University of Washington
Dept. of Electrical Engineering

Homework 1. Due **Friday April 8th, 11:45pm** Electronically

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All homework is due electronically via the link https://canvas.uw.edu/courses/1039754/assignments. Note that the due dates and times might be in the evening. Please submit a PDF file. Doing your homework by hand and then converting to a PDF file (by say taking high quality photos using a digital camera and then converting that to a PDF file) is fine, as there are many jpg to pdf converters on the web. Some of the problems below will require that you look at some of the lecture slides at our web page (http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2016/).

Problem 1. More on min (30 points)

Recall from our class lecture slides that given submodular functions $f : 2^V \to \mathbb{R}$ and $g : 2^V \to \mathbb{R}$, that defining a function $h : 2^V \to \mathbb{R}$ as $h(A) = \min(f(A), g(A))$ is in general not submodular unless it is the case that $f(A) - g(A)$ is either monotonically incresing or decreasing.

Problem 1(a). Submodular min (10 pts) Supposing we have two concave functions $g_1, g_2 : \mathbb{R} \to \mathbb{R}$, and a modular function $m : 2^V \to \mathbb{R}_+$, show that $h : 2^V \to \mathbb{R}$ defined as $h(A) = \min(g_1(m(A)), g_2(m(A)))$ is submodular.

Problem 1(b). Min between two complement submodular functions (10 pts) Suppose f is a monotone submodular function, show that $h(A) = \min\{f(A), f(V \setminus A)\}$ is submodular.

Problem 1(c). Counter example (10 pts) Give an example of a monotone submodular f and g but where $h(A) = \min(f(A), g(A))$ is not submodular.

Problem 2. Common span vs. common index (30 points)

Recall the Venn diagrams from lecture. In that lecture, we considered a matroid rank function, r, and let A and B be any two sets, and a set C indexes the common span amongst A and B, argued that $r(A \cap B) \leq f(C)$, meaning that common span is no less than span of common index.

Define a new function $\text{comspan}_f : 2^V \times 2^V \to 2^{2^V}$ that given a polymatroid function f (or a matrix rank function), takes any $A, B \subseteq V$ and returns a set $C \in \text{comspan}_f(A, B)$, with $C \subseteq V$ that corresponds to the common span of A and B.

Problem 2(a). Common span function Define the necessary and sufficient properties of any C where $C \in \text{comspan}_f(A, B)$ for matrix rank function f.

Problem 2(b). Polymatroid Generalization Argue why this intuition works for any polymatroid (non-negative, monotone non-decreasing, submodular) function f, even one that might not be a matrix rank.

Problem 2(c). Empty or not? Must $\text{comspan}_f(A, B)$ always be non-empty? If so, describe why. If not, describe a very simple case where it is empty.

Problem 2(d). Algorithm Write a procedure that computes any $C \in \text{comspan}_f(A, B)$ given A and B and a polymatroid function. Argue why your algorithm is correct.

Problem 2(e). Algorithm analysis Analyze and provide the computational complexity of your algorithm.

Problem 2(f). *extra credit* Prove lower bound complexity for this problem, and provide a tight algorithm achieving it.

Problem 3. Min of submodular and constant (20 points)

We saw in class that if f is a **nondecreasing** submodular set function on S, and q is a real number, then the function f given by

$$f'(U) = \min\{q, f(U)\} \text{ for } U \subseteq S \tag{1}$$

is submodular.

Problem 3(a). Is monotonicity required? If so, show that monotonicity cannot be removed by a counter example. If not, prove that the submodularity of f' always holds even when f is not monotone.

Problem 3(b). Give an example of a submodular function that is non-negative and **non-increasing** (i.e., decreasing or staying the same).

Problem 3(c). What about non-increasing (decreasing) functions in the min? That is, is submodularity preserved in this case in Equation 1 as well when f is monotone non-increasing? Prove or give a counterexample.

Problem 4. Difference of two submodular functions (20 points)

Recall from our lecture slides, that in a directed graph, we define $E^+(X,Y) = \{(x,y) \in E(G) : x \in X \setminus Y, y \in Y \setminus X\}$ as the edges from X to Y, $\delta^+(X) = E(X, V \setminus X)$ as the edges edges leaving X, and $\delta^-(X) = E(V \setminus X, X)$ as the edges entering X.

Problem 4(a). Consider the set function $f(A) = |\delta^+(A)| - |\delta^+(V \setminus A)|$. Is this function submodular, supermodular, modular, or neither? Determine which one and prove it.