Submodular Functions, Optimization, and Applications to Machine Learning
— Spring Quarter, Lecture 12 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee596b_spring_2016/

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\[
f(A) + f(B) \geq f(A \cup B) + f(A \cap B)
\]

\[
\begin{align*}
&= f(A) + 2f(C) + f(B) \\
&= f(A) + f(C) + f(B) \\
&= f(A \cap B)
\end{align*}
\]
Cumulative Outstanding Reading

- Read chapters 2 and 3 from Fujishige’s book.
- Read chapter 1 from Fujishige’s book.
Announcements, Assignments, and Reminders

- **Homework 4**, soon available at our assignment dropbox ([https://canvas.uw.edu/courses/1039754/assignments](https://canvas.uw.edu/courses/1039754/assignments))

- Homework 3, available at our assignment dropbox ([https://canvas.uw.edu/courses/1039754/assignments](https://canvas.uw.edu/courses/1039754/assignments)), due electronically Monday (5/2) at 11:55pm.

- Homework 2, available at our assignment dropbox ([https://canvas.uw.edu/courses/1039754/assignments](https://canvas.uw.edu/courses/1039754/assignments)), due electronically Monday (4/18) at 11:55pm.

- Homework 1, available at our assignment dropbox ([https://canvas.uw.edu/courses/1039754/assignments](https://canvas.uw.edu/courses/1039754/assignments)), due electronically Friday (4/8) at 11:55pm.

- **Weekly Office Hours**: Mondays, 3:30-4:30, or by skype or google hangout (set up meeting via our discussion board ([https://canvas.uw.edu/courses/1039754/discussion_topics](https://canvas.uw.edu/courses/1039754/discussion_topics))).
Class Road Map - IT-I

- L1(3/28): Motivation, Applications, & Basic Definitions
- L3(4/4): Info theory exs, more apps, definitions, graph/combinatorial examples, matrix rank example, visualization
- L4(4/6): Graph and Combinatorial Examples, matrix rank, Venn diagrams, examples of proofs of submodularity, some useful properties
- L5(4/11): Examples & Properties, Other Defs., Independence
- L6(4/13): Independence, Matroids, Matroid Examples, matroid rank is submodular
- L7(4/18): Matroid Rank, More on Partition Matroid, System of Distinct Reps, Transversals, Transversal Matroid,
- L8(4/20): Transversals, Matroid and representation, Dual Matroids,
- L10(4/27): Matroid and Greedy, Polyhedra, Matroid Polytopes,
- L11(5/2): From Matroids to Polymatroids, Polymatroids
- L12(5/4): Polymatroids, Polymatroids and Greedy, Possible Polytopes, Extreme Points
- L13(5/9):
- L14(5/11):
- L15(5/16):
- L16(5/18):
- L17(5/23):
- L18(5/25):
- L19(6/1):

Finals Week: June 6th-10th, 2016.
**Definition 12.2.1 (subvector)**

A subvector of \( x \) is a subvector of \( x \) if \( y \leq x \) (meaning \( y(e) \leq x(e) \) for all \( e \in E \)).

**Definition 12.2.2 (P-basis)**

Given a compact set \( P \subseteq \mathbb{R}_+^E \), for any \( x \in \mathbb{R}_+^E \), a subvector \( y \) of \( x \) is called a **P-basis** of \( x \) if \( y \) maximal in \( P \).

In other words, \( y \) is a **P-basis** of \( x \) if \( y \) is a maximal \( P \)-contained subvector of \( x \).

Here, by \( y \) being “maximal”, we mean that there exists no \( z > y \) (more precisely, no \( z \geq y + \epsilon 1_e \) for some \( e \in E \) and \( \epsilon > 0 \)) having the properties of \( y \) (the properties of \( y \) being: in \( P \), and a subvector of \( x \)).

In still other words: \( y \) is a **P-basis** of \( x \) if:

1. \( y \leq x \) (\( y \) is a subvector of \( x \)); and
2. \( y \in P \) and \( y + \epsilon 1_e \notin P \) for all \( e \in E \) where \( y(e) < x(e) \) and \( \forall \epsilon > 0 \) (\( y \) is maximal \( P \)-contained).
A vector form of rank

- Recall the definition of rank from a matroid $M = (E, \mathcal{I})$.

  \[
  \text{rank}(A) = \max \{|I| : I \subseteq A, I \in \mathcal{I}\} = \max_{I \in \mathcal{I}} |A \cap I|
  \]  

  (12.1)

- **vector rank**: Given a compact set $P \subseteq \mathbb{R}_+^E$, we can define a form of “vector rank” relative to this $P$ in the following way: Given an $x \in \mathbb{R}^E$, we define the vector rank, relative to $P$, as:

  \[
  \text{rank}(x) = \max (y(E) : y \preceq x, y \in P) = \max_{y \in P} (x \land y)(E)
  \]  

  (12.2)

  where $y \preceq x$ is componentwise inequality ($y_i \leq x_i, \forall i$), and where $(x \land y) \in \mathbb{R}_+^E$ has $(x \land y)(i) = \min(x(i), y(i))$.

  - If $B_x$ is the set of $P$-bases of $x$, then $\text{rank}(x) = \max_{y \in B_x} y(E)$.
  - If $x \in P$, then $\text{rank}(x) = x(E)$ ($x$ is its own unique self $P$-basis).
  - If $x_{\min} = \min_{x \in P} x(E)$, and $x \leq x_{\min}$ what then? $-\infty$?

  - In general, might be hard to compute and/or have ill-defined properties.

Next, we look at an object that restrains and cultivates this form of rank.
A **polymatroid** is a compact set $P \subseteq \mathbb{R}^E_+$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For every $x \in \mathbb{R}^E_+$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any $P$-basis of $x$), has the same component sum $y(E)$.

- Vectors within $P$ (i.e., any $y \in P$) are called **independent**, and any vector outside of $P$ is called **dependent**.
- Since all $P$-bases of $x$ have the same component sum, if $\mathcal{B}_x$ is the set of $P$-bases of $x$, then $\text{rank}(x) = y(E)$ for any $y \in \mathcal{B}_x$. 

Matroid and Polymatroid: side-by-side

A Matroid is:

1. a set system \((E, \mathcal{I})\)
2. empty-set containing \(\emptyset \in \mathcal{I}\)
3. down closed, \(\emptyset \subseteq I' \subseteq I \in \mathcal{I} \Rightarrow I' \in \mathcal{I}\).
4. any maximal set \(I\) in \(\mathcal{I}\), bounded by another set \(A\), has the same matroid rank (any maximal independent subset \(I \subseteq A\) has same size \(|I|\)).

A Polymatroid is:

1. a compact set \(P \subseteq \mathbb{R}_+^E\)
2. zero containing, \(0 \in P\)
3. down monotone, \(0 \leq y \leq x \in P \Rightarrow y \in P\)
4. any maximal vector \(y\) in \(P\), bounded by another vector \(x\), has the same vector rank (any maximal independent subvector \(y \leq x\) has same sum \(y(E)\)).
A polymatroid function is a real-valued function $f$ defined on subsets of $E$ which is normalized, non-decreasing, and submodular. That is we have:

1. $f(\emptyset) = 0$ (normalized)
2. $f(A) \leq f(B)$ for any $A \subseteq B \subseteq E$ (monotone non-decreasing)
3. $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$ for any $A, B \subseteq E$ (submodular)

We can define the polyhedron $P^+_f$ associated with a polymatroid function as follows:

$$P^+_f = \{ y \in \mathbb{R}^E_+ : y(A) \leq f(A) \text{ for all } A \subseteq E \}$$

$$= \{ y \in \mathbb{R}^E : y \geq 0, y(A) \leq f(A) \text{ for all } A \subseteq E \}$$
A polymatroid vs. a polymatroid function’s polyhedron

- Summarizing the above, we have:
  - Given a polymatroid function $f$, its associated polytope is given as
    \[ P_f^+ = \{ y \in \mathbb{R}_+^E : y(A) \leq f(A) \text{ for all } A \subseteq E \} \] (12.10)
  - We also have the definition of a polymatroidal polytope $P$ (compact subset, zero containing, down-monotone, and $\forall x$ any maximal independent subvector $y \leq x$ has same component sum $y(E)$).

- Is there any relationship between these two polytopes?
- In the next theorem, we show that any $P_f^+$-basis has the same component sum, when $f$ is a polymatroid function, and $P_f^+$ satisfies the other properties so that $P_f^+$ is a polymatroid.
A polymatroid function’s polyhedron is a polymatroid.

**Theorem 12.2.1**

Let $f$ be a polymatroid function defined on subsets of $E$. For any $x \in \mathbb{R}^E_+$, and any $P_f^+$-basis $y^x \in \mathbb{R}^E_+$ of $x$, the component sum of $y^x$ is

$$y^x(E) = \text{rank}(x) = \max \{ y(E) : y \leq x, y \in P_f^+ \}$$

$$= \min \{ x(A) + f(E \setminus A) : A \subseteq E \} \quad (12.10)$$

As a consequence, $P_f^+$ is a polymatroid, since r.h.s. is constant w.r.t. $y^x$.

Taking $E \setminus B = \text{supp}(x)$ (so elements $B$ are all zeros in $x$), and for $b \notin B$ we make $x(b)$ is big enough, the r.h.s. min has solution $A^* = B$. We recover submodular function from the polymatroid polyhedron via the following:

$$\text{rank} \left( \frac{1}{\epsilon} \mathbf{1}_{E \setminus B} \right) = f(B) = \max \left\{ y(B) : y \in P_f^+ \right\} \quad (12.11)$$

In fact, we will ultimately see a number of important consequences of this theorem (other than just that $P_f^+$ is a polymatroid).
A polymatroid function’s polyhedron is a polymatroid.

Proof.

- Clearly $0 \in P_f^+$ since $f$ is non-negative.
- Also, for any $y \in P_f^+$ then any $x \leq y$ is also such that $x \in P_f^+$. So, $P_f^+$ is down-monotone.
- Now suppose that we are given an $x \in \mathbb{R}^E_+$, and maximal $y^x \in P_f^+$ with $y^x \leq x$ (i.e., $y^x$ is a $P_f^+$-basis of $x$).
- Goal is to show that any such $y^x$ has $y^x(E) = \text{const}$, dependent only on $x$ and also $f$ (which defines the polytope) but not dependent on $y^x$, the particular $P_f^+$-basis.
- Doing so will thus establish that $P_f^+$ is a polymatroid.
A polymatroid function’s polyhedron is a polymatroid.

...proof continued.

- First trivial case: could have $y^x = x$, which happens if $x(A) \leq f(A), \forall A \subseteq E$ (i.e., $x \in P_f^+$ strictly). In such case,

\[
\min (x(A) + f(E \setminus A) : A \subseteq E) = x(E) + \min (f(E \setminus A) - x(E \setminus A) : A \subseteq E)
\]

(12.10)

\[
= x(E) + \min (f(A) - x(A) : A \subseteq E)
\]

(12.11)

\[
= x(E)
\]

(12.12)

\[
= x(E)
\]

(12.13)

- When $x \in P_f^+$, $y = x$ is clearly the solution to

\[
\max \left( y(E) : y \leq x, y \in P_f^+ \right), \text{ so this is tight, and } \text{rank}(x) = x(E).
\]

- This is a value dependent only on $x$ and not on any of its $P_f^+$-bases.
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

- 2nd trivial case: \( x(A) > f(A), \forall A \subseteq E \) (i.e., \( x \notin P_f^+ \) every direction),

Then for any order \((a_1, a_2, \ldots)\) of the elements and \(A_i \triangleq (a_1, a_2, \ldots, a_i)\), we have \( x(a_i) \geq f(a_i) \geq f(a_i \mid A_{i-1})\), the second inequality by submodularity. This gives

\[
\min (x(A) + f(E \setminus A) : A \subseteq E) = x(E) + \min (f(A) - x(A) : A \subseteq E) = x(E) + \min \left( \sum_i f(a_i \mid A_{i-1}) - \sum_i x(a_i) : A \subseteq E \right) = x(E) + \min \left( \sum_i \left( f(a_i \mid A_{i-1}) - x(a_i) \right) : A \subseteq E \right) = x(E) + f(E) - x(E) = f(E) = \max (y(E) : y \in P_f^+).
\]
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

Assume neither trivial case. Because $y^x \in P_f^+$, we have that $y^x(A) \leq f(A)$ for all $A \subseteq E$. 

\[ y^x(A) \leq f(A) \text{ for all } A \subseteq E. \]
... proof continued.

- Assume neither trivial case. Because $y^x \in P^+_f$, we have that $y^x(A) \leq f(A)$ for all $A \subseteq E$.
- We show that the constant is given by
  \[ y^x(E) = \min \left( x(A) + f(E \setminus A) : A \subseteq E \right) \]  
  (12.1)
A polymatroid function’s polyhedron is a polymatroid.

...proof continued.

- Assume neither trivial case. Because $y^x \in P_f^+$, we have that $y^x(A) \leq f(A)$ for all $A \subseteq E$.

- We show that the constant is given by

$$y^x(E) = \min ( x(A) + f(E \setminus A) : A \subseteq E )$$

\hspace{1cm} (12.1)

- For any $P_f^+$-basis $y^x$ of $x$, and any $A \subseteq E$, we have weak relationship:

\begin{align*}
y^x(E) &= y^x(A) + y^x(E \setminus A) \\
\leq x(A) + f(E \setminus A) .
\end{align*}

\hspace{1cm} (12.2)

\hspace{1cm} (12.3)

This follows since $y^x \leq x$ and since $y^x \in P_f^+$. 

...
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

- Assume neither trivial case. Because \( y^x \in P^+_f \), we have that
  \[ y^x(A) \leq f(A) \] for all \( A \subseteq E \).

- We show that the constant is given by
  \[
  y^x(E) = \min \{ x(A) + f(E \setminus A) : A \subseteq E \} 
  \] (12.1)

- For any \( P^+_f \)-basis \( y^x \) of \( x \), and any \( A \subseteq E \), we have weak relationship:
  \[
  y^x(E) = y^x(A) + y^x(E \setminus A) 
  \leq x(A) + f(E \setminus A). 
  \] (12.2)

  This follows since \( y^x \leq x \) and since \( y^x \in P^+_f \).

- This ensures
  \[
  \max \{ y(E) : y \leq x, y \in P^+_f \} \leq \min \{ x(A) + f(E \setminus A) : A \subseteq E \} 
  \] (12.4)
Assume neither trivial case. Because $y^x \in P_f^+$, we have that $y^x(A) \leq f(A)$ for all $A \subseteq E$.

We show that the constant is given by

$$y^x(E) = \min (x(A) + f(E \setminus A) : A \subseteq E) \quad (12.1)$$

For any $P_f^+$-basis $y^x$ of $x$, and any $A \subseteq E$, we have weak relationship:

$$y^x(E) = y^x(A) + y^x(E \setminus A) \leq x(A) + f(E \setminus A). \quad (12.2)$$

This follows since $y^x \leq x$ and since $y^x \in P_f^+$.

This ensures

$$\max \left( y(E) : y \leq x, y \in P_f^+ \right) \leq \min (x(A) + f(E \setminus A) : A \subseteq E) \quad (12.4)$$

Given an $A$ where equality in Eqn. (12.3) holds, above min result follows.
... proof continued.

For any \( y \in P_f^+ \), call a set \( B \subseteq E \) tight if \( y(B) = f(B) \). The union (and intersection) of tight sets \( B, C \) is again tight, since

\[
f(B) + f(C)
\]
... proof continued.

For any \( y \in P_f^+ \), call a set \( B \subseteq E \) tight if \( y(B) = f(B) \). The union (and intersection) of tight sets \( B, C \) is again tight, since

\[
f(B) + f(C) = y(B) + y(C)
\]

(12.5)
... proof continued.

For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets $B, C$ is again tight, since

$$f(B) + f(C) = y(B) + y(C)$$  \hspace{1cm} (12.5)

$$= y(B \cap C) + y(B \cup C)$$  \hspace{1cm} (12.6)
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

For any \( y \in P_f^+ \), call a set \( B \subseteq E \) tight if \( y(B) = f(B) \). The union (and intersection) of tight sets \( B, C \) is again tight, since

\[
f(B) + f(C) = y(B) + y(C) = y(B \cap C) + y(B \cup C) \leq f(B \cap C) + f(B \cup C)
\]
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

For any \( y \in P_f^+ \), call a set \( B \subseteq E \) **tight** if \( y(B) = f(B) \). The union (and intersection) of tight sets \( B, C \) is again tight, since

\[
\begin{align*}
 f(B) + f(C) & = y(B) + y(C) \\
 & = y(B \cap C) + y(B \cup C) \\
 & \leq f(B \cap C) + f(B \cup C) \\
 & \leq f(B) + f(C)
\end{align*}
\]

Our goal is that \( y(B \cap C) = f(B \cap C) \) and \( y(B \cup C) = f(B \cup C) \).
A polymatroid function’s polyhedron is a polymatroid.

...proof continued.

For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets $B, C$ is again tight, since

\begin{align*}
f(B) + f(C) &= y(B) + y(C) \quad (12.5) \\
&= y(B \cap C) + y(B \cup C) \quad (12.6) \\
&\leq f(B \cap C) + f(B \cup C) \quad (12.7) \\
&\leq f(B) + f(C) \quad (12.8)
\end{align*}

which requires equality everywhere above.
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

For any \( y \in P^+_f \), call a set \( B \subseteq E \) tight if \( y(B) = f(B) \). The union (and intersection) of tight sets \( B, C \) is again tight, since

\[
f(B) + f(C) = y(B) + y(C) = y(B \cap C) + y(B \cup C) \leq f(B \cap C) + f(B \cup C) \leq f(B) + f(C)
\]

which requires equality everywhere above.

Because \( y(A) \leq f(A), \forall A \), this means \( y(B \cap C) = f(B \cap C) \) and \( y(B \cup C) = f(B \cup C) \), so both also are tight.
\[ a + b = a' + b' \]
\[ a \leq a' \quad b \leq b' \]
\[ \Rightarrow a = a' \quad b = b' \]

if not 1 \[ a < a' \quad a \leq a' \Rightarrow b > b' \]
For any $y \in P^+_f$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets $B, C$ is again tight, since

$$f(B) + f(C) = y(B) + y(C)$$

$$= y(B \cap C) + y(B \cup C)$$

$$\leq f(B \cap C) + f(B \cup C)$$

$$\leq f(B) + f(C)$$

which requires equality everywhere above.

Because $y(A) \leq f(A), \forall A$, this means $y(B \cap C) = f(B \cap C)$ and $y(B \cup C) = f(B \cup C)$, so both also are tight.

For $y \in P^+_f$, it will be ultimately useful to define this lattice family of tight sets: $D(y) \triangleq \{A : A \subseteq E, y(A) = f(A)\}$. 

...
A polymatroid function’s polyhedron is a polymatroid.

...proof continued.

- Also, we define \( \text{sat}(y) \) \( \overset{\text{def}}{=} \bigcup \{ T : T \in D(y) \} \).

\[
\Rightarrow y(\text{sat}(y)) = f(\text{sat}(y))
\]

If \( e \in e \in D(y) \)

\[
\Rightarrow e \in \text{sat}(y)
\]
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

- Also, we define \( \text{sat}(y) \overset{\text{def}}{=} \bigcup \{ T : T \in \mathcal{D}(y) \} \).
- Consider again a \( P_f^+ \)-basis \( y^x \) (so maximal).
A polymatroid function’s polyhedron is a polymatroid.

...proof continued.

- Also, we define \( \text{sat}(y) \overset{\text{def}}{=} \bigcup \{ T : T \in \mathcal{D}(y) \} \).
- Consider again a \( P^+_f \)-basis \( y^x \) (so maximal).
- Given a \( e \in E \), either \( y^x(e) \) is cut off due to \( x \) (so \( y^x(e) = x(e) \)) or \( e \) is saturated by \( f \), meaning it is an element of some tight set and \( e \in \text{sat}(y^x) \).

\[
\begin{align*}
\text{e} & \quad \text{e}\_1 \\
\text{y(e, e)} & \quad \text{f(y(e))}
\end{align*}
\]
A polymatroid function’s polyhedron is a polymatroid.

...proof continued.

- Also, we define \( \text{sat}(y) \overset{\text{def}}{=} \bigcup \{ T : T \in \mathcal{D}(y) \} \).
- Consider again a \( P^+_f \)-basis \( y^x \) (so maximal).
- Given a \( e \in E \), either \( y^x(e) \) is cut off due to \( x \) (so \( y^x(e) = x(e) \)) or \( e \) is saturated by \( f \), meaning it is an element of some tight set and \( e \in \text{sat}(y^x) \).
- Let \( E \setminus A = \text{sat}(y^x) \) be the union of all such tight sets (which is also tight, so \( y^x(E \setminus A) = f(E \setminus A) \)).
A polymatroid function’s polyhedron is a polymatroid.

... proof continued.

- Also, we define $\text{sat}(y) \overset{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\}$.
- Consider again a $P_f^+$-basis $y^x$ (so maximal).
- Given a $e \in E$, either $y^x(e)$ is cut off due to $x$ (so $y^x(e) = x(e)$) or $e$ is saturated by $f$, meaning it is an element of some tight set and $e \in \text{sat}(y^x)$.
- Let $E \setminus A = \text{sat}(y^x)$ be the union of all such tight sets (which is also tight, so $y^x(E \setminus A) = f(E \setminus A)$).
- Hence, we have

$$y^x(E) = y^x(A) + y^x(E \setminus A) = x(A) + f(E \setminus A) \quad (12.9)$$
A polymatroid function’s polyhedron is a polymatroid.

...proof continued.

- Also, we define \( \text{sat}(y) \overset{\text{def}}{=} \bigcup \{ T : T \in \mathcal{D}(y) \} \).
- Consider again a \( P_f^+ \)-basis \( y^x \) (so maximal).
- Given a \( e \in E \), either \( y^x(e) \) is cut off due to \( x \) (so \( y^x(e) = x(e) \)) or \( e \) is saturated by \( f \), meaning it is an element of some tight set and \( e \in \text{sat}(y^x) \).
- Let \( E \setminus A = \text{sat}(y^x) \) be the union of all such tight sets (which is also tight, so \( y^x(E \setminus A) = f(E \setminus A) \)).
- Hence, we have

\[
y^x(E) = y^x(A) + y^x(E \setminus A) = x(A) + f(E \setminus A) \tag{12.9}
\]

- So we identified the \( A \) to be the elements that are non-tight, and achieved the min, as desired.
So, when $f$ is a polymatroid function, $P_f^+$ is a polymatroid.
A polymatroid is a polymatroid function’s polytope

- So, when $f$ is a polymatroid function, $P_f^+$ is a polymatroid.
- Is it the case that, conversely, for any polymatroid $P$, there is an associated polymatroidal function $f$ such that $P = P_f^+$?
So, when $f$ is a polymatroid function, $P_f^+$ is a polymatroid.

Is it the case that, conversely, for any polymatroid $P$, there is an associated polymatroidal function $f$ such that $P = P_f^+$?

**Theorem 12.3.1**

For any polymatroid $P$ (compact subset of $\mathbb{R}_+^E$, zero containing, down-monotone, and $\forall x \in \mathbb{R}_+^E$ any maximal independent subvector $y \leq x$ has same component sum $y(E) = \text{rank}(x)$), there is a polymatroid function $f : 2^E \to \mathbb{R}$ (normalized, monotone non-decreasing, submodular) such that $P = P_f^+$ where $P_f^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E \}$. 
Tight sets $\mathcal{D}(y)$ are closed, and max tight set $\text{sat}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{ A : A \subseteq E, \ y(A) = f(A) \} \quad (12.10)$$

**Theorem 12.3.2**

*For any $y \in P_f^+$, with $f$ a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.*
Tight sets $\mathcal{D}(y)$ are closed, and max tight set $\text{sat}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{ A : A \subseteq E, \ y(A) = f(A) \}$$  \hfill (12.10)

**Theorem 12.3.2**

*For any $y \in P_f^+$, with $f$ a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.*

**Proof.**

We have already proven this as part of Theorem 11.4.1
Tight sets $\mathcal{D}(y)$ are closed, and max tight set $\text{sat}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{ A : A \subseteq E, \ y(A) = f(A) \}$$

(12.10)

**Theorem 12.3.2**

For any $y \in P_f^+$, with $f$ a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

**Proof.**

We have already proven this as part of Theorem 11.4.1

Also recall the definition of $\text{sat}(y)$, the maximal set of tight elements relative to $y \in \mathbb{R}_+^E$.

$$\text{sat}(y) \overset{\text{def}}{=} \bigcup \{ T : T \in \mathcal{D}(y) \}$$

(12.11)
Join $\vee$ and meet $\wedge$ for $x, y \in \mathbb{R}^E_+$

- For $x, y \in \mathbb{R}^E_+$, define vectors $x \wedge y \in \mathbb{R}^E_+$ and $x \vee y \in \mathbb{R}^E_+$ such that, for all $e \in E$

\[
(x \vee y)(e) = \max(x(e), y(e)) \quad (12.12)
\]

\[
(x \wedge y)(e) = \min(x(e), y(e)) \quad (12.13)
\]

Hence,

\[
x \vee y \triangleq \left( \max(x(e_1), y(e_1)), \max(x(e_2), y(e_2)), \ldots, \max(x(e_n), y(e_n)) \right)
\]

and similarly

\[
x \wedge y \triangleq \left( \min(x(e_1), y(e_1)), \min(x(e_2), y(e_2)), \ldots, \min(x(e_n), y(e_n)) \right)
\]
Join $\lor$ and meet $\land$ for $x, y \in \mathbb{R}^E_+$

- For $x, y \in \mathbb{R}^E_+$, define vectors $x \land y \in \mathbb{R}^E_+$ and $x \lor y \in \mathbb{R}^E_+$ such that, for all $e \in E$

  $$(x \lor y)(e) = \max(x(e), y(e))$$  \hspace{1cm} (12.12)

  $$(x \land y)(e) = \min(x(e), y(e))$$  \hspace{1cm} (12.13)

  Hence,

  $$x \lor y \triangleq \left( \max(x(e_1), y(e_1)), \max(x(e_2), y(e_2)), \ldots, \max(x(e_n), y(e_n)) \right)$$

  and similarly

  $$x \land y \triangleq \left( \min(x(e_1), y(e_1)), \min(x(e_2), y(e_2)), \ldots, \min(x(e_n), y(e_n)) \right)$$

- From this, we can define things like an lattices, and other constructs.
Vector rank, $\text{rank}(x)$, is submodular

- Recall that the matroid rank function is submodular.
Vector rank, $\text{rank}(x)$, is submodular

- Recall that the matroid rank function is submodular.
- The vector rank function $\text{rank}(x)$ also satisfies a form of submodularity, namely one defined on the real lattice.

\[ \text{rank}(A) = \text{rank}_M(A) \]
Vector rank, $\text{rank}(x)$, is submodular

- Recall that the matroid rank function is submodular.
- The vector rank function $\text{rank}(x)$ also satisfies a form of submodularity, namely one defined on the real lattice.

Theorem 12.3.3 (vector rank and submodularity)

Let $P$ be a polymatroid polytope. The vector rank function $\text{rank} : \mathbb{R}_+^E \to \mathbb{R}$ with $\text{rank}(x) = \max \{ y(E) : y \leq x, y \in P \}$ satisfies, for all $u, v \in \mathbb{R}_+^E$

$$\text{rank}(u) + \text{rank}(v) \geq \text{rank}(u \lor v) + \text{rank}(u \land v) \quad (12.14)$$
Vector rank $\text{rank}(x)$ is submodular, proof

Proof of Theorem 12.3.3.

- Let $a \in \mathbb{R}_+^E$ be a $P$-basis of $u \wedge v$, so $\text{rank}(u \wedge v) = a(E)$. 
Proof of Theorem 12.3.3.

- Let \( a \in \mathbb{R}_+^E \) be a \( P \)-basis of \( u \land v \), so \( \text{rank}(u \land v) = a(E) \).
- By the polymatroid property, \( \exists \) an independent \( b \in P \) such that:
  \[ a \leq b \leq u \lor v \]
Proof of Theorem 12.3.3.

- Let $a \in \mathbb{R}^E_+$ be a $P$-basis of $u \land v$, so $\text{rank}(u \land v) = a(E)$.
- By the polymatroid property, there exists an independent $b \in P$ such that:
  - $a \leq b \leq u \lor v$ and also such that $\text{rank}(b) = b(E) = \text{rank}(u \lor v)$, so $b$ is a $P$-basis of $u \lor v$, and thus $b \leq u \lor v$. 

...
**Vector rank** $\text{rank}(x)$ is submodular, proof

**Proof of Theorem 12.3.3.**

- Let $a \in \mathbb{R}_+^E$ be a $P$-basis of $u \land v$, so $\text{rank}(u \land v) = a(E)$.

- By the polymatroid property, $\exists$ an independent $b \in P$ such that:
  
  $a \leq b \leq u \lor v$ and also such that $\text{rank}(b) = b(E) = \text{rank}(u \lor v)$, so $b$ is a $P$-basis of $u \lor v$, and thus $b \leq u \lor v$.

- Given $e \in E$, if $a(e)$ is maximal due to $P$, then $a(e) = b(e) \leq \min(u(e), v(e))$.
Vector rank \( \text{rank}(x) \) is submodular, proof

Proof of Theorem 12.3.3.

- Let \( a \in \mathbb{R}_+^E \) be a \( P \)-basis of \( u \land v \), so \( \text{rank}(u \land v) = a(E) \).
- By the polymatroid property, \( \exists \) an independent \( b \in P \) such that:
  \( a \leq b \leq u \lor v \) and also such that \( \text{rank}(b) = b(E) = \text{rank}(u \lor v) \), so \( b \) is a \( P \)-basis of \( u \lor v \), and thus \( b \leq u \lor v \).
- Given \( e \in E \), if \( a(e) \) is maximal due to \( P \), then \( a(e) = b(e) \leq \min(u(e), v(e)) \).
- If \( a(e) \) is maximal due to \( (u \land v)(e) \), then
  \[ a(e) = \min(u(e), v(e)) \leq b(e). \]
Vector rank \( \text{rank}(x) \) is submodular, proof

**Proof of Theorem 12.3.3.**

- Let \( a \in \mathbb{R}^E_+ \) be a \( P \)-basis of \( u \land v \), so \( \text{rank}(u \land v) = a(E) \).
- By the polymatroid property, \( \exists \) an independent \( b \in P \) such that: \( a \leq b \leq u \lor v \) and also such that \( \text{rank}(b) = b(E) = \text{rank}(u \lor v) \), so \( b \) is a \( P \)-basis of \( u \lor v \), and thus \( b \leq u \lor v \).
- Given \( e \in E \), if \( a(e) \) is maximal due to \( P \), then \( a(e) = b(e) \leq \min(u(e), v(e)) \).
- If \( a(e) \) is maximal due to \( (u \land v)(e) \), then \( a(e) = \min(u(e), v(e)) \leq b(e) \).
- Therefore, in either case, \( a = b \land (u \land v) \).
Proof of Theorem 12.3.3.

- Let \( a \in \mathbb{R}^E_+ \) be a \( P \)-basis of \( u \land v \), so \( \text{rank}(u \land v) = a(E) \).
- By the polymatroid property, \( \exists \) an independent \( b \in P \) such that: \( a \leq b \leq u \lor v \) and also such that \( \text{rank}(b) = b(E) = \text{rank}(u \lor v) \), so \( b \) is a \( P \)-basis of \( u \lor v \), and thus \( b \leq u \lor v \).
- Given \( e \in E \), if \( a(e) \) is maximal due to \( P \), then \( a(e) = b(e) \leq \min(u(e), v(e)) \).
- If \( a(e) \) is maximal due to \((u \land v)(e)\), then
  \[ a(e) = \min(u(e), v(e)) \leq b(e). \]
- Therefore, in either case, \( a = b \land (u \land v) \ldots \)
- \( \ldots \) and since \( b \leq u \lor v \), we get
  \[ a + b \]
  \[ \text{(12.15)} \]
Vector rank $\text{rank}(x)$ is submodular, proof

Proof of Theorem 12.3.3.

- Let $a \in \mathbb{R}^E_+$ be a $P$-basis of $u \land v$, so $\text{rank}(u \land v) = a(E)$.

- By the polymatroid property, $\exists$ an independent $b \in P$ such that:
  - $a \leq b \leq u \lor v$ and also such that $\text{rank}(b) = b(E) = \text{rank}(u \lor v)$, so $b$ is a $P$-basis of $u \lor v$, and thus $b \leq u \lor v$.

- Given $e \in E$, if $a(e)$ is maximal due to $P$, then $a(e) = b(e) \leq \min(u(e), v(e))$.

- If $a(e)$ is maximal due to $(u \land v)(e)$, then $a(e) = \min(u(e), v(e)) \leq b(e)$.

- Therefore, in either case, $a = b \land (u \land v)$ ...

- ... and since $b \leq u \lor v$, we get

  \[ a + b = b \land u \land v + b \]  

  (12.15)
Proof of Theorem 12.3.3.

Let \( a \in \mathbb{R}^E_+ \) be a \( P \)-basis of \( u \land v \), so \( \text{rank}(u \land v) = a(E) \).

By the polymatroid property, \( \exists \) an independent \( b \in P \) such that:
\( a \leq b \leq u \lor v \) and also such that \( \text{rank}(b) = b(E) = \text{rank}(u \lor v) \), so \( b \) is a \( P \)-basis of \( u \lor v \), and thus \( b \leq u \lor v \).

Given \( e \in E \), if \( a(e) \) is maximal due to \( P \), then \( a(e) = b(e) \leq \min(u(e), v(e)) \).

If \( a(e) \) is maximal due to \( (u \land v)(e) \), then
\( a(e) = \min(u(e), v(e)) \leq b(e) \).

Therefore, in either case, \( a = b \land (u \land v) \) ...

... and since \( b \leq u \lor v \), we get
\[
a + b = b \land u \land v + b = b \land u + b \land v \tag{12.15}
\]
Vector rank \(\text{rank}(\mathbf{x})\) is submodular, proof

Proof of Theorem 12.3.3.

- Let \(a \in \mathbb{R}_+^E\) be a \(P\)-basis of \(u \land v\), so \(\text{rank}(u \land v) = a(E)\).
- By the polymatroid property, \(\exists\) an independent \(b \in P\) such that: 
  \(a \leq b \leq u \lor v\) and also such that \(\text{rank}(b) = b(E) = \text{rank}(u \lor v)\), so \(b\) is a \(P\)-basis of \(u \lor v\), and thus \(b \leq u \lor v\).
- Given \(e \in E\), if \(a(e)\) is maximal due to \(P\), then \(a(e) = b(e) \leq \min(u(e), v(e))\).
- If \(a(e)\) is maximal due to \((u \land v)(e)\), then 
  \(a(e) = \min(u(e), v(e)) \leq b(e)\).
- Therefore, in either case, \(a = b \land (u \land v)\) ...
- ... and since \(b \leq u \lor v\), we get 
  \[a + b = b \land u \land v + b = b \land u + b \land v\] (12.15)

To see this, consider each case where either \(b\) is the minimum, or \(u\) is minimum with \(b \leq v\), or \(v\) is minimum with \(b \leq u\).
Vector rank $\text{rank}(x)$ is submodular, proof

... proof of Theorem 12.3.3.

- $b$ is independent, and $b \wedge u$ and $b \wedge v$ are independent subvectors of $u$ and $v$ respectively, so $(b \wedge u)(E) \leq \text{rank}(u)$ and $(b \wedge v)(E) \leq \text{rank}(v)$. 

\[
\text{rank}(u \vee v) + \text{rank}(u \wedge v) = a(E) + b(E) \leq \text{rank}(u) + \text{rank}(v) \tag{12.18}
\]
Vector rank \( \text{rank}(x) \) is submodular, proof

... proof of Theorem 12.3.3.

- \( b \) is independent, and \( b \land u \) and \( b \land v \) are independent subvectors of \( u \) and \( v \) respectively, so \( (b \land u)(E) \leq \text{rank}(u) \) and \( (b \land v)(E) \leq \text{rank}(v) \).

- Hence,
  \[
  \text{rank}(u \land v) + \text{rank}(u \lor v)
  \]
Vector rank $\text{rank}(x)$ is submodular, proof

... proof of Theorem 12.3.3.

- $b$ is independent, and $b \land u$ and $b \land v$ are independent subvectors of $u$ and $v$ respectively, so $(b \land u)(E) \leq \text{rank}(u)$ and $(b \land v)(E) \leq \text{rank}(v)$.

- Hence,

$$\text{rank}(u \land v) + \text{rank}(u \lor v) = a(E) + b(E) \quad (12.16)$$
Vector rank \( \operatorname{rank}(x) \) is submodular, proof

...proof of Theorem 12.3.3.

- \( b \) is independent, and \( b \wedge u \) and \( b \wedge v \) are independent subvectors of \( u \) and \( v \) respectively, so \( (b \wedge u)(E) \leq \operatorname{rank}(u) \) and \( (b \wedge v)(E) \leq \operatorname{rank}(v) \).

- Hence,

\[
\begin{align*}
\operatorname{rank}(u \wedge v) + \operatorname{rank}(u \vee v) &= a(E) + b(E) \\
&= (b \wedge u)(E) + (b \wedge v)(E)
\end{align*}
\]
Vector rank $\text{rank}(\vec{x})$ is submodular, proof

... proof of Theorem 12.3.3.

- $b$ is independent, and $b \land u$ and $b \land v$ are independent subvectors of $u$ and $v$ respectively, so $(b \land u)(E) \leq \text{rank}(u)$ and $(b \land v)(E) \leq \text{rank}(v)$.

- Hence, 
  \[
  \text{rank}(u \land v) + \text{rank}(u \lor v) = a(E) + b(E) \\
  = (b \land u)(E) + (b \land v)(E) \\
  \leq \text{rank}(u) + \text{rank}(v)
  \]
Note the remarkable similarity between the proof of Theorem 12.3.3 and the proof of Theorem 6.6.1 that the standard matroid rank function is submodular.

\[ f(A) = \text{rank}(I_A) \]

\[ f \text{ submodular.} \]

\[ I_A \cup I_B = I(A \cup B) \]

\[ I_A \cap I_B = I(A \cap B) \]
Note the remarkable similarity between the proof of Theorem 12.3.3 and the proof of Theorem 6.6.1 that the standard matroid rank function is submodular.

Next, we prove Theorem 12.3.1, that any polymatroid polytope $P$ has a polymatroid function $f$ such that $P = P_f^+$. 
Note the remarkable similarity between the proof of Theorem 12.3.3 and the proof of Theorem 6.6.1 that the standard matroid rank function is submodular.

Next, we prove Theorem 12.3.1, that any polymatroid polytope $P$ has a polymatroid function $f$ such that $P = P^+_f$.

Given this result, we can conclude that a polymatroid is really an extremely natural polyhedral generalization of a matroid. This was all realized by Jack Edmonds in the mid 1960s (and published in 1969 in his landmark paper “Submodular Functions, Matroids, and Certain Polyhedra”).
Proof of Theorem 12.3.1.

- We are given a polymatroid $P$. 

Proof of Theorem 12.3.1.

We are given a polymatroid $P$. Define $\max$, $\max \{ x(E) : x \in P \}$, and note that $\max > 0$ when $P$ is non-empty, and $\max = \lim_{\varepsilon \to 1} \rank(\varepsilon E) = \rank(\max E)$. Hence, for any $x \in P$, and $e \in E$, we have $x(e) \leq x(E) \leq \max$. Define a function $f : 2^V \to \mathbb{R}$ as, for any $A \subseteq E$, 

$$f(A) = \rank(\max 1_A).$$

Then $f$ is submodular since 

$$f(A) + f(B) = \rank(\max 1_A) + \rank(\max 1_B) - \rank(\max 1_{A \cap B}) - \rank(\max 1_{A \cup B}).$$

(12.23)
Proof of Theorem 12.3.1.

- We are given a polymatroid $P$.
- Define $\alpha_{\text{max}} \triangleq \max \{ x(E) : x \in P \}$, and note that $\alpha_{\text{max}} > 0$ when $P$ is non-empty, and $\alpha_{\text{max}} = \lim_{\alpha \to \infty} \text{rank}(\alpha \mathbf{1}_E) = \text{rank}(\alpha_{\text{max}} \mathbf{1}_E)$. 

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Proof of Theorem 12.3.1.

- We are given a polymatroid $P$.
- Define $\alpha_{\text{max}} \triangleq \max \{ x(E) : x \in P \}$, and note that $\alpha_{\text{max}} > 0$ when $P$ is non-empty, and $\alpha_{\text{max}} = \lim_{\alpha \to \infty} \text{rank}(\alpha \mathbf{1}_E) = \text{rank}(\alpha_{\text{max}} \mathbf{1}_E)$.
Proof of Theorem 12.3.1

- We are given a polymatroid $P$.
- Define $\alpha_{\text{max}} \triangleq \max \{ x(E) : x \in P \}$, and note that $\alpha_{\text{max}} > 0$ when $P$ is non-empty, and $\alpha_{\text{max}} = \lim_{\alpha \to \infty} \text{rank}(\alpha 1_E) = \text{rank}(\alpha_{\text{max}} 1_E)$.
- Hence, for any $x \in P$, and $\forall e \in E$, we have $x(e) \leq x(E) \leq \alpha_{\text{max}}$. 
Proof of Theorem 12.3.1

We are given a polymatroid $P$.

Define $\alpha_{\text{max}} \triangleq \max \{ x(E) : x \in P \}$, and note that $\alpha_{\text{max}} > 0$ when $P$ is non-empty, and $\alpha_{\text{max}} = \lim_{\alpha \to \infty} \text{rank}(\alpha 1_E) = \text{rank}(\alpha_{\text{max}} 1_E)$.

Hence, for any $x \in P$, and $\forall e \in E$, we have $x(e) \leq x(E) \leq \alpha_{\text{max}}$.

Define a function $f : 2^V \to \mathbb{R}$ as, for any $A \subseteq E$,

$$f(A) \triangleq \text{rank}(\alpha_{\text{max}} 1_A) \quad (12.19)$$
Proof of Theorem 12.3.1

We are given a polymatroid \( P \).

Define \( \alpha_{\text{max}} \triangleq \max \{ x(E) : x \in P \} \), and note that \( \alpha_{\text{max}} > 0 \) when \( P \) is non-empty, and \( \alpha_{\text{max}} = \lim_{\alpha \to \infty} \text{rank}(\alpha 1_E) = \text{rank}(\alpha_{\text{max}} 1_E) \).

Hence, for any \( x \in P \), and \( \forall e \in E \), we have \( x(e) \leq x(E) \leq \alpha_{\text{max}} \).

Define a function \( f : 2^V \to \mathbb{R} \) as, for any \( A \subseteq E \),

\[
f(A) \triangleq \text{rank}(\alpha_{\text{max}} 1_A)
\]  

Then \( f \) is submodular since

\[
f(A) + f(B)
\]
Proof of Theorem 12.3.1.

- We are given a polymatroid $P$.
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- Hence, for any $x \in P$, and $\forall e \in E$, we have $x(e) \leq x(E) \leq \alpha_{\text{max}}$.
- Define a function $f : 2^V \to \mathbb{R}$ as, for any $A \subseteq E$,

$$f(A) \triangleq \text{rank}(\alpha_{\text{max}} 1_A) \quad (12.19)$$

- Then $f$ is submodular since

$$f(A) + f(B) = \text{rank}(\alpha_{\text{max}} 1_A) + \text{rank}(\alpha_{\text{max}} 1_B) \quad (12.20)$$
Proof of Theorem 12.3.1

We are given a polymatroid \( P \).

Define \( \alpha_{\text{max}} \triangleq \max \{ x(E) : x \in P \} \), and note that \( \alpha_{\text{max}} > 0 \) when \( P \) is non-empty, and \( \alpha_{\text{max}} = \lim_{\alpha \to \infty} \text{rank}(\alpha 1_E) = \text{rank}(\alpha_{\text{max}} 1_E) \).

Hence, for any \( x \in P \), and \( \forall e \in E \), we have \( x(e) \leq x(E) \leq \alpha_{\text{max}} \).

Define a function \( f : 2^V \to \mathbb{R} \) as, for any \( A \subseteq E \),

\[
f(A) \triangleq \text{rank}(\alpha_{\text{max}} 1_A)
\]  

(12.19)

Then \( f \) is submodular since

\[
f(A) + f(B) = \text{rank}(\alpha_{\text{max}} 1_A) + \text{rank}(\alpha_{\text{max}} 1_B)
\geq \text{rank}(\alpha_{\text{max}} 1_A \lor \alpha_{\text{max}} 1_B) + \text{rank}(\alpha_{\text{max}} 1_A \land \alpha_{\text{max}} 1_B)
\]

(12.20)
Proof of Theorem 12.3.1

We are given a polymatroid $P$.

Define $\alpha_{\text{max}} \triangleq \max \{ x(E) : x \in P \}$, and note that $\alpha_{\text{max}} > 0$ when $P$ is non-empty, and $\alpha_{\text{max}} = \lim_{\alpha \to \infty} \text{rank}(\alpha 1_E) = \text{rank}(\alpha_{\text{max}} 1_E)$.

Hence, for any $x \in P$, and $\forall e \in E$, we have $x(e) \leq x(E) \leq \alpha_{\text{max}}$.

Define a function $f : 2^V \to \mathbb{R}$ as, for any $A \subseteq E$,

$$f(A) \triangleq \text{rank}(\alpha_{\text{max}} 1_A) \quad (12.19)$$

Then $f$ is submodular since

$$f(A) + f(B) = \text{rank}(\alpha_{\text{max}} 1_A) + \text{rank}(\alpha_{\text{max}} 1_B) \quad (12.20)$$

$$\geq \text{rank}(\alpha_{\text{max}} 1_A \lor \alpha_{\text{max}} 1_B) + \text{rank}(\alpha_{\text{max}} 1_A \land \alpha_{\text{max}} 1_B) \quad (12.21)$$

$$= \text{rank}(\alpha_{\text{max}} 1_{A \cup B}) + \text{rank}(\alpha_{\text{max}} 1_{A \cap B}) \quad (12.22)$$
Proof of Theorem 12.3.1.

We are given a polymatroid $P$.

Define $\alpha_{\text{max}} \triangleq \max \{ x(E) : x \in P \}$, and note that $\alpha_{\text{max}} > 0$ when $P$ is non-empty, and $\alpha_{\text{max}} = \lim_{\alpha \to \infty} \text{rank}(\alpha 1_E) = \text{rank}(\alpha_{\text{max}} 1_E)$.

Hence, for any $x \in P$, and $\forall e \in E$, we have $x(e) \leq x(E) \leq \alpha_{\text{max}}$.

Define a function $f : 2^V \to \mathbb{R}$ as, for any $A \subseteq E$,

$$f(A) \triangleq \text{rank}(\alpha_{\text{max}} 1_A)$$  \hspace{1cm} (12.19)

Then $f$ is submodular since

$$f(A) + f(B) = \text{rank}(\alpha_{\text{max}} 1_A) + \text{rank}(\alpha_{\text{max}} 1_B)$$  \hspace{1cm} (12.20)

$$\geq \text{rank}(\alpha_{\text{max}} 1_A \lor \alpha_{\text{max}} 1_B) + \text{rank}(\alpha_{\text{max}} 1_A \land \alpha_{\text{max}} 1_B)$$  \hspace{1cm} (12.21)

$$= \text{rank}(\alpha_{\text{max}} 1_{A \cup B}) + \text{rank}(\alpha_{\text{max}} 1_{A \cap B})$$  \hspace{1cm} (12.22)

$$= f(A \cup B) + f(A \cap B)$$  \hspace{1cm} (12.23)
Proof of Theorem 12.3.1.

Moreover, we have that $f$ is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).
Proof of Theorem 12.3.1.

Moreover, we have that \( f \) is non-negative, normalized with \( f(\emptyset) = 0 \), and monotone non-decreasing (since rank is monotone).

Hence, \( f \) is a polymatroid function.
Proof of Theorem 12.3.1.

Moreover, we have that $f$ is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).

Hence, $f$ is a polymatroid function.

Consider the polytope $P_f^+$ defined as:

$$P_f^+ = \{ x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E \}$$

(12.24)
Proof of Theorem 12.3.1.

- Moreover, we have that $f$ is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).
- Hence, $f$ is a polymatroid function.
- Consider the polytope $P_f^+$ defined as:
  \[
P_f^+ = \{ x \in \mathbb{R}^E_+ : x(A) \leq f(A), \ \forall A \subseteq E \} \tag{12.24}
\]
- Given an $x \in P$, then for any $A \subseteq E$, $x \leq \alpha_{\text{max}}1_A$, so $x(A) \leq \max \{ z(E) : z \in P, z \leq \alpha_{\text{max}}1_A \} = \text{rank}(\alpha_{\text{max}}1_A) = f(A)$.
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Moreover, we have that $f$ is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).

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Given an $x \in P$, then for any $A \subseteq E$, $x \leq \alpha_{\text{max}} 1_A$, so

$$x(A) \leq \max \{z(E) : z \in P, z \leq \alpha_{\text{max}} 1_A\} = \text{rank}(\alpha_{\text{max}} 1_A) = f(A),$$

therefore $x \in P_f^+$. 

$x(A) \leq f(A) \forall A$
Proof of Theorem 12.3.1.

Moreover, we have that \( f \) is non-negative, normalized with \( f(\emptyset) = 0 \), and monotone non-decreasing (since rank is monotone).

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Consider the polytope \( P_f^+ \) defined as:

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P_f^+ = \{ x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E \} \tag{12.24}
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Given an \( x \in P \), then for any \( A \subseteq E, x \leq \alpha_{\text{max}} 1_A \), so

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x(A) \leq \max \{ z(E) : z \in P, z \leq \alpha_{\text{max}} 1_A \} = \text{rank}(\alpha_{\text{max}} 1_A) = f(A),
\]

therefore \( x \in P_f^+ \).

Hence, \( P \subseteq P_f^+ \).

...
Proof of Theorem 12.3.1.

Moreover, we have that $f$ is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).

Hence, $f$ is a polymatroid function.

Consider the polytope $P_f^+$ defined as:

$$P_f^+ = \{ x \in \mathbb{R}^E_+ : x(A) \leq f(A), \forall A \subseteq E \}$$  \hspace{1cm} (12.24)

Given an $x \in P$, then for any $A \subseteq E$, $x \leq \alpha_{\max} 1_A$, so

$$x(A) \leq \max \{ z(E) : z \in P, z \leq \alpha_{\max} 1_A \} = \text{rank}(\alpha_{\max} 1_A) = f(A),$$

therefore $x \in P_f^+$.

Hence, $P \subseteq P_f^+$.

We will next show that $P_f^+ \subseteq P$ to complete the proof.
Proof of Theorem 12.3.1.

- Let $x \in P_f^+$ be chosen arbitrarily (goal is to show that $x \in P$).
Proof of Theorem 12.3.1

Let \( x \in P_f^+ \) be chosen arbitrarily (goal is to show that \( x \in P \)).

Suppose \( x \notin P \).
Proof of Theorem 12.3.1.

- Let $x \in P_f^+$ be chosen arbitrarily (goal is to show that $x \in P$).
- Suppose $x \notin P$. Then, choose $y$ to be a $P$-basis of $x$ that maximizes the number of $y$ elements strictly less than the corresponding $x$ element. I.e., that maximizes $|N(y)|$, where

$$N(y) = \{e \in E : y(e) < x(e)\} \quad (12.25)$$
Proof of Theorem 12.3.1.

Let \( x \in P_f^+ \) be chosen arbitrarily (goal is to show that \( x \in P \)).

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\[
N(y) = \{ e \in E : y(e) < x(e) \} 
\]  

Choose \( w \) between \( y \) and \( x \), so that

\[
y \leq w \overset{\Delta}{=} (y + x)/2 \leq x
\]  

so \( y \) is also a \( P \)-basis of \( w \).
Proof of Theorem 12.3.1.

- Let \( x \in P_f^+ \) be chosen arbitrarily (goal is to show that \( x \in P \)).
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y \leq w \triangleq (y + x)/2 \leq x \tag{12.26}
\]

so \( y \) is also a \( P \)-basis of \( w \).

- Hence, \( \text{rank}(x) = \text{rank}(w) = y(E) \), and the set of \( P \)-bases of \( w \) are also \( P \)-bases of \( x \).
Proof of Theorem 12.3.1.

For any $A \subseteq E$, define $x_A \in \mathbb{R}^E_+$ as

$$x_A(e) = \begin{cases} x(e) & \text{if } e \in A \\ 0 & \text{else} \end{cases} \quad (12.27)$$

Note this is an analogous definition to $1_A$ but for a non-unity vector.
Proof of Theorem 12.3.1.

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Now, we have

$$y(N(y)) < w(N(y)) \leq f(N(y)) = \text{rank}(\alpha_{\max}1_{N(y)}) \quad (12.28)$$

the last inequality follows since $w \leq x \in P^+_f$, and $y \leq w$. 
Proof of Theorem 12.3.1.

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y(N(y)) < w(N(y)) \leq f(N(y)) = \text{rank}(\alpha_{\max}1_{N(y)})
\]  

(12.28)

the last inequality follows since \( w \leq x \in P_f^+, \text{ and } y \leq w \).

Thus, \( y \wedge x_{N(y)} \) is not a \( P \)-basis of \( w \wedge x_{N(y)} \) since, over \( N(y) \), it is neither tight at \( w \) nor tight at the rank (i.e., not a maximal independent subvector on \( N(y) \)).
Proof of Theorem 12.3.1.

- We can extend $y \wedge x_N(y)$ to be a $P$-basis of $w \wedge x_N(y)$ since $y \wedge x_N(y) < w \wedge x_N(y)$. 

Proof of Theorem 12.3.1.
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- We can extend \( y \land x_{N(y)} \) to be a \( P \)-basis of \( w \land x_{N(y)} \) since \( y \land x_{N(y)} < w \land x_{N(y)} \).
- This \( P \)-basis, in turn, can be extended to be a \( P \)-basis \( \hat{y} \) of \( w \land x \).
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Proof of Theorem 12.3.1.

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- Thus, \( \hat{y} \) is a base of \( x \), which violates the maximality of \( |N(y)| \).
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- Now, we have $\hat{y}(N(y)) > y(N(y))$,
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- hence $\hat{y}(e) < y(e)$ for some $e \notin N(y)$.
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- This contradiction means that we must have had $x \in P$. 

□
Proof of Theorem 12.3.1.

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- Now, we have $\hat{y}(N(y)) > y(N(y))$,
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- hence $\hat{y}(e) < y(e)$ for some $e \notin N(y)$.
- Thus, $\hat{y}$ is a base of $x$, which violates the maximality of $|N(y)|$.
- This contradiction means that we must have had $x \in P$.
- Therefore, $P_f^+ = P$. 

\[ \qed \]
More on polymatroids

Theorem 12.3.4

A polymatroid can equivalently be defined as a pair \((E, P)\) where \(E\) is a finite ground set and \(P \subseteq \mathbb{R}^E_+\) is a compact non-empty set of independent vectors such that

1. every subvector of an independent vector is independent (if \(x \in P\) and \(y \leq x\) then \(y \in P\), i.e., down closed)
Theorem 12.3.4

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1. every subvector of an independent vector is independent (if \(x \in P\) and \(y \leq x\) then \(y \in P\), i.e., down closed)

2. If \(u, v \in P\) (i.e., are independent) and \(u(E) < v(E)\), then there exists a vector \(w \in P\) such that
   \[ u < w \leq u \lor v \]  

Corollary 12.3.5

The independent vectors of a polymatroid form a convex polyhedron in \(\mathbb{R}_+^E\).
Theorem 12.3.4

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2. If \(u, v \in P\) (i.e., are independent) and \(u(E) < v(E)\), then there exists a vector \(w \in P\) such that

\[ u < w \leq u \lor v \]  \hspace{1cm} (12.29)

Corollary 12.3.5

The independent vectors of a polymatroid form a convex polyhedron in \(\mathbb{R}^E_+\).
The next slide comes from lecture 6.
Matroids by bases

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

Theorem 12.3.3 (Matroid (by bases))

Let $E$ be a set and $\mathcal{B}$ be a nonempty collection of subsets of $E$. Then the following are equivalent.

1. $\mathcal{B}$ is the collection of bases of a matroid;
2. if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.
3. If $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.

Properties 2 and 3 are called “exchange properties.”

Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.
More on polymatroids

For any compact set $P$, $b$ is a base of $P$ if it is a maximal subvector within $P$. Recall the bases of matroids. In fact, we can define a polymatroid via vector bases (analogous to how a matroid can be defined via matroid bases).

**Theorem 12.3.6**

A polymatroid can equivalently be defined as a pair $(E,P)$ where $E$ is a finite ground set and $P \subseteq \mathbb{R}^E_+$ is a compact non-empty set of independent vectors such that

1. every subvector of an independent vector is independent (if $x \in P$ and $y \leq x$ then $y \in P$, i.e., down closed)
2. if $b, c$ are bases of $P$ and $d$ is such that $b \wedge c < d < b$, then there exists an $f$, with $d \wedge c < f \leq c$ such that $d \vee f$ is a base of $P$
3. All of the bases of $P$ have the same rank.

Note, all three of the above are required for a polymatroid (a matroid analogy would require the equivalent of only the first two).
A word on terminology & notation

- Recall how a matroid is sometimes given as \((E, r)\) where \(r\) is the rank function.
A word on terminology & notation

- Recall how a matroid is sometimes given as \((E, r)\) where \(r\) is the rank function.

- We mention also that the term “polymatroid” is sometimes not used for the polytope itself, but instead but for the pair \((E, f)\),
A word on terminology & notation

- Recall how a matroid is sometimes given as $(E, r)$ where $r$ is the rank function.
- We mention also that the term “polymatroid” is sometimes not used for the polytope itself, but instead but for the pair $(E, f)$,
- But now we see that $(E, f)$ is equivalent to a polymatroid polytope, so this is sensible.
Where are we going with this?

Consider the right hand side of Theorem 11.4.1:
\[
\min (x(A) + f(E \setminus A) : A \subseteq E)
\]
Where are we going with this?

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\[ \min (x(A) + f(E \setminus A) : A \subseteq E) \]

We are going to study this problem, and approaches that address it, as part of our ultimate goal which is to present strategies for submodular function minimization (that we will ultimately get to, in near future lectures).
Where are we going with this?

- Consider the right hand side of Theorem 11.4.1:
  \[ \min (x(A) + f(E \setminus A) : A \subseteq E) \]

- We are going to study this problem, and approaches that address it, as part of our ultimate goal which is to present strategies for submodular function minimization (that we will ultimately get to, in near future lectures).

- As a bit of a hint on what’s to come, recall that we can write it as:
  \[ x(E) + \min (f(A) - x(A) : A \subseteq E) \] where \( f \) is a polymatroid function.
Another Interesting Fact: Matroids from polymatroid functions

**Theorem 12.3.7**

Given integral polymatroid function $f$, let $(E, \mathcal{F})$ be a set system with ground set $E$ and set of subsets $\mathcal{F}$ such that

$$\forall F \in \mathcal{F}, \forall \emptyset \subset S \subseteq F, |S| \leq f(S) \quad (12.30)$$

Then $M = (E, \mathcal{F})$ is a matroid.

**Proof.**

Exercise

And its rank function is Exercise.
• Considering Theorem 11.4.1, the matroid case is now a special case, where we have that:

**Corollary 12.3.8**

*We have that:*

\[
\max \{ y(E) : y \in P_{\text{ind. set}}(M), y \leq x \} = \min \{ r_M(A) + x(E \setminus A) : A \subseteq E \}
\] (12.31)

where \( r_M \) is the matroid rank function of some matroid.
The next two slides come respectively from Lecture 11 and Lecture 10.
Definition 12.4.1 (polymatroid)

A polymatroid is a compact set $P \subseteq \mathbb{R}^E_+$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For every $x \in \mathbb{R}^E_+$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any $P$-basis of $x$), has the same component sum $y(E)$.

- Vectors within $P$ (i.e., any $y \in P$) are called independent, and any vector outside of $P$ is called dependent.
- Since all $P$-bases of $x$ have the same component sum, if $\mathcal{B}_x$ is the set of $P$-bases of $x$, then $\text{rank}(x) = y(E)$ for any $y \in \mathcal{B}_x$. 
**Theorem 12.4.5**

Let $M = (V, \mathcal{I})$ be a matroid, with rank function $r$, then for any weight function $w \in \mathbb{R}_+^V$, there exists a chain of sets $U_1 \subset U_2 \subset \cdots \subset U_n \subseteq V$ such that

$$\max \{w(I) | I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i)$$ (12.19)

where $\lambda_i \geq 0$ satisfy

$$w = \sum_{i=1}^{n} \lambda_i 1_{U_i}$$ (12.20)
Polymatroidal polyhedron and greedy

- Let \((E, \mathcal{I})\) be a set system and \(w \in \mathbb{R}_+^E\) be a weight vector.
Let \((E, \mathcal{I})\) be a set system and \(w \in \mathbb{R}^E_+\) be a weight vector.

Recall greedy algorithm: Set \(A = \emptyset\), and repeatedly choose \(y \in E \setminus A\) such that \(A \cup \{y\} \in \mathcal{I}\) with \(w(y)\) as large as possible, stopping when no such \(y\) exists.

For a matroid, we saw that set system \((E, \mathcal{I})\) is a matroid iff for each weight function \(w \in \mathbb{R}^E_+\), the greedy algorithm leads to a set \(I \subset I\) of maximum weight \(w(I)\).

Stated succinctly, considering \(\max\{w(I) : I \subset I\}\), then \((E, \mathcal{I})\) is a matroid iff greedy works for this maximization.

Can we also characterize a polymatroid in this way? That is, if we consider \(\max_n w(x) : x \in P^+_f\), where \(P^+_f\) represents the "independent vectors", is it the case that \(P^+_f\) is a polymatroid iff greedy works for this maximization?

Can we, ultimately, even relax things so that \(w \in \mathbb{R}^E_+\)?
Let \((E, \mathcal{I})\) be a set system and \(w \in \mathbb{R}^E_+\) be a weight vector.

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- Stated succinctly, considering \(\max \{w(I) : I \in \mathcal{I}\}\), then \((E, \mathcal{I})\) is a matroid iff greedy works for this maximization.
Let \((E, \mathcal{I})\) be a set system and \(w \in \mathbb{R}^{E}_+\) be a weight vector.

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- That is, if we consider \(\max \{wx : x \in P_f^+\}\), where \(P_f^+\) represents the “independent vectors”, is it the case that \(P_f^+\) is a polymatroid iff greedy works for this maximization?
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Can we, ultimately, even relax things so that \(w \in \mathbb{R}^E\)?
What is the greedy solution in this setting, when \( w \in \mathbb{R}^E \)?
Polymatroidal polyhedron and greedy

- What is the greedy solution in this setting, when $w \in \mathbb{R}^{E}$?
- Sort elements of $E$ w.r.t. $w$ so that, w.l.o.g.
  
  $E = (e_1, e_2, \ldots, e_m)$ with $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$. 

What is the greedy solution in this setting, when \( w \in \mathbb{R}^E \)?

Sort elements of \( E \) w.r.t. \( w \) so that, w.l.o.g.

\[
E = (e_1, e_2, \ldots, e_m) \text{ with } w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m).
\]

Let \( k + 1 \) be the first point (if any) at which we are non-positive, i.e., \( w(e_k) > 0 \) and \( 0 \geq w(e_{k+1}) \).

That is, we have

\[
\begin{align*}
w(e_1) & \geq w(e_2) \geq \cdots \geq w(e_k) > 0 \geq w(e_{k+1}) \geq \cdots \geq w(e_m)
\end{align*}
\]
What is the greedy solution in this setting, when \( w \in \mathbb{R}^E \)?

Sort elements of \( E \) w.r.t. \( w \) so that, w.l.o.g.
\[
E = (e_1, e_2, \ldots, e_m) \text{ with } w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m).
\]

Let \( k + 1 \) be the first point (if any) at which we are non-positive, i.e.,
\[
w(e_k) > 0 \text{ and } 0 \geq w(e_{k+1}).
\]

Next define partial accumulated sets \( E_i \), for \( i = 0 \ldots m \), we have w.r.t. the above sorted order:
\[
E_i \overset{\text{def}}{=} \{e_1, e_2, \ldots, e_i\} \quad (12.33)
\]
(note \( E_0 = \emptyset \), \( f(E_0) = 0 \), and \( E \) and \( E_i \) is always sorted w.r.t \( w \)).
What is the greedy solution in this setting, when \( w \in \mathbb{R}^E \)?

Sort elements of \( E \) w.r.t. \( w \) so that, w.l.o.g.
\[
E = (e_1, e_2, \ldots, e_m) \text{ with } w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m).
\]

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E_i \overset{\text{def}}{=} \{ e_1, e_2, \ldots e_i \}
\]  
(12.33)

(note \( E_0 = \emptyset \), \( f(E_0) = 0 \), and \( E \) and \( E_i \) is always sorted w.r.t \( w \)).

The greedy solution is the vector \( x \in \mathbb{R}_+^E \) with elements defined as:

\[
x(e_1) \overset{\text{def}}{=} f(E_1) = f(e_1) = f(e_1 | E_0) = f(e_1 | \emptyset) 
\]  
(12.34)

\[
x(e_i) \overset{\text{def}}{=} f(E_i) - f(E_{i-1}) = f(e_i | E_{i-1}) \text{ for } i = 2 \ldots k
\]  
(12.35)

\[
x(e_i) \overset{\text{def}}{=} 0 \text{ for } i = k + 1 \ldots m = |E|
\]  
(12.36)
Some Intuition: greedy and gain

- Note $x(e_i) = f(e_i | E_{i-1}) \leq f(e_i | E')$ for any $E' \subseteq E_{i-1}$
Some Intuition: greedy and gain

- Note $x(e_i) = f(e_i|E_{i-1}) \leq f(e_i|E')$ for any $E' \subseteq E_{i-1}$
- So $x(e_1) = f(e_1)$ and this corresponds to $w(e_1) \geq w(e_i)$ for all $i \neq 1$. 

\[ e_1 \rightarrow w(e_1) \geq w(e_i) \]
Some Intuition: greedy and gain

- Note $x(e_i) = f(e_i|E_{i-1}) \leq f(e_i|E')$ for any $E' \subseteq E_{i-1}$
- So $x(e_1) = f(e_1)$ and this corresponds to $w(e_1) \geq w(e_i)$ for all $i \neq 1$.
- Hence, for the largest value of $w$ (namely $w(e_1)$), we use for $x(e_1)$ the largest possible gain value of $e_1$ (namely $f(e_1|\emptyset) \geq f(e_1|A)$ for any $A \subseteq E \setminus \{e_1\}$).

$$x_1 = f(e_1|E_{1-1})$$
Some Intuition: greedy and gain

- Note $x(e_i) = f(e_i|E_{i-1}) \leq f(e_i|E')$ for any $E' \subseteq E_{i-1}$
- So $x(e_1) = f(e_1)$ and this corresponds to $w(e_1) \geq w(e_i)$ for all $i \neq 1$.
- Hence, for the largest value of $w$ (namely $w(e_1)$), we use for $x(e_1)$ the largest possible gain value of $e_1$ (namely $f(e_1|\emptyset) \geq f(e_1|A)$ for any $A \subseteq E \setminus \{e_1\}$).
- For the next largest value of $w$ (namely $w(e_2)$), we use for $x(e_2)$ the next largest gain value of $e_2$ (namely $f(e_2|e_1)$), while still ensuring (as we will soon see in Theorem 12.4.1) that the resulting $x \in P_f$. 
Some Intuition: greedy and gain

- Note \( x(e_i) = f(e_i | E_{i-1}) \leq f(e_i | E') \) for any \( E' \subseteq E_{i-1} \)
- So \( x(e_1) = f(e_1) \) and this corresponds to \( w(e_1) \geq w(e_i) \) for all \( i \neq 1 \).
- Hence, for the largest value of \( w \) (namely \( w(e_1) \)), we use for \( x(e_1) \) the largest possible gain value of \( e_1 \) (namely \( f(e_1 | \emptyset) \geq f(e_1 | A) \) for any \( A \subseteq E \setminus \{e_1\} \)).
- For the next largest value of \( w \) (namely \( w(e_2) \)), we use for \( x(e_2) \) the next largest gain value of \( e_2 \) (namely \( f(e_2 | e_1) \)), while still ensuring (as we will soon see in Theorem 12.4.1) that the resulting \( x \in P_f \).
- This process continues, using the next largest possible gain of \( e_i \) for \( x(e_i) \) while ensuring (as we will show) we do not leave the polytope, given the values we’ve already chosen for \( x(e_{i'}) \) for \( i' < i \).
Theorem 12.4.1

The vector \( x \in \mathbb{R}^E_+ \) as previously defined using the greedy algorithm maximizes \( wx \) over \( P_f^+ \), with \( w \in \mathbb{R}^E_+ \), if \( f \) is submodular.

Proof.
Polymatroidal polyhedron and greedy

**Theorem 12.4.1**

*The vector* $x \in \mathbb{R}_+^E$ *as previously defined using the greedy algorithm maximizes* $wx$ *over* $P_f^+$, *with* $w \in \mathbb{R}_+^E$, *if* $f$ *is submodular.*

**Proof.**

- Consider the LP strong duality equation:

$$\max (wx : x \in P_f^+) = \min \left( \sum_{A \subseteq E} y_A f(A) : y \in \mathbb{R}^{2E}_+, \sum_{A \subseteq E} y_A 1_A \geq w \right)$$

(12.37)
Theorem 12.4.1

The vector \( x \in \mathbb{R}^E_+ \) as previously defined using the greedy algorithm maximizes \( wx \) over \( P_f^+ \), with \( w \in \mathbb{R}^E_+ \), if \( f \) is submodular.

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- Consider the LP strong duality equation:

\[
\max(wx : x \in P_f^+) = \min \left( \sum_{A \subseteq E} y_A f(A) : y \in \mathbb{R}^{2^E}_+, \sum_{A \subseteq E} y_A 1_A \geq w \right)
\]  

(12.37)

- Sort \( E \) by \( w \), and define the following vector \( y \in \mathbb{R}^{2^E}_+ \) as

\[
y_{E_i} \leftarrow w(e_i) - w(e_{i+1}) \text{ for } i = 1 \ldots (m - 1),
\]

(12.38)

\[
y_{E} \leftarrow w(e_m), \text{ and}
\]

(12.39)

\[
y_A \leftarrow 0 \text{ otherwise}
\]

(12.40)
Polymatroidal polyhedron and greedy

Proof.

- We first will see that greedy \( x \in P_f^+ \) (that is \( x(A) \leq f(A), \forall A \)).
Polymatroidal polyhedron and greedy

Proof.

- We first will see that greedy \( x \in P_f^+ \) (that is \( x(A) \leq f(A), \forall A \)).
- Order \( A = (a_1, a_2, \ldots, a_k) \) based on order \( (e_1, e_2, \ldots, e_m) \).

Ordering:

\[
\begin{array}{cccccccccc}
  a_1 & a_2 & a_3 & a_4 & a_5 & \ldots & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} & \ldots & e_m \\
\end{array}
\]

Choose some \( A \).
Proof.

We first will see that greedy $x \in P^+_f$ (that is $x(A) \leq f(A), \forall A$).

Order $A = (a_1, a_2, \ldots, a_k)$ based on order $(e_1, e_2, \ldots, e_m)$.

| $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $\ldots$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_9$ | $e_{10}$ | $e_{11}$ | $\ldots$ | $e_m$

Define $e^{-1} : E \rightarrow \{1, \ldots, m\}$ so that $e^{-1}(e_i) = i$.

This means that with $A = \{a_1, a_2, \ldots, a_k\}$, and $\forall j \leq k$,

$$e^{-1}(a_j) = i \quad \text{(12.41)}$$

and

$$\{a_1, a_2, \ldots, a_j\} \subseteq \{e_1, e_2, \ldots, e_{e^{-1}(a_j)}\} \quad \text{(12.42)}$$

Also recall matlab notation: $a_{1:j} \equiv \{a_1, a_2, \ldots, a_j\}$.

E.g., with $j = 4$ we get $e^{-1}(a_4) = 9$, and

$$\{a_1, a_2, a_3, a_4\} \subseteq \{e_1, e_2, \ldots, e_9\} \quad \text{(12.43)}$$
Polymatroidal polyhedron and greedy

Proof.

- We first will see that greedy $x \in P_f^+$ (that is $x(A) \leq f(A), \forall A$).

- Order $A = (a_1, a_2, \ldots, a_k)$ based on order $(e_1, e_2, \ldots, e_m)$.

- Define $e^{-1} : E \rightarrow \{1, \ldots, m\}$ so that $e^{-1}(e_i) = i$.

- Then, we have $x \in P_f^+$ since for all $A$:

\[
\begin{align*}
f(A) &= \sum_{i=1}^{k} f(a_i | a_{1:i-1}) \quad (12.41) \\
\geq \sum_{i=1}^{k} f(a_i | e_1:e^{-1}(a_i)-1) \quad (12.42) \\
= \sum_{a \in A} f(a | e_1:e^{-1}(a)-1) = x(A) \quad (12.43)
\end{align*}
\]
Proof.

- We first will see that greedy $x \in P_f^+$ (that is $x(A) \leq f(A), \forall A$).
- Order $A = (a_1, a_2, \ldots, a_k)$ based on order $(e_1, e_2, \ldots, e_m)$.

<table>
<thead>
<tr>
<th></th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
<th>$a_4$</th>
<th>$a_5$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>$e_2$</td>
<td>$e_3$</td>
<td>$e_4$</td>
<td>$e_5$</td>
<td>$e_6$</td>
<td>$e_7$</td>
</tr>
</tbody>
</table>

- Define $e^{-1} : E \rightarrow \{1, \ldots, m\}$ so that $e^{-1}(e_i) = i$.
- Then, we have $x \in P_f^+$ since for all $A$:

\[
\begin{align*}
 f(A) &= \sum_{i=1}^{k} f(a_{i \mid a_{1:i-1}}) \quad (12.41) \\
 \geq \sum_{i=1}^{k} f(a_{i \mid e_{1:e^{-1}(a_i)-1}}) \quad (12.42) \\
 &= \sum_{a \in A} f(a \mid e_{1:e^{-1}(a)-1}) = x(A) \quad (12.43)
\end{align*}
\]
Proof.

Next, $y$ is also feasible for the dual constraints in Eq. 12.37 since:
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- Next, we check that $y$ is dual feasible. Clearly, $y \geq 0$, 

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Proof.

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- Next, we check that $y$ is dual feasible. Clearly, $y \geq 0$,
- and also, considering $y$ component wise, for any $i$, we have that

$$
\sum_{A:e_i \in A} y_A = \sum_{j \geq i} y_{E_j} = \sum_{j=i}^{m-1} (w(e_j) - w(e_{j+1})) + w(e_m) = w(e_i).
$$
Proof.

- Next, $y$ is also feasible for the dual constraints in Eq. 12.37 since:
- Next, we check that $y$ is dual feasible. Clearly, $y \geq 0$,
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$$

- Now optimality for $x$ and $y$ follows from strong duality, i.e.:

$$
w_x = \sum_{e \in E} w(e) x(e) = \sum_{i=1}^{m} w(e_i) f(e_i | E_{i-1}) = \sum_{i=1}^{m} w(e_i) \left( f(E_i) - f(E_{i-1}) \right)
$$

$$= \sum_{i=1}^{m-1} f(E_i) \left( w(e_i) - w(e_{i+1}) \right) + f(E) w(e_m) = \sum_{A \subseteq E} y_A f(A) \ldots
$$
Proof.

The equality in prev. Eq. follows via **Abel summation**:

\[
wx = \sum_{i=1}^{m} w_i x_i
\]

(12.44)

\[
= \sum_{i=1}^{m} w_i \left( f(E_i) - f(E_{i-1}) \right)
\]

(12.45)

\[
= \sum_{i=1}^{m} w_i f(E_i) - \sum_{i=1}^{m-1} w_{i+1} f(E_i)
\]

(12.46)

\[
= w_m f(E_m) + \sum_{i=1}^{m-1} (w_i - w_{i+1}) f(E_i)
\]

(12.47)
What about $w \in \mathbb{R}^E$

- When $w$ contains negative elements, we have $x(e_i) = 0$ for $i = k + 1, \ldots, m$, where $k$ is the last positive element of $w$ when it is sorted in decreasing order.
What about $w \in \mathbb{R}^E$

- When $w$ contains negative elements, we have $x(e_i) = 0$ for $i = k + 1, \ldots, m$, where $k$ is the last positive element of $w$ when it is sorted in decreasing order.

- Exercise: show a modification of the previous proof that works for arbitrary $w \in \mathbb{R}^E$
Polymatroidal polyhedron and greedy

Theorem 12.4.1

Conversely, suppose $P_f^+$ is a polytope of form

$$P_f^+ = \{ x \in \mathbb{R}_{+}^E : x(A) \leq f(A), \forall A \subseteq E \},$$

then the greedy solution to $\max(wx : x \in P)$ is optimum only if $f$ is submodular.

Proof.

Choose $A$ and $B$ arbitrarily, and then order elements of $E$ as $(e_1, e_2, \ldots, e_m)$, with $E_i = (e_1, e_2, \ldots, e_i)$, so the following is true:
Theorem 12.4.1

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Proof.

- Choose $A$ and $B$ arbitrarily, and then order elements of $E$ as $(e_1, e_2, \ldots, e_m)$, with $E_i = (e_1, e_2, \ldots, e_i)$, so the following is true:

- For $1 \leq p \leq q \leq m$, define $A = \{e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_p\} = E_p$ and $B = \{e_1, e_2, \ldots, e_k, e_{p+1}, \ldots, e_q\} = E_k \cup (E_q \setminus E_p)$.
Theorem 12.4.1

Conversely, suppose $P_f^+$ is a polytope of form

$$P_f^+ = \left\{ x \in \mathbb{R}^E_+ : x(A) \leq f(A), \forall A \subseteq E \right\},$$

then the greedy solution to

$$\max(wx : x \in P)$$

is optimum only if $f$ is submodular.

Proof.

- Choose $A$ and $B$ arbitrarily, and then order elements of $E$ as $(e_1, e_2, \ldots, e_m)$, with $E_i = (e_1, e_2, \ldots, e_i)$, so the following is true:

- For $1 \leq p \leq q \leq m$, define $A = \{e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_p\} = E_p$ and $B = \{e_1, e_2, \ldots, e_k, e_{p+1}, \ldots, e_q\} = E_k \cup (E_q \setminus E_p)$

- Note, then we have $A \cap B = \{e_1, \ldots, e_k\} = E_k$, and $A \cup B = E_q$. 

Prof. Jeff Bilmes
**Theorem 12.4.1**

Conversely, suppose \( P_f^+ \) is a polytope of form
\[
P_f^+ = \{ x \in \mathbb{R}^E_+ : x(A) \leq f(A), \forall A \subseteq E \},
\]
then the greedy solution to \( \max(wx : x \in P) \) is optimum only if \( f \) is submodular.

**Proof.**

- Choose \( A \) and \( B \) arbitrarily, and then order elements of \( E \) as \((e_1, e_2, \ldots, e_m)\), with \( E_i = (e_1, e_2, \ldots, e_i) \), so the following is true:
- For \( 1 \leq p \leq q \leq m \), define \( A = \{ e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_p \} = E_p \) and \( B = \{ e_1, e_2, \ldots, e_k, e_{p+1}, \ldots, e_q \} = E_k \cup (E_q \setminus E_p) \).
- Note, then we have \( A \cap B = \{ e_1, \ldots, e_k \} = E_k \), and \( A \cup B = E_q \).
- Define \( w \in \{0, 1\}^m \) as:
\[
w \overset{\text{def}}{=} \sum_{i=1}^{q} 1_{e_i} = 1_{A \cup B}
\] (12.48)
Theorem 12.4.1

Conversely, suppose $P_f^+$ is a polytope of form

$$P_f^+ = \{ x \in \mathbb{R}^E_+ : x(A) \leq f(A), \forall A \subseteq E \},$$

then the greedy solution to

$$\max (wx : x \in P)$$

is optimum only if $f$ is submodular.

Proof.

- Choose $A$ and $B$ arbitrarily, and then order elements of $E$ as $(e_1, e_2, \ldots, e_m)$, with $E_i = (e_1, e_2, \ldots, e_i)$, so the following is true:

- For $1 \leq p \leq q \leq m$, define $A = \{ e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_p \} = E_p$ and $B = \{ e_1, e_2, \ldots, e_k, e_{p+1}, \ldots, e_q \} = E_k \cup (E_q \setminus E_p)$

- Note, then we have $A \cap B = \{ e_1, \ldots, e_k \} = E_k$, and $A \cup B = E_q$.

- Define $w \in \{0, 1\}^m$ as:

\[ w \overset{\text{def}}{=} \sum_{i=1}^{q} 1_{e_i} = 1_{A \cup B} \]  

(12.48)

- Suppose optimum solution $x$ is given by the greedy procedure.
Proof.

Then

\[
\sum_{i=1}^{k} x_i = f(E_1) + \sum_{i=2}^{k} (f(E_i) - f(E_{i-1})) = f(E_k) = f(A \cap B)
\]

(12.49)
Proof.

Then

\[
\sum_{i=1}^{k} x_i = f(E_1) + \sum_{i=2}^{k} (f(E_i) - f(E_{i-1})) = f(E_k) = f(A \cap B)
\]

(12.49)

and

\[
\sum_{i=1}^{p} x_i = f(E_1) + \sum_{i=2}^{p} (f(E_i) - f(E_{i-1})) = f(E_p) = f(A)
\]

(12.50)
Polymatroidal polyhedron and greedy

Proof.

Then

\[
\sum_{i=1}^{k} x_i = f(E_1) + \sum_{i=2}^{k} (f(E_i) - f(E_{i-1})) = f(E_k) = f(A \cap B)
\]

\[(12.49)\]

and

\[
\sum_{i=1}^{p} x_i = f(E_1) + \sum_{i=2}^{p} (f(E_i) - f(E_{i-1})) = f(E_p) = f(A)
\]

\[(12.50)\]

and

\[
\sum_{i=1}^{q} x_i = f(E_1) + \sum_{i=2}^{q} (f(E_i) - f(E_{i-1})) = f(E_q) = f(A \cup B)
\]

\[(12.51)\]
Proof.

Thus, we have

\[ x(B) = \sum_{i \in 1, \ldots, k, p+1, \ldots, q} x_i = \sum_{i: e_i \in B} x_i = f(A \cup B) + f(A \cap B) - f(A) \]

(12.52)
Polymatroidal polyhedron and greedy

Proof.

Thus, we have

\[ x(B) = \sum_{i=1,\ldots,k,p+1,\ldots,q} x_i = \sum_{i: e_i \in B} x_i = f(A \cup B) + f(A \cap B) - f(A) \]  

(12.52)

But given that the greedy algorithm gives the optimal solution to \( \max(wx : x \in P_f^+) \), we have that \( x \in P_f^+ \) and thus \( x(B) \leq f(B) \).
Polymatroidal polyhedron and greedy

Proof.

- Thus, we have

\[ x(B) = \sum_{i=1,...,k,p+1,...,q} x_i = \sum_{i:e_i \in B} x_i = f(A \cup B) + f(A \cap B) - f(A) \]  

(12.52)

- But given that the greedy algorithm gives the optimal solution to \( \max(wx : x \in P_f^+) \), we have that \( x \in P_f^+ \) and thus \( x(B) \leq f(B) \).

- Thus,

\[ x(B) = f(A \cup B) + f(A \cap B) - f(A) = \sum_{i:e_i \in B} x_i \leq f(B) \]  

(12.53)

ensuring the submodularity of \( f \), since \( A \) and \( B \) are arbitrary.
Review from Lecture 9

- The next slide comes from lecture 9.
Matroid and the greedy algorithm

- Let \((E, \mathcal{I})\) be an independence system, and we are given a non-negative modular weight function \(w : E \rightarrow \mathbb{R}_+\).

**Algorithm 1:** The Matroid Greedy Algorithm

1. Set \(X \leftarrow \emptyset\);
2. while \(\exists v \in E \setminus X\) s.t. \(X \cup \{v\} \in \mathcal{I}\) do
3. \(v \in \text{argmax} \{w(v) : v \in E \setminus X, X \cup \{v\} \in \mathcal{I}\}\);
4. \(X \leftarrow X \cup \{v\}\);

- Same as sorting items by decreasing weight \(w\), and then choosing items in that order that retain independence.

**Theorem 12.4.7**

Let \((E, \mathcal{I})\) be an independence system. Then the pair \((E, \mathcal{I})\) is a matroid if and only if for each weight function \(w \in \mathcal{R}_+^E\), Algorithm ?? above leads to a set \(I \in \mathcal{I}\) of maximum weight \(w(I)\).
Thus, restating the above results into a single complete theorem, we have a result very similar to what we saw for matroids (i.e., Theorem 10.5.1)

**Theorem 12.4.1**

If \( f : 2^E \to \mathbb{R}_+ \) is given, and \( P \) is a polytope in \( \mathbb{R}_+^E \) of the form

\[
P = \left\{ x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E \right\},
\]

then the greedy solution to the problem \( \max(\mathbf{w}x : x \in P) \) is \( \mathbf{w} \) optimum iff \( f \) is monotone non-decreasing submodular (i.e., iff \( P \) is a polymatroid).
Given an arbitrary submodular function $f : 2^V \rightarrow \mathbb{R}$ (not necessarily a polymatroid function, so it need not be positive, monotone, etc.).
Multiple Polytopes associated with arbitrary $f$

- Given an arbitrary submodular function $f : 2^V \rightarrow R$ (not necessarily a polymatroid function, so it need not be positive, monotone, etc.).
- If $f(\emptyset) \neq 0$, we can set $f'(A) = f(A) - f(\emptyset)$ without destroying submodularity. This also does not change any minima, so we assume all functions are normalized $f(\emptyset) = 0$.

Note that due to constraint $x(\emptyset) \leq f(\emptyset)$, we must have $f(\emptyset) \geq 0$ since if not (i.e., if $f(\emptyset) < 0$), then $P_f^+$ doesn’t exist.

Another form of normalization can do is:

$$f'(A) = \begin{cases} f(A) & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset \end{cases} \quad (12.54)$$

This preserves submodularity due to $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$, and if $A \cap B = \emptyset$ then r.h.s. only gets smaller when $f(\emptyset) \geq 0$. 

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We can define several polytopes:

- $P_f = \{ x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E \}$ (12.55)
- $P_f^+ = \{ x \in \mathbb{R}^E : x \geq 0 \}$ (12.56)
- $B_f = \{ x \in \mathbb{R}^E : x(E) = f(E) \}$ (12.57)

$P_f$ is what is sometimes called the extended polytope (sometimes annotated as $EP_f$).

$P_f^+$ is $P_f$ restricted to the positive orthant.

$B_f$ is called the base polytope.
Multiple Polytopes associated with arbitrary $f$

- Given an arbitrary submodular function $f : 2^V \rightarrow \mathbb{R}$ (not necessarily a polymatroid function, so it need not be positive, monotone, etc.).
- If $f(\emptyset) \neq 0$, we can set $f'(A) = f(A) - f(\emptyset)$ without destroying submodularity. This also does not change any minima, so we assume all functions are normalized $f(\emptyset) = 0$.
- We can define several polytopes:

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Base Polytope in 3D

\[ P_f = \{ x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E \} \]  
\[ B_f = P_f \cap \{ x \in \mathbb{R}^E : x(E) = f(E) \} \]
A polymatroid function’s polyhedron is a polymatroid.

Theorem 12.5.1

Let \( f \) be a submodular function defined on subsets of \( E \). For any \( x \in \mathbb{R}^E \), we have:

\[
rank(x) = \max (y(E) : y \leq x, y \in P_f) = \min (x(A) + f(E \setminus A) : A \subseteq E)
\]  
(12.62)

Essentially the same theorem as Theorem 11.4.1. Taking \( x = 0 \) we get:

Corollary 12.5.2

Let \( f \) be a submodular function defined on subsets of \( E \). \( x \in \mathbb{R}^E \), we have:

\[
rank(0) = \max (y(E) : y \leq 0, y \in P_f) = \min (f(A) : A \subseteq E)
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(12.63)
Proof of Theorem 12.5.1.

Let $y^*$ be the optimal solution of the l.h.s. and let $A \subseteq E$ be any subset.

Thus we have that $y^*(\text{sat}(y^*)) + y^*(E \cap \text{sat}(y^*)) = f(\text{sat}(y^*)) + x(E \cap \text{sat}(y^*))$, strong duality, showing that the two sides are equal for $y^*$. 

Proof of Theorem 12.5.1.

- Let $y^*$ be the optimal solution of the l.h.s. and let $A \subseteq E$ be any subset.

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Also, for any \( e \in E \), if \( y^*(e) < x(e) \) then there must be some reason for this other than the constraint \( y^* \leq x \), namely it must be that \( \exists T \in D(x) \) with \( e \in T \) (i.e., \( e \) is a member of at least one of the tight sets).
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Hence, for all $e \notin \text{sat}(y^*)$ we have $y^*(e) = x(e)$, and moreover $y^*(\text{sat}(y^*)) = f(\text{sat}(y^*))$ by definition.
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Greedy and $P_f$

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Moreover, in polymatroidal case, since the greedy constructed $x$ has $x(E) = f(E)$, we have that the greedy $x \in B_f$.

In fact, we next will see that the greedy $x$ is a vertex of $B_f$. 
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Consider $P_f^+$ and also $C_f^+ \overset{\text{def}}{=} \{ x : x \in \mathbb{R}^E_+, x(e) \leq f(e), \forall e \in E \}$.
Polymatroid extreme points

- The greedy algorithm does more than solve \( \max(wx : x \in P_f^+) \). We can use it to generate vertices of polymatroidal polytopes.
- Consider \( P_f^+ \) and also \( C_f^+ \) defined as \( \{ x : x \in \mathbb{R}_+^E, x(e) \leq f(e), \forall e \in E \} \).
- Then ordering \( A = (a_1, \ldots, a_{|A|}) \) arbitrarily with \( A_i = \{a_1, \ldots, a_i\} \), \( f(A) = \sum_i f(a_i | A_{i-1}) \leq \sum_i f(a_i) \), and hence \( P_f^+ \subseteq C_f^+ \).
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Polymatroids
Polymatroids and Greedy
Possible Polytopes
Extreme Points
Since $w \in \mathbb{R}^E_+$ is arbitrary, it may be that any $e \in E$ is max (i.e., is such that $w(e) > w(e')$ for $e' \in E \setminus \{e\}$).
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Thus, intuitively, any first vertex of the polytope away from the origin might be obtained by advancing along the corresponding axis.
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Recall, base polytope defined as the extreme face of \( P_f \). I.e.,

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B_f = P_f \cap \{ x \in \mathbb{R}^E_+ : x(E) = f(E) \} \tag{12.64}
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- Also, intuitively, we can continue advancing along the skeletal edges of the polytope to reach any other vertex, given the appropriate ordering. If we advance in all dimensions, we’ll reach a vertex in $B_f$, and if we advance only in some dimensions, we’ll reach a vertex in $P_f \setminus B_f$. 
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- We formalize this next:
Polymatroid extreme points

- Given any arbitrary order of $E = (e_1, e_2, \ldots, e_m)$, define $E_i = (e_1, e_2, \ldots, e_i)$. 
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- Given any arbitrary order of $E = (e_1, e_2, \ldots, e_m)$, define $E_i = (e_1, e_2, \ldots, e_i)$.
- As before, a vector $x$ is generated by $E_i$ using the greedy procedure as follows

\[
\begin{align*}
x(e_1) &= f(E_1) = f(e_1) \\
x(e_j) &= f(E_j) - f(E_{j-1}) = f(e_j | E_{j-1}) \text{ for } 2 \leq j \leq i \\
x(e) &= 0 \text{ for } e \in E \setminus E_i
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  \]

  \[
  x(e) = 0 \quad \text{for } e \in E \setminus E_i \quad (12.67)
  \]

- An extreme point of $P_f$ is a point that is not a convex combination of two other distinct points in $P_f$. Equivalently, an extreme point corresponds to setting certain inequalities in the specification of $P_f$ to be equalities, so that there is a unique single point solution.
Theorem 12.6.1

For a given ordering $E = (e_1, \ldots, e_m)$ of $E$ and a given $E_i = (e_1, \ldots, e_i)$ and $x$ generated by $E_i$ using the greedy procedure ($x(e_i) = f(e_i | E_{i-1})$), then $x$ is an extreme point of $P_f$. 
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Proof.

- We already saw that $x \in P_f$ (Theorem 12.4.1).
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Proof.

- We already saw that $x \in P_f$ (Theorem 12.4.1).
- To show that $x$ is an extreme point of $P_f$, note that it is the unique solution of the following system of equations

\[
x(E_j) = f(E_j) \text{ for } 1 \leq j \leq i \leq m \tag{12.68}
\]

\[
x(e) = 0 \text{ for } e \in E \setminus E_i \tag{12.69}
\]

There are $i \leq m$ equations and $i \leq m$ unknowns, and simple Gaussian elimination gives us back the $x$ constructed via the Greedy algorithm!!
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- \( x(E_3) = x(e_1) + x(e_2) + x(e_3) = f(e_1, e_2, e_3) \) so
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And so on . . . , but we see that this is just Gaussian elimination.
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- And so on . . . , but we see that this is just Gaussian elimination.
- Also, since $x \in P_f$, for each $i$, we see that,

$$x(E_j) = f(E_j) \quad \text{for } 1 \leq j \leq i \quad \text{(12.70)}$$

$$x(A) \leq f(A), \forall A \subseteq E \quad \text{(12.71)}$$
Polymatroid extreme points

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Also, since \( x \in P_f \), for each \( i \), we see that,

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x(E_j) = f(E_j) \quad \text{for } 1 \leq j \leq i \tag{12.70}
\]
\[
x(A) \leq f(A), \forall A \subseteq E \tag{12.71}
\]

Thus, the greedy procedure provides a modular function lower bound on \( f \) that is tight on all points \( E_i \) in the order. This can be useful in its own right.
Polymatroid extreme points

some examples
Polymatroid extreme points

Moreover, we have (and will ultimately prove)

**Corollary 12.6.2**

If $x$ is an extreme point of $P_f$ and $B \subseteq E$ is given such that $\text{supp}(x) = \{e \in E : x(e) \neq 0\} \subseteq B \subseteq \bigcup \{A : x(A) = f(A)\} = \text{sat}(x)$, then $x$ is generated using greedy by some ordering of $B$. 
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**Corollary 12.6.2**

*If \( x \) is an extreme point of \( P_f \) and \( B \subseteq E \) is given such that \( \text{supp}(x) = \{ e \in E : x(e) \neq 0 \} \subseteq B \subseteq \bigcup \{ A : x(A) = f(A) \} = \text{sat}(x) \), then \( x \) is generated using greedy by some ordering of \( B \).*

Note, \( \text{sat}(x) = \text{cl}(x) = \bigcup \{ A : x(A) = f(A) \} \) is also called the closure of \( x \) (recall that sets \( A \) such that \( x(A) = f(A) \) are called tight, and such sets are closed under union and intersection, as seen in Lecture 8, Theorem 12.3.2)
Polymatroid extreme points

Moreover, we have (and will ultimately prove)

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- Note, $\text{sat}(x) = \text{cl}(x) = \bigcup \{ A : x(A) = f(A) \}$ *is also called the closure of* $x$ *(...)*

- Thus, $\text{cl}(x)$ *is a tight set.*
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- Thus, $\text{cl}(x)$ is a tight set.

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- For arbitrary $x$, $\text{supp}(x)$ is not necessarily tight, but for an extreme point, $\text{supp}(x)$ is.
Recall
\[ f(e|A) = f(A+e) - f(A) \]

Notice how submodularity, \( f(e|B) \leq f(e|A) \) for \( A \subseteq B \), defines the shape of the polytope.

In fact, we have strictness here \( f(e|B) < f(e|A) \) for \( A \subset B \).

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Intuition: why greedy works with polymatroids

Given $w$, the goal is to find $x = (x(e_1), x(e_2))$ that maximizes $x^T w = x(e_1)w(e_1) + x(e_2)w(e_2)$.

- If $w(e_2) > w(e_1)$ the upper extreme point indicated maximizes $x^T w$ over $x \in P_f^+$.
- If $w(e_2) < w(e_1)$ the lower extreme point indicated maximizes $x^T w$ over $x \in P_f^+$.