Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 8 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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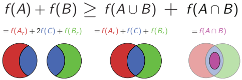
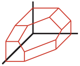
April 23rd, 2014





Cumulative Outstanding Reading

- Read chapters 1 and 2, and sections 3.1-3.2 from Fujishige's book.

Announcements, Assignments, and Reminders

- Homework 1 is out, due Wednesday April 23rd, 11:45pm, electronically via our assignment dropbox (`https://canvas.uw.edu/courses/895956/assignments`).
- All homeworks must be done electronically, only PDF file format accepted.
- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, Combinatorial Geometries
- L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
- L9:
- L10:
- L11:
- L12:
- L13:
- L14:
- L15:
- L16:
- L17:
- L18:
- L19:
- L20:

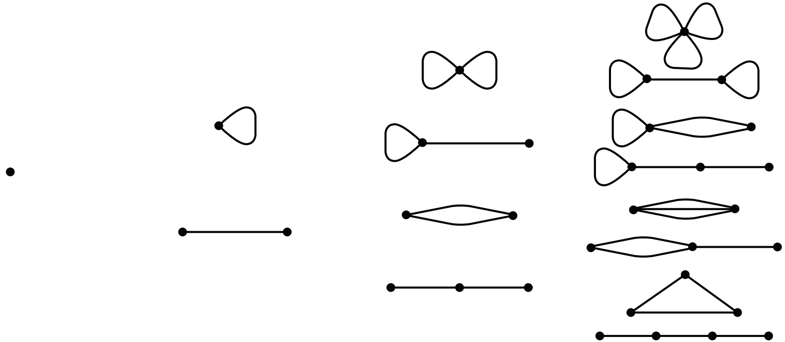
Finals Week: June 9th-13th, 2014.

Matroid Operations

- Matroid restriction/deletion
- Matroid contraction
- Matroid minor (series of deletions & contractions)
- Matroid intersection and its rank (convolution)
- Matroid union and its rank (convolution)

Matroids of three or fewer elements are graphic

- All matroids up to and including three elements are graphic.



(a) The only matroid with zero elements.

(b) The two one-element matroids.

(c) The four two-element matroids.

(d) The eight three-element matroids.

- This is a nice way to show matroids with low ground set sizes. What about matroids that are low rank but with many elements?

Affine Matroids

- Given an $n \times m$ matrix with entries over some field $\mathbb{F}$, we say that a subset $S \subseteq \{1, \dots, m\}$ of indices (with corresponding column vectors $\{v_i : i \in S\}$, with $|S| = k$) is affinely dependent if $m \geq 1$ and there exists elements $\{a_1, \dots, a_k\} \in \mathbb{F}$, not all zero with $\sum_{i=1}^{k} a_i = 0$, such that $\sum_{i=1}^{k} a_i v_i = 0$.
- Otherwise, the set is called affinely independent.
- Concisely: points $\{v_1, v_2, \dots, v_k\}$ are affinely independent if $v_2 - v_1, v_3 - v_1, \dots, v_k - v_1$ are linearly independent.
- Example: in 2D, three collinear points are affinely dependent, three non-collear points are affinely independent, and ≥ 4 non-collinear points are affinely dependent.

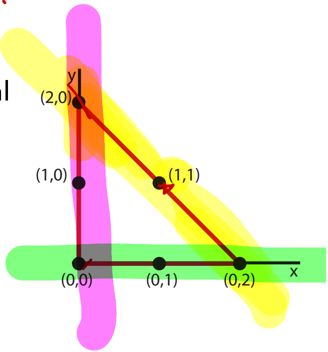
Proposition 8.2.7 (affine matroid)

Let ground set $E = \{1, \dots, m\}$ index column vectors of a matrix, and let $\mathcal{I}$ be the set of subsets X of E such that X indices affinely independent vectors. Then $(E, \mathcal{I})$ is a matroid.

Exercise: prove this.

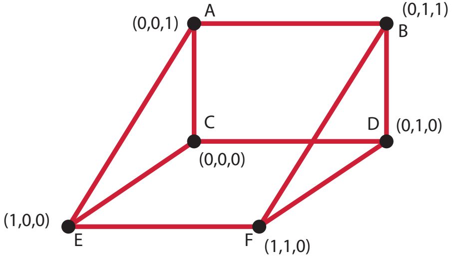
Euclidean Representation of Low-rank Matroids

- Consider the affine matroid with $n \times m = 2 \times 6$ matrix on the field $\mathbb{F} = \mathbb{R}$, and let the elements be $\{(0,0),(1,0),(2,0),(0,1),(0,2),(1,1)\}$.
- We can plot the points in $\mathbb{R}^2$ as on the right:
- Points have rank 1, lines have rank 2, planes have rank 3.
- Flats (points, lines, planes, etc.) have rank equal to one more than their geometric dimension.
- Any two points constitute a line, but lines with only two points are not drawn.
- Lines indicate collinear sets with ≥ 3 points, while any two points have rank 2.
- Dependent sets consist of all subsets with ≥ 4 elements (rank 3), or 3 collinear elements (rank 2). Any two points have rank 2.



Euclidean Representation of Low-rank Matroids

- As another example on the right, a rank 4 matroid



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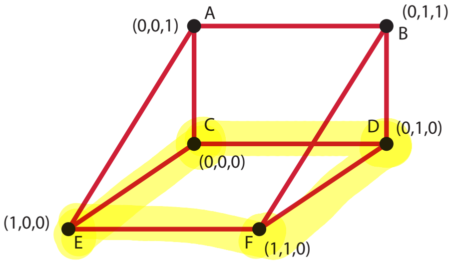

- All sets of 5 points are dependent. The only other sets of dependent points are coplanar ones of size 4. Namely: $\{(0,0,0),(0,1,0),(1,1,0),(1,0,0)\}$, $\{(0,0,0),(0,0,1),(0,1,1),(0,1,0)\}$, and $\{(0,0,1),(0,1,1),(1,1,0),(1,0,0)\}$.

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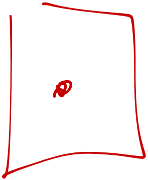
- When they exist, loops are represented in a geometry by a separate box indicating how many loops there are.



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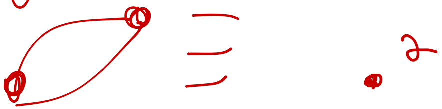
- Parallel elements, when they exist in a matroid, are indicated by a multiplicity next to a point.

Graph Geom



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Theorem 8.3.1

Any matroid of rank $m \leq 4$ can be represented by an affine matroid in $\mathcal{R}^{m-1}$.

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- If diagram has more than one plane, then: 1) any two distinct planes meeting in more than two points do so in a line; 2) any two distinct lines meeting in a point do so in at most one point and lie in on a common plane; 3) any line not lying on a plane intersects it in at most one point.

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- (see Oxley 2011 for more details).

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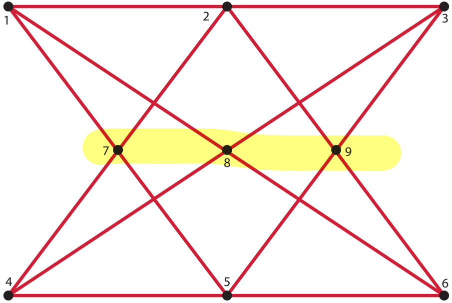

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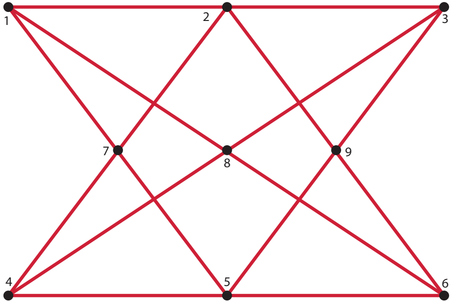
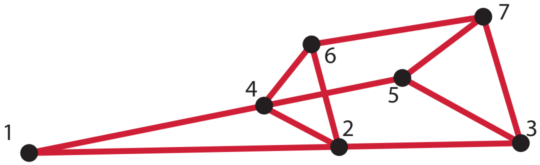


- Called the non-Pappus matroid. Has rank three, but any matric matroid with the above dependencies would require that $\{7, 8, 9\}$ is dependent, hence requiring an additional line in the above.

Euclidean Representation of Low-rank Matroids: A test

- Is this a matroid?



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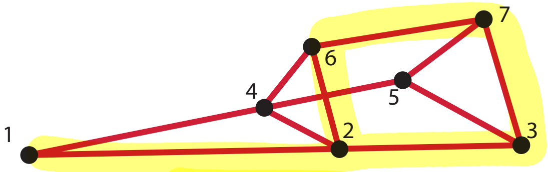

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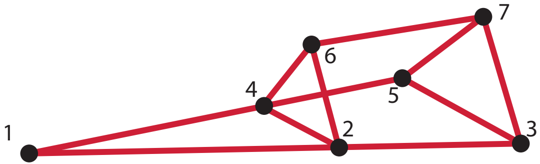

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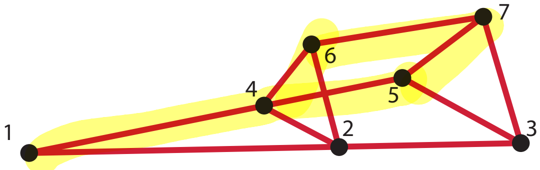

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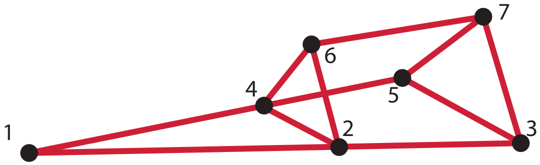

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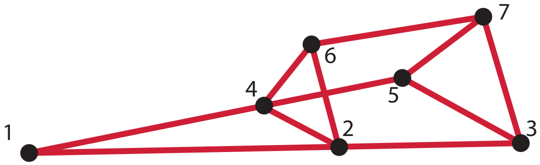

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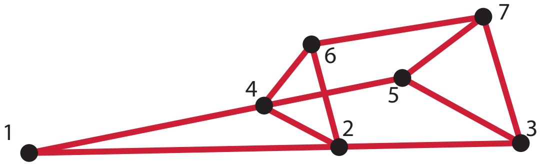

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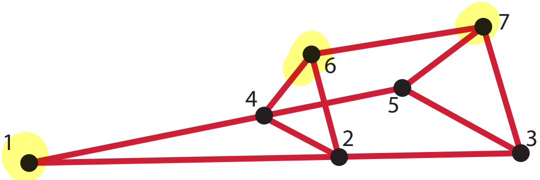

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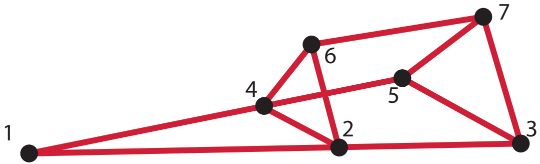

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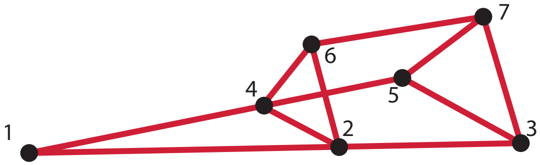

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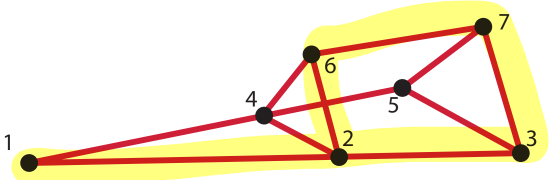

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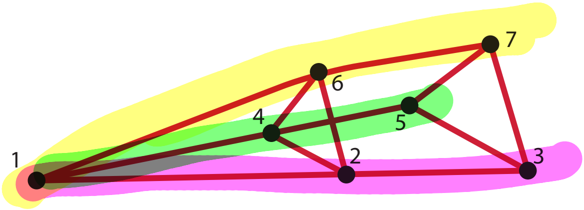
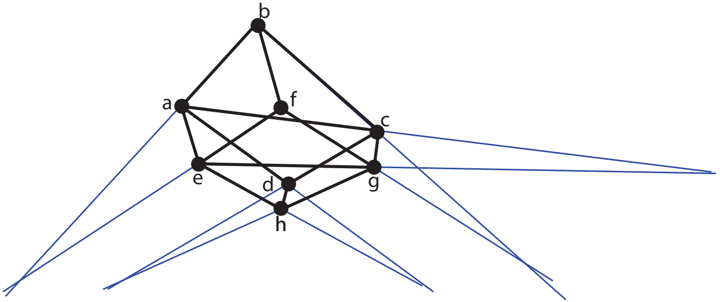


- If we extend the line from 6-7 to 1, then is it a matroid?
- Hence, not all 2D or 3D graphs of points and lines are matroids.

Matroid?

- Consider the following geometry on $|V| = 8$ points with $V = \{a, b, c, d, e, f, g, h\}$.



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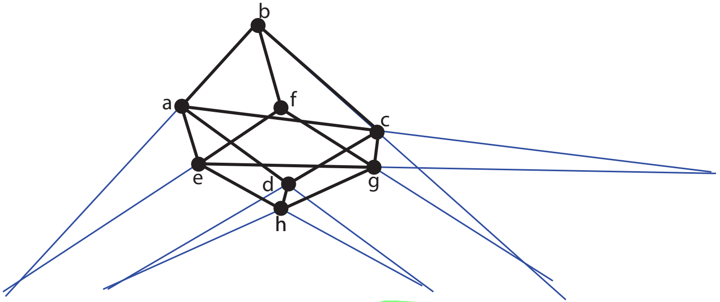
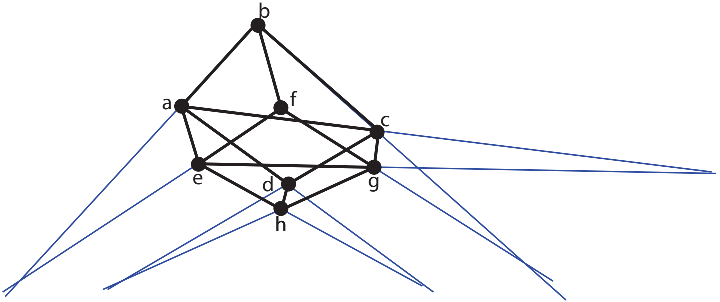


- Note, we are given that the points $\{b, d, h, f\}$ are not coplanar. However, the following sets of points are coplanar: $\{a, b, e, f\}$, $\{d, c, g, h\}$, $\{a, d, h, e\}$, $\{b, c, g, f\}$, $\{b, c, d, a\}$, $\{f, g, h, e\}$, and $\{a, c, g, e\}$.

Matroid?

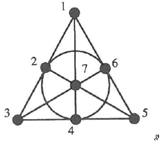
- Consider the following geometry on $|V| = 8$ points with $V = \{a, b, c, d, e, f, g, h\}$.

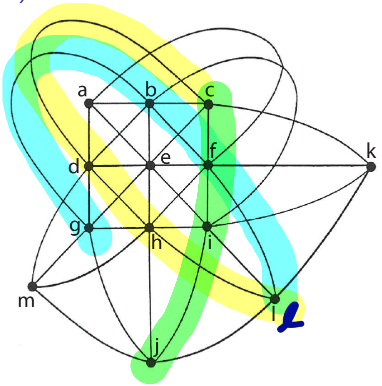


- Note, we are given that the points $\{b, d, h, f\}$ are not coplanar. However, the following sets of points are coplanar: $\{a, b, e, f\}$, $\{d, c, g, h\}$, $\{a, d, h, e\}$, $\{b, c, g, f\}$, $\{b, c, d, a\}$, $\{f, g, h, e\}$, and $\{a, c, g, e\}$.
- Exercise: Is this a matroid? Exercise: If so, is it representable?

Projective Geometries: Other Examples

- Other examples can be more complex, consider the following two matroids (from Oxley, 2011):





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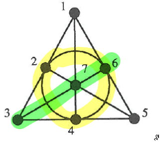

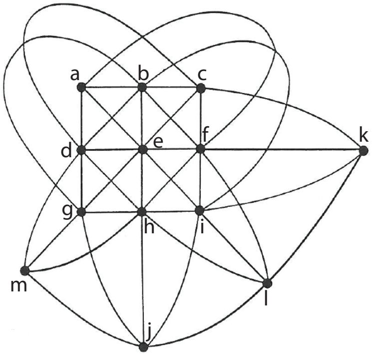

- Right: a matroid (and a 2D depiction of a geometry) over the field $\mathrm{GF}(3) = \{0, 1, 2\}$ mod 3 and is "coordinatizable" in $\mathrm{GF}(3)^3$.

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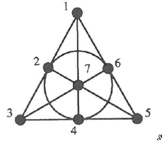

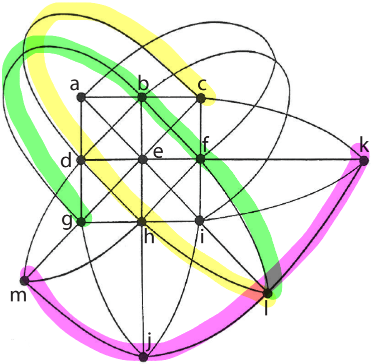

- Right: a matroid (and a 2D depiction of a geometry) over the field GF$(3) = \{0, 1, 2\}$ mod 3 and is "coordinatizable" in GF$(3)^3$.
- Hence, lines (in 2D) which are rank 2 sets may be curved; planes (in 3D) can be twisted.

Matroids, Representation and Equivalence: Summary

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- Not all matroids are linear (i.e., matric) matroids.
- Matroids can be seen as related to projective geometries (and are sometimes called combinatorial geometries).
- Exists much research on different subclasses of matroids, and if/when they are contained in (or isomorphic to) each other.

Matroid Further Reading

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- Greedy is good since it can be made to run very fast $O(n \log n)$.
- Often, however, greedy is heuristic (it might work well in practice, but worst-case performance can be unboundedly poor).
- We will next see that the greedy algorithm working is a defining property of a matroid, and is also a defining property of a polymatroid function.

Matroid and the greedy algorithm

- Let $(E, \mathcal{I})$ be an independence system, and we are given a non-negative modular weight function $w : E \to \mathbb{R}_+$.

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Algorithm 1: The Matroid Greedy Algorithm

Set $X \leftarrow \emptyset$;
while $\exists v \in E \setminus X$ *s.t.* $X \cup \{v\} \in \mathcal{I}$ **do**
  $v \in \operatorname{argmax} \{w(v) : v \in E \setminus X, X \cup \{v\} \in \mathcal{I}\}$;
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Theorem 8.4.1

Let $(E, \mathcal{I})$ be an independence system. Then the pair $(E, \mathcal{I})$ is a matroid if and only if for each weight function $w \in \mathcal{R}_+^E$, Algorithm 1 leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.

Review

- The next slide is from Lecture 5.

Matroids by bases

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

Theorem 8.4.1 (Matroid (by bases))

Let E be a set and $\mathcal{B}$ be a nonempty collection of subsets of E. Then the following are equivalent.

1. *$\mathcal{B}$ is the collection of bases of a matroid;*
2. *if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.*
3. *If $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.*

Properties 2 and 3 are called "exchange properties."
Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.

Matroid and the greedy algorithm

proof of Theorem 8.4.1.

- Assume $(E, \mathcal{I})$ is a matroid and $w : E \to \mathcal{R}_+$ is given.

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- Assume $(E, \mathcal{I})$ is a matroid and $w : E \to \mathcal{R}_+$ is given.
- Let $A = (a_1, a_2, \dots, a_r)$ be the solution returned by greedy, where $r = r(M)$ the rank of the matroid, and we order the elements as they were chosen (so $w(a_1) \geq w(a_2) \geq \dots \geq w(a_r)$).

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- A is a base of M, and let $B = (b_1, \ldots, b_r)$ be <u>any</u> another base of M with elements also ordered decreasing by weight.

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- A is a base of M, and let $B = (b_1, \ldots, b_r)$ be any another base of M with elements also ordered decreasing by weight.
- We next show that not only is $w(A) \geq w(B)$ but that $w(a_i) \geq w(b_i)$ for all i.

...

Matroid and the greedy algorithm

proof of Theorem 8.4.1.

- Assume otherwise, and let k be the first (smallest) integer such that $w(a_k) < w(b_k)$. Hence $w(a_j) \geq w(b_j)$ for $j < k$.

. . .

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proof of Theorem 8.4.1.

- Assume otherwise, and let k be the first (smallest) integer such that $w(a_k) < w(b_k)$. Hence $w(a_j) \geq w(b_j)$ for $j < k$.
- Define independent sets $A_{k-1} = \{a_1, \dots, a_{k-1}\}$ and $B_k = \{b_1, \dots, b_k\}$.

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- Since $|A_{k-1}| < |B_k|$, there exists a $b_i \notin A_{k-1}$ where $A_{k-1} \cup \{b_i\} \in \mathcal{I}$ for some $1 \leq i \leq k$.

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- Since $|A_{k-1}| < |B_k|$, there exists a $b_i \notin A_{k-1}$ where $A_{k-1} \cup \{b_i\} \in \mathcal{I}$ for some $1 \leq i \leq k$.
- But $w(b_i) \geq w(b_k) > w(a_k)$, and so the greedy algorithm would have chosen b_i rather than a_k, contradicting what greedy does.

□

Matroid and the greedy algorithm

converse proof of Theorem 8.4.1.

- Given an independence system $(E, \mathcal{I})$, suppose the greedy algorithm leads to an independent set of max weight for every non-negative weight function. We'll show $(E, \mathcal{I})$ is a matroid.

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- Let $I, J \in \mathcal{I}$ with $|I| < |J|$. Suppose to the contrary, that $I \cup \{z\} \notin \mathcal{I}$ for all $z \in J \setminus I$.

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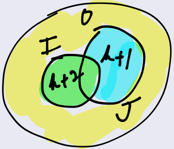
- Let $I, J \in \mathcal{I}$ with $|I| < |J|$. Suppose to the contrary, that $I \cup \{z\} \notin \mathcal{I}$ for all $z \in J \setminus I$.
- Define the following modular weight function w on E, and define $k = |I|$.



$$w(v) = \begin{cases} k+2 & \text{if } v \in I, \\ k+1 & \text{if } v \in J \setminus I, \\ 0 & \text{if } v \in E \setminus (I \cup J) \end{cases} \tag{8.1}$$

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Matroid and the greedy algorithm

converse proof of Theorem 8.4.1.

- Now greedy will, after k iterations, recover I, but it cannot choose any element in $J \setminus I$ by assumption. Thus, greedy chooses a set of weight $k(k + 2)$.

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- On the other hand, J has weight

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- Therefore, there must be a $z \in J \setminus I$ such that $I \cup \{z\} \cancel{\notin} \in \mathcal{I}$, and since I and J are arbitrary, $(E, \mathcal{I})$ must be a matroid.

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- If we stop at a negative value, we'll once again get a maximum weight independent set.
- Exercise: what if we keep going until a base even if we encounter negative values?
- We can instead do as small as possible thus giving us a minimum weight independent set/base.

Summary of Important (for us) Matroid Definitions

Given an independence system, matroids are defined equivalently by any of the following:

- All maximally independent sets have the same size.
- A monotone non-decreasing submodular integral rank function with unit increments.
- The greedy algorithm achieves the maximum weight independent set for all weight functions.

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Definition 8.5.1

A subset $P \subseteq \mathbb{R}^E$ is a polyhedron if there exists an $m \times n$ matrix A and vector $b \in \mathbb{R}^E$ (for some $m \geq 0$) such that

$$P = \{x : Ax \leq b\} \tag{8.3}$$

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- Thus, P is intersection of finitely many affine halfspaces, which are of the form $a_i x \leq b_i$ where a_i is a row vector and b_i a real scalar.

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Definition 8.5.2

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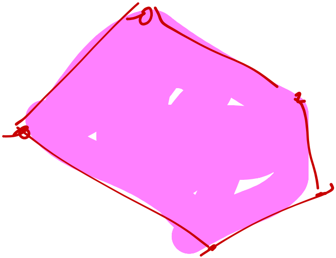

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- We define the convex hull operator as follows:

$$\text{conv}(x_1, x_2, \ldots, x_k) \stackrel{\text{def}}{=} \left\{ \sum_{i=1}^{k} \lambda_i x_i : \forall i, \; \lambda_i \geq 0, \text{ and } \sum_i \lambda_i = 1 \right\} \tag{8.4}$$

Convex Polytope - key representation theorem

- A polytope can be defined in a number of ways, two of which include

Theorem 8.5.3

A subset $P \subseteq \mathbb{R}^E$ is a polytope iff it can be described in either of the following (equivalent) ways:

- *P is the convex hull of a finite set of points.*
- *If it is a bounded intersection of halfspaces, that is there exits matrix A and vector b such that*

$$P = \{x : Ax \leq b\} \tag{8.5}$$

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- This result follows directly from results proven by Fourier, Motzkin, Farkas, and Carátheodory.

Linear Programming

Theorem 8.5.4 (weak duality)

Let A be a matrix and b and c vectors, then

$$\max\{c^\intercal x | Ax \le b\} \le \min\{y^\intercal b : y \ge 0, y^\intercal A = c^\intercal\} \tag{8.6}$$

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Theorem 8.5.5 (strong duality)

Let A be a matrix and b and c vectors, then

$$\max\{c^\intercal x | Ax \leq b\} = \min\{y^\intercal b : y \geq 0, y^\intercal A = c^\intercal\} \tag{8.7}$$

Linear Programming duality forms

There are many ways to construct the dual. For example,

$$\max\{c^\intercal x | x \geq 0, Ax \leq b\} = \min\{y^\intercal b | y \geq 0, y^\intercal A \geq c^\intercal\} \tag{8.8}$$

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$$\min\{c^\intercal x | Ax \geq b\} = \max\{y^\intercal b | y \geq 0, y^\intercal A = c^\intercal\} \tag{8.11}$$

Linear Programming duality forms

How to form the dual in general? We quote V. Vazirani (2001)

Linear Programming duality forms

How to form the dual in general? We quote V. Vazirani (2001)

> *Intuitively, why is [one set of equations] the dual of [another quite different set of equations]? In our experience, this is not the right question to be asked. As stated in Section 12.1, there is a purely mechanical procedure for obtaining the dual of a linear program. Once the dual is obtained, one can devise intuitive, and possibly physical meaningful, ways of thinking about it. Using this mechanical procedure, one can obtain the dual of a complex linear program in a fairly straightforward manner. Indeed, the LP-duality-based approach derives its wide applicability from this fact.*

Also see the text "Convex Optimization" by Boyd and Vandenberghe, chapter 5, for a great discussion on duality.

Vector, modular, incidence

- Recall, any vector $x \in \mathbb{R}^E$ can be seen as a modular function, as for any $A \subseteq E$, we have

$$x(A) = \sum_{a \in A} x_a \tag{8.12}$$

Vector, modular, incidence

- Recall, any vector $x \in \mathbb{R}^E$ can be seen as a modular function, as for any $A \subseteq E$, we have

$$x(A) = \sum_{a \in A} x_a \tag{8.12}$$

- Given an $A \subseteq E$, define the the incidence vector $\mathbf{1}_A \in \{0,1\}^E$ on the unit hypercube as follows:

$$\mathbf{1}_A \stackrel{\text{def}}{=} \left\{ x \in \{0,1\}^E : x_i = 1 \text{ iff } i \in A \right\} \tag{8.13}$$

equivalently,

$$\mathbf{1}_A(j) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } j \in A \\ 0 & \text{if } j \notin A \end{cases} \tag{8.14}$$

Matroid

Slight modification (non unit increment) that is equivalent.

Definition 8.6.1 (Matroid-II)

A set system $(E, \mathcal{I})$ is a Matroid if

(I1') $\emptyset \in \mathcal{I}$

(I2') $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ (or "down-closed")

(I3') $\forall I, J \in \mathcal{I}$, with $|I| > |J|$, then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$

Note (I1)=(I1'), (I2)=(I2'), and we get (I3)≡(I3') using induction.

Independence Polyhedra

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I$.

Independence Polyhedra

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$$P_{\text{ind. set}} = \text{conv}\left\{\bigcup_{I \in \mathcal{I}} \{\mathbf{1}_I\}\right\} \tag{8.15}$$

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$$P_{\text{ind. set}} = \text{conv}\left\{ \bigcup_{I \in \mathcal{I}} \{\mathbf{1}_I\} \right\} \tag{8.15}$$

- Since $\{\mathbf{1}_I : I \in \mathcal{I}\} \subseteq P_{\text{ind. set}}$, we have $\max\{w(I) : I \in \mathcal{I}\} \leq \max\{w^\intercal x : x \in P_{\text{ind. set}}\}$.

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- Now take the rank function r of M, and define the following polyhedron:

$$P_r^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\} \tag{8.16}$$

$$\left(\sum_{a \in A} x(a) \le r(A) \right)$$

Independence Polyhedra

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I$.
- Taking the convex hull, we get the independent set polytope, that is

$$P_{\text{ind. set}} = \text{conv}\left\{\bigcup_{I \in \mathcal{I}} \{\mathbf{1}_I\}\right\} \tag{8.15}$$

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$$P_r^+ = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\right\} \tag{8.16}$$

- Now, take any $x \in P_{\text{ind. set}}$, then we have that $x \in P_r^+$ (or $P_{\text{ind. set}} \subseteq P_r^+$). We show this next.

$P_{\text{ind. set}} \subseteq P_r^+$

- If $x \in P_{\text{ind. set}}$, then

$$x = \sum_i \lambda_i \mathbf{1}_{I_i} \tag{8.17}$$

for some appropriate vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

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- Clearly, for such x, $x \geq 0$.

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$$\leq \sum_i \lambda_i \max_{j : I_j \subseteq A} \mathbf{1}_{I_j}(E) \tag{8.19}$$

$|I_j|$

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$$= r(A) \tag{8.21}$$

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$$= \max_{j : I_j \subseteq A} \mathbf{1}_{I_j}(E) \tag{8.20}$$

$$= r(A) \tag{8.21}$$

- Thus, $x \in P_r^+$ and hence $P_{\text{ind. set}} \subseteq P_r^+$.

Matroid Polyhedron in 2D

$$P_r^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\} \tag{8.22}$$

- Consider this in two dimensions. We have equations of the form:

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \tag{8.23}$$

$$x_1 \leq r(\{v_1\}) \tag{8.24}$$

$$x_2 \leq r(\{v_2\}) \tag{8.25}$$

$$x_1 + x_2 \leq r(\{v_1, v_2\}) \tag{8.26}$$

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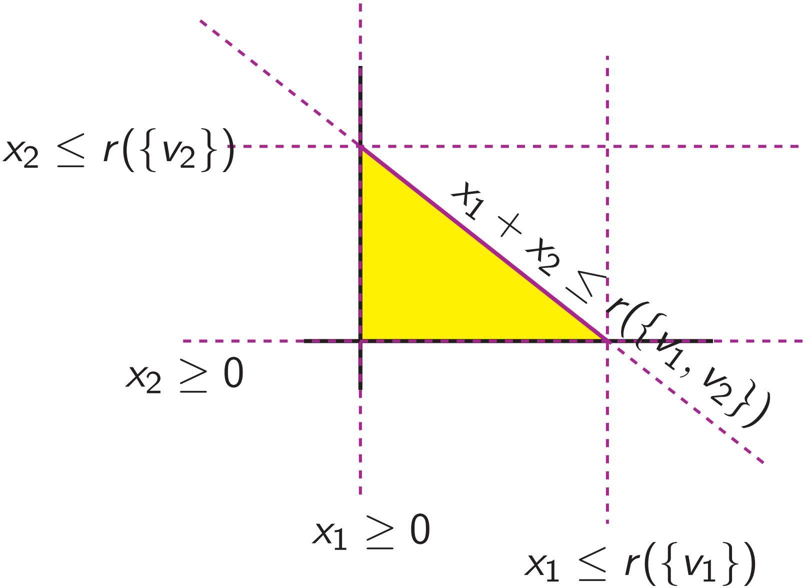
- Because r is submodular, we have

$$r(\{v_1\}) + r(\{v_2\}) \geq r(\{v_1, v_2\}) + r(\emptyset) \tag{8.27}$$

so since $r(\{v_1, v_2\}) \leq r(\{v_1\}) + r(\{v_2\})$, the last inequality is either touching or active.

Matroid Polyhedron in 2D



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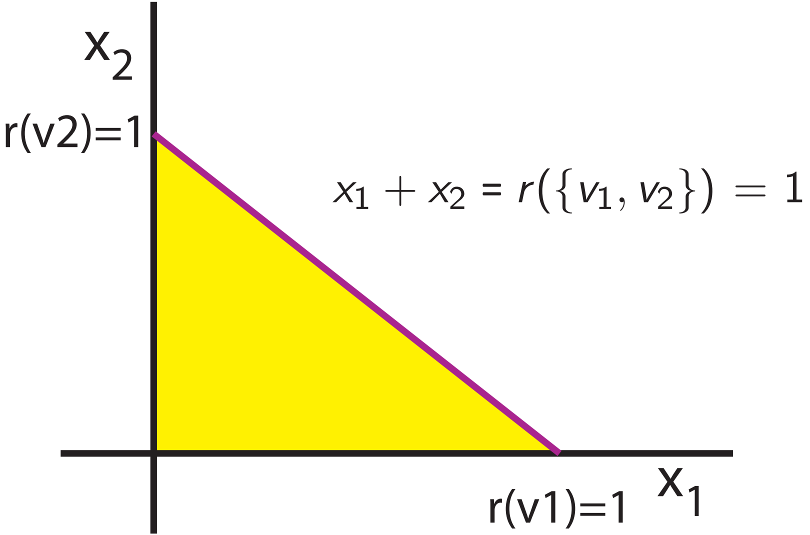

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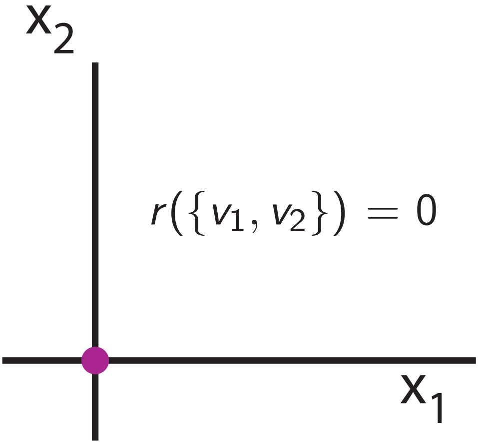

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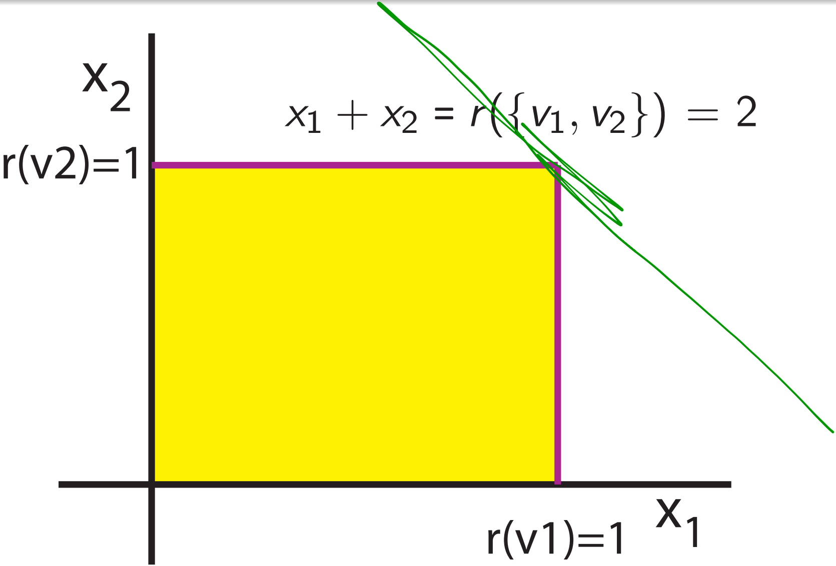
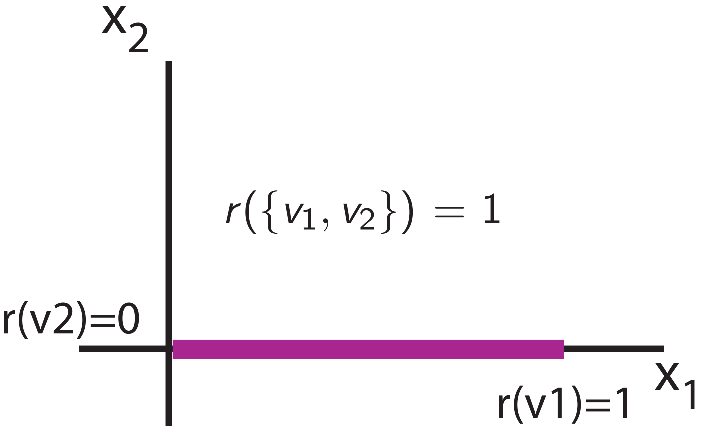
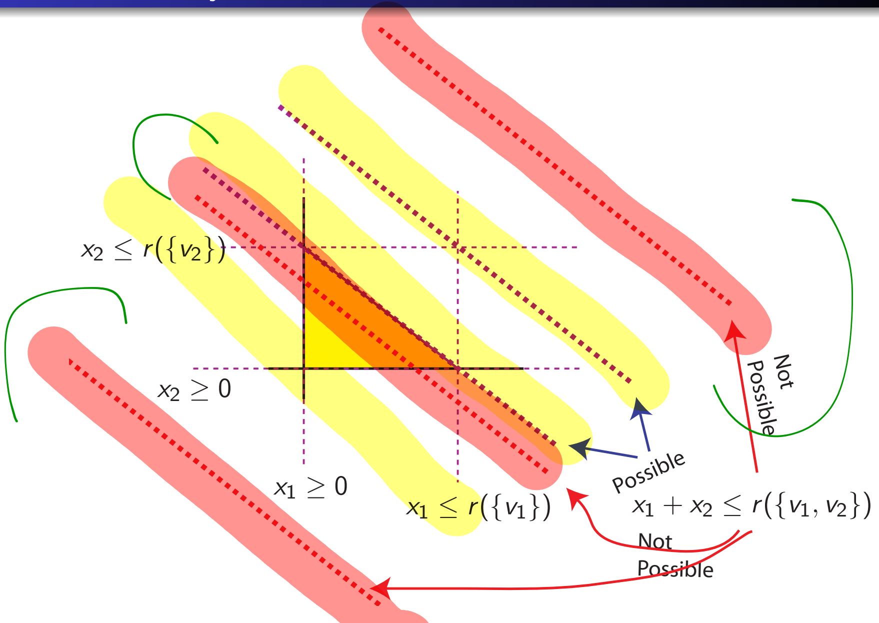


Matroid Polyhedron in 2D

And, if v2 is a loop ...



Matroid Polyhedron in 2D



Matroid Polyhedron in 3D

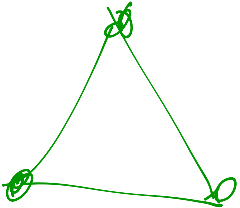
$$P_r^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\} \tag{8.28}$$

- Consider this in three dimensions. We have equations of the form:

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \text{ and } x_3 \geq 0 \tag{8.29}$$
$$x_1 \leq r(\{v_1\}) \tag{8.30}$$
$$x_2 \leq r(\{v_2\}) \tag{8.31}$$
$$x_3 \leq r(\{v_3\}) \tag{8.32}$$
$$x_1 + x_2 \leq r(\{v_1, v_2\}) \tag{8.33}$$
$$x_2 + x_3 \leq r(\{v_2, v_3\}) \tag{8.34}$$
$$x_1 + x_3 \leq r(\{v_1, v_3\}) \tag{8.35}$$
$$x_1 + x_2 + x_3 \leq r(\{v_1, v_2, v_3\}) \tag{8.36}$$

Matroid Polyhedron in 3D

- Consider the simple cycle matroid on a graph consisting of a 3-cycle, $G = (V, E)$ with matroid $M = (E, \mathcal{I})$ where $I \in \mathcal{I}$ is a forest.



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- So any set of either one or two edges is independent, and has rank equal to cardinality.

Matroid Polyhedron in 3D

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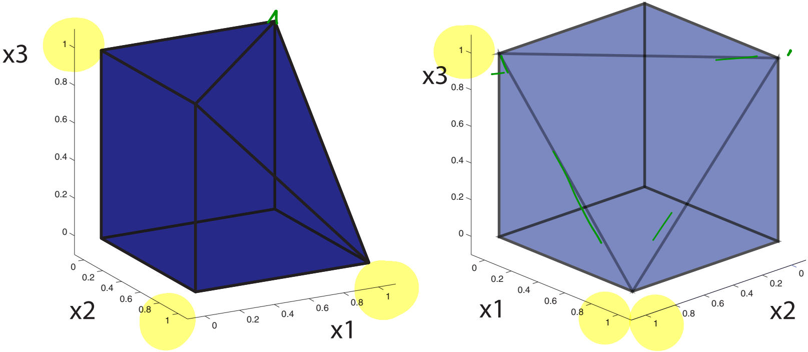
- So any set of either one or two edges is independent, and has rank equal to cardinality.
- The set of three edges is dependent, and has rank 2.

Matroid Polyhedron in 3D

Two view of P_r^+ associated with a matroid
$(\{e_1, e_2, e_3\}, \{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\})$.



Matroid Polyhedron in 3D

P_r^+ associated with the “free” matroid in 3D.

Matroid Polyhedron in 3D

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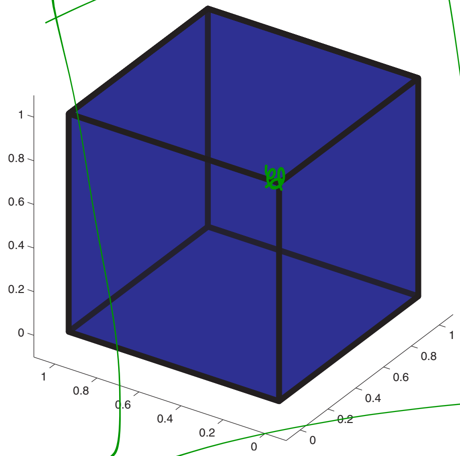

Another Polytope in 3D

Thought question: what kind of polytope might this be?

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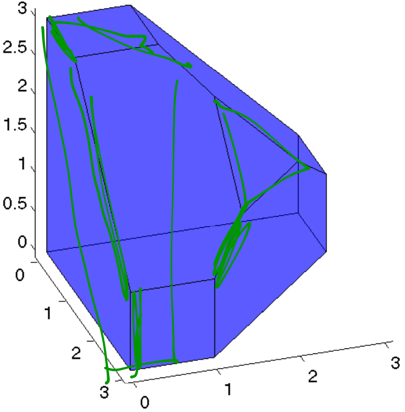

Matroid Independence Polyhedron

- So recall from a moment ago, that we have that

$$P_{\text{ind. set}} = \operatorname{conv}\left\{\cup_{I\in\mathcal{I}}\{\mathbf{1}_I\}\right\}$$
$$\subseteq P_r^+ = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\right\} \tag{8.37}$$

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- In fact, the two polyhedra are identical (and thus both are polytopes).

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- In fact, the two polyhedra are identical (and thus both are polytopes).
- We'll show this in the next few theorems.

Maximum weight independent set via greedy weighted rank

Theorem 8.6.1

Let $M = (V, \mathcal{I})$ be a matroid, with rank function r, then for any weight function $w \in \mathbb{R}_+^V$, there exists a chain of sets $U_1 \subset U_2 \subset \cdots \subset U_n \subseteq V$ such that

$$\max \{w(I) | I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i) \tag{8.38}$$

where $\lambda_i \geq 0$ satisfy

$$w = \sum_{i=1}^{n} \lambda_i \mathbf{1}_{U_i} \tag{8.39}$$

Maximum weight independent set via weighted rank

Proof.

- Firstly, note that for any such $w \in \mathbb{R}^E$, we have

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = (w_1 - w_2)\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + (w_2 - w_3)\begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + (w_{n-1} - w_n)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix} + (w_n)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \tag{8.40}$$

...

Maximum weight independent set via weighted rank

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$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = (w_1 - w_2)\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + (w_2 - w_3)\begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + (w_{n-1} - w_n)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix} + (w_n)\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \tag{8.40}$$

- If we can take w in decreasing order $(w_1 \geq w_2 \geq \cdots \geq w_n)$, then each coefficient of the vectors is non-negative (except possibly the last one, w_n).

...

Maximum weight independent set via weighted rank

Proof.

- Now, again assuming $w \in \mathbb{R}_+^E$, order the elements of V as $(v_1, v_2, \dots, v_n)$ such that $w(v_1) \geq w(v_2) \geq \dots \geq w(v_n)$

. . .

Maximum weight independent set via weighted rank

Proof.

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- Define the sets U_i based on this order as follows, for $i = 0, \dots, n$

$$U_i \stackrel{\text{def}}{=} \{v_1, v_2, \dots, v_i\} \tag{8.41}$$

Note that

$$\mathbf{1}_{U_0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \mathbf{1}_{U_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \mathbf{1}_{U_\ell} = \begin{pmatrix} \left.\begin{matrix} 1 \\ 1 \\ \vdots \\ 1 \end{matrix}\right\} \ell\times \\ \left.\begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}\right\} (n-\ell)\times \end{pmatrix}, \text{ etc.}$$

Maximum weight independent set via weighted rank

Proof.

- Now, again assuming $w \in \mathbb{R}_+^E$, order the elements of V as $(v_1, v_2, \ldots, v_n)$ such that $w(v_1) \geq w(v_2) \geq \cdots \geq w(v_n)$
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$$U_i \stackrel{\text{def}}{=} \{v_1, v_2, \ldots, v_i\} \tag{8.41}$$

- Define the set I as those elements where the rank increases, i.e.:

$$I \stackrel{\text{def}}{=} \{v_i | r(U_i) > r(U_{i-1})\} \tag{8.42}$$

...

Maximum weight independent set via weighted rank

Proof.

- Now, again assuming $w \in \mathbb{R}_+^E$, order the elements of V as $(v_1, v_2, \ldots, v_n)$ such that $w(v_1) \geq w(v_2) \geq \cdots \geq w(v_n)$
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- Therefore, I is the output of the greedy algorithm for $\max\{w(I) | I \in \mathcal{I}\}$. *since items v_i are ordered decreasing by $w(v_i)$, and we only choose the ones that increase the rank, which means they don't violate independence.*

...

Maximum weight independent set via weighted rank

Proof.

- Now, again assuming $w \in \mathbb{R}_+^E$, order the elements of V as $(v_1, v_2, \ldots, v_n)$ such that $w(v_1) \geq w(v_2) \geq \cdots \geq w(v_n)$
- Define the sets U_i based on this order as follows, for $i = 0, \ldots, n$

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- Define the set I as those elements where the rank increases, i.e.:

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- Therefore, I is the output of the greedy algorithm for $\max\{w(I) | I \in \mathcal{I}\}$.
- And therefore, I is a maximum weight independent set (even a base, actually).

...

Maximum weight independent set via weighted rank

Proof.

- Now, we define λ_i as follows

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- Since we took $v_1, v_2, \dots$ in decreasing order, for all i, and since $w \in \mathbb{R}_+^E$, we have $\lambda_i \geq 0$

□

Linear Program LP

Consider the linear programming primal problem

$$\begin{aligned} \text{maximize} \quad & w^{\intercal} x \\ \text{subject to} \quad & x_v \geq 0 && (v \in V) \\ & x(U) \leq r(U) && (\forall U \subseteq V) \end{aligned} \tag{8.47}$$

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Thanks to strong duality, the solutions to these are equal to each other.

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$$\max w^\intercal x \text{ such that } x \in P_r^+ \tag{8.50}$$

where, again, $P_r^+ = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\right\}$.

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- Therefore, since $P_{\text{ind. set}} \subseteq P_r^+$, the above problem can only have a larger solution. I.e.,

$$\max w^\intercal x \text{ s.t. } x \in P_{\text{ind. set}} \leq \max w^\intercal x \text{ s.t. } x \in P_r^+. \tag{8.51}$$

Polytope equivalence

- Hence, we have the following relations:

$$\max\{w(I) : I \in \mathcal{I}\} \leq \max\{w^\intercal x : x \in P_{\text{ind. set}}\} \quad (8.52)$$

$$\leq \max\{w^\intercal x : x \in P_r^+\} \quad (8.53)$$

$$\stackrel{\text{def}}{=} \alpha_{\text{min}} = \min\left\{\sum_{U \subseteq V} y_U r(U) : y \geq 0, \sum_{U \subseteq V} y_U \mathbf{1}_U \geq w\right\} \quad (8.54)$$

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$$\max\{w(I) : I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i) \tag{8.55}$$

for the chain of U_i's and $\lambda_i \geq 0$ that satisfies $w = \sum_{i=1}^n \lambda_i \mathbf{1}_{U_i}$ (i.e., the r.h.s. of Eq. 8.55 is feasible w.r.t. the dual LP).

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- Therefore, we also have

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- That is, we have just proven:

Theorem 8.6.2

$$P_r^+ = P_{\textit{ind. set}} \tag{8.57}$$

Polytope Equivalence (Summarizing the above)

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I$.

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The LP problem $\max\{w^\intercal x : x \in P_{\text{ind. set}}\}$ can be solved exactly using the greedy algorithm.

Note that this LP problem has an exponential number of constraints (since $P_{\text{ind. set}}$ is described as the intersection of an exponential number of half spaces).

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- This means that if LP problems have certain structure, they can be solved much easier than immediately implied by the equations.

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- What does this look like?

Spanning set polytope

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Theorem 8.6.5

The spanning set polytope is determined by the following equations:

$$0 \le x_e \le 1 \qquad \text{for } e \in E \tag{8.64}$$

$$x(A) \ge r(E) - r(E \setminus A) \qquad \text{for } A \subseteq E \tag{8.65}$$

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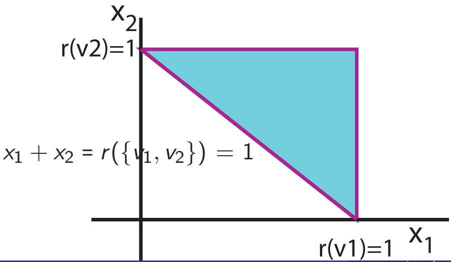
- Example of spanning set polytope in 2D.



Spanning set polytope

Proof.

- Recall that any A is spanning in M iff $E \setminus A$ is independent in M^* (the dual matroid).

...

Spanning set polytope

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- For any $x \in \mathbb{R}^E$, we have that

$$x \in P_{\text{spanning}}(M) \Leftrightarrow 1 - x \in P_{\text{ind. set}}(M^*) \tag{8.66}$$

as we show next ...

...

Spanning set polytope

. . . proof continued.

- This follows since if $x \in P_{\text{spanning}}(M)$, we can represent x as a convex combination:

$$x = \sum_i \lambda_i \mathbf{1}_{A_i} \tag{8.67}$$

where A_i is spanning in M.

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- Consider

$$\mathbf{1} - x = \mathbf{1}_E - x = \mathbf{1}_E - \sum_i \lambda_i \mathbf{1}_{A_i} = \sum_i \lambda_i \mathbf{1}_{E \setminus A_i}, \tag{8.68}$$

which follows since $\sum_i \lambda_i \mathbf{1} = \mathbf{1}_E$, so $\mathbf{1} - x$ is a convex combination of independent sets in M^* and so $\mathbf{1} - x \in P_{\text{ind. set}}(M^*)$.

. . .

Spanning set polytope

... proof continued.

- which means, from the definition of $P_{\text{ind. set}}(M^*)$, that

$$\mathbf{1} - x \geq 0 \tag{8.69}$$

$$\mathbf{1}_A - x(A) = |A| - x(A) \leq r_{M^*}(A) \text{ for } A \subseteq E \tag{8.70}$$

And we know the dual rank function is

$$r_{M^*}(A) = |A| + r_M(E \setminus A) - r_M(E) \tag{8.71}$$

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- giving

$$x(A) \geq r_M(E) - r_M(E \setminus A) \text{ for all } A \subseteq E \tag{8.72}$$

$\square$