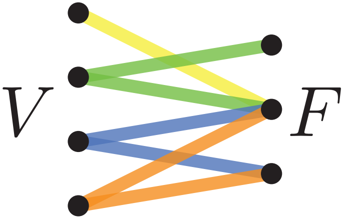

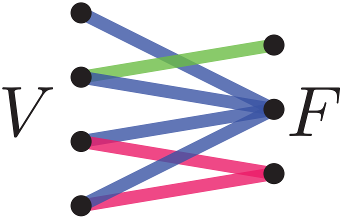

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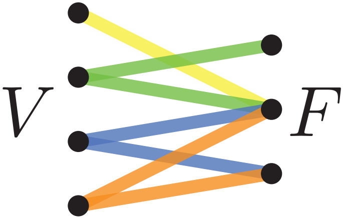

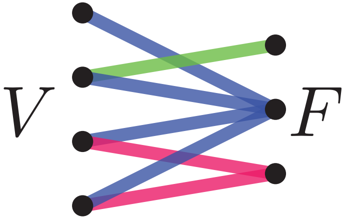

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- In bipartite graph case, therefore, can be solved in polynomial time.

Matroid Intersection and Network Communication

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- This is again a matroid intersection problem.

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- Then a Hamiltonian cycle exists iff there is an n-element intersection of M_1, M_2, and M_3.

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- But bipartite graph example gives us hope for 2 matroids, as in that case we can easily solve $\max |X|$ s.t. $x \in \mathcal{I}_1 \cap \mathcal{I}_2$.

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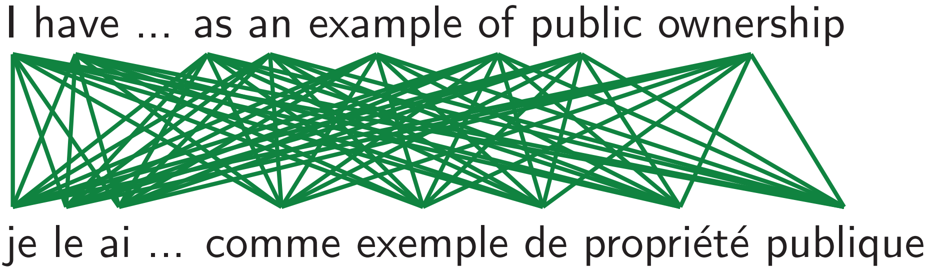
- Consider bipartite graph $G = (E, F, V)$ where E and F are the left/right set of nodes, respectively, and V is the set of edges.
- E corresponds to, say, an English language sentence and F corresponds to a French language sentence — goal is to form a matching (an alignment) between the two.

Greedy over > 1 matroids: Multiple Language Alignment

- Consider English string and French string, set up as a bipartite graph.


I have ... as an example of public ownership
je le ai ... comme exemple de propriété publique

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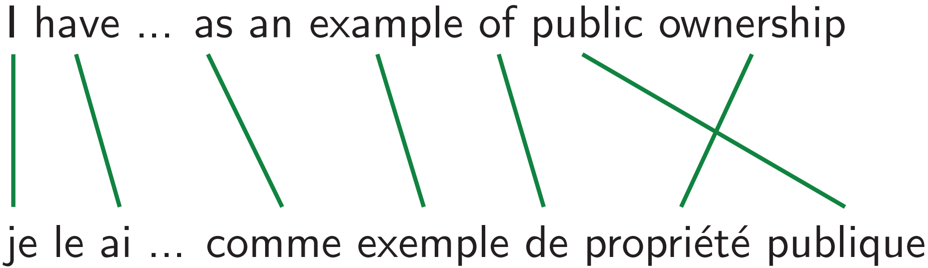
- One possible alignment, a matching, with score as sum of edge weights.



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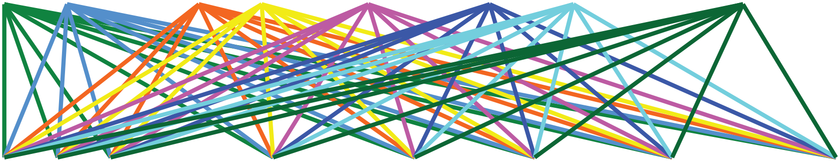

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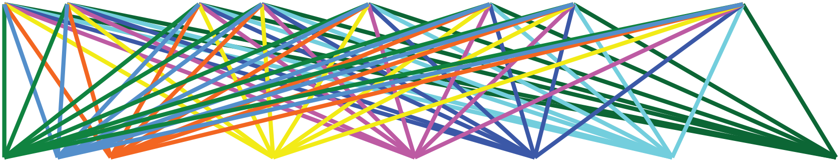

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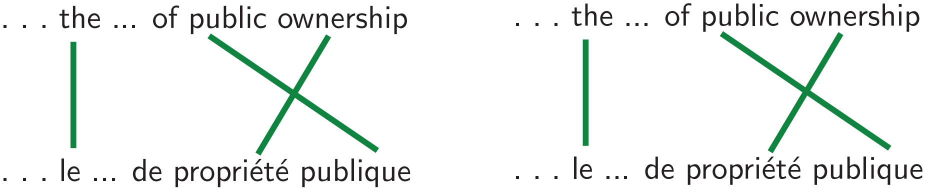

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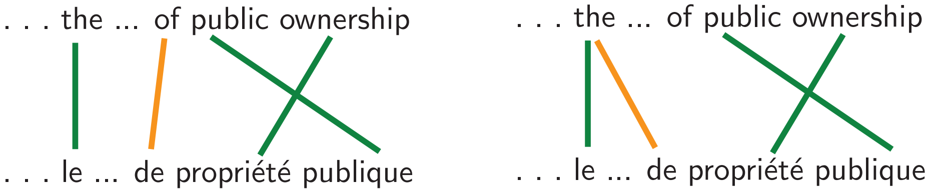
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- Maximizing submodular function subject to multiple matroid constraints addresses this problem.

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- We can solve this via submodular maximization subject to multiple matroid independence constraints as we next describe ...

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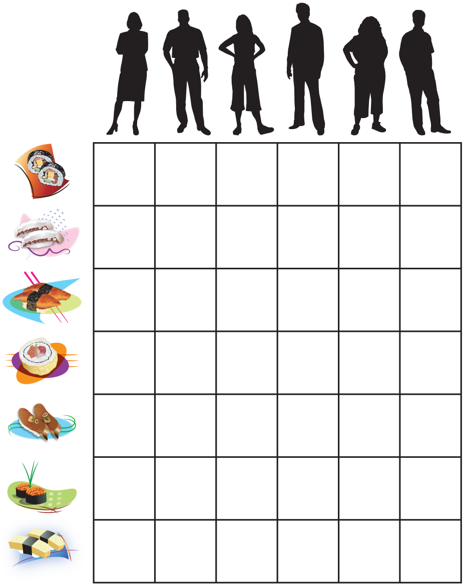
Submodular Social Welfare



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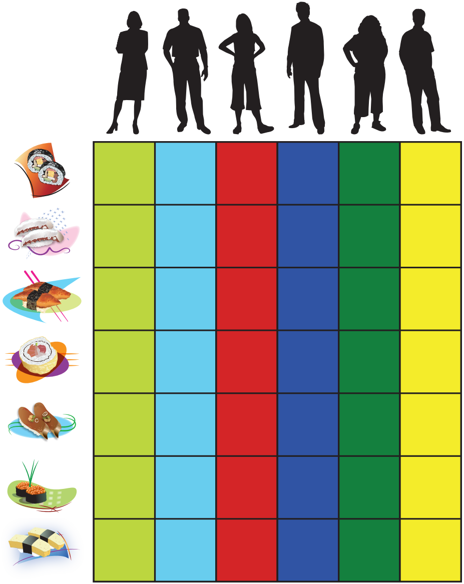

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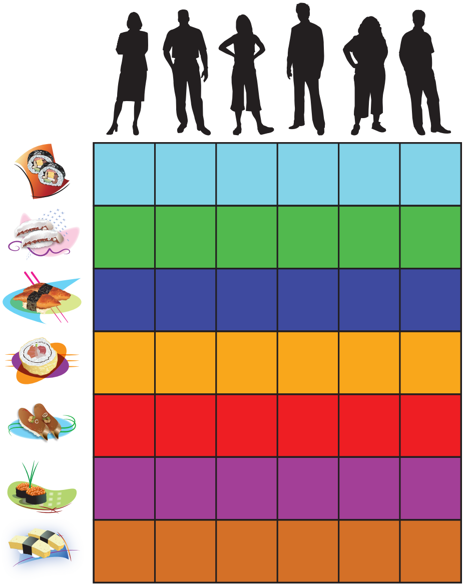

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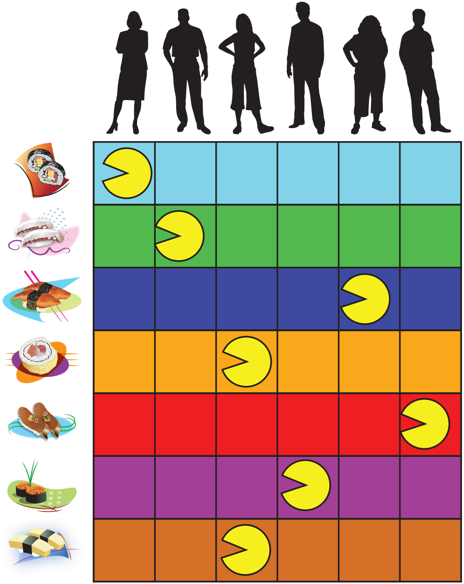

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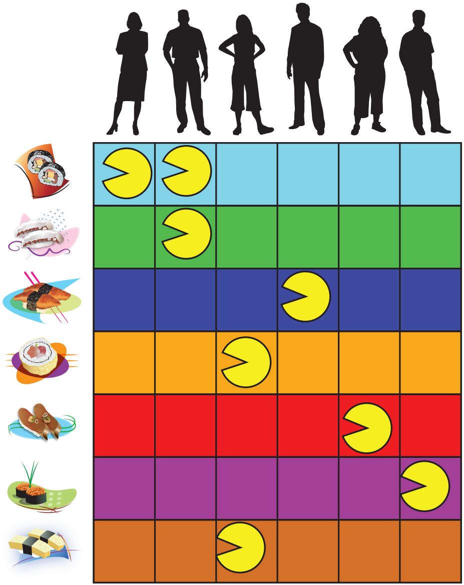

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- Partition matroid partitions: $E_{e_1} \cup E_{e_2} \cup E_{e_3} \cup E_{e_4} \cup E_{e_5} \cup E_{e_6} \cup E_{e_7}$.
- independent allocation
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- Important: A knapsack constraint yields an independence system (down closed) but it is not a matroid!
- $c(e)$ may be seen as the cost of item e and if $c(e) = 1$ for all e, then we recover the cardinality constraint we saw earlier.

Monotone Submodular over Knapsack Constraint

- Greedy can be seen as choosing the best gain: Starting with $S_0 = \emptyset$, we repeat the following greedy step

$$S_{i+1} = S_i \cup \left\{ \operatorname*{argmax}_{v \in V \setminus S_i} \Big(f(S_i \cup \{v\}) - f(S_i) \Big) \right\} \tag{14.28}$$

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- Core idea in knapsack case: Greedy can be extended to choose next whatever looks cost-normalized best, i.e., Starting some initial set S_0, we repeat the following cost-normalized greedy step

$$S_{i+1} = S_i \cup \left\{ \operatorname*{argmax}_{v \in V \setminus S_i} \frac{f(S_i \cup \{v\}) - f(S_i)}{c(v)} \right\} \tag{14.29}$$

  which we repeat until $c(S_{i+1}) > b$ and then take S_i as the solution.

A Knapsack Constraint

- There are a number of ways of getting approximation bounds using this strategy.
- If we run the normalized greedy procedure starting with $S_0 = \emptyset$, and compare the solution found with the max of the singletons $\max_{v \in V} f(\{v\})$, choosing the max, then we get a $(1 - e^{-1/2}) \approx 0.39$ approximation, in $O(n^2)$ time (Minoux trick also possible for further speed)
- Partial enumeration: On the other hand, we can get a $(1 - e^{-1}) \approx 0.63$ approximation in $O(n^5)$ time if we run the above procedure starting from all sets of cardinality three (so restart for all S_0 such that $|S_0| = 3$), and compare that with the best singleton and pairwise solution.
- Extending something similar to this to d simultaneous knapsack constraints is possible as well.

Local Search Algorithms

From J. Vondrak

- Local search involves switching up to t elements, as long as it provides a (non-trivial) improvement; can iterate in several phases. Some examples follow:
- $1/3$ approximation to unconstrained non-monotone maximization [Feige, Mirrokni, Vondrak, 2007]
- $1/(k + 2 + \frac{1}{k} + \delta_t)$ approximation for non-monotone maximization subject to k matroids [Lee, Mirrokni, Nagarajan, Sviridenko, 2009]
- $1/(k + \delta_t)$ approximation for monotone submodular maximization subject to $k \geq 2$ matroids [Lee, Sviridenko, Vondrak, 2010].

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- Thus, any approximation algorithm must be for unipolar submodular functions. E.g., non-negative but otherwise arbitrary submodular functions.
- We may get a $(\frac{1}{3} - \frac{\epsilon}{n})$ approximation for maximizing non-monotone non-negative submodular functions, with most $O(\frac{1}{\epsilon}n^3 \log n)$ function calls using approximate local maxima.

Submodularity and local optima

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- Similarly, given $v_1, v_2 \notin S$, and $f(S + v_1) \leq f(S)$ and $f(S + v_2) \leq f(S)$. Submodularity requires $f(S + v_1) + f(S + v_2) \geq f(S) + f(S + v_1 + v_2)$ which requires $f(S + v_1 + v_2) \leq f(S)$.

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- This is the approach that yields the $(\frac{1}{3} - \frac{\epsilon}{n})$ approximation algorithm.

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Algorithm 6: Randomized Linear-time non-monotone submodular max

```
Set $L \leftarrow \emptyset$ ; $U \leftarrow V$      /* Lower $L$, upper $U$. Invariant: $L \subseteq U$ */ ;
Order elements of $V = (v_1, v_2, \ldots, v_n)$ arbitrarily ;
for $i \leftarrow 0 \ldots |V|$ do
    $a \leftarrow [f(v_i|L)]_+$; $b \leftarrow [-f(U|U \setminus \{v_i\})]_+$ ;
    if $a = b = 0$ then $p \leftarrow 1/2$ ;
    ;
    else $p \leftarrow a/(a+b)$;
    ;
    if Flip of coin with $\Pr(heads) = p$ draws heads then
        $L \leftarrow L \cup \{v_i\}$ ;
    Otherwise /* if the coin drew tails, an event with prob. $1 - p$ */
        $U \leftarrow U \setminus \{v\}$
```

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- In practice, run it multiple times, each with a different random permutation of the elements, and then take the cumulative best.
- It may be possible to choose the random order smartly to get better results in practice.

More general still: multiple constraints different types

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- Often the computational costs of the algorithms are prohibitive (e.g., exponential in k) with large constants, so these algorithms might not scale.
- On the other hand, these algorithms offer deep and interesting intuition into submodular functions, beyond what we have covered here.

Some results on submodular maximization

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- We can recover $1 - 1/e$ approximation using the continuous greedy algorithm on the multilinear extension and then using pipage rounding to re-integerize the solution (see J. Vondrak's publications).
- More general constraints are possible too, as we see on the next table (for references, see Jan Vondrak's publications `http://theory.stanford.edu/~jvondrak/`).

Submodular Max Summary - 2012: From J. Vondrak

Monotone Maximization

Constraint	Approximation	Hardness	Technique
$\lvert S\rvert \leq k$	$1 - 1/e$	$1 - 1/e$	greedy
matroid	$1 - 1/e$	$1 - 1/e$	multilinear ext.
$O(1)$ knapsacks	$1 - 1/e$	$1 - 1/e$	multilinear ext.
k matroids	$k + \epsilon$	$k/\log k$	local search
k matroids and $O(1)$ knapsacks	$O(k)$	$k/\log k$	multilinear ext.

Nonmonotone Maximization

Constraint	Approximation	Hardness	Technique
Unconstrained	$1/2$	$1/2$	combinatorial
matroid	$1/e$	0.48	multilinear ext.
$O(1)$ knapsacks	$1/e$	0.49	multilinear ext.
k matroids	$k + O(1)$	$k/\log k$	local search
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- Most of the approaches for submodular max have not used such an approach, probably due to the difficulty in computing the "concave extension" of a submodular function (the convex extension is easy, namely the Lovász extension).
- A paper by Chekuri, Vondrak, and Zenklusen (2011) make some progress on this front using multilinear extensions.

Multilinear extension

Definition 14.5.3

For a set function $f : 2^V \to \mathbb{R}$, define its multilinear extension $F : [0,1]^V \to \mathbb{R}$ by

$$F(x) = \sum_{S \subseteq V} f(S) \prod_{i \in S} x_i \prod_{j \in V \setminus S} (1 - x_j) \tag{14.30}$$

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- Another connection between submodularity and convexity/concavity

Multilinear extension

- Moreover, we have

Lemma 14.5.5

Let $F : [0,1]^V \to \mathbb{R}$ *be multilinear extension of set function* $f : 2^V \to \mathbb{R}$*, then*

- *If* f *is monotone non-decreasing, then* F *is non-decreasing along any line of direction* $d \in \mathbb{R}^E$ *with* $d \geq 0$
- *If* f *is submodular, then* F *is concave along any line of direction* $d \geq 0$*, and is convex along any line of direction* $\mathbf{1}_v - \mathbf{1}_w$ *for any* $v, w \in V$*.*

- Another connection between submodularity and convexity/concavity
- but note, unlike the Lovász extension, this function is neither.

Submodular Max and polyhedral approaches

- Basic idea: Given a set of constraints $\mathcal{I}$, we form a polytope $P_{\mathcal{I}}$ such that $\{\mathbf{1}_I : I \in \mathcal{I}\} \subseteq P_{\mathcal{I}}$
- We find $\max_{x \in P_{\mathcal{I}}} F(x)$ where $F(x)$ is the multi-linear extension of f, to find a fractional solution x^*
- We then round x^* to a point on the hypercube, thus giving us a solution to the discrete problem.

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- In practice, one needs to do Monte-Carlo methods to estimate the multilinear extension (so further approximations would apply).