Submodular Functions, Optimization, and Applications to Machine Learning
— Spring Quarter, Lecture 6 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/

Prof. Jeff Bilmes
University of Washington, Seattle
Department of Electrical Engineering
http://melodi.ee.washington.edu/~bilmes

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\[ f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \]

- \[ f(A) + 2f(C) + f(B) \]
- \[ f(A) + f(C) + f(B) \]
- \[ f(A \cap B) \]

Cumulative Outstanding Reading

- Read chapter 1 from Fujishige’s book.
- Read chapter 2 from Fujishige’s book.
If you have any questions about anything, please ask them via our discussion board
(https://canvas.uw.edu/courses/1216339/discussion_topics).
Class Road Map - EE563

- L1(3/26): Motivation, Applications, & Basic Definitions,
- L3(4/2): Info theory exs, more apps, definitions, graph/combinatorial examples
- L4(4/4): Graph and Combinatorial Examples, Matrix Rank, Examples and Properties, visualizations
- L5(4/9): More Examples/Properties/Other SubmodularDefs., Independence,
- L6(4/11): Matroids, Matroid Examples, Matroid Rank, Partition/Laminar Matroids
- L7(4/16):
- L8(4/18):
- L9(4/23):
- L10(4/25):
- L11(4/30):
- L12(5/2):
- L13(5/7):
- L14(5/9):
- L15(5/14):
- L16(5/16):
- L17(5/21):
- L18(5/23):
- L–(5/28): Memorial Day (holiday)
- L19(5/30):

Last day of instruction, June 1st. Finals Week: June 2-8, 2018.
Composition of non-decreasing submodular and non-decreasing concave

Theorem 6.2.1

Given two functions, one defined on sets

\[ f : 2^V \rightarrow \mathbb{R} \]  \hspace{1cm} (6.1)

and another continuous valued one:

\[ \phi : \mathbb{R} \rightarrow \mathbb{R} \]  \hspace{1cm} (6.2)

the composition formed as \( h = \phi \circ f : 2^V \rightarrow \mathbb{R} \) (defined as \( h(S) = \phi(f(S)) \)) is nondecreasing submodular, if \( \phi \) is non-decreasing concave and \( f \) is nondecreasing submodular.
Let $f$ and $g$ both be submodular functions on subsets of $V$ and let $(f - g)(\cdot)$ be either monotone non-decreasing or monotone non-increasing. Then $h : 2^V \to \mathbb{R}$ defined by

$$h(A) = \min(f(A), g(A))$$  \hspace{1cm} (6.1)

is submodular.

**Proof.**

If $h(A)$ agrees with $f$ on both $X$ and $Y$ (or $g$ on both $X$ and $Y$), and since

$$h(X) + h(Y) = f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$$ \hspace{1cm} (6.2)

or

$$h(X) + h(Y) = g(X) + g(Y) \geq g(X \cup Y) + g(X \cap Y),$$ \hspace{1cm} (6.3)

the result (Equation (6.1) being submodular) follows since

$$f(X) + f(Y) \geq \min(f(X \cup Y), g(X \cup Y)) + \min(f(X \cap Y), g(X \cap Y))$$ \hspace{1cm} (6.4)
Arbitrary functions: difference between submodular funcs.

**Theorem 6.2.1**

Given an arbitrary set function $h$, it can be expressed as a difference between two submodular functions (i.e., $\forall h \in 2^V \rightarrow \mathbb{R}$, \exists f, g s.t. $\forall A, h(A) = f(A) - g(A)$ where both $f$ and $g$ are submodular).

**Proof.**

Let $h$ be given and arbitrary, and define:

$$\alpha \triangleq \min_{X,Y: X \not\subseteq Y, Y \not\subseteq X} \left( h(X) + h(Y) - h(X \cup Y) - h(X \cap Y) \right)$$  \hspace{1cm} (6.4)

If $\alpha \geq 0$ then $h$ is submodular, so by assumption $\alpha < 0$. Now let $f$ be an arbitrary strict submodular function and define

$$\beta \triangleq \min_{X,Y: X \not\subseteq Y, Y \not\subseteq X} \left( f(X) + f(Y) - f(X \cup Y) - f(X \cap Y) \right).$$  \hspace{1cm} (6.5)

Strict means that $\beta > 0$. ...
Many (Equivalent) Definitions of Submodularity

\( f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \), \( \forall A, B \subseteq V \)  
(6.16)

\( f(j|S) \geq f(j|T), \forall S \subseteq T \subseteq V \), with \( j \in V \setminus T \)  
(6.17)

\( f(C|S) \geq f(C|T), \forall S \subseteq T \subseteq V \), with \( C \subseteq V \setminus T \)  
(6.18)

\( f(j|S) \geq f(j|S \cup \{k\}), \forall S \subseteq V \) with \( j \in V \setminus (S \cup \{k\}) \)  
(6.19)

\( f(A \cup B | A \cap B) \leq f(A | A \cap B) + f(B | A \cap B), \forall A, B \subseteq V \)  
(6.20)

\( f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S) - \sum_{j \in S \setminus T} f(j|S \cup T - \{j\}), \forall S, T \subseteq V \)  
(6.21)

\( f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S), \forall S \subseteq T \subseteq V \)  
(6.22)

\( f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|S \setminus \{j\}) + \sum_{j \in T \setminus S} f(j|S \cap T) \forall S, T \subseteq V \)  
(6.23)

\( f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|S \setminus \{j\}), \forall T \subseteq S \subseteq V \)  
(6.24)
On Rank

- Let \( \text{rank} : 2^V \rightarrow \mathbb{Z}_+ \) be the rank function.
- In general, \( \text{rank}(A) \leq |A| \), and vectors in \( A \) are linearly independent if and only if \( \text{rank}(A) = |A| \).
- If \( A, B \) are such that \( \text{rank}(A) = |A| \) and \( \text{rank}(B) = |B| \), with \( |A| < |B| \), then the space spanned by \( B \) is greater, and we can find a vector in \( B \) that is linearly independent of the space spanned by vectors in \( A \).
- To stress this point, note that the above condition is \( |A| < |B| \), not \( A \subseteq B \) which is sufficient (to be able to find an independent vector) but not required.
- In other words, given \( A, B \) with \( \text{rank}(A) = |A| \) & \( \text{rank}(B) = |B| \), then \( |A| < |B| \iff \exists \text{ an } b \in B \text{ such that } \text{rank}(A \cup \{b\}) = |A| + 1 \).
Spanning trees/forests & incidence matrices

- A directed version of the graph (right) and its adjacency matrix (below).
- Orientation can be arbitrary.
- Note, rank of this matrix is 7.

```
1  2  3  4  5  6  7  8  9 10 11 12
1 -1  1  0  0  0  0  0  0  0  0  0
2  1  0 -1  0  1  0  0  0  0  0  0
3  0 -1  0  1  0 -1  0  0  0  0  0  0
4  0  0  1 -1  0  0  1 -1  0  0  0  0
5  0  0  0  0  0  1 -1  0  0  1  0  0
6  0  0  0  0  0  0  0  1 -1  0 -1  0
7  0  0  0  0  0  0  0  0  0  0  1  0
8  0  0  0  0  0  0  0  0  0 -1  1 -1
```
So $V$ is set of column vector indices of a matrix.
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Let $\mathcal{I} = \{I_1, I_2, \ldots\}$ be a set of all subsets of $V$ such that for any $I \in \mathcal{I}$, the vectors indexed by $I$ are linearly independent.
From Matrix Rank → Matroid

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- Let $\mathcal{I} = \{I_1, I_2, \ldots\}$ be a set of all subsets of $V$ such that for any $I \in \mathcal{I}$, the vectors indexed by $I$ are linearly independent.
- Given a set $B \in \mathcal{I}$ of linearly independent vectors, then any subset $A \subseteq B$ is also linearly independent.
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$$ A \subseteq B \text{ and } B \in \mathcal{I} \Rightarrow A \in \mathcal{I} \quad (6.1) $$
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(6.1)

- maxInd: Inclusionwise maximal independent subsets (or bases) of any set $B \subseteq V$.

$$\text{maxInd}(B) \triangleq \{A \subseteq B : A \in \mathcal{I} \text{ and } \forall v \in B \setminus A, A \cup \{v\} \notin \mathcal{I}\}$$

(6.2)

set of basis of $B$. 

\begin{center}
\begin{tikzpicture}
  \draw[->] (0,0) -- (2,0);
  \draw[->] (0,0) -- (0,2);
  \draw[->] (0,0) -- (1,1);
  \node at (2,0) {$A$};
  \node at (0,2) {$B$};
\end{tikzpicture}
\end{center}
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Let $\mathcal{I} = \{I_1, I_2, \ldots\}$ be a set of all subsets of $V$ such that for any $I \in \mathcal{I}$, the vectors indexed by $I$ are linearly independent.

Given a set $B \in \mathcal{I}$ of linearly independent vectors, then any subset $A \subseteq B$ is also linearly independent. Hence, $\mathcal{I}$ is down-closed or “subclusive”, under subsets. In other words,

$$A \subseteq B \text{ and } B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$$  \hspace{1cm} (6.1)

**maxInd:** Inclusionwise maximal independent subsets (or bases) of any set $B \subseteq V$.

$$\text{maxInd}(B) \overset{\Delta}{=} \{A \subseteq B : A \in \mathcal{I} \text{ and } \forall v \in B \setminus A, A \cup \{v\} \notin \mathcal{I}\}$$  \hspace{1cm} (6.2)

Given any set $B \subseteq V$ of vectors, all maximal (by set inclusion) subsets of linearly independent vectors are the same size. That is, for all $B \subseteq V$,

$$\forall A_1, A_2 \in \text{maxInd}(B), \ |A_1| = |A_2| = \text{rank}(B)$$  \hspace{1cm} (6.3)
Let $\mathcal{I} = \{I_1, I_2, \ldots\}$ be the set of sets as described above.
Let $\mathcal{I} = \{I_1, I_2, \ldots\}$ be the set of sets as described above. Thus, for all $I \in \mathcal{I}$, the matrix rank function has the property

$$r(I) = |I|$$

(6.4)

and for any $B \notin \mathcal{I}$,

$$r(B) = \max \{|A| : A \subseteq B \text{ and } A \in \mathcal{I}\} < |B|$$

(6.5)

Since all maximally independent subsets of a set are the same size, the rank function is well defined.
Matroids abstract the notion of linear independence of a set of vectors to general algebraic properties.
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In a matroid, there is an underlying ground set, say $E$ (or $V$), and a collection of subsets $\mathcal{I} = \{I_1, I_2, \ldots\}$ of $E$ that correspond to independent elements.
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There are many definitions of matroids that are mathematically equivalent, we’ll see some of them here.
Definition 6.3.1 (set system)

A (finite) ground set \( E \) and a set of subsets of \( E \), \( \emptyset \neq \mathcal{I} \subseteq 2^E \) is called a set system, notated \((E, \mathcal{I})\).

Set systems can be arbitrarily complex since, as stated, there is no systematic method (besides exponential-cost exhaustive search) to determine if a given set \( S \subseteq E \) has \( S \in \mathcal{I} \).
Independence System

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- Set systems can be arbitrarily complex since, as stated, there is no systematic method (besides exponential-cost exhaustive search) to determine if a given set $S \subseteq E$ has $S \in \mathcal{I}$.

- One useful property is “heredity.” Namely, a set system is a hereditary set system if for any $A \subset B \in \mathcal{I}$, we have that $A \in \mathcal{I}$.” 

\((\text{down closed})\)
Definition 6.3.2 (independence (or hereditary) system)

A set system \((V, \mathcal{I})\) is an independence system if

\[
\emptyset \in \mathcal{I} \quad \text{(emptyset containing)} \quad (I1)
\]

and

\[
\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I} \quad \text{(subclusive)} \quad (I2)
\]

- Property (I2) called “down monotone,” “down closed,” or “subclusive”
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- Example: \(E = \{1, 2, 3, 4\}\). With \(\mathcal{I} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 4\}\}\).
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- Example: \(E = \{1, 2, 3, 4\}\). With \(\mathcal{I} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 4\}\}\).
- Then \((E, \mathcal{I})\) is a set system, but not an independence system since it is not down closed (i.e., we have \(\{1, 2\} \in \mathcal{I}\) but not \(\{2\} \in \mathcal{I}\)).
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- With \(\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}\), then \((E, \mathcal{I})\) is now an independence (hereditary) system.
Given any set of linearly independent vectors $A$, any subset $B \subset A$ will also be linearly independent.
Independence System

Given any set of linearly independent vectors $A$, any subset $B \subset A$ will also be linearly independent.

Given any forest $G_f$ that is an edge-induced sub-graph of a graph $G$, any sub-graph of $G_f$ is also a forest.
### Independence System

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\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\end{pmatrix}
= 
\begin{pmatrix}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
\end{pmatrix}
\tag{6.6}
\]

- Given any set of linearly independent vectors \( A \), any subset \( B \subseteq A \) will also be linearly independent.
- Given any forest \( G_f \) that is an edge-induced sub-graph of a graph \( G \), any sub-graph of \( G_f \) is also a forest.
- So these both constitute independence systems.
Independent set definition of a matroid is perhaps most natural. Note, if \( J \in \mathcal{I} \), then \( J \) is said to be an **independent set**.

**Definition 6.3.3 (Matroid)**

A set system \((E, \mathcal{I})\) is a **Matroid** if

(I1) \( \emptyset \in \mathcal{I} \)

(I2) \( \forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I} \) (down-closed or subclusive)

(I3) \( \forall I, J \in \mathcal{I}, \text{ with } |I| = |J| + 1, \text{ then there exists } x \in I \setminus J \text{ such that } J \cup \{x\} \in \mathcal{I} \).

Why is (I1) is not redundant given (I2)?
Independent set definition of a matroid is perhaps most natural. Note, if \( J \in \mathcal{I} \), then \( J \) is said to be an independent set.

**Definition 6.3.3 (Matroid)**

A set system \((E, \mathcal{I})\) is a **Matroid** if

1. \( \emptyset \in \mathcal{I} \)
2. \( \forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I} \) (down-closed or subclusive)
3. \( \forall I, J \in \mathcal{I}, \text{ with } |I| = |J| + 1, \text{ then there exists } x \in I \setminus J \text{ such that } J \cup \{x\} \in \mathcal{I} \).

Why is (I1) is not redundant given (I2)? **Because without (I1) could have a non-matroid where \( \mathcal{I} = \{\} \).**
On Matroids

- Abstract properties of linear dependence (Hassler Whitney, 1935), but already then found instances of objects with those properties not based on a matrix.
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- Crapo & Rota preferred the term “combinatorial geometry”, or more specifically a “pregeometry” and said that pregeometries are “often described by the ineffably cacaphonic [sic] term ’matroid’, which we prefer to avoid in favor of the term ’pregeometry’.”
Slight modification (non unit increment) that is equivalent.

**Definition 6.3.4 (Matroid-II)**

A set system \((E, \mathcal{I})\) is a **Matroid** if

1. \((I1')\)  \(\emptyset \in \mathcal{I}\)
2. \((I2')\)  \(\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}\) (down-closed or subclusive)
3. \((I3')\)  \(\forall I, J \in \mathcal{I}, \text{ with } |I| > |J|, \text{ then there exists } x \in I \setminus J \text{ such that } J \cup \{x\} \in \mathcal{I}\)

Note \((I1) = (I1'), (I2) = (I2'),\) and we get \((I3) \equiv (I3')\) using induction.
Matroids, independent sets, and bases

- **Independent sets:** Given a matroid $M = (E, \mathcal{I})$, a subset $A \subseteq E$ is called independent if $A \in \mathcal{I}$ and otherwise $A$ is called dependent.
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- **A base of $U \subseteq E$**: For $U \subseteq E$, a subset $B \subseteq U$ is called a base of $U$ if $B$ is inclusionwise maximally independent subset of $U$. That is, $B \in \mathcal{I}$ and there is no $Z \in \mathcal{I}$ with $B \subset Z \subseteq U$.

  $B$ is a **base** of $U$

  whenever $B \in \text{maxInd}(U)$. 

Prof. Jeff Bilmes
EE563/Spring 2018/Submodularity - Lecture 6 - April 11th, 2018

F20/47 (pg.44/169)
Matroids, independent sets, and bases

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- **A base of $U \subseteq E$:** For $U \subseteq E$, a subset $B \subseteq U$ is called a base of $U$ if $B$ is inclusionwise maximally independent subset of $U$. That is, $B \in I$ and there is no $Z \in I$ with $B \subset Z \subseteq U$.

- **A base of a matroid:** If $U = E$, then a “base of $E$” is just called a base of the matroid $M$ (this corresponds to a basis in a linear space, or a spanning forest in a graph, or a spanning tree in a connected graph).
Proposition 6.3.5

In a matroid $M = (E, I)$, for any $U \subseteq E(M)$, any two bases of $U$ have the same size.
Proposition 6.3.5

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Matroids - important property

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Matroids - rank

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- We can a bit more formally define the rank function this way.

\[ \mathcal{I} = \{ I_1, \ldots, I_3 \} \subseteq \mathcal{I} \]
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Definition 6.3.7 (matroid rank function)

The rank function of a matroid is a function $r : 2^E \rightarrow \mathbb{Z}_+$ defined by

$$r(A) = \max \{|X| : X \subseteq A, X \in \mathcal{I}\} = \max_{X \in \mathcal{I}} |A \cap X|$$  \hspace{1cm} (6.7)
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- From the above, we immediately see that $r(A) \leq |A|$.
Graph $G = (V, E)$. 

Any $(V, I) = M = (v(m), i(m))$
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- From the above, we immediately see that $r(A) \leq |A|$.
- Moreover, if $r(A) = |A|$, then $A \in \mathcal{I}$, meaning $A$ is independent (in this case, $A$ is a self base).
Matroids, other definitions using matroid rank $r : 2^E \to \mathbb{Z}_+$

**Definition 6.3.8 (closed/flat/subspace)**

A subset $A \subseteq E$ is **closed** (equivalently, a **flat** or a **subspace**) of matroid $M$ if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

**Definition:** A **hyperplane** is a flat of rank $r(M) - 1$. 

A **closed set** is a set that doesn't increment rank.
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Definition 6.3.9 (closure)

Given $A \subseteq E$, the closure (or span) of $A$, is defined by $\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}$.
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Therefore, a closed set $A$ has $\text{span}(A) = A$.

Definition 6.3.10 (circuit)  
A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise-minimal dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).
Matroids by bases

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

**Theorem 6.3.11 (Matroid (by bases))**

Let $E$ be a set and $\mathcal{B}$ be a nonempty collection of subsets of $E$. Then the following are equivalent.

1. $\mathcal{B}$ is the collection of bases of a matroid;
2. if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.
3. If $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.

Properties 2 and 3 are called “exchange properties.”
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Properties 2 and 3 are called “exchange properties.”

Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.
A set is independent if and only if it contains no circuit. Therefore, it is not surprising that circuits can also characterize a matroid.

**Theorem 6.3.12 (Matroid by circuits)**

Let $E$ be a set and $C$ be a collection of subsets of $E$ that satisfy the following three properties:

1. (C1): $\emptyset \notin C$
2. (C2): if $C_1, C_2 \in C$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.
3. (C3): if $C_1, C_2 \in C$ with $C_1 \neq C_2$, and $e \in C_1 \cap C_2$, then there exists a $C_3 \in C$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$. 
Several circuit definitions for matroids.

**Theorem 6.3.13 (Matroid by circuits)**

*Let $E$ be a set and $C$ be a collection of nonempty subsets of $E$, such that no two sets in $C$ are contained in each other. Then the following are equivalent.*

1. $C$ is the collection of circuits of a matroid;
2. if $C, C' \in C$, and $x \in C \cap C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $C$;
3. if $C, C' \in C$, and $x \in C \cap C'$, and $y \in C \setminus C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $C$ containing $y$;
Several circuit definitions for matroids.

**Theorem 6.3.13 (Matroid by circuits)**

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Again, think about this for a moment in terms of linear spaces and matrices, and spanning trees.
Uniform Matroid

Given $E$, consider $\mathcal{I}$ to be all subsets of $E$ that are at most size $k$. That is $\mathcal{I} = \{ A \subseteq E : |A| \leq k \}.$
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- Then $(E, \mathcal{I})$ is a matroid called a $k$-uniform matroid.
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- Note, if $I, J \in \mathcal{I}$, and $|I| < |J| \leq k$, and $j \in J$ such that $j \notin I$, then $j$ is such that $|I + j| \leq k$ and so $I + j \in \mathcal{I}$. 

\[ r(A) = \begin{cases} |A| & \text{if } |A| \leq k \\ k & \text{if } |A| > k \end{cases} \]

Note, this function is submodular. Not surprising since $r(A) = \min(|A|, k)$ which is a non-decreasing concave function applied to a modular function.
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- Rank function

$$r(A) = \begin{cases} |A| & \text{if } |A| \leq k \\ k & \text{if } |A| > k \end{cases}$$  \hspace{1cm} (6.8)
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Closure function

\[
    \text{span}(A) = \begin{cases} 
        A & \text{if } |A| < k, \\
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(6.9)

- A “free” matroid sets $k = |E|$, so everything is independent.
Linear (or Matric) Matroid

- Let $\mathbf{X}$ be an $n \times m$ matrix and $E = \{1, \ldots, m\}$
- Let $\mathcal{I}$ consists of subsets of $E$ such that if $A \in \mathcal{I}$, and $A = \{a_1, a_2, \ldots, a_k\}$ then the vectors $x_{a_1}, x_{a_2}, \ldots, x_{a_k}$ are linearly independent.
- The rank function is just the rank of the space spanned by the corresponding set of vectors.
- Rank is submodular, it is intuitive that it satisfies the diminishing returns property (a given vector can only become linearly dependent in a greater context, thereby no longer contributing to rank).
- Called both linear matroids and matric matroids.
Let $G = (V, E)$ be a graph. Consider $(E, I)$ where the edges of the graph $E$ are the ground set and $A \in I$ if the edge-induced graph $G(V, A)$ by $A$ does not contain any cycle.
Cycle Matroid of a graph: Graphic Matroids

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$\mathcal{I}$ contains all forests.
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- $I$ contains all forests.
- Bases are spanning forests (spanning trees if $G$ is connected).
**Cycle Matroid of a graph: Graphic Matroids**

- Let $G = (V, E)$ be a graph. Consider $(E, \mathcal{I})$ where the edges of the graph $E$ are the ground set and $A \in \mathcal{I}$ if the edge-induced graph $G'(V, A)$ by $A$ does not contain any cycle.
- Then $M = (E, \mathcal{I})$ is a matroid.
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- Rank function $r(A)$ is the size of the largest spanning forest contained in $G(V, A)$. 

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Rank function $r(A)$ is the size of the largest spanning forest contained in $G(V, A)$.

Recall from earlier, $r(A) = |V(G)| - k_G(A)$, where for $A \subseteq E(G)$, we define $k_G(A)$ as the number of connected components of the edge-induced spanning subgraph $(V(G), A)$, and that $k_G(A)$ is supermodular, so $|V(G)| - k_G(A)$ is submodular.

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- Closure function adds all edges between the vertices adjacent to any edge in $A$. Closure of a spanning forest is $G$. 
Example: graphic matroid

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Let $V$ be our ground set.

Partition Matroid
Partition Matroid

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Let $V = V_1 \cup V_2 \cup \cdots \cup V_\ell$ be a partition of $V$ into $\ell$ blocks (i.e., disjoint sets). Define a set of subsets of $V$ as

$$\mathcal{I} = \{ X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell \}.$$  \hfill (6.10)

where $k_1, \ldots, k_\ell$ are fixed “limit” parameters, $k_i \geq 0$. Then $M = (V, \mathcal{I})$ is a matroid.
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- Note that a $k$-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$.
- Parameters associated with a partition matroid: $\ell$ and $k_1, k_2, \ldots, k_\ell$ although often the $k_i$’s are all the same.
- We’ll show that property (I3’) in Def 6.3.4 holds. First note, for any $X \subseteq V$, $|X| = \sum_{i=1}^{\ell} |X \cap V_i|$ since $\{V_1, V_2, \ldots, V_\ell\}$ is a partition.
Partition Matroid

- Let $V$ be our ground set.
- Let $V = V_1 \cup V_2 \cup \cdots \cup V_\ell$ be a partition of $V$ into $\ell$ blocks (i.e., disjoint sets). Define a set of subsets of $V$ as

$$\mathcal{I} = \{X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell\}. \quad (6.10)$$

where $k_1, \ldots, k_\ell$ are fixed “limit” parameters, $k_i \geq 0$. Then $M = (V, \mathcal{I})$ is a matroid.

- Note that a $k$-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$.

- Parameters associated with a partition matroid: $\ell$ and $k_1, k_2, \ldots, k_\ell$ although often the $k_i$’s are all the same.

- We’ll show that property (I3’) in Def 6.3.4 holds. First note, for any $X \subseteq V$, $|X| = \sum_{i=1}^{\ell} |X \cap V_i|$ since $\{V_1, V_2, \ldots, V_\ell\}$ is a partition.

- If $X, Y \in \mathcal{I}$ with $|Y| > |X|$, then there must be at least one $i$ with $|Y \cap V_i| > |X \cap V_i|$. Therefore, adding one element $e \in V_i \cap (Y \setminus X)$ to $X$ won’t break independence.
Ground set of objects, $V = \{ \}$
Partition Matroid

Partition of $V$ into six blocks, $V_1, V_2, \ldots, V_6$
Partition Matroid

Limit associated with each block, \( \{k_1, k_2, \ldots, k_6\} \)
Partition Matroid

Independent subset but not maximally independent.
Partition Matroid

Maximally independent subset, what is called a base.
Partition Matroid

Not independent since over limit in set six.
Lemma 6.5.1

The rank function $r : 2^E \to \mathbb{Z}_+$ of a matroid is submodular, that is

$$r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$$
Lemma 6.5.1

The rank function $r : 2^E \rightarrow \mathbb{Z}_+$ of a matroid is submodular, that is

$$r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$$

Proof.

1. Let $X \in \mathcal{I}$ be an inclusionwise maximal set with $X \subseteq A \cap B$
   
   - $\text{rank}(X) = |X|$,
   - $\forall x \in (A \cap B) \setminus X$, $X + r \notin \mathcal{I}$.
The rank function \( r : 2^E \rightarrow \mathbb{Z}_+ \) of a matroid is submodular, that is
\[
 r(A) + r(B) \geq r(A \cup B) + r(A \cap B)
\]

Proof.

1. Let \( X \in \mathcal{I} \) be an inclusionwise maximal set with \( X \subseteq A \cap B \).
2. Let \( Y \in \mathcal{I} \) be inclusionwise maximal set with \( X \subseteq Y \subseteq A \cup B \).

We can find such a \( Y \supseteq X \) because the following. Let \( Y' \in \mathcal{I} \) be any inclusionwise maximal set with \( Y' \subseteq A \cup B \), which might not have \( X \subseteq Y' \). Starting from \( Y \leftarrow X \subseteq A \cup B \), since \( |Y'| \geq |X| \), there exists a \( y \in Y' \setminus X \subseteq A \cup B \) such that \( X + y \in \mathcal{I} \) but since \( y \in A \cup B \), also \( X + y \in A \cup B \) — we then add \( y \) to \( Y \). We can keep doing this while \( |Y'| > |X| \) since this is a matroid. We end up with an inclusionwise maximal set \( Y \) with \( Y \in \mathcal{I} \) and \( X \subseteq Y \).

- \( X \subseteq Y \)
- \( \text{rank}(Y) = 17 \)
- \( \forall v \in (A \cup B) \setminus Y \),
  \[ Y + v \notin \mathcal{I} \]
Lemma 6.5.1

The rank function $r : 2^E \to \mathbb{Z}_+$ of a matroid is submodular, that is

$$r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$$

Proof.

1. Let $X \in \mathcal{I}$ be an inclusionwise maximal set with $X \subseteq A \cap B$.
2. Let $Y \in \mathcal{I}$ be inclusionwise maximal set with $X \subseteq Y \subseteq A \cup B$.
3. Since $M$ is a matroid, we know that $r(A \cap B) = r(X) = |X|$, and $r(A \cup B) = r(Y) = |Y|$. Also, for any $U \in \mathcal{I}$, $r(A) \geq |A \cap U|$.

Proof continued:

$$A \cup U \in \mathcal{I}$$

if $U \in \mathcal{I}$.
Matroids - rank function is submodular

Lemma 6.5.1

The rank function \( r : 2^E \to \mathbb{Z}_+ \) of a matroid is submodular, that is

\[
r(A) + r(B) \geq r(A \cup B) + r(A \cap B)
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Proof.

1. Let \( X \in \mathcal{I} \) be an inclusionwise maximal set with \( X \subseteq A \cap B \).
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4. Then we have (since \( X \subseteq A \cap B \), \( X \subseteq Y \), and \( Y \subseteq A \cup B \)),

\[
r(A) + r(B)
\]

\( (6.11) \)
Matroids - rank function is submodular

Lemma 6.5.1

The rank function \( r : 2^E \to \mathbb{Z}_+ \) of a matroid is submodular, that is
\[
\text{for any } A, B \subseteq E, \quad r(A) + r(B) \geq r(A \cup B) + r(A \cap B)
\]

Proof.

1. Let \( X \in \mathcal{I} \) be an inclusionwise maximal set with \( X \subseteq A \cap B \).
2. Let \( Y \in \mathcal{I} \) be inclusionwise maximal set with \( X \subseteq Y \subseteq A \cup B \).
3. Since \( M \) is a matroid, we know that \( r(A \cap B) = r(X) = |X| \), and \( r(A \cup B) = r(Y) = |Y| \). Also, for any \( U \in \mathcal{I} \), \( r(A) \geq |A \cap U| \).
4. Then we have (since \( X \subseteq A \cap B \), \( X \subseteq Y \), and \( Y \subseteq A \cup B \)),
   \[
   r(A) + r(B) \geq |Y \cap A| + |Y \cap B|
   \]
   (6.11)
Lemma 6.5.1

The rank function $r : 2^E \rightarrow \mathbb{Z}_+$ of a matroid is submodular, that is

$$r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$$

Proof.

1. Let $X \in \mathcal{I}$ be an inclusionwise maximal set with $X \subseteq A \cap B$.
2. Let $Y \in \mathcal{I}$ be inclusionwise maximal set with $X \subseteq Y \subseteq A \cup B$.
3. Since $M$ is a matroid, we know that $r(A \cap B) = r(X) = |X|$, and $r(A \cup B) = r(Y) = |Y|$. Also, for any $U \in \mathcal{I}$, $r(A) \geq |A \cap U|$.
4. Then we have (since $X \subseteq A \cap B$, $X \subseteq Y$, and $Y \subseteq A \cup B$),

$$r(A) + r(B) \geq |Y \cap A| + |Y \cap B| \tag{6.11}$$

$$= |Y \cap (A \cap B)| + |Y \cap (A \cup B)| \tag{6.12}$$

$$= m(A \cap B) + m(A \cup B)$$
**Lemma 6.5.1**

The rank function \( r : 2^E \to \mathbb{Z}_+ \) of a matroid is submodular, that is

\[
r(A) + r(B) \geq r(A \cup B) + r(A \cap B)
\]

**Proof.**

1. Let \( X \in \mathcal{I} \) be an inclusionwise maximal set with \( X \subseteq A \cap B \).
2. Let \( Y \in \mathcal{I} \) be inclusionwise maximal set with \( X \subseteq Y \subseteq A \cup B \).
3. Since \( M \) is a matroid, we know that \( r(A \cap B) = r(X) = |X| \), and \( r(A \cup B) = r(Y) = |Y| \). Also, for any \( U \in \mathcal{I} \), \( r(A) \geq |A \cap U| \).
4. Then we have (since \( X \subseteq A \cap B \), \( X \subseteq Y \), and \( Y \subseteq A \cup B \)),

\[
r(A) + r(B) \geq |Y \cap A| + |Y \cap B| \tag{6.11}
\]

\[
= |Y \cap (A \cap B)| + |Y \cap (A \cup B)| \tag{6.12}
\]

\[
\geq |X| + |Y| = r(A \cap B) + r(A \cup B) \tag{6.13}
\]
A matroid is defined from its rank function

**Theorem 6.5.2 (Matroid from rank)**

Let $E$ be a set and let $r : 2^E \rightarrow \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with $r$ being its rank function if and only if for all $A, B \subseteq E$:

(R1) $\forall A \subseteq E \ 0 \leq r(A) \leq |A|$ (non-negative cardinality bounded)

(R2) $r(A) \leq r(B)$ whenever $A \subseteq B \subseteq E$ (monotone non-decreasing)

(R3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ for all $A, B \subseteq E$ (submodular)

From above, $r(\emptyset) = 0$. Let $v \not\in A$, then by monotonicity and submodularity, $r(A) \leq r(A \cup \{v\}) \leq r(A) + r(\{v\})$ which gives only two possible values to $r(A \cup \{v\})$. 
A matroid is defined from its rank function

**Theorem 6.5.2 (Matroid from rank)**

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From above, $r(\emptyset) = 0$. Let $v \notin A$, then by monotonicity and submodularity, $r(A) \leq r(A \cup \{v\}) \leq r(A) + r(\{v\})$ which gives only two possible values to $r(A \cup \{v\})$.

Hence, unit increment (if $r(A) = k$, then either $r(A \cup \{v\}) = k$ or $r(A \cup \{v\}) = k + 1$) holds.
A matroid is defined from its rank function

**Theorem 6.5.2 (Matroid from rank)**

Let $E$ be a set and let $r : 2^E \rightarrow \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with $r$ being its rank function if and only if for all $A, B \subseteq E$:

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- From above, $r(\emptyset) = 0$. Let $v \notin A$, then by monotonicity and submodularity, $r(A) \leq r(A \cup \{v\}) \leq r(A) + r(\{v\})$ which gives only two possible values to $r(A \cup \{v\})$.
- Hence, unit increment (if $r(A) = k$, then either $r(A \cup \{v\}) = k$ or $r(A \cup \{v\}) = k + 1$) holds.
- Thus, submodularity, non-negative monotone non-decreasing, and unit increment of rank is necessary and sufficient to define a matroid.
A matroid is defined from its rank function \( M = (E, \mathcal{I}) = (E, r) \)

**Theorem 6.5.2 (Matroid from rank)**

Let \( E \) be a set and let \( r : 2^E \to \mathbb{Z}_+ \) be a function. Then \( r(\cdot) \) defines a matroid with \( r \) being its rank function if and only if for all \( A, B \subseteq E \):

1. (R1) \( \forall A \subseteq E \ 0 \leq r(A) \leq |A| \) (non-negative cardinality bounded)
2. (R2) \( r(A) \leq r(B) \) whenever \( A \subseteq B \subseteq E \) (monotone non-decreasing)
3. (R3) \( r(A \cup B) + r(A \cap B) \leq r(A) + r(B) \) for all \( A, B \subseteq E \) (submodular)

- From above, \( r(\emptyset) = 0 \). Let \( v \notin A \), then by monotonicity and submodularity, \( r(A) \leq r(A \cup \{v\}) \leq r(A) + r(\{v\}) \) which gives only two possible values to \( r(A \cup \{v\}) \).
- Hence, unit increment (if \( r(A) = k \), then either \( r(A \cup \{v\}) = k \) or \( r(A \cup \{v\}) = k + 1 \) holds.
- Thus, submodularity, non-negative monotone non-decreasing, and unit increment of rank is necessary and sufficient to define a matroid.
- Can refer to matroid as \((E, r)\), \( E \) is ground set, \( r \) is rank function.
Proof of Theorem 6.5.2 (matroid from rank).

Given a matroid $M = (E, \mathcal{I})$, we see its rank function as defined in Eq. 6.7 satisfies (R1), (R2), and, as we saw in Lemma 6.5.1, (R3) too.

$$r(A) = \max_{I \in \mathcal{I}} |A \cup I|$$

...
Proof of Theorem 6.5.2 (matroid from rank).

- Given a matroid $M = (E, \mathcal{I})$, we see its rank function as defined in Eq. 6.7 satisfies (R1), (R2), and, as we saw in Lemma 6.5.1, (R3) too.

- Next, assume we have (R1), (R2), and (R3). Define $\mathcal{I} = \{X \subseteq E : r(X) = |X|\}$. We will show that $(E, \mathcal{I})$ is a matroid.

...
Given a matroid $M = (E, \mathcal{I})$, we see its rank function as defined in Eq. 6.7 satisfies (R1), (R2), and, as we saw in Lemma 6.5.1, (R3) too.

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First, $\emptyset \in \mathcal{I}$. 

...
Proof of Theorem 6.5.2 (matroid from rank).

- Given a matroid $M = (E, I)$, we see its rank function as defined in Eq. 6.7 satisfies (R1), (R2), and, as we saw in Lemma 6.5.1, (R3) too.

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- First, $\emptyset \in I$.

- Also, if $Y \in I$ and $X \subseteq Y$ then by submodularity,
Proof of Theorem 6.5.2 (matroid from rank).

- Given a matroid \( M = (E, \mathcal{I}) \), we see its rank function as defined in Eq. 6.7 satisfies (R1), (R2), and, as we saw in Lemma 6.5.1, (R3) too.

- Next, assume we have (R1), (R2), and (R3). Define \( \mathcal{I} = \{ X \subseteq E : r(X) = |X| \} \). We will show that \( (E, \mathcal{I}) \) is a matroid.

- First, \( \emptyset \in \mathcal{I} \).

- Also, if \( Y \in \mathcal{I} \) and \( X \subseteq Y \) then by submodularity,

\[
r(X) \geq r(Y) - r(Y \setminus X)
\]

(6.14)

\[
r(X) + r(Y \setminus X) \geq r(X \cup (Y \setminus X)) + r(X \cap (Y \setminus X))
\]

\[
\Rightarrow \quad \Rightarrow \quad \geq r(Y) + 0
\]

\[
\ldots
\]

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Proof of Theorem 6.5.2 (matroid from rank).

- Given a matroid $M = (E, \mathcal{I})$, we see its rank function as defined in Eq. 6.7 satisfies (R1), (R2), and, as we saw in Lemma 6.5.1, (R3) too.

- Next, assume we have (R1), (R2), and (R3). Define $\mathcal{I} = \{ X \subseteq E : r(X) = |X| \}$. We will show that $(E, \mathcal{I})$ is a matroid.

- First, $\emptyset \in \mathcal{I}$.

- Also, if $Y \in \mathcal{I}$ and $X \subseteq Y$ then by submodularity,

$$r(X) \geq r(Y) - r(Y \setminus X) + r(\emptyset)$$  \hspace{1cm} (6.14)

...
Proof of Theorem 6.5.2 (matroid from rank).

- Given a matroid $M = (E, I)$, we see its rank function as defined in Eq. 6.7 satisfies (R1), (R2), and, as we saw in Lemma 6.5.1, (R3) too.
- Next, assume we have (R1), (R2), and (R3). Define $I = \{ X \subseteq E : r(X) = |X| \}$. We will show that $(E, I)$ is a matroid.
- First, $\emptyset \in I$.
- Also, if $Y \in I$ and $X \subseteq Y$ then by submodularity,

  $r(X) \geq r(Y) - r(Y \setminus X) + r(\emptyset)$  \hspace{1cm} (6.14)

  $\geq |Y| - |Y \setminus X|$  \hspace{1cm} (6.15)

  $r(Y \setminus X) \leq |Y \setminus X|$
Proof of Theorem 6.5.2 (matroid from rank).

- Given a matroid $M = (E, \mathcal{I})$, we see its rank function as defined in Eq. 6.7 satisfies (R1), (R2), and, as we saw in Lemma 6.5.1, (R3) too.

- Next, assume we have (R1), (R2), and (R3). Define $\mathcal{I} = \{ X \subseteq E : r(X) = |X| \}$. We will show that $(E, \mathcal{I})$ is a matroid.

- First, $\emptyset \in \mathcal{I}$.

- Also, if $Y \in \mathcal{I}$ and $X \subseteq Y$ then by submodularity,

\[
\begin{align*}
  r(X) &\geq r(Y) - r(Y \setminus X) + r(\emptyset) \\
  &\geq |Y| - |Y \setminus X| \\
  &= |X|
\end{align*}
\] (6.14) (6.15) (6.16)

...
Proof of Theorem 6.5.2 (matroid from rank).

- Given a matroid $M = (E, I)$, we see its rank function as defined in Eq. 6.7 satisfies (R1), (R2), and, as we saw in Lemma 6.5.1, (R3) too.

- Next, assume we have (R1), (R2), and (R3). Define $I = \{X \subseteq E : r(X) = |X|\}$. We will show that $(E, I)$ is a matroid.

- First, $\emptyset \in I$.

- Also, if $Y \in I$ and $X \subseteq Y$ then by submodularity,

\[
\begin{align*}
    r(X) &\geq r(Y) - r(Y \setminus X) + r(\emptyset) \\
    &\geq |Y| - |Y \setminus X| \\
    &= |X|
\end{align*}
\]

\[r(X) \leq |X| \tag{6.16}\]

implies $r(X) = |X|$, and thus $X \in I$. 

...
Proof of Theorem 6.5.2 (matroid from rank) cont.

Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $1 \leq k \leq |B|$).
Proof of Theorem 6.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $1 \leq k \leq |B|$).

- Suppose, to the contrary, that $\forall b \in B \setminus A, A + b \notin \mathcal{I}$, which means for all such $b, r(A + b) = r(A) = |A| < |A| + 1$. Then

$\begin{align*}
A + b &\neq A \\
\end{align*}$
Proof of Theorem 6.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $1 \leq k \leq |B|$).

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$$r(B) \leq r(A \cup B) \quad (6.17)$$
Matroids from rank

Proof of Theorem 6.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $1 \leq k \leq |B|$).

- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A| < |A| + 1$. Then

$$r(B) \leq r(A \cup B)$$

$$\leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A)$$

$$\geq r(X \cup Y) + r(X \cup \{b_1\})$$

$$\geq r(X \cup Y) + r(X \cup Y)$$

$$\geq r(A \cup B)$$

This gives a contradiction since $B \notin \mathcal{I}$.
Proof of Theorem 6.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $1 \leq k \leq |B|$).

- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A| < |A| + 1$. Then

\[
\begin{align*}
 r(B) & \leq r(A \cup B) \\
 & \leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) \\
 & = r(A \cup (B \setminus \{b_1\})) \\
 & \quad \text{ (by (6.20))} \\
 & \quad \text{ (by (6.21))}
\end{align*}
\]
Matroids from rank

Proof of Theorem 6.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $1 \leq k \leq |B|$).

- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A| < |A| + 1$. Then

$$r(B) \leq r(A \cup B) \quad (6.17)$$

$$\leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) \quad (6.18)$$

$$= r(A \cup (B \setminus \{b_1\})) \quad (6.19)$$

$$\leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) \quad (6.20)$$
Proof of Theorem 6.5.2 (matroid from rank) cont.

Let \( A, B \in \mathcal{I} \), with \( |A| < |B| \), so \( r(A) = |A| < r(B) = |B| \). Let \( B \setminus A = \{b_1, b_2, \ldots, b_k\} \) (note \( 1 \leq k \leq |B| \)).

Suppose, to the contrary, that \( \forall b \in B \setminus A, A + b \notin \mathcal{I} \), which means for all such \( b \), \( r(A + b) = r(A) = |A| < |A| + 1 \). Then

\[
\begin{align*}
 r(B) & \leq r(A \cup B) \\
 & \leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) \\
 & = r(A \cup (B \setminus \{b_1\})) \\
 & \leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) \\
 & = r(A \cup (B \setminus \{b_1, b_2\}))
\end{align*}
\] (6.17) (6.18) (6.19) (6.20) (6.21)
Proof of Theorem 6.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $1 \leq k \leq |B|$).

- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A| < |A| + 1$. Then

$$r(B) \leq r(A \cup B) \leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A)$$  \hspace{1cm} (6.17)

$$= r(A \cup (B \setminus \{b_1\})) \leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A)$$  \hspace{1cm} (6.18)

$$= \ldots \leq r(A) = |A| < |B|$$  \hspace{1cm} (6.22)
Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $1 \leq k \leq |B|$).

Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A| < |A| + 1$. Then

\[
\begin{align*}
r(B) &\leq r(A \cup B) \\
&\leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) \\
&= r(A \cup (B \setminus \{b_1\})) \\
&\leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) \\
&= r(A \cup (B \setminus \{b_1, b_2\})) \\
&\leq \ldots \leq r(A) = |A| < |B|
\end{align*}
\]

giving a contradiction since $B \in \mathcal{I}$.
Another way of using function $r$ to define a matroid.

**Theorem 6.5.3 (Matroid from rank II)**

Let $E$ be a finite set and let $r : 2^E \to \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with $r$ being its rank function if and only if for all $X \subseteq E$, and $x, y \in E$:

(R1') $r(\emptyset) = 0$;

(R2') $r(X) \leq r(X \cup \{y\}) \leq r(X) + 1$;

(R3') If $r(X \cup \{x\}) = r(X \cup \{y\}) = r(X)$, then $r(X \cup \{x, y\}) = r(X)$. 

Prof. Jeff Bilmes
Theorem 6.5.4 (Matroid by submodular functions)

Let \( f : 2^E \rightarrow \mathbb{Z} \) be a integer valued monotone non-decreasing submodular function. Define a set of sets as follows:

\[
C(f) = \left\{ C \subseteq E : C \text{ is non-empty, is inclusionwise-minimal, and has } f(C) < |C| \right\}
\]

(6.23)

Then \( C(f) \) is the collection of circuits of a matroid on \( E \).

Inclusionwise-minimal in this case means that if \( C \in C(f) \), then there exists no \( C' \subseteq C \) with \( C' \in C(f) \) (i.e., \( C' \subset C \) would either be empty or have \( f(C') \geq |C'| \)). Also, recall inclusionwise-minimal in Definition 6.3.10, the definition of a circuit.
Summarizing what we’ve so far seen, we saw that it is possible to uniquely define a matroid based on any of:

- Independence (define the independent sets).
Summarizing: Many ways to define a Matroid

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Summarizing: Many ways to define a Matroid

Summarizing what we’ve so far seen, we saw that it is possible to uniquely define a matroid based on any of:

- Independence (define the independent sets).
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- Circuit axioms
- Closure axioms (we didn’t see this, but it is possible)
- Rank axioms (normalized, monotone, cardinality bounded, non-negative integral, submodular)
- Matroids by integral submodular functions.
Maximization problems for matroids

- Given a matroid $M = (E, I)$ and a modular cost function $c : E \to \mathbb{R}$, the task is to find an $X \in I$ such that $c(X) = \sum_{x \in X} c(x)$ is maximum.
- This seems remarkably similar to the max spanning tree problem.

\[
\text{Compute } \max_{I \in I} c(I) \text{.}
\]
Minimization problems for matroids

- Given a matroid $M = (E, I)$ and a modular cost function $c : E \to \mathbb{R}$, the task is to find a basis $B \in \mathcal{B}$ such that $c(B)$ is minimized.

- This sounds like a set cover problem (find the minimum cost covering set of sets).

$$\min_{A} c(A) \quad \text{s.t.} \quad \bigcup_{a \in A} V_a = \bigcup_{a \in A}$$
What is the partition matroid’s rank function?

\[
\text{Partition Matroid} \\

\text{What is the partition matroid’s rank function?}
\]
Partition Matroid

- What is the partition matroid’s rank function?
- A partition matroids rank function:

\[ r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i) \]  

(6.24)

which we also immediately see is submodular using properties we spoke about last week. That is:

\[ r(x) \leq 2 \Rightarrow |x| \]

\[ r(x) = 2 \Rightarrow |x| \]
What is the partition matroid’s rank function?

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which we also immediately see is submodular using properties we spoke about last week. That is:

1. $|A \cap V_i|$ is submodular (in fact modular) in $A$
2. $\min(\text{submodular}(A), k_i)$ is submodular in $A$ since $|A \cap V_i|$ is monotone.
What is the partition matroid’s rank function?

A partition matroids rank function:

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r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i)
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3. Sums of submodular functions are submodular.
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1. \(|A \cap V_i|\) is submodular (in fact modular) in \(A\)
2. \(\min(\text{submodular}(A), k_i)\) is submodular in \(A\) since \(|A \cap V_i|\) is monotone.
3. Sums of submodular functions are submodular.

\(r(A)\) is also non-negative integral monotone non-decreasing, so it defines a matroid (the partition matroid).
Example: 2-partition matroid rank function: Given natural numbers $a, b \in \mathbb{Z}_+ \text{ with } a < b$, and any set $R \subseteq V$ with $|R| = b$. 

Figure showing partition blocks and partition matroid limits.
Example: 2-partition matroid rank function: Given natural numbers $a, b \in \mathbb{Z}_+ \text{ with } a < b$, and any set $\mathcal{R} \subseteq V$ with $|\mathcal{R}| = b$.

Create two-block partition $V = (\mathcal{R}, \bar{\mathcal{R}})$, where $\bar{\mathcal{R}} = V \setminus \mathcal{R}$ so $|\bar{\mathcal{R}}| = |V| - b$. Gives 2-partition matroid rank function as follows:

\begin{align*}
    r(A) &= \min(|A \cap \mathcal{R}|, a) + \min(|A \cap \bar{\mathcal{R}}|, |\bar{\mathcal{R}}|) \\
    &= \min(|A \cap \mathcal{R}|, a) + |A \cap \bar{\mathcal{R}}| \\
    &= \min(|A \cap \bar{\mathcal{R}} + |A \cap \mathcal{R}|, |A \cap \bar{\mathcal{R}}| + a) \\
    &= \min(|A|, |A \cap \bar{\mathcal{R}}| + a)
\end{align*}
From 2-partition matroid rank to truncated matroid rank

- **Example:** 2-partition matroid rank function: Given natural numbers $a, b \in \mathbb{Z}_+$ with $a < b$, and any set $R \subseteq V$ with $|R| = b$.

- **Create two-block partition** $V = (R, \bar{R})$, where $\bar{R} = V \setminus R$ so $|\bar{R}| = |V| - b$. Gives 2-partition matroid rank function as follows:

$$r(A) = \min(|A \cap R|, a) + \min(|A \cap \bar{R}|, |\bar{R}|)$$

$$= \min(|A \cap R|, a) + |A \cap \bar{R}|$$

$$= \min(|A \cap \bar{R}| + |A \cap R|, |A \cap \bar{R}| + a)$$

$$= \min(|A|, |A \cap \bar{R}| + a)$$

- **Figure showing partition blocks and partition matroid limits.**

Since $|\bar{R}| = |V| - b$

the limit on $\bar{R}$ is vacuous.

$a < |R| = b$
Example: 2-partition matroid rank function: Given natural numbers $a, b \in \mathbb{Z}_+ \text{ with } a < b$, and any set $R \subseteq V$ with $|R| = b$.

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$$r(A) = \min(|A \cap R|, a) + \min(|A \cap \bar{R}|, |\bar{R}|)$$  \hspace{1cm} (6.25)

$$= \min(|A \cap R|, a) + |A \cap \bar{R}|$$  \hspace{1cm} (6.26)

$$= \min(|A \cap \bar{R}| + |A \cap R|, |A \cap \bar{R}| + a)$$  \hspace{1cm} (6.27)

$$= \min(|A|, |A \cap \bar{R}| + a)$$  \hspace{1cm} (6.28)

Figure showing partition blocks and partition matroid limits.

Since $|\bar{R}| = |V| - b$
the limit on $\bar{R}$ is vacuous.

$a < |R| = b$
Define truncated matroid rank function. Start with 2-partition matroid rank $r(A) = \min(|A \cap R|, a) + \min(|A \cap \bar{R}|, |\bar{R}|)$, $a < b$. Define:

$$f_R(A) = \min \left\{ r(A), b \right\}$$ (6.29)

$$= \min \left\{ \min(|A|, |A \cap \bar{R}| + a), b \right\}$$ (6.30)

$$= \min \left\{ |A|, a + |A \cap \bar{R}|, b \right\}$$ (6.31)
Define **truncated matroid rank** function. Start with 2-partition matroid rank \( r(A) = \min(|A \cap R|, a) + \min(|A \cap \bar{R}|, |\bar{R}|) \), \( a < b \). Define:

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- Defines a matroid \( M = (V, f_R) = (V, \mathcal{I}) \) (Goemans et. al.) with

\[
\mathcal{I} = \{ I \subseteq V : |I| \leq b \text{ and } |I \cap R| \leq a \},
\]

(6.32)
Truncated Matroid Rank Function

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- Defines a matroid \( M = (V, f_R) = (V, \mathcal{I}) \) (Goemans et. al.) with \( \mathcal{I} = \{ I \subseteq V : |I| \leq b \text{ and } |I \cap R| \leq a \} \),

Useful for showing hardness of constrained submodular minimization. Consider sets \( B \subseteq V \) with \( |B| = b \). Recall \( R \) fixed, \( |R| = b \).
Truncated Matroid Rank Function

- Define truncated matroid rank function. Start with 2-partition matroid rank \( r(A) = \min(|A \cap R|, a) + \min(|A \cap \bar{R}|, |\bar{R}|), \ a < b \). Define:

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\[
\min \left\{ |B|, a + |B \cap \bar{R}|, b \right\}
\]

\[
\leq a + |B \cap \bar{R}|
\]
Truncated Matroid Rank Function

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\[
\begin{align*}
  f_R(A) &= \min \left\{ r(A), b \right\} \\
  &= \min \left\{ \min(|A|, |A \cap \overline{R}| + a), b \right\} \\
  &= \min \left\{ |A|, a + |A \cap \overline{R}|, b \right\}
\end{align*}
\]

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Consider sets \( B \subseteq V \) with \( |B| = b \).

- For \( R \), we have \( f_R(R) = \min(b, a, b) = a < b \).
- For any \( B \) with \( |B \cap R| \leq a \), \( f_R(B) = b \).
- For any \( B \) with \( |B \cap R| = \ell \), with \( a \leq \ell \leq b \), \( f_R(B) = a + b - \ell \).
Truncated Matroid Rank Function

- Define truncated matroid rank function. Start with 2-partition matroid rank \( r(A) = \min(|A \cap R|, a) + \min(|A \cap \bar{R}|, |\bar{R}|), a < b \). Define:
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  f_R(A) = \min \left\{ r(A), b \right\} = \min \left\{ \min(|A|, |A \cap \bar{R}| + a), b \right\} = \min \left\{ |A|, a + |A \cap \bar{R}|, b \right\}
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Useful for showing hardness of constrained submodular minimization.

Consider sets \( B \subseteq V \) with \( |B| = b \).

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- For any \( B \) with \( |B \cap R| = \ell \), with \( a \leq \ell \leq b \), \( f_R(B) = a + b - \ell \).
- \( R \), the set with minimum valuation amongst size-\( b \) sets, is hidden within an exponentially larger set of size-\( b \) sets with larger valuation.
A partition matroid can be viewed using a bipartite graph.

Letting $V$ denote the ground set, and $V_1, V_2, \ldots$ the partition, the bipartite graph is $G = (V, I, E)$ where $V$ is the ground set, $I$ is a set of “indices”, and $E$ is the set of edges.

$I = (I_1, I_2, \ldots, I_\ell)$ is a set of $k = \sum_{i=1}^{\ell} k_i$ nodes, grouped into $\ell$ clusters, where there are $k_i$ nodes in the $i^{th}$ group $I_i$, and $|I_i| = k_i$.

$(v, i) \in E(G)$ iff $v \in V_j$ and $i \in I_j$. 
Example where $\ell = 5$,

$$(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3).$$
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$$(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3).$$

Recall, $\Gamma : 2^V \to \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of $X$ is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.
Partition Matroid, rank as matching

Example where $\ell = 5$, $(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3)$.

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Here, for $X \subseteq V$, we have $\Gamma(X) = \{i \in I : (v, i) \in E(G) \text{ and } v \in X\}$. 
Matroids

Matroid Examples

Matroid Rank

More on Partition Matroid

Partition Matroid, rank as matching

Example where $\ell = 5$,

$$(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3).$$

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Here, for $X \subseteq V$, we have $\Gamma(X) = \{i \in I : (v, i) \in E(G) \text{ and } v \in X\}$.

For such a constructed bipartite graph, the rank function of a partition matroid is

$$r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i) = \text{the maximum matching involving } X.$$
Laminar Family and Laminar Matroid

- We can define a matroid with structures richer than just partitions.
Laminar Family and Laminar Matroid

- We can define a matroid with structures richer than just partitions.
- A set system \((V, \mathcal{F})\) is called a \textit{laminar} family if for any two sets \(A, B \in \mathcal{F}\), at least one of the three sets \(A \cap B\), \(A \setminus B\), or \(B \setminus A\) is empty.

\[ A \quad B \quad B \quad A \quad AB \]

Exercise: what is the rank function here?
Laminar Family and Laminar Matroid

- We can define a matroid with structures richer than just partitions.
- A set system \((V, \mathcal{F})\) is called a **laminar** family if for any two sets \(A, B \in \mathcal{F}\), at least one of the three sets \(A \cap B\), \(A \setminus B\), or \(B \setminus A\) is empty.

- Family is laminar \(\exists\) no two properly intersecting members: \(\forall A, B \in \mathcal{F}\), either \(A, B\) disjoint \((A \cap B = \emptyset)\) or comparable \((A \subseteq B\) or \(B \subseteq A\)).
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Suppose we have a laminar family \(\mathcal{F}\) of subsets of \(V\) and an integer \(k_A\) for every set \(A \in \mathcal{F}\).
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Suppose we have a laminar family \(\mathcal{F}\) of subsets of \(V\) and an integer \(k_A\) for every set \(A \in \mathcal{F}\). Then \((V, \mathcal{I})\) defines a matroid where

\[
\mathcal{I} = \{ I \subseteq E : |I \cap A| \leq k_A \text{ for all } A \in \mathcal{F} \} \tag{6.33}
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We can define a matroid with structures richer than just partitions.

A set system \((V, \mathcal{F})\) is called a **laminar** family if for any two sets \(A, B \in \mathcal{F}\), at least one of the three sets \(A \cap B\), \(A \setminus B\), or \(B \setminus A\) is empty.

![Laminar example](image)

Family is laminar \(\exists\) no two properly intersecting members: \(\forall A, B \in \mathcal{F}\), either \(A, B\) disjoint \((A \cap B = \emptyset)\) or comparable \((A \subseteq B\) or \(B \subseteq A\)).

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Exercise: what is the rank function here?