Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 9 —

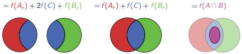
http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

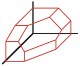
Prof. Jeff Bilmes

University of Washington, Seattle
Department of Electrical Engineering
http://melodi.ee.washington.edu/~bilmes

April 28th, 2014

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$





Submodular Functions, Optimization, and Applications to Machine Learning

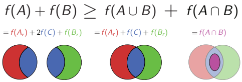
— Spring Quarter, Lecture 9 —

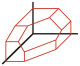
http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

Prof. Jeff Bilmes

University of Washington, Seattle
Department of Electrical Engineering
http://melodi.ee.washington.edu/~bilmes

April 28th, 2014





Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 9 —

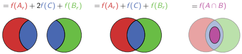
http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

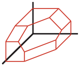
Prof. Jeff Bilmes

University of Washington, Seattle
Department of Electrical Engineering
http://melodi.ee.washington.edu/~bilmes

April 28th, 2014

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$





Cumulative Outstanding Reading

- Read chapters 1 and 2, and sections 3.1-3.2 from Fujishige's book.
- Good references for today: Schrijver-2003, Oxley-1992/2011, Welsh-1973, Goemans-2010, Cunningham-1984, Edmonds-1969.

Announcements, Assignments, and Reminders

- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, Combinatorial Geometries
- L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
- L9: From Matroid Polytopes to Polymatroids.
- L10:
- L11:
- L12:
- L13:
- L14:
- L15:
- L16:
- L17:
- L18:
- L19:
- L20:

Finals Week: June 9th-13th, 2014.

Matroid and the greedy algorithm

- Let $(E, \mathcal{I})$ be an independence system, and we are given a non-negative modular weight function $w : E \to \mathbb{R}_+$.

Algorithm 1: The Matroid Greedy Algorithm

Set $X \leftarrow \emptyset$;
while $\exists v \in E \setminus X$ *s.t.* $X \cup \{v\} \in \mathcal{I}$ **do**
  $v \in \operatorname{argmax} \{w(v) : v \in E \setminus X,\ X \cup \{v\} \in \mathcal{I}\}$;
  $X \leftarrow X \cup \{v\}$;

- Same as sorting items by decreasing weight w, and then choosing items in that order that retain independence.

Theorem 9.2.2

*Let $(E, \mathcal{I})$ be an independence system. Then the pair $(E, \mathcal{I})$ is a matroid if and only if for each weight function $w \in \mathcal{R}_+^E$, Algorithm **??** leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.*

Matroid Polyhedron in 2D

$$P_r^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\} \tag{9.10}$$

- Consider this in two dimensions. We have equations of the form:

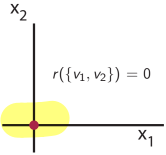
$$x_1 \geq 0 \text{ and } x_2 \geq 0 \tag{9.11}$$
$$x_1 \leq r(\{v_1\}) \tag{9.12}$$
$$x_2 \leq r(\{v_2\}) \tag{9.13}$$
$$x_1 + x_2 \leq r(\{v_1, v_2\}) \tag{9.14}$$

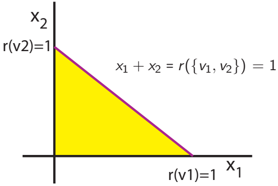
- Because r is submodular, we have

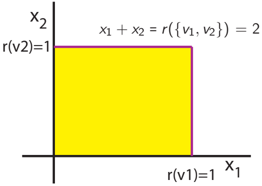
$$r(\{v_1\}) + r(\{v_2\}) \geq r(\{v_1, v_2\}) + r(\emptyset) \tag{9.15}$$

so since $r(\{v_1, v_2\}) \leq r(\{v_1\}) + r(\{v_2\})$, the last inequality is either touching or active.

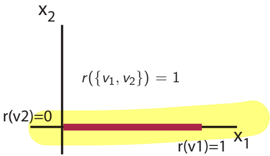
Matroid Polyhedron in 2D

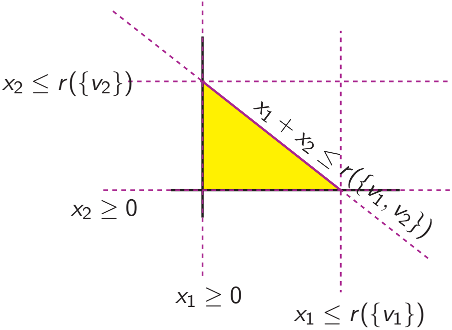






And, if v2 is a loop ...





Independence Polyhedra

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I$.
- Taking the convex hull, we get the independent set polytope, that is

$$P_{\text{ind. set}} = \text{conv}\left\{ \bigcup_{I \in \mathcal{I}} \{\mathbf{1}_I\} \right\} \tag{9.10}$$

- Since $\{\mathbf{1}_I : I \in \mathcal{I}\} \subseteq P_{\text{ind. set}}$, we have $\max\{w(I) : I \in \mathcal{I}\} \leq \max\{w^\intercal x : x \in P_{\text{ind. set}}\}$.
- Now take the rank function r of M, and define the following polyhedron:

$$P_r^+ = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \right\} \tag{9.11}$$

- Now, take any $x \in P_{\text{ind. set}}$, then we have that $x \in P_r^+$ (or $P_{\text{ind. set}} \subseteq P_r^+$). We show this next.

$P_{\text{ind. set}} \subseteq P_r^+$

- If $x \in P_{\text{ind. set}}$, then

$$x = \sum_i \lambda_i \mathbf{1}_{I_i} \tag{9.10}$$

  for some appropriate vector $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$.
- Clearly, for such x, $x \geq 0$.
- Now, for any $A \subseteq E$,

$$x(A) = x^\intercal \mathbf{1}_A = \sum_i \lambda_i \mathbf{1}_{I_i}{}^\intercal \mathbf{1}_A \tag{9.11}$$

$$\leq \sum_i \lambda_i \max_{j : I_j \subseteq A} \mathbf{1}_{I_j}(E) \tag{9.12}$$

$$= \max_{j : I_j \subseteq A} \mathbf{1}_{I_j}(E) \tag{9.13}$$

$$= r(A) \tag{9.14}$$

- Thus, $x \in P_r^+$ and hence $P_{\text{ind. set}} \subseteq P_r^+$.

Matroid Independence Polyhedron

- So recall from a moment ago, that we have that

$$\begin{aligned} P_{\text{ind. set}} &= \text{conv}\left\{\cup_{I \in \mathcal{I}}\{\mathbf{1}_I\}\right\} \\ &\subseteq P_r^+ = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\right\} \end{aligned} \tag{9.19}$$

- In fact, the two polyhedra are identical (and thus both are polytopes).
- We'll show this in the next few theorems.

Maximum weight independent set via greedy weighted rank

Theorem 9.2.6

Let $M = (V, \mathcal{I})$ be a matroid, with rank function r, then for any weight function $w \in \mathbb{R}_+^V$, there exists a chain of sets $U_1 \subset U_2 \subset \cdots \subset U_n \subseteq V$ such that

$$\max \{w(I) | I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i) \tag{9.19}$$

where $\lambda_i \geq 0$ satisfy

$$w = \sum_{i=1}^{n} \lambda_i \mathbf{1}_{U_i} \tag{9.20}$$

Maximum weight independent set via weighted rank

Proof.

- Firstly, note that for any such $w \in \mathbb{R}^E$, we have

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = (w_1 - w_2)\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + (w_2 - w_3)\begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + (w_{n-1} - w_n)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix} + (w_n)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \tag{9.19}$$

- If we can take w in decreasing order ($w_1 \geq w_2 \geq \cdots \geq w_n$), then each coefficient of the vectors is non-negative (except possibly the last one, w_n).

...

Maximum weight independent set via weighted rank

Proof.

- Now, again assuming $w \in \mathbb{R}_+^E$, order the elements of V as $(v_1, v_2, \dots, v_n)$ such that $w(v_1) \geq w(v_2) \geq \dots \geq w(v_n)$
- Define the sets U_i based on this order as follows, for $i = 0, \dots, n$

$$U_i \stackrel{\text{def}}{=} \{v_1, v_2, \dots, v_i\} \tag{9.20}$$

- Define the set I as those elements where the rank increases, i.e.:

$$I \stackrel{\text{def}}{=} \{v_i | r(U_i) > r(U_{i-1})\} \tag{9.21}$$

- Therefore, I is the output of the greedy algorithm for $\max\{w(I) | I \in \mathcal{I}\}$.
- And therefore, I is a maximum weight independent set (even a base, actually).

...

Maximum weight independent set via weighted rank

Proof.

- Now, we define λ_i as follows

$$\lambda_i \stackrel{\text{def}}{=} w(v_i) - w(v_{i+1}) \text{ for } i = 1, \dots, n-1 \tag{9.22}$$

$$\lambda_n \stackrel{\text{def}}{=} w(v_n) \tag{9.23}$$

- And the weight of the independent set $w(I)$ is given by

$$w(I) = \sum_{v \in I} w(v) = \sum_{i=1}^{n} w(v_i)\big(r(U_i) - r(U_{i-1})\big) \tag{9.24}$$

$$= w(v_n)r(U_n) + \sum_{i=1}^{n-1} \big(w(v_i) - w(v_{i+1})\big) r(U_i) = \sum_{i=1}^{n} \lambda_i r(U_i) \tag{9.25}$$

- Since we took $v_1, v_2, \dots$ in decreasing order, for all i, and since $w \in \mathbb{R}_+^E$, we have $\lambda_i \geq 0$

□

Linear Program LP

Consider the linear programming primal problem

$$\begin{aligned} \text{maximize} \quad & w^\intercal x \\ \text{subject to} \quad & x_v \geq 0 && (v \in V) \\ & x(U) \leq r(U) && (\forall U \subseteq V) \end{aligned} \tag{9.1}$$

Linear Program LP

Consider the linear programming primal problem

$$\begin{array}{rlr} \text{maximize} & w^\intercal x & \\ \text{subject to} & x_v \geq 0 & (v \in V) \\ & x(U) \leq r(U) & (\forall U \subseteq V) \end{array} \tag{9.1}$$

And its convex dual (note $y \in \mathbb{R}_+^{2^n}$, y_U is a scalar element within this exponentially big vector):

$$\begin{array}{rlr} \text{minimize} & \sum_{U \subseteq V} y_U r(U), & \\ \text{subject to} & y_U \geq 0 & (\forall U \subseteq V) \\ & \sum_{U \subseteq V} y_U \mathbf{1}_U \geq w & \end{array} \tag{9.2}$$

Linear Program LP

Consider the linear programming primal problem

$$\begin{array}{lll} \text{maximize} & w^\intercal x & \\ \text{subject to} & x_v \geq 0 & (v \in V) \\ & x(U) \leq r(U) & (\forall U \subseteq V) \end{array} \tag{9.1}$$

And its convex dual (note $y \in \mathbb{R}_+^{2^n}$, y_U is a scalar element within this exponentially big vector):

$$\begin{array}{lll} \text{minimize} & \sum_{U \subseteq V} y_U r(U), & \\ \text{subject to} & y_U \geq 0 & (\forall U \subseteq V) \\ & \sum_{U \subseteq V} y_U \mathbf{1}_U \geq w & \end{array} \tag{9.2}$$

Thanks to strong duality, the solutions to these are equal to each other.

Linear Program LP

- Consider the linear programming primal problem

$$\begin{aligned} \text{maximize} \quad & w^\intercal x \\ \text{s.t.} \quad & x_v \geq 0 && (v \in V) \\ & x(U) \leq r(U) && (\forall U \subseteq V) \end{aligned} \tag{9.3}$$

Linear Program LP

- Consider the linear programming primal problem

$$\begin{array}{rlll} \text{maximize} & w^\intercal x & & \\ \text{s.t.} & x_v \geq 0 & (v \in V) & \\ & x(U) \leq r(U) & (\forall U \subseteq V) & \end{array} \tag{9.3}$$

- This is identical to the problem

$$\max w^\intercal x \text{ such that } x \in P_r^+ \tag{9.4}$$

where, again, $P_r^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\}$.

Linear Program LP

- Consider the linear programming primal problem

$$\begin{array}{rlll} \text{maximize} & w^\intercal x & & \\ \text{s.t.} & x_v \geq 0 & (v \in V) & \\ & x(U) \leq r(U) & (\forall U \subseteq V) & \end{array} \tag{9.3}$$

- This is identical to the problem

$$\max w^\intercal x \text{ such that } x \in P_r^+ \tag{9.4}$$

where, again, $P_r^+ = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\right\}$.

- Therefore, since $P_{\text{ind. set}} \subseteq P_r^+$, the above problem can only have a larger solution. I.e.,

$$\max w^\intercal x \text{ s.t. } x \in P_{\text{ind. set}} \leq \max w^\intercal x \text{ s.t. } x \in P_r^+. \tag{9.5}$$

Polytope equivalence

- Hence, we have the following relations:

$$\max\{w(I) : I \in \mathcal{I}\} \leq \max\{w^\intercal x : x \in P_{\text{ind. set}}\} \tag{9.6}$$

$$\leq \max\left\{w^\intercal x : x \in P_r^+\right\} \tag{9.7}$$

$$\stackrel{\text{def}}{=} \alpha_{\text{min}} = \min\left\{\sum_{U \subseteq V} y_U r(U) : y \geq 0, \sum_{U \subseteq V} y_U \mathbf{1}_U \geq w\right\} \tag{9.8}$$

Polytope equivalence

- Hence, we have the following relations:

$$\max\{w(I) : I \in \mathcal{I}\} \leq \max\{w^\intercal x : x \in P_{\text{ind. set}}\} \tag{9.6}$$

$$\leq \max\left\{w^\intercal x : x \in P_r^+\right\} \tag{9.7}$$

$$\stackrel{\text{def}}{=} \alpha_{\text{min}} = \min\left\{\sum_{U\subseteq V} y_U r(U) : y \geq 0, \sum_{U\subseteq V} y_U \mathbf{1}_U \geq w\right\} \tag{9.8}$$

- Theorem 8.6.1 states that

$$\max\{w(I) : I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i) \tag{9.9}$$

for the chain of U_i's and $\lambda_i \geq 0$ that satisfies $w = \sum_{i=1}^{n} \lambda_i \mathbf{1}_{U_i}$ (i.e., the r.h.s. of Eq. 9.9 is feasible w.r.t. the dual LP).

Polytope equivalence

- Hence, we have the following relations:

$$\max\{w(I) : I \in \mathcal{I}\} \le \max\{w^\intercal x : x \in P_{\text{ind. set}}\} \tag{9.6}$$

$$\le \max\{w^\intercal x : x \in P_r^+\} \tag{9.7}$$

$$\stackrel{\text{def}}{=} \alpha_{\text{min}} = \min\left\{\sum_{U \subseteq V} y_U r(U) : y \ge 0, \sum_{U \subseteq V} y_U \mathbf{1}_U \ge w\right\} \tag{9.8}$$

- Theorem 8.6.1 states that

$$\max\{w(I) : I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i) \tag{9.9}$$

for the chain of U_i's and $\lambda_i \ge 0$ that satisfies $w = \sum_{i=1}^n \lambda_i \mathbf{1}_{U_i}$ (i.e., the r.h.s. of Eq. 9.9 is feasible w.r.t. the dual LP).

- Therefore, we also have

$$\max\{w(I) : I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i) \ge \alpha_{\text{min}} \tag{9.10}$$

Polytope equivalence

- Hence, we have the following relations:

$$\max\{w(I) : I \in \mathcal{I}\} \le \max\{w^\intercal x : x \in P_{\text{ind. set}}\} \tag{9.6}$$

$$\le \max\{w^\intercal x : x \in P_r^+\} \tag{9.7}$$

$$\stackrel{\text{def}}{=} \alpha_{\min} = \min\left\{\sum_{U \subseteq V} y_U r(U) : y \ge 0, \sum_{U \subseteq V} y_U \mathbf{1}_U \ge w\right\} \tag{9.8}$$

- Therefore, all the inequalities above are equalities.

Polytope equivalence

- Hence, we have the following relations:

$$\max\{w(I) : I \in \mathcal{I}\} = \max\{w^\intercal x : x \in P_{\text{ind. set}}\} \tag{9.6}$$

$$= \max\left\{w^\intercal x : x \in P_r^+\right\} \tag{9.7}$$

$$\stackrel{\text{def}}{=} \alpha_{\min} = \min\left\{\sum_{U \subseteq V} y_U r(U) : y \geq 0, \sum_{U \subseteq V} y_U \mathbf{1}_U \geq w\right\} \tag{9.8}$$

- Therefore, all the inequalities above are equalities.
- And since $w \in \mathbb{R}_+^E$ is an arbitrary direction into the positive orthant, we see that $P_r^+ = P_{\text{ind. set}}$

Polytope equivalence

- Hence, we have the following relations:

$$\max\{w(I) : I \in \mathcal{I}\} = \max\{w^\intercal x : x \in P_{\text{ind. set}}\} \tag{9.6}$$

$$= \max\left\{w^\intercal x : x \in P_r^+\right\} \tag{9.7}$$

$$\stackrel{\text{def}}{=} \alpha_{\min} = \min\left\{\sum_{U \subseteq V} y_U r(U) : y \geq 0, \sum_{U \subseteq V} y_U \mathbf{1}_U \geq w\right\} \tag{9.8}$$

- Therefore, all the inequalities above are equalities.
- And since $w \in \mathbb{R}_+^E$ is an arbitrary direction into the positive orthant, we see that $P_r^+ = P_{\text{ind. set}}$
- That is, we have just proven:

Theorem 9.3.1

$$P_r^+ = P_{\textit{ind. set}} \tag{9.11}$$

Polytope Equivalence (Summarizing the above)

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I$.

Polytope Equivalence (Summarizing the above)

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I$.
- Taking the convex hull, we get the independent set polytope, that is

$$P_{\text{ind. set}} = \text{conv}\left\{\cup_{I \in \mathcal{I}}\{\mathbf{1}_I\}\right\} \tag{9.12}$$

Polytope Equivalence (Summarizing the above)

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I$.
- Taking the convex hull, we get the independent set polytope, that is

$$P_{\text{ind. set}} = \text{conv}\left\{\cup_{I \in \mathcal{I}}\{\mathbf{1}_I\}\right\} \tag{9.12}$$

- Now take the rank function r of M, and define the following polyhedron:

$$P_r^+ = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\right\} \tag{9.13}$$

Polytope Equivalence (Summarizing the above)

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I$.
- Taking the convex hull, we get the independent set polytope, that is

$$P_{\text{ind. set}} = \text{conv}\left\{\cup_{I \in \mathcal{I}}\{\mathbf{1}_I\}\right\} \tag{9.12}$$

- Now take the rank function r of M, and define the following polyhedron:

$$P_r^+ = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\right\} \tag{9.13}$$

Theorem 9.3.2

$$P_r^+ = P_{\textit{ind. set}} \tag{9.14}$$

Greedy solves a linear programming problem

- So we can describe the independence polytope of a matroid using the set of inequalities (an exponential number of them).

Greedy solves a linear programming problem

- So we can describe the independence polytope of a matroid using the set of inequalities (an exponential number of them).
- In fact, considering equations starting at Eq 9.6, the LP problem with exponential number of constraints $\max\{w^\intercal x : x \in P_r^+\}$ is identical to the maximum weight independent set problem in a matroid, and since greedy solves the latter problem exactly, we have also proven:

Greedy solves a linear programming problem

- So we can describe the independence polytope of a matroid using the set of inequalities (an exponential number of them).
- In fact, considering equations starting at Eq 9.6, the LP problem with exponential number of constraints $\max\{w^\intercal x : x \in P_r^+\}$ is identical to the maximum weight independent set problem in a matroid, and since greedy solves the latter problem exactly, we have also proven:

Theorem 9.3.3

The LP problem $\max\{w^\intercal x : x \in P_r^+\}$ can be solved exactly using the greedy algorithm.

Note that this LP problem has an exponential number of constraints (since P_r^+ is described as the intersection of an exponential number of half spaces).

Greedy solves a linear programming problem

- So we can describe the independence polytope of a matroid using the set of inequalities (an exponential number of them).
- In fact, considering equations starting at Eq 9.6, the LP problem with exponential number of constraints $\max \{w^\intercal x : x \in P_r^+\}$ is identical to the maximum weight independent set problem in a matroid, and since greedy solves the latter problem exactly, we have also proven:

Theorem 9.3.3

The LP problem $\max \{w^\intercal x : x \in P_r^+\}$ can be solved exactly using the greedy algorithm.

Note that this LP problem has an exponential number of constraints (since P_r^+ is described as the intersection of an exponential number of half spaces).

- This means that if LP problems have certain structure, they can be solved much easier than immediately implied by the equations.

Base Polytope Equivalence

- Consider convex hull of indicator vectors just of the bases of a matroid, rather than all of the independent sets.

Base Polytope Equivalence

- Consider convex hull of indicator vectors just of the bases of a matroid, rather than all of the independent sets.
- Consider a polytope defined by the following constraints:

$$x \geq 0 \tag{9.15}$$

$$x(A) \leq r(A) \; \forall A \subseteq V \tag{9.16}$$

$$x(V) = r(V) \tag{9.17}$$

Base Polytope Equivalence

- Consider convex hull of indicator vectors just of the bases of a matroid, rather than all of the independent sets.
- Consider a polytope defined by the following constraints:

$$x \geq 0 \tag{9.15}$$

$$x(A) \leq r(A) \; \forall A \subseteq V \tag{9.16}$$

$$x(V) = r(V) \tag{9.17}$$

- Note the third requirement, $x(V) = r(V)$. $= \sum_{v \in V} x(v)$

Base Polytope Equivalence

- Consider convex hull of indicator vectors just of the bases of a matroid, rather than all of the independent sets.
- Consider a polytope defined by the following constraints:

$$x \geq 0 \tag{9.15}$$

$$x(A) \leq r(A) \; \forall A \subseteq V \tag{9.16}$$

$$x(V) = r(V) \tag{9.17}$$

- Note the third requirement, $x(V) = r(V)$.
- By essentially the same argument as above (Exercise:), we can shown that the convex hull of the incidence vectors of the bases of a matroid is a polytope that can be described by Eq. 9.15- 9.17 above.

Base Polytope Equivalence

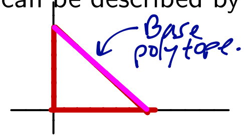
- Consider convex hull of indicator vectors just of the bases of a matroid, rather than all of the independent sets.
- Consider a polytope defined by the following constraints:

$$x \geq 0 \tag{9.15}$$

$$x(A) \leq r(A)\ \forall A \subseteq V \tag{9.16}$$

$$x(V) = r(V) \tag{9.17}$$

- Note the third requirement, $x(V) = r(V)$.
- By essentially the same argument as above (Exercise:), we can shown that the convex hull of the incidence vectors of the bases of a matroid is a polytope that can be described by Eq. 9.15- 9.17 above.
- What does this look like?



Spanning set polytope

- Recall, a set A is spanning in a matroid $M = (E, \mathcal{I})$ if $r(A) = r(E)$.

Spanning set polytope

- Recall, a set A is spanning in a matroid $M = (E, \mathcal{I})$ if $r(A) = r(E)$.
- Consider convex hull of incidence vectors of spanning sets of a matroid M, and call this $P_{\text{spanning}}(M)$.

Spanning set polytope

- Recall, a set A is spanning in a matroid $M = (E, \mathcal{I})$ if $r(A) = r(E)$.
- Consider convex hull of incidence vectors of spanning sets of a matroid M, and call this $P_{\text{spanning}}(M)$.

Theorem 9.3.4

The spanning set polytope is determined by the following equations:

$$0 \leq x_e \leq 1 \quad \text{for } e \in E \tag{9.18}$$

$$x(A) \geq r(E) - r(E \setminus A) \quad \text{for } A \subseteq E \tag{9.19}$$

Spanning set polytope

- Recall, a set A is spanning in a matroid $M = (E, \mathcal{I})$ if $r(A) = r(E)$.
- Consider convex hull of incidence vectors of spanning sets of a matroid M, and call this $P_{\text{spanning}}(M)$.

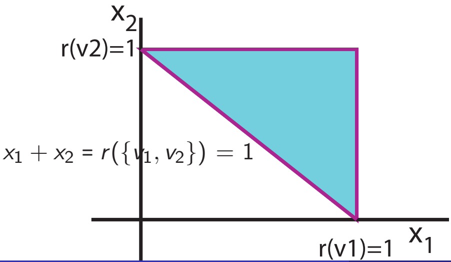
Theorem 9.3.4

The spanning set polytope is determined by the following equations:

$$0 \leq x_e \leq 1 \qquad \text{for } e \in E \tag{9.18}$$

$$x(A) \geq r(E) - r(E \setminus A) \qquad \text{for } A \subseteq E \tag{9.19}$$

- Example of spanning set polytope in 2D.



Spanning set polytope

Proof.

- Recall that any A is spanning in M iff $E \setminus A$ is independent in M^* (the dual matroid).

...

Spanning set polytope

Proof.

- Recall that any A is spanning in M iff $E \setminus A$ is independent in M^* (the dual matroid).
- For any $x \in \mathbb{R}^E$, we have that

$$x \in P_{\text{spanning}}(M) \Leftrightarrow 1 - x \in P_{\text{ind. set}}(M^*) \tag{9.20}$$

as we show next ...

...

Spanning set polytope

. . . proof continued.

- This follows since if $x \in P_{\text{spanning}}(M)$, we can represent x as a convex combination:

$$x = \sum_i \lambda_i \mathbf{1}_{A_i} \tag{9.21}$$

where A_i is spanning in M.

. . .

Spanning set polytope

. . . proof continued.

- This follows since if $x \in P_{\text{spanning}}(M)$, we can represent x as a convex combination:

$$x = \sum_i \lambda_i \mathbf{1}_{A_i} \tag{9.21}$$

where A_i is spanning in M.

- Consider

$$\mathbf{1} - x = \mathbf{1}_E - x = \mathbf{1}_E - \sum_i \lambda_i \mathbf{1}_{A_i} = \sum_i \lambda_i \mathbf{1}_{E \setminus A_i}, \tag{9.22}$$

which follows since $\sum_i \lambda_i \mathbf{1} = \mathbf{1}_E$, so $\mathbf{1} - x$ is a convex combination of independent sets in M^* and so $\mathbf{1} - x \in P_{\text{ind. set}}(M^*)$.

. . .

Spanning set polytope

. . . proof continued.

- which means, from the definition of $P_{\text{ind. set}}(M^*)$, that

$$\mathbf{1} - x \geq 0 \tag{9.23}$$

$$\mathbf{1}_A - x(A) = \cancel{|A|} - x(A) \leq r_{M^*}(A) \text{ for } A \subseteq E \tag{9.24}$$

And we know the dual rank function is

$$r_{M^*}(A) = \cancel{|A|} + r_M(E \setminus A) - r_M(E) \tag{9.25}$$

. . .

Spanning set polytope

. . . proof continued.

- which means, from the definition of $P_{\text{ind. set}}(M^*)$, that

$$\mathbf{1} - x \geq 0 \tag{9.23}$$

$$\mathbf{1}_A - x(A) = |A| - x(A) \leq r_{M^*}(A) \text{ for } A \subseteq E \tag{9.24}$$

And we know the dual rank function is

$$r_{M^*}(A) = |A| + r_M(E \setminus A) - r_M(E) \tag{9.25}$$

- giving

$$x(A) \geq r_M(E) - r_M(E \setminus A) \text{ for all } A \subseteq E \tag{9.26}$$

□

Matroids

where are we going with this?

- We've been discussing results about matroids (independence polytope, etc.).

Matroids

where are we going with this?

- We've been discussing results about matroids (independence polytope, etc.).
- By now, it is clear that matroid rank functions are special cases of submodular functions. We ultimately will be reviewing submodular function minimization procedures, but in some cases it it worth showing a result for a general submodular function first.

Matroids
where are we going with this?

- We've been discussing results about matroids (independence polytope, etc.).
- By now, it is clear that matroid rank functions are special cases of submodular functions. We ultimately will be reviewing submodular function minimization procedures, but in some cases it it worth showing a result for a general submodular function first.
- Henceforth, we will skip between submodular functions and matroids, each lecture talking less about matroids specifically and taking more about submodular functions more generally ...

Maximal points in a set

- Regarding sets, a subset X of S is a maximal subset of S possessing a given property $\mathfrak{P}$ if X possesses property $\mathfrak{P}$ and no set properly containing X (i.e., any $X' \supset X$ with $X' \setminus X \subseteq V \setminus X$) possesses $\mathfrak{P}$.

$\forall v \notin X$, either

$X + v \notin S$

or $\mathfrak{P}(X) = F$.

Maximal points in a set

- Regarding sets, a subset X of S is a maximal subset of S possessing a given property $\mathfrak{P}$ if X possesses property $\mathfrak{P}$ and no set properly containing X (i.e., any $X' \supset X$ with $X' \setminus X \subseteq V \setminus X$) possesses $\mathfrak{P}$.
- Given any compact (essentially closed & bounded) set $P \subseteq \mathbb{R}^E$, we say that a vector x is maximal within P if it is the case that for any $\epsilon > 0$, and for all $e \in E$, we have that

$$x + \epsilon \mathbf{1}_e \notin P \tag{9.27}$$

Maximal points in a set

- Regarding sets, a subset X of S is a maximal subset of S possessing a given property $\mathfrak{P}$ if X possesses property $\mathfrak{P}$ and no set properly containing X (i.e., any $X' \supset X$ with $X' \setminus X \subseteq V \setminus X$) possesses $\mathfrak{P}$.
- Given any compact (essentially closed & bounded) set $P \subseteq \mathbb{R}^E$, we say that a vector x is maximal within P if it is the case that for any $\epsilon > 0$, and for all $e \in E$, we have that

$$x + \epsilon \mathbf{1}_e \notin P \tag{9.27}$$

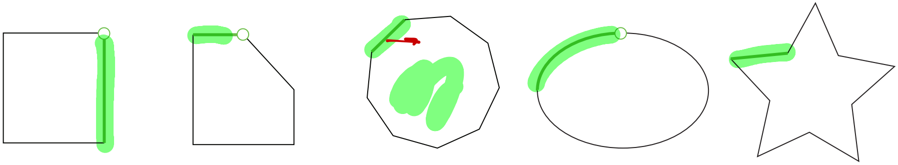
- Examples of maximal regions (in red)

Maximal points in a set

- Regarding sets, a subset X of S is a maximal subset of S possessing a given property $\mathfrak{P}$ if X possesses property $\mathfrak{P}$ and no set properly containing X (i.e., any $X' \supset X$ with $X' \setminus X \subseteq V \setminus X$) possesses $\mathfrak{P}$.
- Given any compact (essentially closed & bounded) set $P \subseteq \mathbb{R}^E$, we say that a vector x is maximal within P if it is the case that for any $\epsilon > 0$, and for all $e \in E$, we have that

$$x + \epsilon \mathbf{1}_e \notin P \tag{9.27}$$

- Examples of non-maximal regions (in green)



Review

- The next slide comes from Lecture 5.

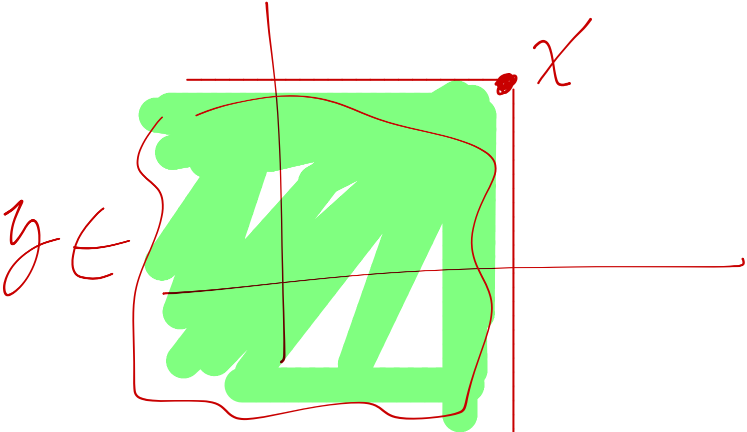
Matroids, independent sets, and bases

- Independent sets: Given a matroid $M = (E, \mathcal{I})$, a subset $A \subseteq E$ is called independent if $A \in \mathcal{I}$ and otherwise A is called dependent.
- A base of $U \subseteq E$: For $U \subseteq E$, a subset $B \subseteq U$ is called a base of U if B is inclusionwise maximally independent subset of U. That is, $B \in \mathcal{I}$ and there is no $Z \in \mathcal{I}$ with $B \subset Z \subseteq U$.
- A base of a matroid: If $U = E$, then a "base of E" is just called a base of the matroid M (this corresponds to a basis in a linear space).

P-basis of x given compact set $P \subseteq \mathbb{R}_+^E$

Definition 9.4.1 (subvector)

y is a <u>subvector</u> of x if $y \leq x$ (meaning $y(e) \leq x(e)$ for all $e \in E$).



P-basis of x given compact set $P \subseteq \mathbb{R}_+^E$

Definition 9.4.1 (subvector)

y is a subvector of x if $y \leq x$ (meaning $y(e) \leq x(e)$ for all $e \in E$).

Definition 9.4.2 (P-basis)

Given a compact set $P \subseteq \mathcal{R}_+^E$, for any $x \in \mathbb{R}_+^E$, a subvector y of x is called a P-basis of x if y maximal in P.
In other words, y is a P-basis of x if y is a maximal P-contained subvector of x.

P-basis of x given compact set $P \subseteq \mathbb{R}_+^E$

Definition 9.4.1 (subvector)

y is a subvector of x if $y \leq x$ (meaning $y(e) \leq x(e)$ for all $e \in E$).

Definition 9.4.2 (P-basis)

Given a compact set $P \subseteq \mathcal{R}_+^E$, for any $x \in \mathbb{R}_+^E$, a subvector y of x is called a P-basis of x if y maximal in P.
In other words, y is a P-basis of x if y is a maximal P-contained subvector of x.

Here, by y being "maximal", we mean that there exists no $z > y$ (more precisely, no $z \geq y + \epsilon \mathbf{1}_e$ for some $e \in E$ and $\epsilon > 0$) having the properties of y (the properties of y being: in P, and a subvector of x).

P-basis of x given compact set $P \subseteq \mathbb{R}_+^E$

Definition 9.4.1 (subvector)

y is a subvector of x if $y \leq x$ (meaning $y(e) \leq x(e)$ for all $e \in E$).

Definition 9.4.2 (P-basis)

Given a compact set $P \subseteq \mathcal{R}_+^E$, for any $x \in \mathbb{R}_+^E$, a subvector y of x is called a P-basis of x if y maximal in P.
In other words, y is a P-basis of x if y is a maximal P-contained subvector of x.

Here, by y being "maximal", we mean that there exists no $z > y$ (more precisely, no $z \geq y + \epsilon \mathbf{1}_e$ for some $e \in E$ and $\epsilon > 0$) having the properties of y (the properties of y being: in P, and a subvector of x).
In still other words: y is a P-basis of x if:

P-basis of x given compact set $P \subseteq \mathbb{R}_+^E$

Definition 9.4.1 (subvector)

y is a subvector of x if $y \leq x$ (meaning $y(e) \leq x(e)$ for all $e \in E$).

Definition 9.4.2 (P-basis)

Given a compact set $P \subseteq \mathcal{R}_+^E$, for any $x \in \mathbb{R}_+^E$, a subvector y of x is called a P-basis of x if y maximal in P.
In other words, y is a P-basis of x if y is a maximal P-contained subvector of x.

Here, by y being "maximal", we mean that there exists no $z > y$ (more precisely, no $z \geq y + \epsilon \mathbf{1}_e$ for some $e \in E$ and $\epsilon > 0$) having the properties of y (the properties of y being: in P, and a subvector of x).
In still other words: y is a P-basis of x if:

1. $y \leq x$ (y is a subvector of x); and

P-basis of x given compact set $P \subseteq \mathbb{R}_+^E$

Definition 9.4.1 (subvector)

y is a subvector of x if $y \leq x$ (meaning $y(e) \leq x(e)$ for all $e \in E$).

Definition 9.4.2 (P-basis)

Given a compact set $P \subseteq \mathcal{R}_+^E$, for any $x \in \mathbb{R}_+^E$, a subvector y of x is called a P-basis of x if y maximal in P.
In other words, y is a P-basis of x if y is a maximal P-contained subvector of x.

Here, by y being "maximal", we mean that there exists no $z > y$ (more precisely, no $z \geq y + \epsilon \mathbf{1}_e$ for some $e \in E$ and $\epsilon > 0$) having the properties of y (the properties of y being: in P, and a subvector of x).
In still other words: y is a P-basis of x if:

1. $y \leq x$ (y is a subvector of x); and
2. $y \in P$ and $y + \epsilon \mathbf{1}_e \notin P$ for all $e \in E, \epsilon > 0$ (y is maximal P-contained).

A vector form of rank

- Recall the definition of rank from a matroid $M = (E, \mathcal{I})$.

$$\text{rank}(A) = \max\{|I| : I \subseteq A, I \in \mathcal{I}\} \tag{9.28}$$

A vector form of rank

- Recall the definition of rank from a matroid $M = (E, \mathcal{I})$.

$$\text{rank}(A) = \max \{|I| : I \subseteq A, I \in \mathcal{I}\} \tag{9.28}$$

- vector rank: Given a compact set $P \subseteq \mathcal{R}_+^E$, we can define a form of "vector rank" relative to this P in the following way: Given an $x \in \mathbb{R}^E$, we define the vector rank, relative to P, as:

$$\text{rank}(x) = \max (y(E) : y \leq x, y \in P) \tag{9.29}$$

where $y \leq x$ is componentwise inequality ($y_i \leq x_i, \forall i$).

A vector form of rank

- Recall the definition of rank from a matroid $M = (E, \mathcal{I})$.

$$\text{rank}(A) = \max\{|I| : I \subseteq A, I \in \mathcal{I}\} \tag{9.28}$$

- vector rank: Given a compact set $P \subseteq \mathcal{R}_+^E$, we can define a form of "vector rank" relative to this P in the following way: Given an $x \in \mathbb{R}^E$, we define the vector rank, relative to P, as:

$$\text{rank}(x) = \max\left(y(E) : y \leq x, y \in P\right) \tag{9.29}$$

where $y \leq x$ is componentwise inequality ($y_i \leq x_i, \forall i$).

- If $\mathcal{B}_x$ is the set of P-bases of x, than $\text{rank}(x) = \max_{y \in \mathcal{B}_x} y(E)$.

A vector form of rank

- Recall the definition of rank from a matroid $M = (E, \mathcal{I})$.

$$\text{rank}(A) = \max\{|I| : I \subseteq A, I \in \mathcal{I}\} \tag{9.28}$$

- vector rank: Given a compact set $P \subseteq \mathcal{R}_+^E$, we can define a form of "vector rank" relative to this P in the following way: Given an $x \in \mathbb{R}^E$, we define the vector rank, relative to P, as:

$$\text{rank}(x) = \max\left(y(E) : y \leq x, y \in P\right) \tag{9.29}$$

  where $y \leq x$ is componentwise inequality $(y_i \leq x_i, \forall i)$.
- If $\mathcal{B}_x$ is the set of P-bases of x, than $\text{rank}(x) = \max_{y \in \mathcal{B}_x} y(E)$.
- If $x \in P$, then $\text{rank}(x) = x(E)$ (x is its own unique self P-basis).

A vector form of rank

- Recall the definition of rank from a matroid $M = (E, \mathcal{I})$.

$$\text{rank}(A) = \max \{|I| : I \subseteq A, I \in \mathcal{I}\} \tag{9.28}$$

- vector rank: Given a compact set $P \subseteq \mathcal{R}_+^E$, we can define a form of "vector rank" relative to this P in the following way: Given an $x \in \mathbb{R}^E$, we define the vector rank, relative to P, as:

$$\text{rank}(x) = \max \left(y(E) : y \leq x, y \in P \right) \tag{9.29}$$

  where $y \leq x$ is componentwise inequality ($y_i \leq x_i, \forall i$).
- If $\mathcal{B}_x$ is the set of P-bases of x, than $\text{rank}(x) = \max_{y \in \mathcal{B}_x} y(E)$.
- If $x \in P$, then $\text{rank}(x) = x(E)$ (x is its own unique self P-basis).
- In general, this might be hard to compute and/or have ill-defined properties. We next look at an object that restrains and cultivates this form of rank.

Polymatroidal polyhedron (or a "polymatroid")

Definition 9.4.3 (polymatroid)

A polymatroid is a compact set $P \subseteq \mathbb{R}_+^E$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For every $x \in \mathbb{R}_+^E$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any P-basis of x), has the same component sum $y(E)$

Polymatroidal polyhedron (or a "polymatroid")

Definition 9.4.3 (polymatroid)

A polymatroid is a compact set $P \subseteq \mathbb{R}_+^E$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For every $x \in \mathbb{R}_+^E$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any P-basis of x), has the same component sum $y(E)$

- Condition 3 restated: That is for any two distinct maximal vectors $y^1, y^2 \in P$, with $y^1 \leq x$ & $y^2 \leq x$, with $y^1 \neq y^2$, we must have $y^1(E) = y^2(E)$.

Polymatroidal polyhedron (or a "polymatroid")

Definition 9.4.3 (polymatroid)

A polymatroid is a compact set $P \subseteq \mathbb{R}_+^E$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For every $x \in \mathbb{R}_+^E$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any P-basis of x), has the same component sum $y(E)$

- Condition 3 restated: That is for any two distinct maximal vectors $y^1, y^2 \in P$, with $y^1 \leq x$ & $y^2 \leq x$, with $y^1 \neq y^2$, we must have $y^1(E) = y^2(E)$.
- Condition 3 restated (again): For every vector $x \in \mathbb{R}_+^E$, every maximal independent subvector y of x has the same component sum $y(E) = \text{rank}(x)$.

Polymatroidal polyhedron (or a "polymatroid")

Definition 9.4.3 (polymatroid)

A polymatroid is a compact set $P \subseteq \mathbb{R}_+^E$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For every $x \in \mathbb{R}_+^E$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any P-basis of x), has the same component sum $y(E)$

- Condition 3 restated: That is for any two distinct maximal vectors $y^1, y^2 \in P$, with $y^1 \leq x$ & $y^2 \leq x$, with $y^1 \neq y^2$, we must have $y^1(E) = y^2(E)$.
- Condition 3 restated (again): For every vector $x \in \mathbb{R}_+^E$, every maximal independent subvector y of x has the same component sum $y(E) = \text{rank}(x)$.
- Condition 3 restated (yet again): All P-bases of x have the same component sum.

Polymatroidal polyhedron (or a "polymatroid")

Definition 9.4.3 (polymatroid)

A polymatroid is a compact set $P \subseteq \mathbb{R}_+^E$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For every $x \in \mathbb{R}_+^E$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any P-basis of x), has the same component sum $y(E)$

- Vectors within P (i.e., any $y \in P$) are called independent, and any vector outside of P is called dependent.

Polymatroidal polyhedron (or a "polymatroid")

Definition 9.4.3 (polymatroid)

A polymatroid is a compact set $P \subseteq \mathbb{R}_+^E$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For every $x \in \mathbb{R}_+^E$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any P-basis of x), has the same component sum $y(E)$

- Vectors within P (i.e., any $y \in P$) are called independent, and any vector outside of P is called dependent.
- Since all P-bases of x have the same component sum, if $\mathcal{B}_x$ is the set of P-bases of x, than $\text{rank}(x) = y(E)$ for any $y \in \mathcal{B}_x$.

Matroid and Polymatroid: side-by-side

A Matroid is:

A Polymatroid is:

Matroid and Polymatroid: side-by-side

A Matroid is:

1. a set system $(E, \mathcal{I})$

A Polymatroid is:

1. a compact set $P \subseteq \mathbb{R}_+^E$

Matroid and Polymatroid: side-by-side

A Matroid is:

1. a set system $(E, \mathcal{I})$
2. empty-set containing $\emptyset \in \mathcal{I}$

A Polymatroid is:

1. a compact set $P \subseteq \mathbb{R}_+^E$
2. zero containing, $\mathbf{0} \in P$

Matroid and Polymatroid: side-by-side

A Matroid is:

1. a set system $(E, \mathcal{I})$
2. empty-set containing $\emptyset \in \mathcal{I}$
3. down closed, $\emptyset \subseteq I' \subseteq I \in \mathcal{I} \Rightarrow I' \in \mathcal{I}$.

A Polymatroid is:

1. a compact set $P \subseteq \mathbb{R}_+^E$
2. zero containing, $\mathbf{0} \in P$
3. down monotone, $0 \leq y \leq x \in P \Rightarrow y \in P$

Matroid and Polymatroid: side-by-side

A Matroid is:

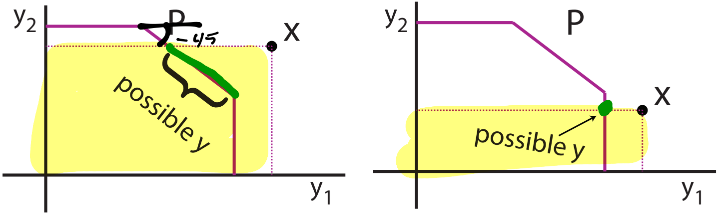
1. a set system $(E, \mathcal{I})$
2. empty-set containing $\emptyset \in \mathcal{I}$
3. down closed, $\emptyset \subseteq I' \subseteq I \in \mathcal{I} \Rightarrow I' \in \mathcal{I}$.
4. any maximal set I in $\mathcal{I}$, bounded by another set A, has the same matroid rank (any maximal independent subset $I \subseteq A$ has same size $|I|$).

A Polymatroid is:

1. a compact set $P \subseteq \mathbb{R}_+^E$
2. zero containing, $\mathbf{0} \in P$
3. down monotone, $0 \leq y \leq x \in P \Rightarrow y \in P$
4. any maximal vector y in P, bounded by another vector x, has the same vector rank (any maximal independent subvector $y \leq x$ has same sum $y(E)$).

.

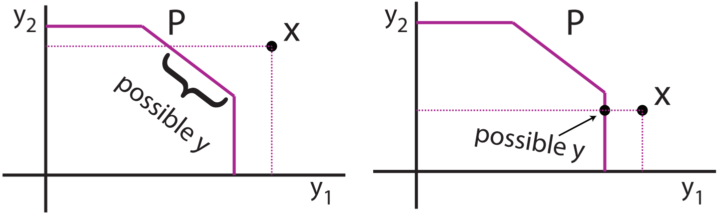
Polymatroidal polyhedron (or a "polymatroid")



Left: $\exists$ multiple maximal $y \leq x$ Right: $\exists$ only one maximal $y \leq x$,

- Polymatroid condition here: $\forall$ maximal $y \in P$, with $y \leq x$ (which here means $y_1 \leq x_1$ and $y_2 \leq x_2$), we just have $y(E) = y_1 + y_2 =$ const.

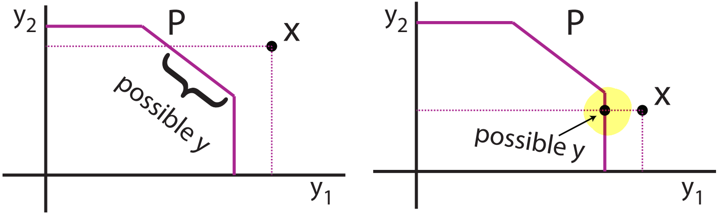
Polymatroidal polyhedron (or a "polymatroid")



Left: $\exists$ multiple maximal $y \leq x$ Right: $\exists$ only one maximal $y \leq x$,

- Polymatroid condition here: $\forall$ maximal $y \in P$, with $y \leq x$ (which here means $y_1 \leq x_1$ and $y_2 \leq x_2$), we just have $y(E) = y_1 + y_2 =$ const.
- On the left, we see there are multiple possible maximal $y \in P$ such that $y \leq x$. Each such y must have the same value $y(E)$.

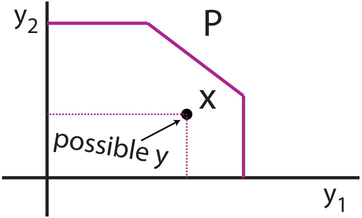
Polymatroidal polyhedron (or a "polymatroid")



Left: $\exists$ multiple maximal $y \leq x$ Right: $\exists$ only one maximal $y \leq x$,

- Polymatroid condition here: $\forall$ maximal $y \in P$, with $y \leq x$ (which here means $y_1 \leq x_1$ and $y_2 \leq x_2$), we just have $y(E) = y_1 + y_2 =$ const.
- On the left, we see there are multiple possible maximal $y \in P$ such that $y \leq x$. Each such y must have the same value $y(E)$.
- On the right, there is only one maximal $y \in P$. Since there is only one, the condition on the same value of $y(E), \forall y$ is vacuous.

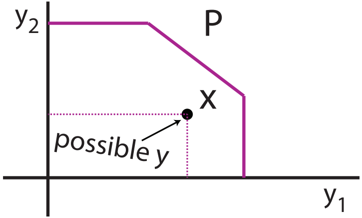
Polymatroidal polyhedron (or a "polymatroid")



$\exists$ only one maximal $y \leq x$.

- If $x \in P$ already, then x is its own P-basis, i.e., it is a self P-basis.

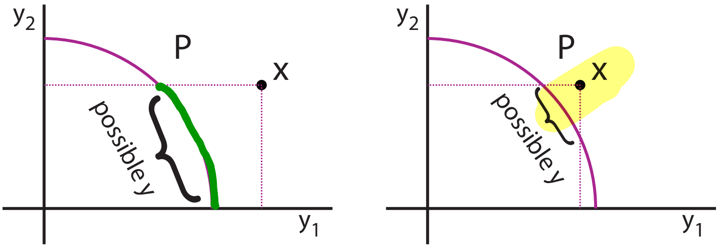
Polymatroidal polyhedron (or a "polymatroid")



$\exists$ only one maximal $y \leq x$.

- If $x \in P$ already, then x is its own P-basis, i.e., it is a self P-basis.
- In a matroid, a base of A is the maximally contained independent set. If A is already independent, then A is a self-base of A (as we saw in Lecture 5)

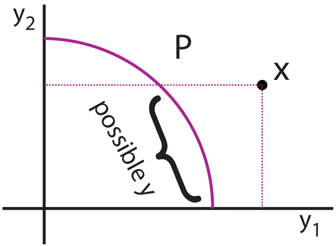
Polymatroid as well?

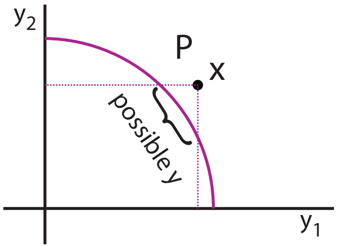


Left and right: $\exists$ multiple maximal $y \leq x$ as indicated.

- On the left, we see there are multiple possible maximal such $y \in P$ that are $y \leq x$. Each such y must have the same value $y(E)$, but since the equation for the curve is $y_1^2 + y_2^2 =$ const. $\neq y_1 + y_2$, we see this is not a polymatroid.

Polymatroid as well? no

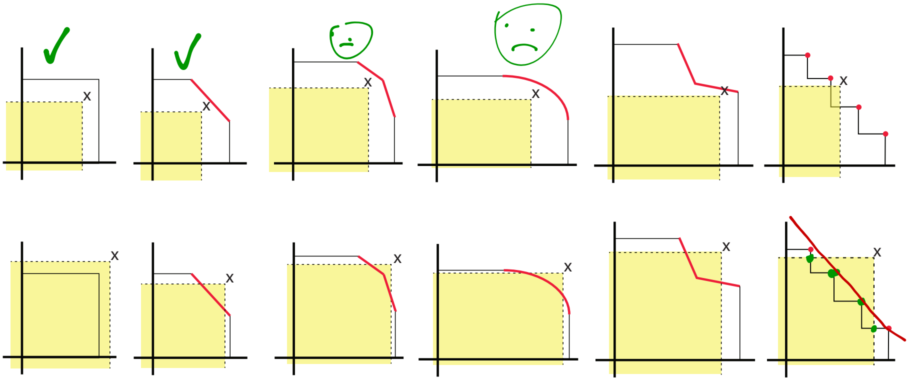




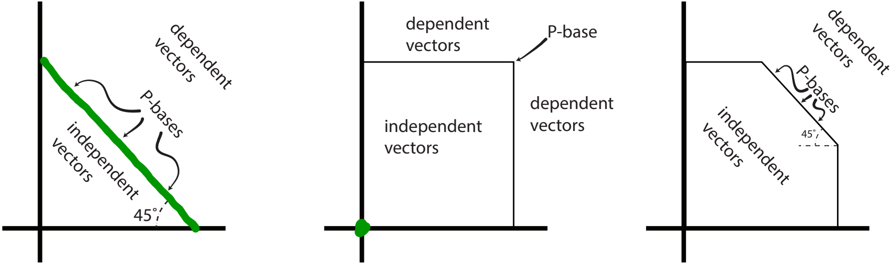
Left and right: $\exists$ multiple maximal $y \leq x$ as indicated.

- On the left, we see there are multiple possible maximal such $y \in P$ that are $y \leq x$. Each such y must have the same value $y(E)$, but since the equation for the curve is $y_1^2 + y_2^2 =$ const. $\neq y_1 + y_2$, we see this is not a polymatroid.
- On the right, we have a similar situation, just the set of potential values that must have the $y(E)$ condition changes, but the values of course are still not constant.

Other examples: Polymatroid or not?



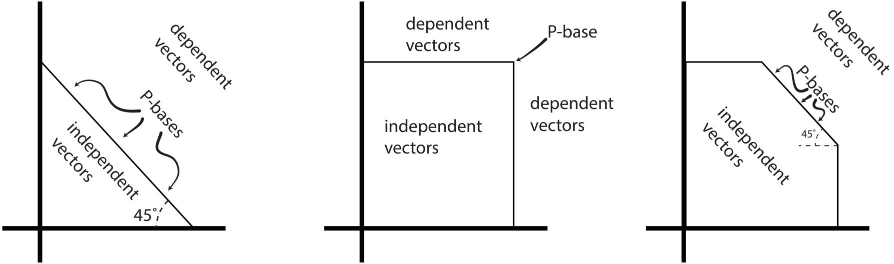
Some possible polymatroid forms in 2D



It appears that we have three possible forms of polymatroid in 2D, when neither of the elements $\{v_1, v_2\}$ are self-dependent.

1. On the left: full dependence between v_1 and v_2

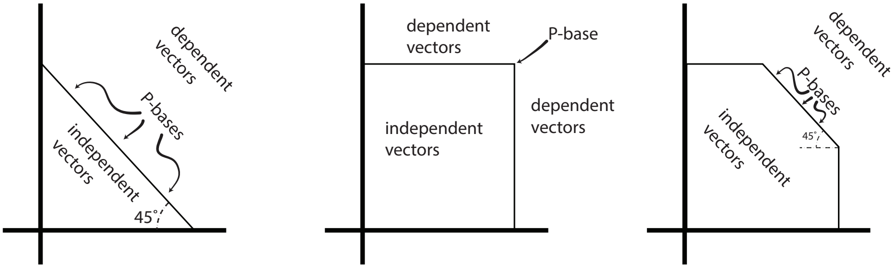
Some possible polymatroid forms in 2D



It appears that we have three possible forms of polymatroid in 2D, when neither of the elements $\{v_1, v_2\}$ are self-dependent.

1. On the left: full dependence between v_1 and v_2
2. In the middle: full independence between v_1 and v_2

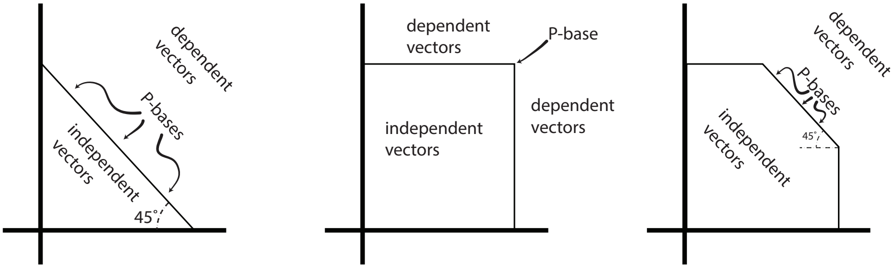
Some possible polymatroid forms in 2D



It appears that we have three possible forms of polymatroid in 2D, when neither of the elements $\{v_1, v_2\}$ are self-dependent.

1. On the left: full dependence between v_1 and v_2
2. In the middle: full independence between v_1 and v_2
3. On the right: partial independence between v_1 and v_2

Some possible polymatroid forms in 2D

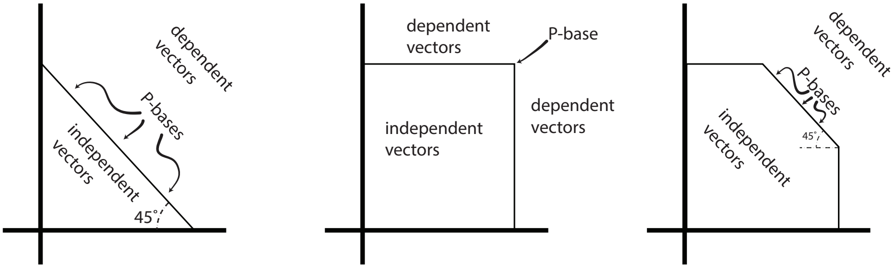


It appears that we have three possible forms of polymatroid in 2D, when neither of the elements $\{v_1, v_2\}$ are self-dependent.

1. On the left: full dependence between v_1 and v_2
2. In the middle: full independence between v_1 and v_2
3. On the right: partial independence between v_1 and v_2

- The P-bases (or single P-base in the middle case) are as indicated.

Some possible polymatroid forms in 2D

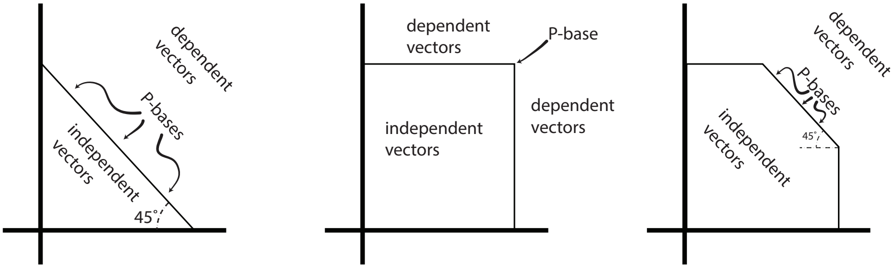


It appears that we have three possible forms of polymatroid in 2D, when neither of the elements $\{v_1, v_2\}$ are self-dependent.

1. On the left: full dependence between v_1 and v_2
2. In the middle: full independence between v_1 and v_2
3. On the right: partial independence between v_1 and v_2

- The P-bases (or single P-base in the middle case) are as indicated.
- Independent vectors are those within or on the boundary of the polytope. Dependent vectors are exterior to the polytope.

Some possible polymatroid forms in 2D



It appears that we have three possible forms of polymatroid in 2D, when neither of the elements $\{v_1, v_2\}$ are self-dependent.

1. On the left: full dependence between v_1 and v_2
2. In the middle: full independence between v_1 and v_2
3. On the right: partial independence between v_1 and v_2

- The P-bases (or single P-base in the middle case) are as indicated.
- Independent vectors are those within or on the boundary of the polytope. Dependent vectors are exterior to the polytope.
- The set of P-bases for a polytope is called the base polytope.

Polymatroidal polyhedron (or a "polymatroid")

- Note that if x contains any zeros (i.e., suppose that $x \in \mathbb{R}_+^E$ has $E \setminus S$ s.t. $x(E \setminus S) = 0$, so S indicates the non-zero elements, or $S = \text{supp}(x)$), then this also forces $y(E \setminus S) = 0$, so that $y(E) = y(S)$. This is true either for $x \in P$ or $x \notin P$.

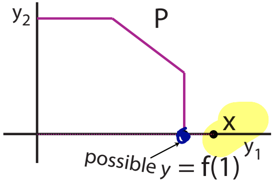
for $y \leq x$, $y \in \mathbb{R}_+^E$

Polymatroidal polyhedron (or a "polymatroid")

- Note that if x contains any zeros (i.e., suppose that $x \in \mathbb{R}_+^E$ has $E \setminus S$ s.t. $x(E \setminus S) = 0$, so S indicates the non-zero elements, or $S = \mathrm{supp}(x)$), then this also forces $y(E \setminus S) = 0$, so that $y(E) = y(S)$. This is true either for $x \in P$ or $x \notin P$.
- Therefore, in this case, it is the non-zero elements of x, corresponding to elements S (i.e., the support $\mathrm{supp}(x)$ of x), determine the common component sum.

Polymatroidal polyhedron (or a "polymatroid")

- Note that if x contains any zeros (i.e., suppose that $x \in \mathbb{R}_+^E$ has $E \setminus S$ s.t. $x(E \setminus S) = 0$, so S indicates the non-zero elements, or $S = \text{supp}(x)$), then this also forces $y(E \setminus S) = 0$, so that $y(E) = y(S)$. This is true either for $x \in P$ or $x \notin P$.
- Therefore, in this case, it is the non-zero elements of x, corresponding to elements S (i.e., the support $\text{supp}(x)$ of x), determine the common component sum.
- For the case of either $x \notin P$ or right at the boundary of P, we might give a "name" to this component sum, lets say $f(S)$ for any given set S of non-zero elements of x. We could name rank$(\frac{1}{\epsilon}\mathbf{1}_S) \triangleq f(S)$ for ϵ very small. What kind of function might f be?



Polymatroid function and its polyhedron.

Definition 9.4.4

A polymatroid function is a real-valued function f defined on subsets of E which is normalized, non-decreasing, and submodular. That is we have

1. $f(\emptyset) = 0$ (normalized)
2. $f(A) \leq f(B)$ for any $A \subseteq B \subseteq E$ (monotone non-decreasing)
3. $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$ for any $A, B \subseteq E$ (submodular)

We can define the polyhedron P_f^+ associated with a polymatroid function as follows

$$P_f^+ = \{y \in \mathbb{R}_+^E : y(A) \leq f(A) \text{ for all } A \subseteq E\} \tag{9.30}$$

$$= \{y \in \mathbb{R}^E : y \geq 0, y(A) \leq f(A) \text{ for all } A \subseteq E\} \tag{9.31}$$

Associated polyhedron with a polymatroid function

$$P_f^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\} \tag{9.32}$$

- Consider this in three dimensions. We have equations of the form:

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \text{ and } x_3 \geq 0 \tag{9.33}$$
$$x_1 \leq f(\{v_1\}) \tag{9.34}$$
$$x_2 \leq f(\{v_2\}) \tag{9.35}$$
$$x_3 \leq f(\{v_3\}) \tag{9.36}$$
$$x_1 + x_2 \leq f(\{v_1, v_2\}) \tag{9.37}$$
$$x_2 + x_3 \leq f(\{v_2, v_3\}) \tag{9.38}$$
$$x_1 + x_3 \leq f(\{v_1, v_3\}) \tag{9.39}$$
$$x_1 + x_2 + x_3 \leq f(\{v_1, v_2, v_3\}) \tag{9.40}$$

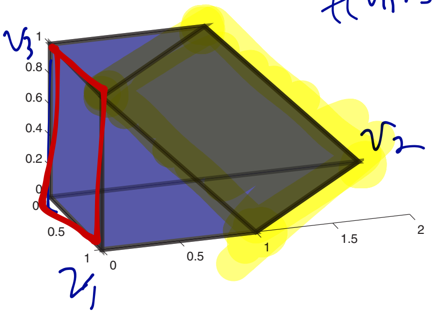
Associated polyhedron with a polymatroid function

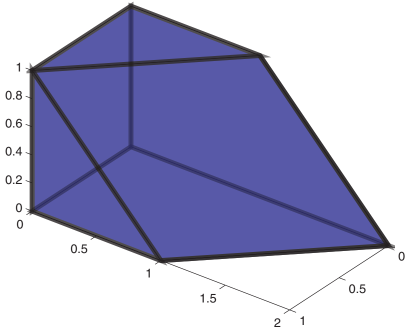
- Consider the asymmetric graph cut function on the simple chain graph $v_1 - v_2 - v_3$. That is, $f(S) = |\{(v,s) \in E(G) : v \in V, s \in S\}|$ is count of any edges within S or between S and $V \setminus S$, so that $\delta(S) = f(S) + f(V \setminus S) - f(V)$ is the standard graph cut.

Associated polyhedron with a polymatroid function

- Consider the asymmetric graph cut function on the simple chain graph $v_1 - v_2 - v_3$. That is, $f(S) = |\{(v,s) \in E(G) : v \in V, s \in S\}|$ is count of any edges within S or between S and $V \setminus S$, so that $\delta(S) = f(S) + f(V \setminus S) - f(V)$ is the standard graph cut.
- Observe: P_f^+ (at two views):

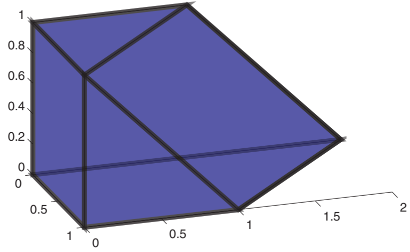
$f(v_1, v_3) = 2$

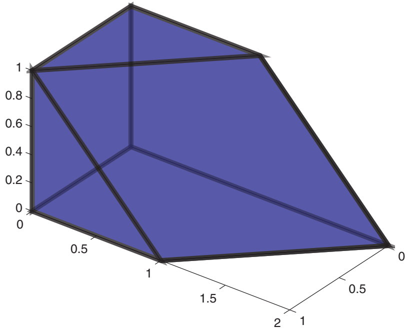




Associated polyhedron with a polymatroid function

- Consider the asymmetric graph cut function on the simple chain graph $v_1 - v_2 - v_3$. That is, $f(S) = |\{(v,s) \in E(G) : v \in V, s \in S\}|$ is count of any edges within S or between S and $V \setminus S$, so that $\delta(S) = f(S) + f(V \setminus S) - f(V)$ is the standard graph cut.
- Observe: P_f^+ (at two views):





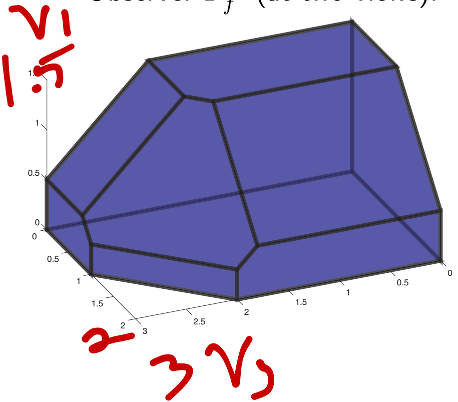
- which axis is which?

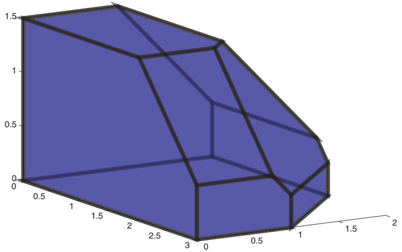
Associated polyhedron with a polymatroid function

- Consider: $f(\emptyset) = 0$, $f(\{v_1\}) = 1.5$, $f(\{v_2\}) = 2$, $f(\{v_1, v_2\}) = 2.5$, $f(\{v_3\}) = 3$, $f(\{v_3, v_1\}) = 3.5$, $f(\{v_3, v_2\}) = 4$, $f(\{v_3, v_2, v_1\}) = 4.3$.

Associated polyhedron with a polymatroid function

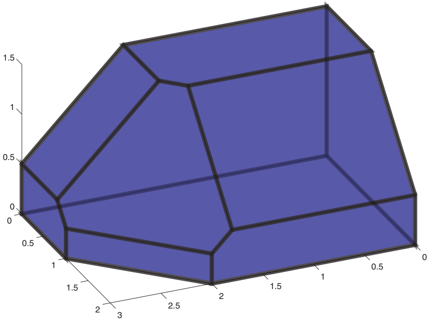
- Consider: $f(\emptyset) = 0$, $f(\{v_1\}) = 1.5$, $f(\{v_2\}) = 2$, $f(\{v_1, v_2\}) = 2.5$, $f(\{v_3\}) = 3$, $f(\{v_3, v_1\}) = 3.5$, $f(\{v_3, v_2\}) = 4$, $f(\{v_3, v_2, v_1\}) = 4.3$.
- Observe: P_f^+ (at two views):

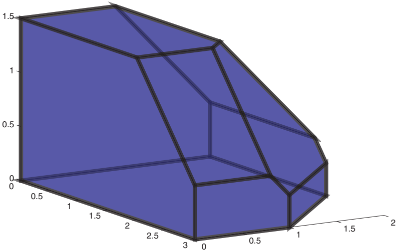




Associated polyhedron with a polymatroid function

- Consider: $f(\emptyset) = 0$, $f(\{v_1\}) = 1.5$, $f(\{v_2\}) = 2$, $f(\{v_1, v_2\}) = 2.5$, $f(\{v_3\}) = 3$, $f(\{v_3, v_1\}) = 3.5$, $f(\{v_3, v_2\}) = 4$, $f(\{v_3, v_2, v_1\}) = 4.3$.
- Observe: P_f^+ (at two views):





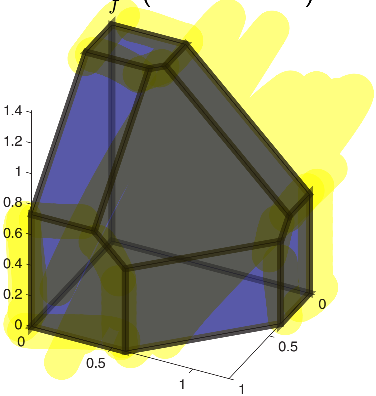
- which axis is which?

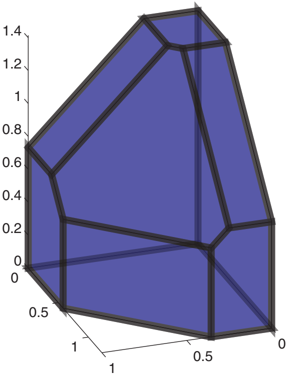
Associated polyhedron with a polymatroid function

- Consider modular function $w : V \to \mathbb{R}_+$ as $w = (1, 1.5, 2)^\intercal$, and then the submodular function $f(S) = \sqrt{w(S)}$.

Associated polyhedron with a polymatroid function

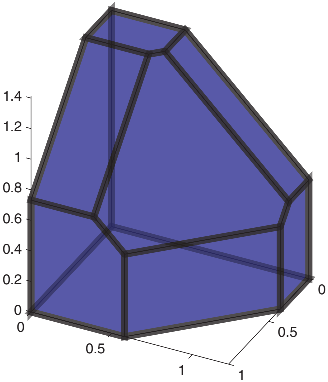
- Consider modular function $w : V \to \mathbb{R}_+$ as $w = (1, 1.5, 2)^\intercal$, and then the submodular function $f(S) = \sqrt{w(S)}$.
- Observe: P_f^+ (at two views):

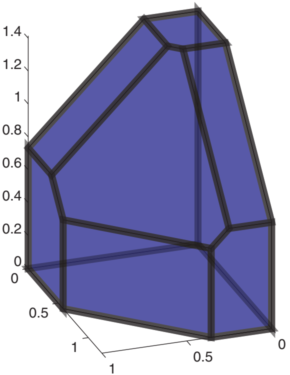




Associated polyhedron with a polymatroid function

- Consider modular function $w : V \to \mathbb{R}_+$ as $w = (1, 1.5, 2)^\intercal$, and then the submodular function $f(S) = \sqrt{w(S)}$.
- Observe: P_f^+ (at two views):





- which axis is which?

Associated polytope with a non-submodular function

- Consider function on integers: $g(0) = 0$, $g(1) = 3$, $g(2) = 4$, and $g(3) = 5.5$.

Associated polytope with a non-submodular function

- Consider function on integers: $g(0) = 0$, $g(1) = 3$, $g(2) = 4$, and $g(3) = 5.5$. Is $f(S) = g(|S|)$ submodular?

Associated polytope with a non-submodular function

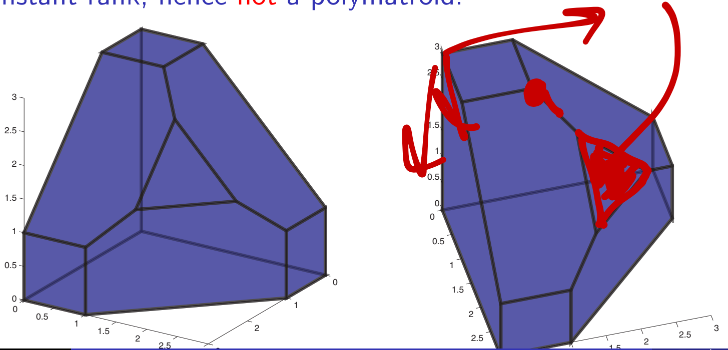
- Consider function on integers: $g(0) = 0$, $g(1) = 3$, $g(2) = 4$, and $g(3) = 5.5$. Is $f(S) = g(|S|)$ submodular? $f(S) = g(|S|)$ is not submodular since $f(\{e_1, e_3\}) + f(\{e_1, e_2\}) = 4 + 4 = 8$ but $f(\{e_1, e_2, e_3\}) + f(\{e_1\}) = 5.5 + 3 = 8.5$.

Associated polytope with a non-submodular function

- Consider function on integers: $g(0) = 0$, $g(1) = 3$, $g(2) = 4$, and $g(3) = 5.5$. Is $f(S) = g(|S|)$ submodular? $f(S) = g(|S|)$ is not submodular since $f(\{e_1, e_3\}) + f(\{e_1, e_2\}) = 4 + 4 = 8$ but $f(\{e_1, e_2, e_3\}) + f(\{e_1\}) = 5.5 + 3 = 8.5$. Alternatively, consider concavity violation, $1 = g(1+1) - g(1) < g(2+1) - g(2) = 1.5$.

Associated polytope with a non-submodular function

- Consider function on integers: $g(0) = 0$, $g(1) = 3$, $g(2) = 4$, and $g(3) = 5.5$. Is $f(S) = g(|S|)$ submodular? $f(S) = g(|S|)$ is not submodular since $f(\{e_1, e_3\}) + f(\{e_1, e_2\}) = 4 + 4 = 8$ but $f(\{e_1, e_2, e_3\}) + f(\{e_1\}) = 5.5 + 3 = 8.5$. Alternatively, consider concavity violation, $1 = g(1 + 1) - g(1) < g(2 + 1) - g(2) = 1.5$.
- Observe: P_f^+ (at two views), maximal independent subvectors not constant rank, hence not a polymatroid.



A polymatroid vs. a polymatroid function's polyhedron

- Summarizing the above, we have:

A polymatroid vs. a polymatroid function's polyhedron

- Summarizing the above, we have:
  - Given a polymatroid function f, its associated polytope is given as

$$P_f^+ = \{y \in \mathbb{R}_+^E : y(A) \leq f(A) \text{ for all } A \subseteq E\} \tag{9.41}$$

A polymatroid vs. a polymatroid function's polyhedron

- Summarizing the above, we have:
  - Given a polymatroid function f, its associated polytope is given as

$$P_f^+ = \left\{ y \in \mathbb{R}_+^E : y(A) \leq f(A) \text{ for all } A \subseteq E \right\} \tag{9.41}$$

  - We also have the definition of a polymatroid polytope (compact subset, zero containing, down-monotone, and $\forall x$ any maximal independent subvector $y \leq x$ has same component sum $y(E)$).

A polymatroid vs. a polymatroid function's polyhedron

- Summarizing the above, we have:
  - Given a polymatroid function f, its associated polytope is given as

$$P_f^+ = \{y \in \mathbb{R}_+^E : y(A) \leq f(A) \text{ for all } A \subseteq E\} \tag{9.41}$$

  - We also have the definition of a polymatroid polytope (compact subset, zero containing, down-monotone, and $\forall x$ any maximal independent subvector $y \leq x$ has same component sum $y(E)$).
- Is there any relationship between these two polytopes?

A polymatroid vs. a polymatroid function's polyhedron

- Summarizing the above, we have:
  - Given a polymatroid function f, its associated polytope is given as

$$P_f^+ = \{y \in \mathbb{R}_+^E : y(A) \leq f(A) \text{ for all } A \subseteq E\} \tag{9.41}$$

  - We also have the definition of a polymatroid polytope (compact subset, zero containing, down-monotone, and $\forall x$ any maximal independent subvector $y \leq x$ has same component sum $y(E)$).
- Is there any relationship between these two polytopes?
- In the next theorem, we show that any P_f^+-basis has the same component sum, when f is a polymatroid function, and P_f^+ satisfies the other properties so that P_f^+ is a polymatroid.

A polymatroid function's polyhedron is a polymatroid.

Theorem 9.4.5

Let f be a polymatroid function defined on subsets of E. For any $x \in \mathbb{R}_+^E$, and any P_f^+-basis $y^x \in \mathbb{R}_+^E$ of x, the component sum of y^x is

$$y^x(E) = \textit{rank}(x) = \max\Big(y(E) : y \le x, y \in P_f^+\Big)$$
$$= \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{9.42}$$

As a consequence, P_f^+ is a polymatroid, since r.h.s. is constant w.r.t. y^x.

A polymatroid function's polyhedron is a polymatroid.

Theorem 9.4.5

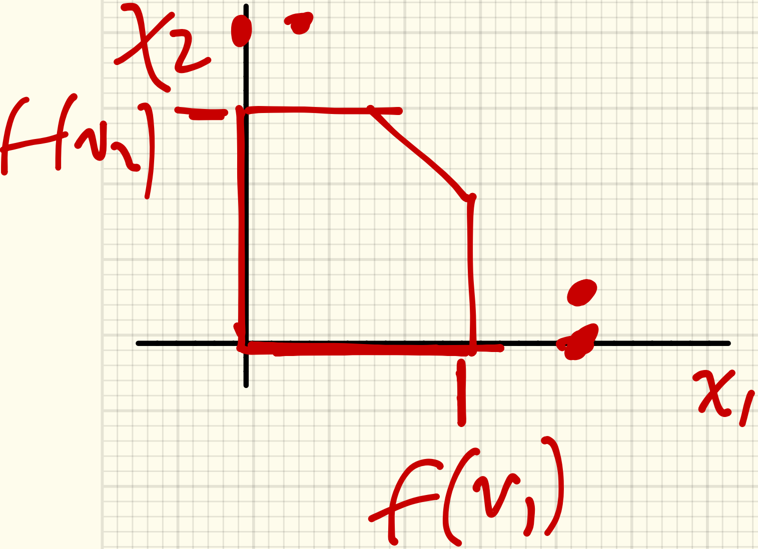
Let f be a polymatroid function defined on subsets of E. For any $x \in \mathbb{R}_+^E$, and any P_f^+-basis $y^x \in \mathbb{R}_+^E$ of x, the component sum of y^x is

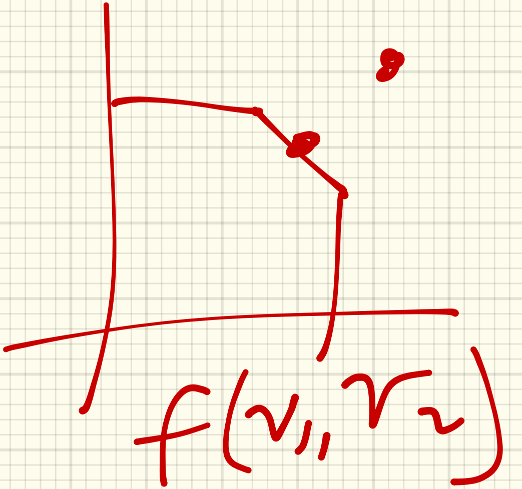
$$y^x(E) = \mathit{rank}(x) = \max\left(y(E) : y \leq x, y \in P_f^+\right)$$
$$= \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{9.42}$$

As a consequence, P_f^+ is a polymatroid, since r.h.s. is constant w.r.t. y^x.

By taking $B = \text{supp}(x)$ (so elements $E \setminus B$ are zero in x), and for $b \in B$, $x(b)$ is big enough, the r.h.s. min has solution $A^* = E \setminus B$. We recover submodular function from the polymatroid polyhedron via the following:

$$f(B) = \max\left\{y(B) : y \in P_f^+\right\} \tag{9.43}$$


x_2
$f(v_2)$
x_1
$f(v_1)$


$f(v_1, v_2)$

A polymatroid function's polyhedron is a polymatroid.

Theorem 9.4.5

Let f be a polymatroid function defined on subsets of E. For any $x \in \mathbb{R}_+^E$, and any P_f^+-basis $y^x \in \mathbb{R}_+^E$ of x, the component sum of y^x is

$$y^x(E) = \textit{rank}(x) = \max\left(y(E) : y \leq x, y \in P_f^+\right) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \quad (9.42)$$

As a consequence, P_f^+ is a polymatroid, since r.h.s. is constant w.r.t. y^x.

By taking $B = \text{supp}(x)$ (so elements $E \setminus B$ are zero in x), and for $b \in B$, $x(b)$ is big enough, the r.h.s. min has solution $A^* = E \setminus B$. We recover submodular function from the polymatroid polyhedron via the following:

$$f(B) = \max\left\{y(B) : y \in P_f^+\right\} \quad (9.43)$$

In fact, we will ultimately see a number of important consequences of this theorem (other than just that P_f^+ is a polymatroid)

A polymatroid function's polyhedron is a polymatroid.

Proof.

- Clearly $0 \in P_f^+$ since f is non-negative.

...

A polymatroid function's polyhedron is a polymatroid.

Proof.

- Clearly $0 \in P_f^+$ since f is non-negative.
- Also, for any $y \in P_f^+$ then any $x <= y$ is also such that $x \in P_f^+$. So, P_f^+ is down-monotone.

...

A polymatroid function's polyhedron is a polymatroid.

Proof.

- Clearly $0 \in P_f^+$ since f is non-negative.
- Also, for any $y \in P_f^+$ then any $x <= y$ is also such that $x \in P_f^+$. So, P_f^+ is down-monotone.
- Now suppose that we are given an $x \in \mathbb{R}_+^E$, and maximal $y^x \in P_f^+$ with $y^x \leq x$ (i.e., y^x is a P_f^+-basis of x).

...

A polymatroid function's polyhedron is a polymatroid.

Proof.

- Clearly $0 \in P_f^+$ since f is non-negative.
- Also, for any $y \in P_f^+$ then any $x <= y$ is also such that $x \in P_f^+$. So, P_f^+ is down-monotone.
- Now suppose that we are given an $x \in \mathbb{R}_+^E$, and maximal $y^x \in P_f^+$ with $y^x \leq x$ (i.e., y^x is a P_f^+-basis of x).
- Goal is to show that any such y^x has $y^x(E) = \text{const}$, dependent only on x and also f (which defines the polytope) but not dependent on y^x, the particular P-basis.

...

A polymatroid function's polyhedron is a polymatroid.

Proof.

- Clearly $0 \in P_f^+$ since f is non-negative.
- Also, for any $y \in P_f^+$ then any $x <= y$ is also such that $x \in P_f^+$. So, P_f^+ is down-monotone.
- Now suppose that we are given an $x \in \mathbb{R}_+^E$, and maximal $y^x \in P_f^+$ with $y^x \leq x$ (i.e., y^x is a P_f^+-basis of x).
- Goal is to show that any such y^x has $y^x(E) = \text{const}$, dependent only on x and also f (which defines the polytope) but not dependent on y^x, the particular P-basis.
- Doing so will thus establish that P_f^+ is a polymatroid.

...

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- First trivial case: could have $y^x = x$, which happens if $x(A) < f(A), \forall A \subseteq E$ (i.e., $x \in P_f^+$ strictly). In such case, $\min\left(x(A) + f(E \setminus A) : A \subseteq E\right) = x(E)$.

. . .

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- First trivial case: could have $y^x = x$, which happens if $x(A) < f(A), \forall A \subseteq E$ (i.e., $x \in P_f^+$ strictly). In such case, $\min(x(A) + f(E \setminus A) : A \subseteq E) = x(E)$.
- 2nd trivial case is when $x(A) > f(A), \forall A \subseteq E$ (i.e., $x \notin P_f^+$ strictly), meaning $\min(x(A) + f(E \setminus A) : A \subseteq E) = f(E) = y^x(E)$.

. . .

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- Assume neither trivial case. Because $y^x \in P_f^+$, we have that $y^x(A) \leq f(A)$ for all A.

. . .

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- Assume neither trivial case. Because $y^x \in P_f^+$, we have that $y^x(A) \le f(A)$ for all A.
- We show that the constant is given by

$$y^x(E) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{9.44}$$

. . .

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- Assume neither trivial case. Because $y^x \in P_f^+$, we have that $y^x(A) \le f(A)$ for all A.
- We show that the constant is given by

$$y^x(E) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{9.44}$$

- For any P_f^+-basis y^x of x, and any $A \subseteq E$, we have that

$$y^x(E) = y^x(A) + y^x(E \setminus A) \tag{9.45}$$
$$\le x(A) + f(E \setminus A). \tag{9.46}$$

This follows since $y^x \le x$ and since $y^x \in P_f^+$.

. . .

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- Assume neither trivial case. Because $y^x \in P_f^+$, we have that $y^x(A) \leq f(A)$ for all A.
- We show that the constant is given by

$$y^x(E) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{9.44}$$

- For any P_f^+-basis y^x of x, and any $A \subseteq E$, we have that

$$y^x(E) = y^x(A) + y^x(E \setminus A) \tag{9.45}$$

$$\leq x(A) + f(E \setminus A). \tag{9.46}$$

This follows since $y^x \leq x$ and since $y^x \in P_f^+$.

- Given one A where equality holds, the above min result follows.

. . .

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets B, C is again tight, since

$$f(B) + f(C)$$

. . .

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets B, C is again tight, since

$$f(B) + f(C) = y(B) + y(C) \tag{9.47}$$

. . .

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets B, C is again tight, since

$$f(B) + f(C) = y(B) + y(C) \tag{9.47}$$

$$= y(B \cap C) + y(B \cup C) \tag{9.48}$$

. . .

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets B, C is again tight, since

$$f(B) + f(C) = y(B) + y(C) \tag{9.47}$$
$$= y(B \cap C) + y(B \cup C) \tag{9.48}$$
$$\leq f(B \cap C) + f(B \cup C) \tag{9.49}$$

. . .

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets B, C is again tight, since

$$f(B) + f(C) = y(B) + y(C) \tag{9.47}$$
$$= y(B \cap C) + y(B \cup C) \tag{9.48}$$
$$\leq f(B \cap C) + f(B \cup C) \tag{9.49}$$
$$\leq f(B) + f(C) \tag{9.50}$$

. . .

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets B, C is again tight, since

$$
\begin{aligned}
f(B) + f(C) &= y(B) + y(C) && (9.47)\\
&= y(B \cap C) + y(B \cup C) && (9.48)\\
&\le f(B \cap C) + f(B \cup C) && (9.49)\\
&\le f(B) + f(C) && (9.50)
\end{aligned}
$$

which requires equality everywhere above.

. . .

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets B, C is again tight, since

$$f(B) + f(C) = y(B) + y(C) \tag{9.47}$$
$$= y(B \cap C) + y(B \cup C) \tag{9.48}$$
$$\leq f(B \cap C) + f(B \cup C) \tag{9.49}$$
$$\leq f(B) + f(C) \tag{9.50}$$

  which requires equality everywhere above.
- Because $y(B) \leq f(B), \forall B$, this means $y(B \cap C) = f(B \cap C)$ and $y(B \cup C) = f(B \cup C)$, so both also are tight.

. . .

A polymatroid function's polyhedron is a polymatroid.

... proof continued.

- For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets B, C is again tight, since

$$
\begin{aligned}
f(B) + f(C) &= y(B) + y(C) && (9.47)\\
&= y(B \cap C) + y(B \cup C) && (9.48)\\
&\leq f(B \cap C) + f(B \cup C) && (9.49)\\
&\leq f(B) + f(C) && (9.50)
\end{aligned}
$$

  which requires equality everywhere above.

- Because $y(B) \leq f(B), \forall B$, this means $y(B \cap C) = f(B \cap C)$ and $y(B \cup C) = f(B \cup C)$, so both also are tight.
- For $y \in P_f^+$, it will be ultimately useful to define this lattice family of tight sets: $\mathcal{D}(y) \triangleq \{A : A \subseteq E,\ y(A) = f(A)\}$.

...

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- Also, define $\mathrm{sat}(y) \stackrel{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\}$

$\square$

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- Also, define $\mathrm{sat}(y) \stackrel{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\}$
- Consider again a P_f^+-basis y^x (so maximal).

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- Also, define $\mathrm{sat}(y) \stackrel{\mathrm{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\}$
- Consider again a P_f^+-basis y^x (so maximal).
- Given a $e \in E$, either $y^x(e)$ is cut off due to x (so $y^x(e) = x(e)$) or e is saturated by f, meaning it is an element of some tight set and $e \in \mathrm{sat}(y^x)$.

□

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- Also, define $\text{sat}(y) \stackrel{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\}$
- Consider again a P_f^+-basis y^x (so maximal).
- Given a $e \in E$, either $y^x(e)$ is cut off due to x (so $y^x(e) = x(e)$) or e is saturated by f, meaning it is an element of some tight set and $e \in \text{sat}(y^x)$.
- Let $E \setminus A = \text{sat}(y^x)$ be the union of all such tight sets (which is also tight, so $y(E \setminus A) = f(E \setminus A)$).

$\square$

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- Also, define $\text{sat}(y) \stackrel{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\}$
- Consider again a P_f^+-basis y^x (so maximal).
- Given a $e \in E$, either $y^x(e)$ is cut off due to x (so $y^x(e) = x(e)$) or e is saturated by f, meaning it is an element of some tight set and $e \in \text{sat}(y^x)$.
- Let $E \setminus A = \text{sat}(y^x)$ be the union of all such tight sets (which is also tight, so $y(E \setminus A) = f(E \setminus A)$).
- Hence, we have

$$y(E) = y(A) + y(E \setminus A) = x(A) + f(E \setminus A) \tag{9.51}$$

□

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- Also, define $\mathrm{sat}(y) \stackrel{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\}$
- Consider again a P_f^+-basis y^x (so maximal).
- Given a $e \in E$, either $y^x(e)$ is cut off due to x (so $y^x(e) = x(e)$) or e is saturated by f, meaning it is an element of some tight set and $e \in \mathrm{sat}(y^x)$.
- Let $E \setminus A = \mathrm{sat}(y^x)$ be the union of all such tight sets (which is also tight, so $y(E \setminus A) = f(E \setminus A)$).
- Hence, we have

$$y(E) = y(A) + y(E \setminus A) = x(A) + f(E \setminus A) \tag{9.51}$$

- So we identified the A to be the elements that are non-tight, and achieved the min, as desired. □

A polymatroid is a polymatroid function's polytope

- So, when f is a polymatroid function, P_f^+ is a polymatroid.

A polymatroid is a polymatroid function's polytope

- So, when f is a polymatroid function, P_f^+ is a polymatroid.
- Is it the case that, conversely, for any polymatroid P, there is an associated polymatroidal function f such that $P = P_f^+$?

A polymatroid is a polymatroid function's polytope

- So, when f is a polymatroid function, P_f^+ is a polymatroid.
- Is it the case that, conversely, for any polymatroid P, there is an associated polymatroidal function f such that $P = P_f^+$?

Theorem 9.4.6

For any polymatroid P (compact subset of $\mathbb{R}_+^E$, zero containing, down-monotone, and $\forall x \in \mathbb{R}_+^E$ any maximal independent subvector $y \leq x$ has same component sum $y(E) = \text{rank}(x)$), there is a polymatroid function $f : 2^E \to \mathbb{R}$ (normalized, monotone non-decreasing, submodular) such that $P = P_f^+$ where $P_f^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\}$.

First, a bit on $\mathcal{D}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{A : A \subseteq E,\ y(A) = f(A)\} \tag{9.52}$$

Theorem 9.4.7

For any $y \in P_f^+$, with f a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

First, a bit on $\mathcal{D}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{A : A \subseteq E,\ y(A) = f(A)\} \tag{9.52}$$

Theorem 9.4.7

For any $y \in P_f^+$, with f a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

Proof.

We have already proven this as part of Theorem 9.4.5 □

First, a bit on $\mathcal{D}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{A : A \subseteq E,\ y(A) = f(A)\} \tag{9.52}$$

Theorem 9.4.7

For any $y \in P_f^+$, with f a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

Proof.

We have already proven this as part of Theorem 9.4.5 □

Also recall the definition of $\mathrm{sat}(y)$, the maximal set of tight elements relative to $y \in \mathbb{R}_+^E$.

$$\mathrm{sat}(y) \stackrel{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\} \tag{9.53}$$

Next, a bit on rank(x), join and meet for $x, y \in \mathbb{R}_+^E$

- For $x, y \in \mathbb{R}_+^E$, define vectors $x \wedge y \in \mathbb{R}_+^E$ and $x \vee y \in \mathbb{R}_+^E$ such that, for all $e \in E$

$$(x \vee y)(e) = \max(x(e), y(e)) \tag{9.54}$$

$$(x \wedge y)(e) = \min(x(e), y(e)) \tag{9.55}$$

Hence,

$$x \vee y = (\max(x(e_1), y(e_1)), \max(x(e_2), y(e_2)), \dots, \max(x(e_n), y(e_n)))$$

and similarly

$$x \wedge y = (\min(x(e_1), y(e_1)), \min(x(e_2), y(e_2)), \dots, \min(x(e_n), y(e_n)))$$

Next, a bit on rank(x), join and meet for $x, y \in \mathbb{R}_+^E$

- For $x, y \in \mathbb{R}_+^E$, define vectors $x \wedge y \in \mathbb{R}_+^E$ and $x \vee y \in \mathbb{R}_+^E$ such that, for all $e \in E$

$$(x \vee y)(e) = \max(x(e), y(e)) \tag{9.54}$$
$$(x \wedge y)(e) = \min(x(e), y(e)) \tag{9.55}$$

Hence,

$$x \vee y = (\max(x(e_1), y(e_1)), \max(x(e_2), y(e_2)), \dots, \max(x(e_n), y(e_n)))$$

and similarly

$$x \wedge y = (\min(x(e_1), y(e_1)), \min(x(e_2), y(e_2)), \dots, \min(x(e_n), y(e_n)))$$

- From this, we can define things like an lattices, and other constructs.

Next, a bit on rank(x)

- Recall that the matroid rank function is submodular.

Next, a bit on rank(x)

- Recall that the matroid rank function is submodular.
- The vector rank function rank(x) also satisfies a form of submodularity.

Next, a bit on rank(x)

- Recall that the matroid rank function is submodular.
- The vector rank function rank(x) also satisfies a form of submodularity.

Theorem 9.4.8 (vector rank and submodularity)

Let P be a polymatroid polytope. The vector rank function rank $: \mathbb{R}_+^E \to \mathbb{R}$ with rank$(x) = \max\left(y(E) : y \leq x, y \in P\right)$ satisfies, for all $u, v \in \mathbb{R}_+^E$

$$\text{rank}(u) + \text{rank}(v) \geq \text{rank}(u \vee v) + \text{rank}(u \wedge v) \tag{9.56}$$

Next, a bit on rank(x)

Proof of Theorem 9.4.8.

- Let a be a P-basis of $u \wedge v$, so rank$(u \wedge v) = a(E)$.

...

Next, a bit on rank(x)

Proof of Theorem 9.4.8.

- Let a be a P-basis of $u \wedge v$, so rank$(u \wedge v) = a(E)$.
- By the polymatroid property, $\exists$ an independent $b \in P$ such that: $a \leq b \leq u \vee v$

...

Next, a bit on rank(x)

Proof of Theorem 9.4.8.

- Let a be a P-basis of $u \wedge v$, so rank$(u \wedge v) = a(E)$.
- By the polymatroid property, $\exists$ an independent $b \in P$ such that: $a \leq b \leq u \vee v$ and also such that rank$(b) = b(E) =$ rank$(u \vee v)$.

...

Next, a bit on rank(x)

Proof of Theorem 9.4.8.

- Let a be a P-basis of $u \wedge v$, so rank$(u \wedge v) = a(E)$.
- By the polymatroid property, $\exists$ an independent $b \in P$ such that: $a \leq b \leq u \vee v$ and also such that rank$(b) = b(E) =$ rank$(u \vee v)$.
- Given $e \in E$, if $a(e)$ is maximal due to P, then then $a(e) = b(e) \leq \min(u(e), v(e))$.

...

Next, a bit on rank(x)

Proof of Theorem 9.4.8.

- Let a be a P-basis of $u \wedge v$, so rank$(u \wedge v) = a(E)$.
- By the polymatroid property, $\exists$ an independent $b \in P$ such that: $a \leq b \leq u \vee v$ and also such that rank$(b) = b(E) =$ rank$(u \vee v)$.
- Given $e \in E$, if $a(e)$ is maximal due to P, then then $a(e) = b(e) \leq \min(u(e), v(e))$.
  If $a(e)$ is maximal due to $(u \wedge v)(e)$, then $a(e) = \min(u(e), v(e)) \leq b(e)$.

...

Next, a bit on rank(x)

Proof of Theorem 9.4.8.

- Let a be a P-basis of $u \wedge v$, so rank$(u \wedge v) = a(E)$.
- By the polymatroid property, $\exists$ an independent $b \in P$ such that: $a \leq b \leq u \vee v$ and also such that rank$(b) = b(E) =$ rank$(u \vee v)$.
- Given $e \in E$, if $a(e)$ is maximal due to P, then then $a(e) = b(e) \leq \min(u(e), v(e))$.
  If $a(e)$ is maximal due to $(u \wedge v)(e)$, then $a(e) = \min(u(e), v(e)) \leq b(e)$.
  Therefore, $a = b \wedge (u \wedge u)$.

...

Next, a bit on rank(x)

Proof of Theorem 9.4.8.

- Let a be a P-basis of $u \wedge v$, so rank$(u \wedge v) = a(E)$.
- By the polymatroid property, $\exists$ an independent $b \in P$ such that: $a \leq b \leq u \vee v$ and also such that rank$(b) = b(E) =$ rank$(u \vee v)$.
- Given $e \in E$, if $a(e)$ is maximal due to P, then then $a(e) = b(e) \leq \min(u(e), v(e))$.
  If $a(e)$ is maximal due to $(u \wedge v)(e)$, then $a(e) = \min(u(e), v(e)) \leq b(e)$.
  Therefore, $a = b \wedge (u \wedge u)$.
- Since $a = b \wedge (u \wedge v)$

...

Next, a bit on rank(x)

Proof of Theorem 9.4.8.

- Let a be a P-basis of $u \wedge v$, so rank$(u \wedge v) = a(E)$.
- By the polymatroid property, $\exists$ an independent $b \in P$ such that: $a \leq b \leq u \vee v$ and also such that rank$(b) = b(E) =$ rank$(u \vee v)$.
- Given $e \in E$, if $a(e)$ is maximal due to P, then then $a(e) = b(e) \leq \min(u(e), v(e))$.
  If $a(e)$ is maximal due to $(u \wedge v)(e)$, then $a(e) = \min(u(e), v(e)) \leq b(e)$.
  Therefore, $a = b \wedge (u \wedge u)$.
- Since $a = b \wedge (u \wedge v)$ and since $b \leq u \vee v$, we get

$$a + b \tag{9.57}$$

...

Next, a bit on rank(x)

Proof of Theorem 9.4.8.

- Let a be a P-basis of $u \wedge v$, so rank$(u \wedge v) = a(E)$.
- By the polymatroid property, $\exists$ an independent $b \in P$ such that: $a \leq b \leq u \vee v$ and also such that rank$(b) = b(E) =$ rank$(u \vee v)$.
- Given $e \in E$, if $a(e)$ is maximal due to P, then then $a(e) = b(e) \leq \min(u(e), v(e))$.
  If $a(e)$ is maximal due to $(u \wedge v)(e)$, then $a(e) = \min(u(e), v(e)) \leq b(e)$.
  Therefore, $a = b \wedge (u \wedge u)$.
- Since $a = b \wedge (u \wedge v)$ and since $b \leq u \vee v$, we get

$$a + b = b + b \wedge u \wedge v \tag{9.57}$$

...

Next, a bit on rank(x)

Proof of Theorem 9.4.8.

- Let a be a P-basis of $u \wedge v$, so $\text{rank}(u \wedge v) = a(E)$.
- By the polymatroid property, $\exists$ an independent $b \in P$ such that: $a \leq b \leq u \vee v$ and also such that $\text{rank}(b) = b(E) = \text{rank}(u \vee v)$.
- Given $e \in E$, if $a(e)$ is maximal due to P, then then $a(e) = b(e) \leq \min(u(e), v(e))$.
  If $a(e)$ is maximal due to $(u \wedge v)(e)$, then $a(e) = \min(u(e), v(e)) \leq b(e)$.
  Therefore, $a = b \wedge (u \wedge u)$.
- Since $a = b \wedge (u \wedge v)$ and since $b \leq u \vee v$, we get

$$a + b = b + b \wedge u \wedge v = b \wedge u + b \wedge v \tag{9.57}$$

...

Next, a bit on rank(x)

Proof of Theorem 9.4.8.

- Let a be a P-basis of $u \wedge v$, so rank$(u \wedge v) = a(E)$.
- By the polymatroid property, $\exists$ an independent $b \in P$ such that: $a \leq b \leq u \vee v$ and also such that rank$(b) = b(E) =$ rank$(u \vee v)$.
- Given $e \in E$, if $a(e)$ is maximal due to P, then then $a(e) = b(e) \leq \min(u(e), v(e))$. If $a(e)$ is maximal due to $(u \wedge v)(e)$, then $a(e) = \min(u(e), v(e)) \leq b(e)$. Therefore, $a = b \wedge (u \wedge u)$.
- Since $a = b \wedge (u \wedge v)$ and since $b \leq u \vee v$, we get

$$a + b = b + b \wedge u \wedge v = b \wedge u + b \wedge v \tag{9.57}$$

To see this, consider each case where either b is the minimum, or u is minimum with $b \leq v$, or v is minimum with $b \leq u$.

...

Next, a bit on rank(x)

. . . proof of Theorem 9.4.8.

- But $b \wedge u$ and $b \wedge v$ are independent subvectors of u and v respectively, so $(b \wedge u)(E) \leq \text{rank}(u)$ and $(b \wedge v)(E) \leq \text{rank}(v)$.

$\square$

Next, a bit on rank(x)

. . . proof of Theorem 9.4.8.

- But $b \wedge u$ and $b \wedge v$ are independent subvectors of u and v respectively, so $(b \wedge u)(E) \leq \text{rank}(u)$ and $(b \wedge v)(E) \leq \text{rank}(v)$.
- Hence,

$$\text{rank}(u \wedge v) + \text{rank}(u \vee v)$$

$\square$

Next, a bit on rank(x)

. . . proof of Theorem 9.4.8.

- But $b \wedge u$ and $b \wedge v$ are independent subvectors of u and v respectively, so $(b \wedge u)(E) \leq \text{rank}(u)$ and $(b \wedge v)(E) \leq \text{rank}(v)$.
- Hence,

$$\text{rank}(u \wedge v) + \text{rank}(u \vee v) = a(E) + b(E) \tag{9.58}$$

□

Next, a bit on rank(x)

. . . proof of Theorem 9.4.8.

- But $b \wedge u$ and $b \wedge v$ are independent subvectors of u and v respectively, so $(b \wedge u)(E) \leq \text{rank}(u)$ and $(b \wedge v)(E) \leq \text{rank}(v)$.
- Hence,

$$\text{rank}(u \wedge v) + \text{rank}(u \vee v) = a(E) + b(E) \tag{9.58}$$

$$= (b \wedge u)(E) + (b \wedge v)(E) \tag{9.59}$$

$\square$

Next, a bit on $\text{rank}(x)$

. . . proof of Theorem 9.4.8.

- But $b \wedge u$ and $b \wedge v$ are independent subvectors of u and v respectively, so $(b \wedge u)(E) \leq \text{rank}(u)$ and $(b \wedge v)(E) \leq \text{rank}(v)$.
- Hence,

$$\text{rank}(u \wedge v) + \text{rank}(u \vee v) = a(E) + b(E) \tag{9.58}$$

$$= (b \wedge u)(E) + (b \wedge v)(E) \tag{9.59}$$

$$\leq \text{rank}(u) + \text{rank}(v) \tag{9.60}$$

□

A polymatroid function's polyhedron vs. a polymatroid.

- Note the remarkable similarity between the proof of Theorem 9.4.8 and the proof of Theorem 5.5.1 that the standard matroid rank function is submodular.

A polymatroid function's polyhedron vs. a polymatroid.

- Note the remarkable similarity between the proof of Theorem 9.4.8 and the proof of Theorem 5.5.1 that the standard matroid rank function is submodular.
- Next, we prove Theorem 9.4.6, that any polymatroid polytope P has a polymatroid function f such that $P = P_f^+$.

A polymatroid function's polyhedron vs. a polymatroid.

- Note the remarkable similarity between the proof of Theorem 9.4.8 and the proof of Theorem 5.5.1 that the standard matroid rank function is submodular.
- Next, we prove Theorem 9.4.6, that any polymatroid polytope P has a polymatroid function f such that $P = P_f^+$.
- Given this result, we can conclude that a polymatroid is really an extremely natural polyhedral generalization of a matroid. This was all realized by Jack Edmonds in the mid 1960s (and published in 1969 in his landmark paper "Submodular Functions, Matroids, and Certain Polyhedra").

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- We are given a polymatroid P.

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- We are given a polymatroid P.
- Define $\alpha_{\text{max}} \triangleq \max\{x(E) : x \in P\}$, and note that $\alpha_{\text{max}} > 0$ when P is non-empty, and $\alpha_{\text{max}} = \text{rank}(\infty \mathbf{1}_E) = \text{rank}(\alpha_{\text{max}} \mathbf{1}_E)$.

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- We are given a polymatroid P.
- Define $\alpha_{\max} \triangleq \max\{x(E) : x \in P\}$, and note that $\alpha_{\max} > 0$ when P is non-empty, and $\alpha_{\max} = \text{rank}(\infty \mathbf{1}_E) = \text{rank}(\alpha_{\max}\mathbf{1}_E)$.
- Hence, for any $x \in P$, $x(e) \leq \alpha_{\max}, \forall e \in E$.

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- We are given a polymatroid P.
- Define $\alpha_{\text{max}} \triangleq \max\{x(E) : x \in P\}$, and note that $\alpha_{\text{max}} > 0$ when P is non-empty, and $\alpha_{\text{max}} = \text{rank}(\infty \mathbf{1}_E) = \text{rank}(\alpha_{\text{max}} \mathbf{1}_E)$.
- Hence, for any $x \in P$, $x(e) \le \alpha_{\text{max}}, \forall e \in E$.
- Define a function $f : 2^V \to \mathbb{R}$ as, for any $A \subseteq E$,

$$f(A) \triangleq \text{rank}(\alpha_{\text{max}} \mathbf{1}_A) \tag{9.61}$$

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- We are given a polymatroid P.
- Define $\alpha_{\text{max}} \triangleq \max\{x(E) : x \in P\}$, and note that $\alpha_{\text{max}} > 0$ when P is non-empty, and $\alpha_{\text{max}} = \text{rank}(\infty \mathbf{1}_E) = \text{rank}(\alpha_{\text{max}} \mathbf{1}_E)$.
- Hence, for any $x \in P$, $x(e) \leq \alpha_{\text{max}}, \forall e \in E$.
- Define a function $f : 2^V \to \mathbb{R}$ as, for any $A \subseteq E$,

$$f(A) \triangleq \text{rank}(\alpha_{\text{max}} \mathbf{1}_A) \tag{9.61}$$

- Then f is submodular since

$f(A) + f(B)$

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- We are given a polymatroid P.
- Define $\alpha_{\text{max}} \triangleq \max\{x(E) : x \in P\}$, and note that $\alpha_{\text{max}} > 0$ when P is non-empty, and $\alpha_{\text{max}} = \text{rank}(\infty \mathbf{1}_E) = \text{rank}(\alpha_{\text{max}} \mathbf{1}_E)$.
- Hence, for any $x \in P$, $x(e) \leq \alpha_{\text{max}}, \forall e \in E$.
- Define a function $f : 2^V \to \mathbb{R}$ as, for any $A \subseteq E$,

$$f(A) \triangleq \text{rank}(\alpha_{\text{max}} \mathbf{1}_A) \tag{9.61}$$

- Then f is submodular since

$$f(A) + f(B) = \text{rank}(\alpha_{\text{max}} \mathbf{1}_A) + \text{rank}(\alpha_{\text{max}} \mathbf{1}_B) \tag{9.62}$$

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- We are given a polymatroid P.
- Define $\alpha_{\max} \triangleq \max\{x(E) : x \in P\}$, and note that $\alpha_{\max} > 0$ when P is non-empty, and $\alpha_{\max} = \text{rank}(\infty \mathbf{1}_E) = \text{rank}(\alpha_{\max}\mathbf{1}_E)$.
- Hence, for any $x \in P$, $x(e) \leq \alpha_{\max}, \forall e \in E$.
- Define a function $f : 2^V \to \mathbb{R}$ as, for any $A \subseteq E$,

$$f(A) \triangleq \text{rank}(\alpha_{\max}\mathbf{1}_A) \tag{9.61}$$

- Then f is submodular since

$$f(A) + f(B) = \text{rank}(\alpha_{\max}\mathbf{1}_A) + \text{rank}(\alpha_{\max}\mathbf{1}_B) \tag{9.62}$$

$$\geq \text{rank}(\alpha_{\max}\mathbf{1}_A \vee \alpha_{\max}\mathbf{1}_B) + \text{rank}(\alpha_{\max}\mathbf{1}_A \wedge \alpha_{\max}\mathbf{1}_B) \tag{9.63}$$

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- We are given a polymatroid P.
- Define $\alpha_{\max} \triangleq \max\{x(E) : x \in P\}$, and note that $\alpha_{\max} > 0$ when P is non-empty, and $\alpha_{\max} = \text{rank}(\infty \mathbf{1}_E) = \text{rank}(\alpha_{\max}\mathbf{1}_E)$.
- Hence, for any $x \in P$, $x(e) \leq \alpha_{\max}, \forall e \in E$.
- Define a function $f : 2^V \to \mathbb{R}$ as, for any $A \subseteq E$,

$$f(A) \triangleq \text{rank}(\alpha_{\max}\mathbf{1}_A) \tag{9.61}$$

- Then f is submodular since

$$f(A) + f(B) = \text{rank}(\alpha_{\max}\mathbf{1}_A) + \text{rank}(\alpha_{\max}\mathbf{1}_B) \tag{9.62}$$

$$\geq \text{rank}(\alpha_{\max}\mathbf{1}_A \vee \alpha_{\max}\mathbf{1}_B) + \text{rank}(\alpha_{\max}\mathbf{1}_A \wedge \alpha_{\max}\mathbf{1}_B) \tag{9.63}$$

$$= \text{rank}(\alpha_{\max}\mathbf{1}_{A\cup B}) + \text{rank}(\alpha_{\max}\mathbf{1}_{A\cap B}) \tag{9.64}$$

...

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- We are given a polymatroid P.
- Define $\alpha_{\text{max}} \triangleq \max \{x(E) : x \in P\}$, and note that $\alpha_{\text{max}} > 0$ when P is non-empty, and $\alpha_{\text{max}} = \text{rank}(\infty \mathbf{1}_E) = \text{rank}(\alpha_{\text{max}} \mathbf{1}_E)$.
- Hence, for any $x \in P$, $x(e) \leq \alpha_{\text{max}}, \forall e \in E$.
- Define a function $f : 2^V \to \mathbb{R}$ as, for any $A \subseteq E$,

$$f(A) \triangleq \text{rank}(\alpha_{\text{max}} \mathbf{1}_A) \tag{9.61}$$

- Then f is submodular since

$$f(A) + f(B) = \text{rank}(\alpha_{\text{max}} \mathbf{1}_A) + \text{rank}(\alpha_{\text{max}} \mathbf{1}_B) \tag{9.62}$$

$$\geq \text{rank}(\alpha_{\text{max}} \mathbf{1}_A \vee \alpha_{\text{max}} \mathbf{1}_B) + \text{rank}(\alpha_{\text{max}} \mathbf{1}_A \wedge \alpha_{\text{max}} \mathbf{1}_B) \tag{9.63}$$

$$= \text{rank}(\alpha_{\text{max}} \mathbf{1}_{A \cup B}) + \text{rank}(\alpha_{\text{max}} \mathbf{1}_{A \cap B}) \tag{9.64}$$

$$= f(A \cup B) + f(A \cap B) \tag{9.65}$$

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- Moreover, we have that f is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).

...

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- Moreover, we have that f is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).
- Hence, f is a polymatroid function.

...

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- Moreover, we have that f is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).
- Hence, f is a polymatroid function.
- Consider the polytope P_f^+ defined as:

$$P_f^+ = \left\{ x \in \mathbb{R}_+^E : x(A) \le f(A), \ \forall A \subseteq E \right\} \tag{9.66}$$

...

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- Moreover, we have that f is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).
- Hence, f is a polymatroid function.
- Consider the polytope P_f^+ defined as:

$$P_f^+ = \left\{ x \in \mathbb{R}_+^E : x(A) \leq f(A),\ \forall A \subseteq E \right\} \tag{9.66}$$

- Given an $x \in P$, then for any $A \subseteq E$, $x(A) \leq \max\left\{ z(E) : z \in P, z \leq \alpha_{\text{max}} \mathbf{1}_A \right\} = \text{rank}(\alpha_{\text{max}} \mathbf{1}_A) = f(A)$,

...

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- Moreover, we have that f is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).
- Hence, f is a polymatroid function.
- Consider the polytope P_f^+ defined as:

$$P_f^+ = \left\{x \in \mathbb{R}_+^E : x(A) \leq f(A), \ \forall A \subseteq E\right\} \tag{9.66}$$

- Given an $x \in P$, then for any $A \subseteq E$, $x(A) \leq \max\left\{z(E) : z \in P, z \leq \alpha_{\text{max}} \mathbf{1}_A\right\} = \text{rank}(\alpha_{\text{max}} \mathbf{1}_A) = f(A)$, therefore $x \in P_f^+$.

...

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- Moreover, we have that f is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).
- Hence, f is a polymatroid function.
- Consider the polytope P_f^+ defined as:

$$P_f^+ = \left\{x \in \mathbb{R}_+^E : x(A) \leq f(A), \ \forall A \subseteq E\right\} \tag{9.66}$$

- Given an $x \in P$, then for any $A \subseteq E$, $x(A) \leq \max\left\{z(E) : z \in P, z \leq \alpha_{\max}\mathbf{1}_A\right\} = \text{rank}(\alpha_{\max}\mathbf{1}_A) = f(A)$, therefore $x \in P_f^+$.
- Hence, $P \subseteq P_f^+$.

...

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- Moreover, we have that f is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).
- Hence, f is a polymatroid function.
- Consider the polytope P_f^+ defined as:

$$P_f^+ = \left\{x \in \mathbb{R}_+^E : x(A) \leq f(A), \; \forall A \subseteq E\right\} \tag{9.66}$$

- Given an $x \in P$, then for any $A \subseteq E$, $x(A) \leq \max\left\{z(E) : z \in P, z \leq \alpha_{\mathsf{max}} \mathbf{1}_A\right\} = \mathsf{rank}(\alpha_{\mathsf{max}} \mathbf{1}_A) = f(A)$, therefore $x \in P_f^+$.
- Hence, $P \subseteq P_f^+$.
- We will next show that $P_f^+ \subseteq P$ to complete the proof.

...

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- Let $x \in P_f^+$ be chosen arbitrarily (goal is to show that $x \in P$).

...

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- Let $x \in P_f^+$ be chosen arbitrarily (goal is to show that $x \in P$).
- Suppose $x \notin P$.

...

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- Let $x \in P_f^+$ be chosen arbitrarily (goal is to show that $x \in P$).
- Suppose $x \notin P$. Then, choose y to be a P-basis of x that maximizes the number of y elements strictly less than the corresponding x element. I.e., that maximizes $|N(y)|$, where

$$N(y) = \{e \in E : y(e) < x(e)\} \tag{9.67}$$

...

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- Let $x \in P_f^+$ be chosen arbitrarily (goal is to show that $x \in P$).
- Suppose $x \notin P$. Then, choose y to be a P-basis of x that maximizes the number of y elements strictly less than the corresponding x element. I.e., that maximizes $|N(y)|$, where

$$N(y) = \{e \in E : y(e) < x(e)\} \tag{9.67}$$

- Choose w between y and x, so that

$$y \leq w \triangleq (y + x)/2 \leq x \tag{9.68}$$

so y is also a P-basis of w.

...

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- Let $x \in P_f^+$ be chosen arbitrarily (goal is to show that $x \in P$).
- Suppose $x \notin P$. Then, choose y to be a P-basis of x that maximizes the number of y elements strictly less than the corresponding x element. I.e., that maximizes $|N(y)|$, where

$$N(y) = \{e \in E : y(e) < x(e)\} \tag{9.67}$$

- Choose w between y and x, so that

$$y \leq w \triangleq (y + x)/2 \leq x \tag{9.68}$$

so y is also a P-basis of w.

- Hence, $\text{rank}(x) = \text{rank}(w)$, and the set of P-bases of w are also P-bases of x.

. . .

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- For any $A \subseteq E$, define $x_A \in \mathbb{R}_+^E$ as

$$x_A(e) = \begin{cases} x(e) & \text{if } e \in A \\ 0 & \text{else} \end{cases} \tag{9.69}$$

note this is an analogous definition to $\mathbf{1}_A$ but for a non-unity vector.

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- For any $A \subseteq E$, define $x_A \in \mathbb{R}_+^E$ as

$$x_A(e) = \begin{cases} x(e) & \text{if } e \in A \\ 0 & \text{else} \end{cases} \tag{9.69}$$

note this is an analogous definition to $\mathbf{1}_A$ but for a non-unity vector.

- Now, we have

$$y(N(y)) < w(N(y)) \leq f(N(y)) = \text{rank}(\alpha_{\max}\mathbf{1}_{N(y)}) \tag{9.70}$$

the last inequality follows since $w \leq x \in P_f^+$.

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- For any $A \subseteq E$, define $x_A \in \mathbb{R}_+^E$ as

$$x_A(e) = \begin{cases} x(e) & \text{if } e \in A \\ 0 & \text{else} \end{cases} \tag{9.69}$$

note this is an analogous definition to $\mathbf{1}_A$ but for a non-unity vector.

- Now, we have

$$y(N(y)) < w(N(y)) \leq f(N(y)) = \text{rank}(\alpha_{\text{max}} \mathbf{1}_{N(y)}) \tag{9.70}$$

the last inequality follows since $w \leq x \in P_f^+$.

- Thus, $y \wedge x_{N(y)}$ is not a P-basis of $w \wedge x_{N(y)}$ since, over $N(y)$, it is neither tight at w nor tight at the rank (i.e., not a maximal independent subvector on $N(y)$).

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- We can extend $y \wedge x_{N(y)}$ to be a P-basis of $w \wedge x_{N(y)}$.

$\square$

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- We can extend $y \wedge x_{N(y)}$ to be a P-basis of $w \wedge x_{N(y)}$.
- This P-basis, in turn, can be extended to be a P-basis $\hat{y}$ of w.

$\square$

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- We can extend $y \wedge x_{N(y)}$ to be a P-basis of $w \wedge x_{N(y)}$.
- This P-basis, in turn, can be extended to be a P-basis $\hat{y}$ of w.
- Now, we have $\hat{y}(N(y)) > y(N(y))$,

$\square$

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- We can extend $y \wedge x_{N(y)}$ to be a P-basis of $w \wedge x_{N(y)}$.
- This P-basis, in turn, can be extended to be a P-basis $\hat{y}$ of w.
- Now, we have $\hat{y}(N(y)) > y(N(y))$,
- and also that $\hat{y}(E) = y(E)$ (since both are P-bases),

$\square$

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- We can extend $y \wedge x_{N(y)}$ to be a P-basis of $w \wedge x_{N(y)}$.
- This P-basis, in turn, can be extended to be a P-basis $\hat{y}$ of w.
- Now, we have $\hat{y}(N(y)) > y(N(y))$,
- and also that $\hat{y}(E) = y(E)$ (since both are P-bases),
- hence $\hat{y}(e) < y(e)$ for some $e \notin N(y)$.

$\square$

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- We can extend $y \wedge x_{N(y)}$ to be a P-basis of $w \wedge x_{N(y)}$.
- This P-basis, in turn, can be extended to be a P-basis $\hat{y}$ of w.
- Now, we have $\hat{y}(N(y)) > y(N(y))$,
- and also that $\hat{y}(E) = y(E)$ (since both are P-bases),
- hence $\hat{y}(e) < y(e)$ for some $e \notin N(y)$.
- Thus, $\hat{y}$ is a base of x, which violates the maximality of $|N(y)|$.

$\square$

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- We can extend $y \wedge x_{N(y)}$ to be a P-basis of $w \wedge x_{N(y)}$.
- This P-basis, in turn, can be extended to be a P-basis $\hat{y}$ of w.
- Now, we have $\hat{y}(N(y)) > y(N(y))$,
- and also that $\hat{y}(E) = y(E)$ (since both are P-bases),
- hence $\hat{y}(e) < y(e)$ for some $e \notin N(y)$.
- Thus, $\hat{y}$ is a base of x, which violates the maximality of $|N(y)|$.
- This contradiction means that we must have had $x \in P$.

$\square$

Proof of Theorem 9.4.6

Proof of Theorem 9.4.6.

- We can extend $y \wedge x_{N(y)}$ to be a P-basis of $w \wedge x_{N(y)}$.
- This P-basis, in turn, can be extended to be a P-basis $\hat{y}$ of w.
- Now, we have $\hat{y}(N(y)) > y(N(y))$,
- and also that $\hat{y}(E) = y(E)$ (since both are P-bases),
- hence $\hat{y}(e) < y(e)$ for some $e \notin N(y)$.
- Thus, $\hat{y}$ is a base of x, which violates the maximality of $|N(y)|$.
- This contradiction means that we must have had $x \in P$.
- Therefore, $P_f^+ = P$.

$\square$

More on polymatroids

Theorem 9.4.9

A polymatroid can equivalently be defined as a pair (E, P) where E is a finite ground set and $P \subseteq R_+^E$ is a compact non-empty set of independent vectors such that

1. *every subvector of an independent vector is independent (if $x \in P$ and $y \leq x$ then $y \in P$, i.e., down closed)*

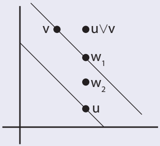
More on polymatroids

Theorem 9.4.9

A polymatroid can equivalently be defined as a pair (E, P) where E is a finite ground set and $P \subseteq R_+^E$ is a compact non-empty set of independent vectors such that

1. *every subvector of an independent vector is independent (if $x \in P$ and $y \leq x$ then $y \in P$, i.e., down closed)*
2. *If $u, v \in P$ (i.e., are independent) and $u(E) < v(E)$, then there exists a vector $w \in P$ such that*

$$u < w \leq u \vee v \tag{9.71}$$



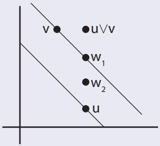
More on polymatroids

Theorem 9.4.9

A polymatroid can equivalently be defined as a pair (E, P) where E is a finite ground set and $P \subseteq R_+^E$ is a compact non-empty set of independent vectors such that

1. *every subvector of an independent vector is independent (if $x \in P$ and $y \leq x$ then $y \in P$, i.e., down closed)*
2. *If $u, v \in P$ (i.e., are independent) and $u(E) < v(E)$, then there exists a vector $w \in P$ such that*

$$u < w \leq u \vee v \qquad (9.71)$$



Corollary 9.4.10

The independent vectors of a polymatroid form a convex polyhedron in $\mathbb{R}_+^E$.

More on polymatroids

For any compact set P, b is a base of P if it is a maximal subvector within P. Recall the bases of polymatroids. In fact, we can define a polymatroid via vector bases (analogous to how a matroid can be defined via matroid bases).

Theorem 9.4.11

A polymatroid can equivalently be defined as a pair (E, P) where E is a finite ground set and $P \subseteq R_+^E$ is a compact non-empty set of independent vectors such that

1. *every subvector of an independent vector is independent (if $x \in P$ and $y \leq x$ then $y \in P$, i.e., down closed)*
2. *if b, c are bases of P and d is such that $b \wedge c < d < b$, then there exists an f, with $d \wedge c < f \leq c$ such that $d \vee f$ is a base of P*
3. *All of the bases of P have the same rank.*

Note, all three of the above are required for a polymatroid (a matroid analogy would require the equivalent of only the first two).

also, a word on terminology

- Recall how a matroid is sometimes given as (E, r) where r is the rank function.

also, a word on terminology

- Recall how a matroid is sometimes given as (E, r) where r is the rank function.
- We mention also that the term "polymatroid" is sometimes not used for the polytope itself, but instead but for the pair (E, f),

also, a word on terminology

- Recall how a matroid is sometimes given as (E, r) where r is the rank function.
- We mention also that the term "polymatroid" is sometimes not used for the polytope itself, but instead but for the pair (E, f),
- But now we see that (E, f) is equivalent to a polymatroid polytope, so this is sensible.

Where are we going with this?

- Consider the right hand side of Theorem 9.4.5:
  $\min\left(x(A) + f(E \setminus A) : A \subseteq E\right)$

Where are we going with this?

- Consider the right hand side of Theorem 9.4.5: $\min\left(x(A) + f(E \setminus A) : A \subseteq E\right)$
- We are going to study this problem, and approaches that address it, as part of our ultimate goal which is to present strategies for submodular function minimization (that we will ultimately get to, in near future lectures).

Where are we going with this?

- Consider the right hand side of Theorem 9.4.5: $\min\left(x(A) + f(E \setminus A) : A \subseteq E\right)$
- We are going to study this problem, and approaches that address it, as part of our ultimate goal which is to present strategies for submodular function minimization (that we will ultimately get to, in near future lectures).
- As a bit of a hint on what's to come, note that we can write it as: $x(E) + \min\left(f(A) - x(A) : A \subseteq E\right)$ where f is a polymatroid function.

Another Interesting Fact: Matroids from polymatroid functions

Theorem 9.4.12

Given integral polymatroid function f, let $(E, \mathcal{F})$ be a set system with ground set E and set of subsets $\mathcal{F}$ such that

$$\forall F \in \mathcal{F}, \;\; \forall \emptyset \subset S \subseteq F, |S| \leq f(S) \tag{9.72}$$

Then $M = (E, \mathcal{F})$ is a matroid.

Proof.

Exercise $\square$

And its rank function is Exercise.

Matroid instance of Theorem 9.4.5

- Considering Theorem 9.4.5, the matroid case is now a special case, where we have that:

Corollary 9.4.13

We have that:

$$\max\{y(E) : y \in P_{\textit{ind. set}}(M), y \leq x\} = \min\{r_M(A) + x(E \setminus A) : A \subseteq E\} \tag{9.73}$$

where r_M is the matroid rank function of some matroid.