Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 7 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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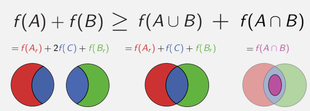

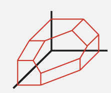

Cumulative Outstanding Reading

- Read chapters 1 and 2, and sections 3.1-3.2 from Fujishige's book.

Announcements, Assignments, and Reminders

- Homework 1 is out, due Wednesday April 23rd, 11:45pm, electronically via our assignment dropbox (`https://canvas.uw.edu/courses/895956/assignments`).
- All homeworks must be done electronically, only PDF file format accepted.
- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, Combinatorial Geometries
- L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
- L9: From Matroid Polytopes to Polymatroids.
- L10: Polymatroids and Submodularity
- L11: More properties of polymatroids, SFM special cases
- L12:
- L13:
- L14:
- L15:
- L16:
- L17:
- L18:
- L19:
- L20:

Finals Week: June 9th-13th, 2014.

System of Distinct Representatives

- Let $(V, \mathcal{V})$ be a set system (i.e., $\mathcal{V} = (V_k : i \in I)$ where $V_i \subseteq V$ for all i), and I is an index set. Hence, $|I| = |\mathcal{V}|$.
- A family $(v_i : i \in I)$ with $v_i \in V$ is said to be a system of distinct representatives of $\mathcal{V}$ if $\exists$ a bijection $\pi : I \leftrightarrow I$ such that $v_i \in V_{\pi(i)}$ and $v_i \neq v_j$ for all $i \neq j$.
- In a system of distinct representatives, there is a requirement for the representatives to be distinct. Lets re-state (and rename) this as a:

Definition 7.2.1 (transversal)

Given a set system $(V, \mathcal{V})$ as defined above, a set $T \subseteq V$ is a transversal of $\mathcal{V}$ if there is a bijection $\pi : T \leftrightarrow I$ such that

$$x \in V_{\pi(x)} \text{ for all } x \in T \tag{7.1}$$

- Note that due to $\pi : T \leftrightarrow I$ being a bijection, all of I and T are "covered" (so this makes things distinct automatically).

When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
- Given a set system $(V, \mathcal{V})$ with $\mathcal{V} = (V_i : i \in I)$, and $V_i \subseteq V$ for all i. Then, for any $J \subseteq I$, let

$$V(J) = \cup_{j \in J} V_j \tag{7.1}$$

so $|V(J)| : 2^I \to \mathbb{Z}_+$ is the set cover func. (we know is submodular).
- We have

Theorem 7.2.1 (Hall's theorem)

Given a set system $(V, \mathcal{V})$, the family of subsets $\mathcal{V} = (V_i : i \in I)$ has a transversal $(v_i : i \in I)$ iff for all $J \subseteq I$

$$|V(J)| \geq |J| \tag{7.2}$$

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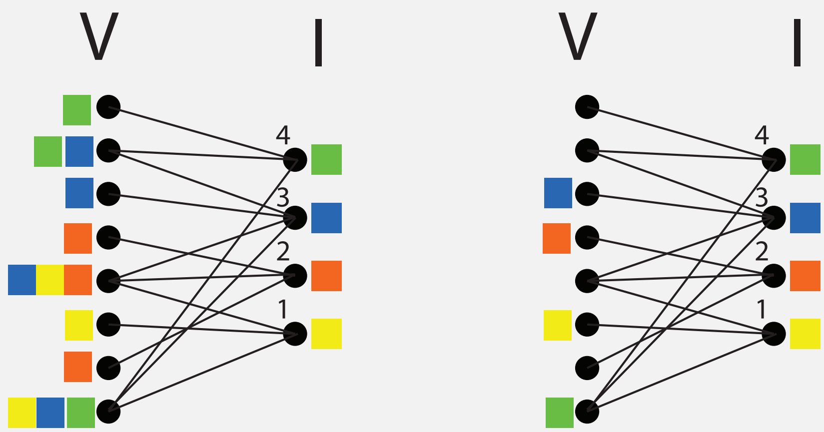
so $|V(J)| : 2^I \to \mathbb{Z}_+$ is the set cover func. (we know is submodular).
- Hall's theorem $(\forall J \subseteq I, |V(J)| \geq |J|)$ as a bipartite graph.



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so $|V(J)| : 2^I \to \mathbb{Z}_+$ is the set cover func. (we know is submodular).
- Moreover, we have

Theorem 7.2.2 (Rado's theorem (1942))

If $M = (V, r)$ is a matroid on V with rank function r, then the family of subsets $(V_i : i \in I)$ of V has a transversal $(v_i : i \in I)$ that is independent in M iff for all $J \subseteq I$

$$r(V(J)) \geq |J| \tag{7.3}$$

- Note, a transversal T independent in M means that $r(T) = |T|$.

Application's of Hall's theorem

- Consider a set of jobs I and a set of applicants V to the jobs. If an applicant $v \in V$ is qualified for job $i \in I$, we add edge (v, i) to the bipartite graph $G = (V, I, E)$.
- We wish all jobs to be filled, and hence Hall's condition ($\forall J \subseteq I, |V(J)| \geq |J|$) is a necessary and sufficient condition for this to be possible.
- Note if $|V| = |I|$, then Hall's theorem is the Marriage Theorem (Frobenious 1917), where an edge (v, i) in the graph indicate compatibility between two individuals $v \in V$ and $i \in I$ coming from two separate groups V and I.
- If $\forall J \subseteq I, |V(J)| \geq |J|$, then all individuals in each group can be matched with a compatible mate.

More general conditions for existence of transversals

Theorem 7.2.1 (Polymatroid transversal theorem)

If $\mathcal{V} = (V_i : i \in I)$ is a finite family of non-empty subsets of V, and $f : 2^V \to \mathbb{Z}_+$ is a non-negative, integral, monotone non-decreasing, and submodular function, then $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that

$$f(\cup_{i \in J}\{v_i\}) \geq |J| \text{ for all } J \subseteq I \tag{7.1}$$

if and only if

$$f(V(J)) \geq |J| \text{ for all } J \subseteq I \tag{7.2}$$

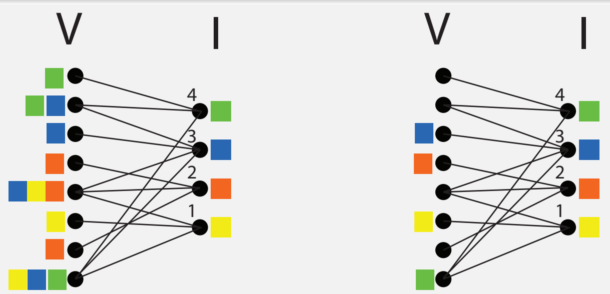
- Given Theorem 7.2.1, we immediately get Theorem 7.2.1 by taking $f(S) = |S|$ for $S \subseteq V$. *In which case, Eq. 7.7 requires the system of representatives to be distinct.*
- We get Theorem 7.2.2 by taking $f(S) = r(S)$ for $S \subseteq V$, the rank function of the matroid. *where, Eq. 7.7 insists the system of representatives*

Transversals and Bipartite Matchings

- Transversals correspond exactly to matchings in bipartite graphs.
- Given a set system $(V, \mathcal{V})$, with $\mathcal{V} = (V_i : i \in I)$, we can define a bipartite graph $G = (V, I, E)$ associated with $\mathcal{V}$ that has edge set $\{(v, i) : v \in V, i \in I, v \in V_i\}$.
- A matching in this graph is a set of edges no two of which that have a common endpoint. In fact, we easily have:

Lemma 7.2.4

A subset $T \subseteq V$ is a partial transversal of $\mathcal{V}$ iff there is a matching in (V, I, E) in which every edge has one endpoint in T (T matched into I).



Partial Transversals Are Independent Sets in a Matroid

In fact, we have

Theorem 7.2.4

Let $(V, \mathcal{V})$ where $\mathcal{V} = (V_1, V_2, \ldots, V_\ell)$ be a subset system. Let $I = \{1, \ldots, \ell\}$. Let $\mathcal{I}$ be the set of partial transversals of $\mathcal{V}$. Then $(V, \mathcal{I})$ is a matroid.

Proof.

- We note that $\emptyset \in \mathcal{I}$ since the empty set is a transversal of the empty subfamily of $\mathcal{V}$, thus (I1') holds.
- We already saw that if T is a partial transversal of $\mathcal{V}$, and if $T' \subseteq T$, then T' is also a partial transversal. So (I2') holds.
- Suppose that T_1 and T_2 are partial transversals of $\mathcal{V}$ such that $|T_1| < |T_2|$. Exercise: show that (I3') holds.

□

Representable

Definition 7.2.4 (Matroid isomorphism)

Two matroids M_1 and M_2 respectively on ground sets V_1 and V_2 are isomorphic if there is a bijection $\pi : V_1 \to V_2$ which preserves independence (equivalently, rank, circuits, and so on).

- Let $\mathbb{F}$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $\mathbb{F}$, such as a Galois field GF(p) where p is prime (such as GF(2)).
  Succinctly: A field is a set with $+$, $*$, closure, associativity, commutativity, and additive and multiplictaive identities and inverses.
- We can more generally define matroids on a field.

Definition 7.2.5 (linear matroids on a field)

Let $\mathbf{X}$ be an $n \times m$ matrix and $E = \{1, \dots, m\}$, where $\mathbf{X}_{ij} \in \mathbb{F}$ for some field, and let $\mathcal{I}$ be the set of subsets of E such that the columns of $\mathbf{X}$ are linearly independent over $\mathbb{F}$.

Logistics
Review

Representable

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  Succinctly: A field is a set with $+$, $*$, closure, associativity, commutativity, and additive and multiplictaive identities and inverses.
- We can more generally define matroids on a field.

Definition 7.2.6 (representable (as a linear matroid))

Any matroid isomorphic to a linear matroid on a field is called representable over $\mathbb{F}$

Matroids, other definitions using matroid rank $r : 2^V \to \mathbb{Z}_+$

Definition 7.2.1 (closed/flat/subspace)

A subset $A \subseteq E$ is closed (equivalently, a flat or a subspace) of matroid M if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

A hyperplane is a flat of rank $r(M) - 1$.

Definition 7.2.2 (closure)

Given $A \subseteq E$, the closure (or span) of A, is defined by $\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}$.

Therefore, a closed set A has $\text{span}(A) = A$.

Definition 7.2.3 (circuit)

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise-minimal dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

Spanning Sets

- We have the following definitions:

Definition 7.2.6 (spanning set of a set)

Given a matroid $\mathcal{M} = (V, \mathcal{I})$, and a set $Y \subseteq V$, then any set $X \subseteq Y$ such that $r(X) = r(Y)$ is called a spanning set of Y.

Definition 7.2.7 (spanning set of a matroid)

Given a matroid $\mathcal{M} = (V, \mathcal{I})$, any set $A \subseteq V$ such that $r(A) = r(V)$ is called a spanning set of the matroid.

- A base of a matroid is a minimal spanning set (and it is independent) but supersets of a base are also spanning.
- V is always trivially spanning.
- Consider the terminology: "spanning tree in a graph", comes from spanning in a matroid sense.

Dual of a Matroid

- Given a matroid $M = (V, \mathcal{I})$, a dual matroid $M^* = (V, \mathcal{I}^*)$ can be defined on the same ground set V, but using a very different set of independent sets $\mathcal{I}^*$.
- We define the set of sets $\mathcal{I}^*$ for M^* as follows:

$$\mathcal{I}^* = \{A \subseteq V : V \setminus A \text{ is a spanning set of } M\} \tag{7.21}$$

- That is, a set A is independent in the dual matroid M^* if removal of A from V does not decrease the rank in M:

$$\mathcal{I}^* = \{A \subseteq V : \text{rank}_M(V \setminus A) = \text{rank}_M(V)\} \tag{7.22}$$

- In other words, a set $A \subseteq V$ is independent in the dual M^* (i.e., $A \in \mathcal{I}^*$) if its complement is spanning in M (residual $V \setminus A$ must contain a base in M).
- Dual of the dual: Note, we have that $(M^*)^* = M$.

Dual of a Matroid: Bases

- Since the smallest spanning sets are bases, the bases of M (when $V \setminus I$ is as small as possible while still spanning) are complements of the bases of M^* (where I is as large as possible while still being independent).
- In fact, we have that

Theorem 7.3.1 (Dual matroid bases)

Let $M = (V, \mathcal{I})$ be a matroid and $\mathcal{B}(M)$ be the set of bases of M. Then define

$$\mathcal{B}^*(M) = \{V \setminus B : B \in \mathcal{B}(M)\}. \tag{7.1}$$

Then $\mathcal{B}^*(M)$ is the set of basis of M^* (that is, $\mathcal{B}^*(M) = \mathcal{B}(M^*)$.

Dual of a Matroid: Terminology

- $\mathcal{B}^*(M)$, the bases of M^*, are called cobases of M.
- The circuits of M^* are called cocircuits of M.
- The hyperplanes of M^* are called cohyperplanes of M.
- The independent sets of M^* are called coindependent sets of M.
- The spanning sets of M^* are called cospanning sets of M.

Proposition 7.3.2 (from Oxley 2011)

Let $M = (V, \mathcal{I})$ be a matroid, and let $X \subseteq V$. Then

1. *X is independent in M iff $V \setminus X$ is cospanning in M (spanning in M^*).*
2. *X is spanning in M iff $V \setminus X$ is coindependent in M (independent in M^*).*
3. *X is a hyperplane in M iff $V \setminus X$ is a cocircuit in M (circuit in M^*).*
4. *X is a circuit in M iff $V \setminus X$ is a cohyperplane in M (hyperplane in M^*).*

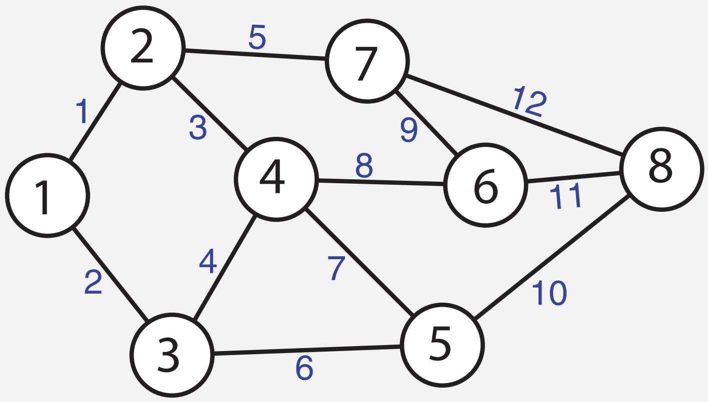
Example duality: graphic matroid

- Using a graphic/cycle matroid, we can already see how dual matroid concepts demonstrates the extraordinary flexibility and power that a matroid can have.
- Recall, in cycle matroid, a spanning set of G is any set of edges that are incident to all nodes (i.e., any superset of a spanning forest), a minimal spanning set is a spanning tree, and a circuit has a nice visual interpretation (a cycle in the graph).
- A cut in a graph G is a set of edges, the removal of which increases the number of connected components. I.e., $X \subseteq E(G)$ is a cut in G if $k(G) < k(G \setminus X)$.
- A minimal cut in G is a cut $X \subseteq E(G)$ such that $X \setminus \{x\}$ is not a cut for any $x \in X$.
- A cocycle (a cocircuit in a graphic matroid) is a “minimal cut” in the graph. Cocycle matroid sometimes called a “cut matroid”.
- All dependent sets in a cocycle matroid are cuts (i.e., a dependent set is a minimal cut or contains one).

Example: cocycle matroid (sometimes "cut matroid")

- The dual of the cycle matroid is called the cocycle matroid. Recall, $\mathcal{I}^* = \{A \subseteq V : V \setminus A \text{ is a spanning set of } M\}$
- It consists of all sets of edges the complement of which contains a spanning tree — i.e., an independent set can't consist of edges that, if removed, would render the graph non-spanning.

A graph G



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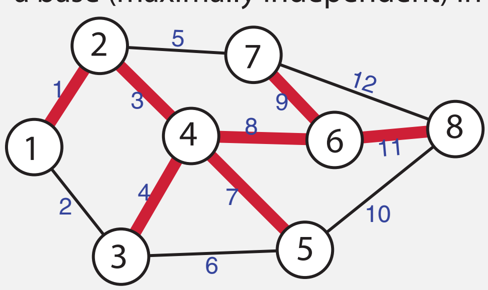
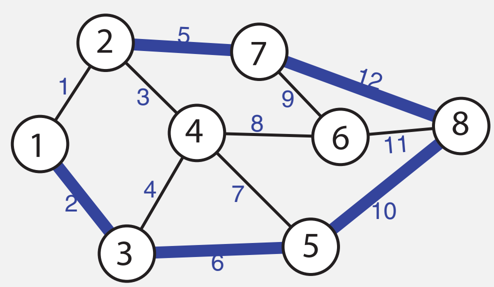
Minimally spanning in M (and thus a base (maximally independent) in M)



Maximally independent in M* (thus a base, minimally spanning, in M*)



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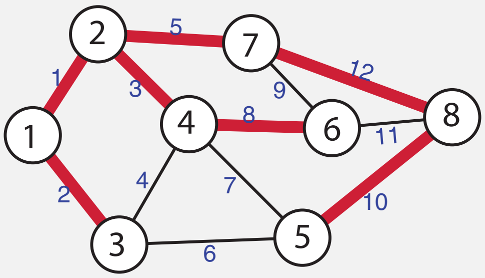

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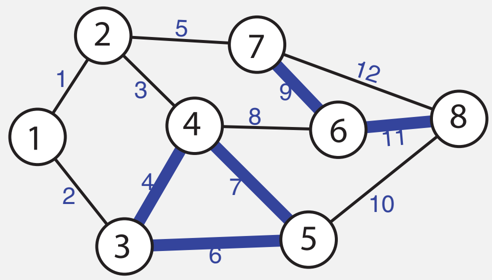

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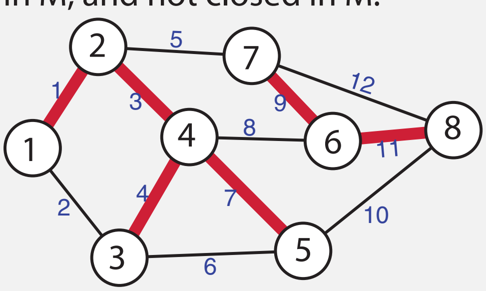
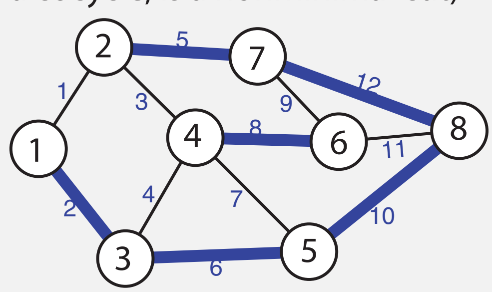
Independent but not spanning in M, and not closed in M.



Dependent in M* (contains a cocycle, is a nonminimal cut)



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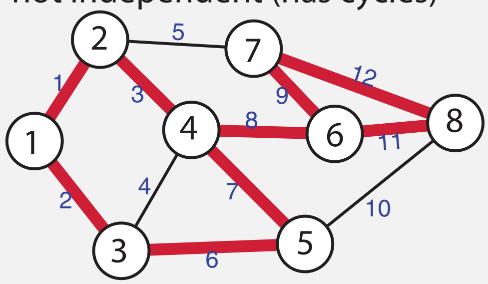
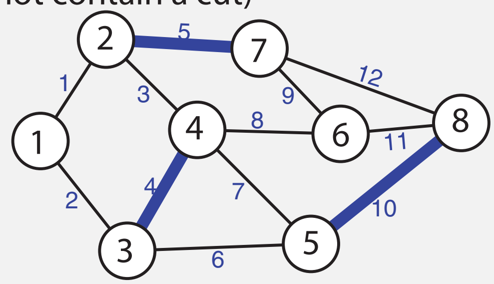
Spanning in M, but not a base, and not independent (has cycles)



Independent in M* (does not contain a cut)



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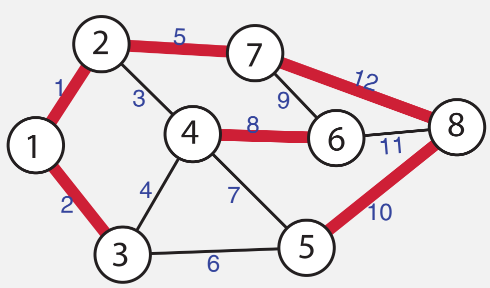

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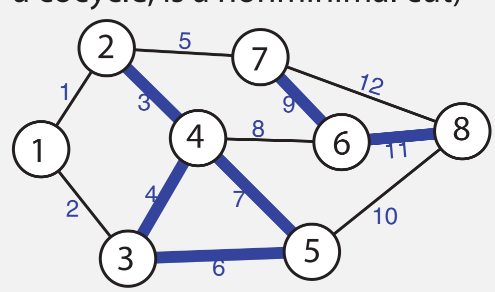

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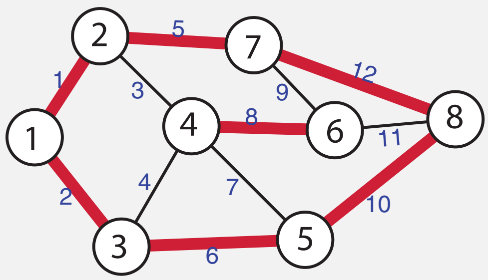
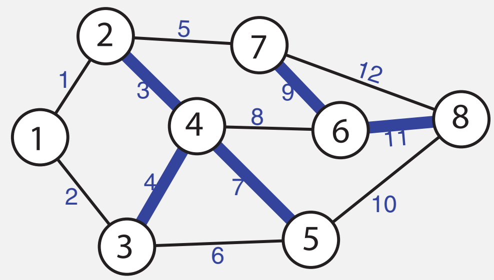
A hyperplane in M, dependent but not spanning in M



A cycle in M* (minimally dependent in M*, a cocycle, or a minimal cut)



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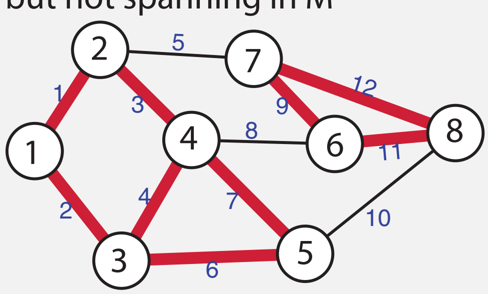

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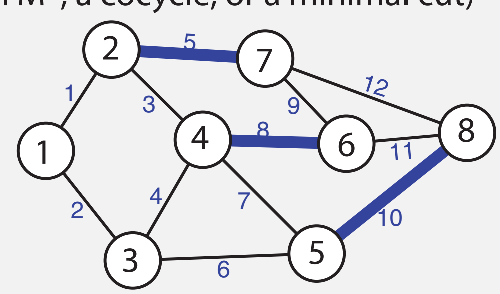

The dual of a matroid is (indeed) a matroid

Theorem 7.3.3

Given matroid $M = (V, \mathcal{I})$, let $M^* = (V, \mathcal{I}^*)$ be as previously defined. Then M^* is a matroid.

Proof.

- Clearly $\emptyset \in I^*$, so (I1') holds.
- Also, if $I \subseteq J \in \mathcal{I}^*$, then clearly also $I \in \mathcal{I}^*$ since if $V \setminus J$ is spanning in M, so must $V \setminus I$. Therefore, (I2') holds.

...

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Proof.

- Consider $I, J \in \mathcal{I}^*$ with $|I| < |J|$. We need to show that there is some member $v \in J \setminus I$ such that $I + v$ is independent in M^*, which means that $V \setminus (I + v) = (V \setminus I) \setminus v$ is still spanning in M. That is, removing v from $V \setminus I$ doesn't make $(V \setminus I) \setminus v$ not spanning.
- Since $V \setminus J$ is spanning in M, $V \setminus J$ contains some base (say $B \subseteq V \setminus J$) of M. Also, $V \setminus I$ contains a base of M, say $B' \subseteq V \setminus I$.
- Since $B \setminus I \subseteq V \setminus I$, and $B \setminus I$ is independent in M, we can choose the base B' of M s.t. $B \setminus I \subseteq B' \subseteq V \setminus I$.
- Since B and J are disjoint, we have both: 1) $B \setminus I$ and $J \setminus I$ are disjoint; and 2) $B \cap I \subseteq I \setminus J$. Also note, B' and I are disjoint.

...

The dual of a matroid is (indeed) a matroid

Theorem 7.3.3

Given matroid $M = (V, \mathcal{I})$, let $M^* = (V, \mathcal{I}^*)$ be as previously defined. Then M^* is a matroid.

Proof.

- Now $J \setminus I \not\subseteq B'$, since otherwise (i.e., assuming $J \setminus I \subseteq B'$):

$$|B| = |B \cap I| + |B \setminus I| \tag{7.2}$$
$$\leq |I \setminus J| + |B \setminus I| \tag{7.3}$$
$$< |J \setminus I| + |B \setminus I| \leq |B'| \tag{7.4}$$

  which is a contradiction. *The last inequality on the right follows since $J \setminus I \subseteq B'$ (by assumption) and $B \setminus I \subseteq B'$ implies that $(J \setminus I) \cup (B \setminus I) \subseteq B'$, but since J and B are disjoint, we have that $|J \setminus I| + |B \setminus I| \leq B'$.*
- Therefore, $J \setminus I \not\subseteq B'$, and there is a $v \in J \setminus I$ s.t. $v \notin B'$.
- So B' is disjoint with $I \cup \{v\}$, means $B' \subseteq V \setminus (I \cup \{v\})$, or

Matroid Duals and Representability

Theorem 7.3.4

Let M be an $\mathbb{F}$-representable matroid (i.e., one that can be represented by a finite sized matrix over field $\mathbb{F}$). Then M^* is also $\mathbb{F}$-representable.

Theorem 7.3.5

Let M be a graphic matroid (i.e., one that can be represented by a graph $G = (V, E)$). Then M^* is not necessarily also graphic.

Dual Matroid Rank

Theorem 7.3.6

The rank function r_{M^*} of the dual matroid M^* may be specified in terms of the rank r_M in matroid M as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \tag{7.5}$$

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2. *I.e., $|X|$ is modular, complement $f(V \setminus X)$ is submodular if f is submodular, $r_M(V)$ is a constant, and summing submodular functions and a constant preserves submodularity.*
- Non-negativity integral follows since $|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V)$. *The right inequality follows since r_M is submodular.*
- Monotone non-decreasing follows since, as X increases by one, $|X|$ always increases by 1, while $r_M(V \setminus X)$ decreases by one or zero.
- Therefore, r_{M^*} is the rank function of a matroid. That it is the dual

Dual Matroid Rank

Theorem 7.3.6

The rank function r_{M^*} of the dual matroid M^* may be specified in terms of the rank r_M in matroid M as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \tag{7.5}$$

Proof.

A set X is independent in (V, r_{M^*}) if and only if

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) = |X| \tag{7.6}$$

or

$$r_M(V \setminus X) = r_M(V) \tag{7.7}$$

But a subset X is independent in M^* only if $V \setminus X$ is spanning in M (by the definition of the dual matroid). □

Matroid restriction/deletion

- Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then

$$\mathcal{I}_Y = \{Z : Z \subseteq Y, Z \in \mathcal{I}\} \tag{7.8}$$

  is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid with rank $r(M_Y) = r(Y)$.
- This is called the restriction of M to Y, and is often written $M|Y$.
- If $Y = V \setminus X$, then we have

$$\mathcal{I}_Y = \{Z : Z \cap X = \emptyset, Z \in \mathcal{I}\} \tag{7.9}$$

  is considered a deletion of X from M, and is often written $M \setminus Z$.
- Hence, $M|Y = M \setminus (V \setminus Y)$.
- The rank function is of the same form. I.e., $r_Y : 2^Y \to \mathbb{Z}_+$, where $r_Y(Z) = r(Z)$ for $Z \subseteq Y$.

Matroid contraction

- Contraction is dual to deletion, and is like a forced inclusion of contained base, but with a similar ground set removal. Contracting Z is written M/Z.
- Let $Z \subseteq V$ and let X be a base of Z. Then a subset I of $V \setminus Z$ is independent in M/Z iff $I \cup X$ is independent in M.
- In fact, it is the case $M/Z = (M^* \setminus Z)^*$ (Exercise: show why).
- The rank function takes the form

$$r_{M/Z}(Y) = r(Y \cup Z) - r(Z) = r(Y|Z) \tag{7.10}$$

- So given $I \subseteq V \setminus Z$ and X is a base of Z, $r_{M/Z}(I) = |I|$ is identical to $r(I \cup Z) = |I| + r(Z) = |I| + |X|$ but $r(I \cup Z) = r(I \cup X)$. This implies $r(I \cup X) = |I| + |X|$, or $I \cup X$ is independent in M.
- A minor of a matroid is any matroid obtained via a series of deletions and contractions of some matroid.

Matroid Intersection

- Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$.
- While $(V, \mathcal{I}_1 \cap \mathcal{I}_2)$ is typically not a matroid (Exercise: show graphical example.), we might be interested in finding the maximum size common independent set. That is, find $\max |X|$ such that both $X \in \mathcal{I}_1$ and $X \in \mathcal{I}_2$.

Theorem 7.4.1

Let M_1 and M_2 be given as above, with rank functions r_1 and r_2. Then the size of the maximum size set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by

$$(r_1 * r_2)(V) \triangleq \min_{X \subseteq V} \Big(r_1(X) + r_2(V \setminus X) \Big) \tag{7.11}$$

This is an instance of the convolution of two submodular functions, f_1 and f_2 that, evaluated at $Y \subseteq V$, is written as:

$$(f_1 * f_2)(Y) = \min_{X \subseteq Y} \Big(f_1(X) + f_2(Y \setminus X) \Big) \tag{7.12}$$

Convolution and Hall's Theorem

- Recall Hall's theorem, that a transversal exists iff for all $X \subseteq V$, we have $|\Gamma(X)| \geq |X|$.
- $\Leftrightarrow \quad |\Gamma(X)| - |X| \geq 0, \forall X$
- $\Leftrightarrow \quad \min_X |\Gamma(X)| - |X| \geq 0$
- $\Leftrightarrow \quad \min_X |\Gamma(X)| + |V| - |X| \geq |V|$
- $\Leftrightarrow \quad \min_X \Big(|\Gamma(X)| + |V \setminus X| \Big) \geq |V|$
- $\Leftrightarrow \quad [\Gamma(\cdot) * |\cdot|](V) \geq |V|$
- So Hall's theorem can be expressed as convolution.
- Note, in general, convolution of two submodular functions does not preserve submodularity (but in certain special cases it does).

Matroid Union

Definition 7.4.2

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, $\dots$, $M_k = (V_k, \mathcal{I}_k)$ be matroids. We define the union of matroids as $M_1 \vee M_2 \vee \dots \vee M_k = (V_1 \uplus V_2 \uplus \dots \uplus V_k, \mathcal{I}_1 \vee \mathcal{I}_2 \vee \dots \vee \mathcal{I}_k)$, where

$$I_1 \vee \mathcal{I}_2 \vee \dots \vee \mathcal{I}_k = \{I_1 \uplus I_2 \uplus \dots \uplus I_k | I_1 \in \mathcal{I}_1, \dots, I_k \in \mathcal{I}_k\} \quad (7.13)$$

Note $A \uplus B$ designates the disjoint union of A and B.

Theorem 7.4.3

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, $\dots$, $M_k = (V_k, \mathcal{I}_k)$ be matroids, with rank functions $r_1, \dots, r_k$. Then the union of these matroids is still a matroid, having rank function

$$r(Y) = \min_{X \subseteq Y} \Big(|Y \setminus X| + r_1(X \cap V_1) + \dots + r_k(X \cap V_k) \Big) \quad (7.14)$$

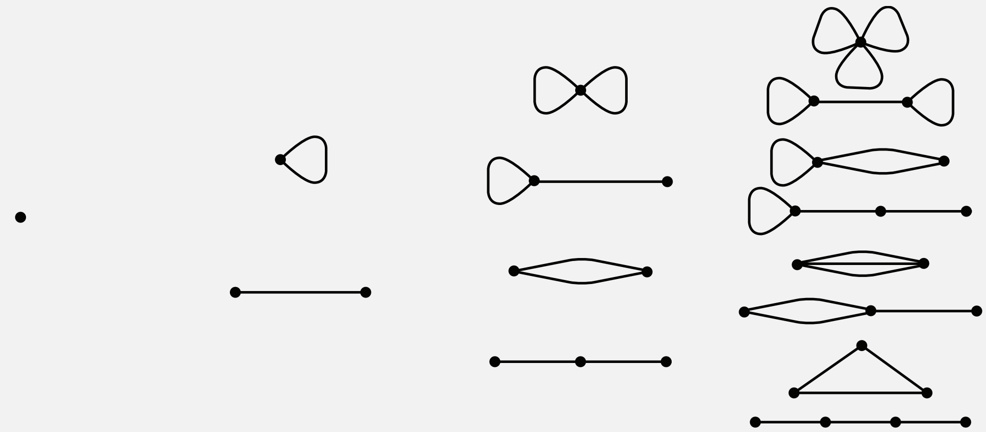
for any $Y \subseteq V_1 \cup \dots V_k$.

Exercise: Matroid Union, and Matroid duality

Exercise: Describe $M \vee M^*$.

Matroids of three or fewer elements are graphic

- All matroids up to and including three elements are graphic.



(a) The only matroid with zero elements. (b) The two one-element matroids. (c) The four two-element matroids. (d) The eight three-element matroids.

- This is a nice way to show matroids with low ground set sizes. What about matroids that are low rank but with many elements?

Affine Matroids

- Given an $n \times m$ matrix with entries over some field $\mathbb{F}$, we say that a subset $S \subseteq \{1, \dots, m\}$ of indices (with corresponding column vectors $\{v_i : i \in S\}$, with $|S| = k$) is affinely dependent if $m \geq 1$ and there exists elements $\{a_1, \dots, a_k\} \in \mathbb{F}$, not all zero with $\sum_{i=1}^k a_i = 0$, such that $\sum_{i=1}^k a_i v_i = 0$.
- Otherwise, the set is called affinely independent.
- Concisely: points $\{v_1, v_2, \dots, v_k\}$ are affinely independent if $v_2 - v_1, v_3 - v_1, \dots, v_k - v_1$ are linearly independent.
- Example: in 2D, three collinear points are affinely dependent, three non-collear points are affinely independent, and ≥ 4 non-collinear points are affinely dependent.

Proposition 7.5.1 (affine matroid)

Let ground set $E = \{1, \dots, m\}$ index column vectors of a matrix, and let $\mathcal{I}$ be the set of subsets X of E such that X indices affinely independent vectors. Then $(E, \mathcal{I})$ is a matroid.

Exercise: prove this.

Euclidean Representation of Low-rank Matroids

- Consider the affine matroid with $n \times m = 2 \times 6$ matrix on the field $\mathbb{F} = \mathbb{R}$, and let the elements be $\{(0,0),(1,0),(2,0),(0,1),(0,2),(1,1)\}$.
- We can plot the points in $\mathbb{R}^2$ as on the right:
- Points have rank 1, lines have rank 2, planes have rank 3.
- Flats (points, lines, planes, etc.) have rank equal to one more than their geometric dimension.
- Any two points constitute a line, but lines with only two points are not drawn.
- Lines indicate collinear sets with ≥ 3 points, while any two points have rank 2.
- Dependent sets consist of all subsets with ≥ 4 elements (rank 3), or 3 collinear elements (rank 2). Any two points have rank 2.