EE596A Submodular Functions **University of Washington**
Spring 2016 **Dept. of Electrical Engineering**

Homework 3. Due **Monday, 5/2/2016, 11:59pm** Electronically

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All homework is due electronically via the link https://canvas.uw.edu/courses/1039754/assignments. Note that the due dates and times might be in the evening. Please submit a PDF file. Doing your homework by hand and then converting to a PDF file (by say taking high quality photos using a digital camera and then converting that to a PDF file) is fine, as there are many jpg to pdf converters on the web. Some of the problems below will require that you look at some of the lecture slides at our web page (http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2016/).

Problem 1. Submodular Definition (15 points)

Firstly, see the slides for lecture 5. You are to prove Eq. (5.58) and Eq. (5.59).

Problem 1(a). Eq. (5.58) Lecture 5 Prove that a function f is submodular if and only if the following is true:

$$f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|S \setminus \{j\}) + \sum_{j \in T \setminus S} f(j|S \cap T) \; \forall S, T \subseteq V \tag{1}$$

Problem 1(b). Eq. (5.59) Lecture 5 Prove that a function f is submodular if and only if the following is true:

$$f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|S \setminus \{j\}) \; \forall T \subseteq S \subseteq V \tag{2}$$

Problem 1(c). Weaker but no weaker Consider a modified version of Equation (1) as follows:

$$f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|(T \cup S) \setminus \{j\}) + \sum_{j \in T \setminus S} f(j|S \cap T) \; \forall S, T \subseteq V \tag{3}$$

Note that Equation (3) is a weaker version of Equation (1).

Problem 1(c)-i. Proof Even so, prove that a function is submodular if and only if Equation (3) holds.

Problem 1(c)-ii. Weaker but no weaker Given two general theorems, results, conditions, or propositions P_1 and P_2, we say that P_1 is **stronger** than P_2 (and correspondingly P_2 is **weaker**) if $P_1 \Rightarrow P_2$, but not necessarily vice verse. Suppose that there exists a proposition Q such that $Q \Leftrightarrow P_1$, then if $P_1 \Rightarrow P_2$ this does not ensure us that $Q \Leftrightarrow P_2$. In other words, if P_2 is weaker than P_1 then P_2 is not equivalent to the same things that P_1 is equivalent to.

Why do we say that Equation (3) is weaker than Equation (1) but no weaker? Why is it that, in particular, starting from the stonger Equation (1) and then weakening it to Equation (3) is still both necessary and sufficient for submodularity?

Problem 2. Equal Size of Bases (15 points)

In a matroid $M = (E, \mathcal{I})$, show that for any $U \subseteq E(M)$, any two bases of U have the same size.

Problem 3. Matroid by submodular functions (15 points)

Let $f : 2^E \to \mathbb{Z}$ be an integer valued monotone non-decreasing submodular function. Define a set of sets as follows:

$$\mathcal{C}(f) = \{C \subseteq E : C \neq \emptyset, \text{is inclusionwise minimal, and } f(C) < |C|\}. \tag{4}$$

Prove that $\mathcal{C}(f)$ is the collection of circuits of a matroid on E.

Problem 4. Set functions defined on a bipartite graph (25 points) You are given a weighted complete bipartite graph $G(V, U, E, w)$, where V is a set of items, U is a set of objects, and E is the set of all edges connecting between an item $v \in V$ and a object $u \in U$. Let $e_{u,v}$ denote the edge between u and v, and $w_{u,v} \geq 0$ denote the weight associated with this edge.

Let $f : 2^V \to \mathbb{R}$ be a set function defined on the ground set V. Consider the following set functions, decide for each instance whether it is submodular, supermodular, modular, or none. Please justify your answer.

Problem 4(a). Relevance function (5 points) Let f be defined as: for any subset $A \subseteq V$, $f(A)$ valuates the sum of all edge weights whose edge is incident to an item in A. Or, mathematically, f is written as $f(A) = \sum_{u \in U} \sum_{v \in A} w_{v,u}$.

Problem 4(b). Set cover function (5 points) Assuming $w_{v,u} \in \{0, 1\}$ for all $v \in V$ and $u \in U$. Let $f(A) = \sum_{u \in U} \min\{\sum_{v \in A} w_{v,u}, 1\}$.

Problem 4(c). Generalized set cover function (5 points) Assuming $w_{v,u} \geq 0, \forall v \in V, u \in U$. Let $f(A) = \sum_{u \in U} \min\{\sum_{v \in A} w_{v,u}, k_u\}$, where $k_u \geq 0, \forall u \in U$ are given constants.

Problem 4(d). Maximum coverage function (5 points) Let $f(A) = \sum_{u \in U} \max_{v \in A} w_{v,u}$.

Problem 4(e). Probabilistic coverage function (5 points) Assuming that $0 \leq w_{vu} \leq 1$, let $f(A) = \sum_{u \in U} \Big(1 - \prod_{v \in A}(1 - w_{v,u})\Big)$.

Problem 5. Truncated Matroid Rank (15 points)

Fix a set $R \subseteq V$ with $|R| = b$, and choose $a \in \mathbb{Z}_+$ with $a < b$. Consider the following function:

$$f_R(A) = \min\big\{|A|, a + |A \cap \bar{R}|, b\big\} \tag{5}$$

Problem 5(a). Matroid Rank (7.5 points) Prove that this function is a matroid rank function (this was mentioned in the lecture slides).

Problem 5(b). Matroid Rank (7.5 points) If this is a matroid rank function, there must be a matroid $M = (V, \mathcal{I})$. Your task here is to clearly and succinctly define, characterize, and justify the independent sets $\mathcal{I}$ of this matroid.

Problem 6. Matroids (15 points)

Recall from our lecture slides that the three the axioms for a matroid (I1), (I2), and (I3). Recall also from lecture that we stated that we can replace (I3) with the following condition.

Proposition 1. *(I3') For any $U \subseteq E(M)$, any two bases of U have the same size.*

Problem 6(a). In this problem, you are to show that this is true (namely that I1', I2', and I3' (as given in lecture slides) define a matroid.

Problem 7. matroid (15 points)

Prove the following theorem.

Theorem 2. *If $\mathcal{V} = (V_i : I \in I)$ is a finite family of non-empty subsets of V, and $f : 2^V \to \mathbb{Z}_+$ is a non-negative, integral, monotone non-decreasing, and submodular function, then for any non-negative integer $d \in \mathbb{Z}_+$, $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that*

$$f(\cup_{i \in J}\{v_i\}) \geq |J| - d \textit{ for all } J \subseteq I \tag{6}$$

if and only if

$$f(V(J)) \geq |J| - d \textit{ for all } J \subseteq I \tag{7}$$

You may use any of the theorems from the slides in this class (you may find it useful to look at the slides from Lectures 6 and 7).

Problem 8. Matroids, Partition vs. Graphic (15 points)

In class we defined a partition matroid based on a partition $V = \{V_1, V_2, \ldots, V_\ell\}$ of the ground set V into ℓ blocks, along with ℓ non-negative integer limits $k_1, k_2, \ldots, k_\ell \in \mathbb{Z}_+$.

Can any partition be represented as a graphic matroid? If so, prove it constructively by forming a graphic matroid for any partition matroid that is isomorphic to the partition matroid. If not, prove it.

Problem 9. Matroids, Partition vs. Graphic (15 points)

Recall from Lecture 6, the original definition of a matroid.

Definition 3 (Matroid)**.** *A set system $(E, \mathcal{I})$ is a Matroid if*

(I1) $\emptyset \in \mathcal{I}$

(I2) $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ *(down-closed or subclusive)*

(I3) $\forall I, J \in \mathcal{I}$, *with* $|I| = |J| + 1$, *then there exists* $x \in I \setminus J$ *such that* $J \cup \{x\} \in \mathcal{I}$.

We had another definition that, in lecture, we said was equivalent, namely:

Definition 4 (Matroid)**.** *A set system $(V, \mathcal{I})$ is a Matroid if*

(I1') $\emptyset \in \mathcal{I}$ *(emptyset containing)*

(I2') $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ *(down-closed or subclusive)*

(I3') $\forall X \subseteq V$, *and* $I_1, I_2 \in$ *maxInd*(X), *we have* $|I_1| = |I_2|$ *(all maximally independent subsets of X have the same size).*

Your task is to prove that the two definitions are equivalent. I.e., prove that given a set system $(V, \mathcal{I})$, that:

1. if properties I1, I2, and I3 hold, then I1', I2', and I3' hold.
2. if properties I1', I2', and I3' hold, then I1, I2, and I3 hold.

Problem 10. Matroids, Partition vs. Graphic (15 points)

Suppose we have a laminar family $\mathcal{F}$ of subsets of V and an integer k_A for every set $A \in \mathcal{F}$. Then $(V, \mathcal{I})$ defines a matroid where

$$\mathcal{I} = \{I \subseteq E : |I \cap A| \leq k_A \text{ for all } A \in \mathcal{F}\} \tag{8}$$

What is the rank function for this matroid? Prove that your answer is correct.

Problem 11. Transversal matroids (15 points)

Recall from lecture that a transversal matroid has rank

$$r(A) = \min_{J \subseteq I} \left(|V(J) \cap A| - |J| + |I|\right) \tag{9}$$

and that, therefore, this function is submodular. Note that the rank function $r(A)$ is a minimum over a set of modular functions. Is it true in general that the min over a set of modular functions is submodular?
