EE596A Submodular Functions | **University of Washington**
Spring 2016 | **Dept. of Electrical Engineering**

Homework 2. Due 4/18/2016 11:55pm Electronically

Prof: J. Bilmes <bilmes@ee.washington.edu>
TA: K. Wei <kaiwei@uw.edu>

4/11/2016

All homework is due electronically via the link https://canvas.uw.edu/courses/1039754/assignments. Note that the due dates and times might be in the evening. Please submit a PDF file. Doing your homework by hand and then converting to a PDF file (by say taking high quality photos using a digital camera and then converting that to a PDF file) is fine, as there are many jpg to pdf converters on the web. Some of the problems below will require that you look at some of the lecture slides at our web page (http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2016/).

Problem 1. Composition with modular functions (84 points, 6 points each 14 problems) Let $m_1, m_2 : V \to \mathbb{R}_{++}$ be two normalized modular set functions, i.e., $m_1(\emptyset) = m_2(\emptyset) = 0$. Decide whether each set function below is modular, submodular, supermodular, or none. Please briefly justify your answers.

Problem 1(a).

$$f(A) = m_1(A) - m_2(A) \tag{1}$$

Problem 1(b).

$$f(A) = m_1(A)m_2(A) \tag{2}$$

Problem 1(c).

$$f(A) = \frac{m_1(A)}{m_2(A) + 1} \tag{3}$$

Problem 1(d).

$$f(A) = (1 + m_1(A)) \log(1 + m_1(A)) \tag{4}$$

Problem 1(e).

$$f(A) = \max_{a \in A} m_1(a) \tag{5}$$

Problem 1(f).

$$f(A) = \min_{a \in A} m_1(a) \tag{6}$$

Problem 1(g).

$$f(A) = \max\{m_1(A), m_2(A)\} \tag{7}$$

Problem 1(h).

$$f(A) = \min\{m_1(A), m_2(A)\} \tag{8}$$

Problem 1(i).

$$f(A) = |m_1(A) - m_2(A)| \tag{9}$$

Problem 1(j).

$$f(A) = \log(1 + m_1(A)) \tag{10}$$

Problem 1(k).

$$f(A) = 1 - \alpha^{-m_1(A)} \text{ with } \alpha > 1 \tag{11}$$

Problem 1(l).

$$f(A) = m_1(A) + m_2(V \setminus A) \tag{12}$$

Problem 1(m).

$$f(A) = \prod_{a \in A} m_1(a) \tag{13}$$

where we assume that $m_1(a) \in [0, 1], \forall a \in V$.

Problem 1(n). Given modular function $m : V \to \mathbb{R}_+$,

$$f(A) = 1/(m(A) + \epsilon) \tag{14}$$

where $\epsilon > 0$ is a constant.

Problem 2. $\mathscr{S}_n$ has 2^V (40 points)

Let V be the ground set of size $n = |V|$. We claim that submodular functions have 2^n degrees of freedom. This means that the set of submodular functions live on a non-zero measure subset of $\mathbb{R}^{2^n}$, one that is not a lower-dimensional manifold and hence one that can not be projected into a space of dimension lower than 2^n.

Recall that a cone $\mathcal{K}$ is a set such that $\mathbf{0} \in \mathcal{K}$ and $\theta x \in \mathcal{K}$ whenever $x \in \mathcal{K}$ and $\theta \geq 0$. A convex cone is a cone that is convex (i.e., for any $x, y \in \mathcal{K}$ we have $(1 - \lambda)x + \lambda y \in \mathcal{K}$ for all $0 \leq \lambda \leq 1$).

Another way to state the property about submodular functions is as follows. Let $\mathscr{S}_n$ be the space of all submodular functions over a ground set V of size $n = |V|$. That is, any $f \in \mathscr{S}_n$ is a function $f : 2^V \to \mathbb{R}$ that is submodular. We know that $\mathscr{S}_n \subseteq \mathbb{R}^{2^n}$ and moreover $\mathscr{S}_n$ is a convex cone — this follows since any $f_1, f_2 \in \mathscr{S}_n$, we have $\alpha f_1 + \beta f_2 \in \mathscr{S}_n$ for any $\alpha, \beta \geq 0$. Equivalently, we can see that $\mathscr{S}_n$ is a cone since it is the intersection of a set of half-spaces defined in $\mathbb{R}^{2^n}$, one half space for each A, B via the inequality $f(A) + f(B) - f(A \cup B) - f(A \cap B) \geq 0$.

Your task is to prove that $\mathcal{S}_n$ can not be embedded in a space of dimension less than 2^n (in other words, in the general case, submodular functions have 2^n degrees of freedom).

Problem 2(a). Comparable sets (5 points) Two sets A, B are said to be *comparable* if either $A \subseteq B$ or $B \subseteq A$. The two subsets are otherwise called *non-comparable*. Note that if A, B are non-comparable, then neither can be empty. Given any arbitrary set function $f : 2^V \to \mathbb{R}$, and any two comparable sets A, B, prove that the following always holds.

$$f(A) + f(B) = f(A \cup B) + f(A \cap B) \tag{15}$$

Problem 2(b). square-root of cardinality is strict (15 points) We say a set function is strictly submodular if the following holds for any two non-comparable subsets $A, B \subseteq V$.

$$f(A) + f(B) > f(A \cup B) + f(A \cap B). \tag{16}$$

Please show that the square root of cardinality set function $f : 2^V \to \mathbb{R}$, i.e., $f(A) = \sqrt{|A|}$ for $A \subseteq V$ is strictly submodular.

Problem 2(c). 2^n degrees of freedom (20 points) Using the above, show that submodular functions, in the general case, have 2^n degrees of freedom.

Problem 3. Inequalities and how many (30 points)

There are many equivalent definitions of submodularity, some of which we saw in class. One is the classic $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ for all $A, B \subseteq V$ where $|V| = n$. We called this definition "submodular concave".

Another, which we called "four-points", says that for any $A \subseteq V$, and any $j, k \in V \setminus A$, we must have $f(A + j) + f(A + k) \geq f(A + j + k) + f(A)$.

Problem 3(a). submodular concave How many non-vacuous inequalties (i.e., ones that can be strict for some submodular function) are there in the "submdoular concave" definition?

Problem 3(b). four points How many non-vacuous inequalities are there in the submodular four-points definition? (hint, recall the 3-dimensional hypercube figure from class).

Problem 4. Geometric Means (25 points)

Given two polymatroid (normalized, monotone non-decreasing, submodular) functions $f_1, f_2 : 2^V \to \mathbb{R}_+$, consider the following function:

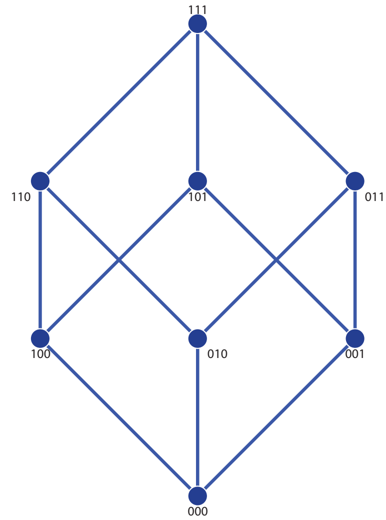
$$f(A) = \sqrt{f_1(A) f_2(A)} \tag{17}$$

What kind of function is $f(A)$? Submodular, supermodular, modular, or neither. Give very clear and concise proof if so, or if not a very clean and specific counter example.

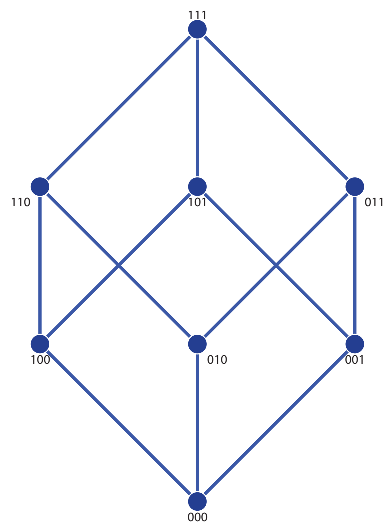
Problem 5. Distinctness of subadditive, submodular, monotone non-decreasing, and non-negative (40 points)

For this next problem, you are to use the 3-D hypercube and label the points on this hypercube. Your task is to, by labeling the vertices of the hypercube (that are given below), show example functions that prove the distinctness of the properties subadditive, submodular, monotone non-decreasing, and non-negative.

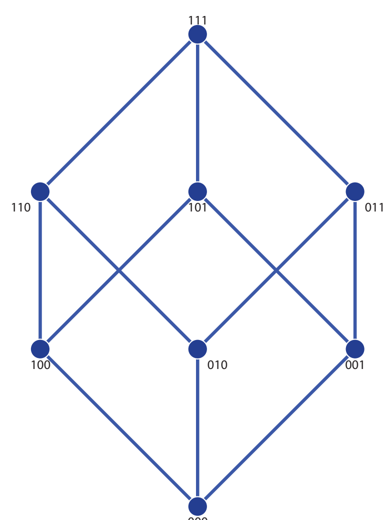
Problem 5(a). subadditive but not submodular, monotone non-decreasing, or non-negative. Label the vertices in the figure below:



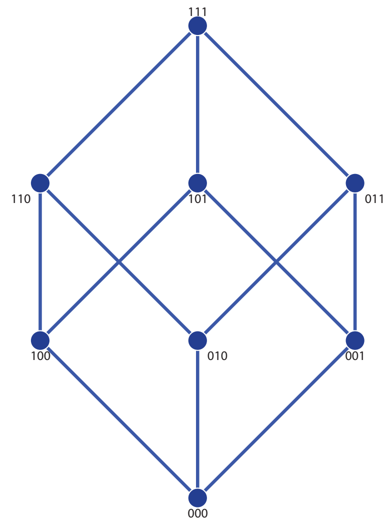
Problem 5(b). submodular, but not subadditive, monotone non-decreasing, or non-negative. Label the vertices in the figure below:



Problem 5(c). monotone non-decreasing, but not submodular, subadditive, or non-negative. Label the vertices in the figure below:



Problem 5(d). non-negative, but not monotone non-decreasing, submodular, or subadditive. Label the vertices in the figure below:


111
110
101
011
100
010
001
000