Submodular Functions, Optimization, 
and Applications to Machine Learning
— Spring Quarter, Lecture 7 —
http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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\[ f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \]
Cumulative Outstanding Reading

- Read chapters 1 and 2, and sections 3.1-3.2 from Fujishige’s book.
Announcements, Assignments, and Reminders

- Homework 1 is out, due Wednesday April 23rd, 11:45pm, electronically via our assignment dropbox (https://canvas.uw.edu/courses/895956/assignments).
- All homeworks must be done electronically, only PDF file format accepted.
- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).
Class Road Map - IT-I

- **L1** (3/31): Motivation, Applications, & Basic Definitions
- **L2**: (4/2): Applications, Basic Definitions, Properties
- **L3**: More examples and properties (e.g., closure properties), and examples, spanning trees
- **L4**: proofs of equivalent definitions, independence, start matroids
- **L5**: matroids, basic definitions and examples
- **L6**: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- **L7**: Dual Matroids, other matroid properties, matroids and greedy
- **L8**:
- **L9**:
- **L10**:
- **L11**:
- **L12**:
- **L13**:
- **L14**:
- **L15**:
- **L16**:
- **L17**:
- **L18**:
- **L19**:
- **L20**:

**Finals Week**: June 9th-13th, 2014.
System of Distinct Representatives

- Let \((V, \mathcal{V})\) be a set system (i.e., \(\mathcal{V} = (V_i : i \in I)\) where \(V_i \subseteq V\) for all \(i\)), and \(I\) is an index set. Hence, \(|I| = |\mathcal{V}|\).

- A family \((v_i : i \in I)\) with \(v_i \in V\) is said to be a system of distinct representatives of \(\mathcal{V}\) if \(\exists\) a bijection \(\pi : I \leftrightarrow I\) such that \(v_i \in V_{\pi(i)}\) and \(v_i \neq v_j\) for all \(i \neq j\).

- In a system of distinct representatives, there is a requirement for the representatives to be distinct. Let's re-state (and rename) this as a:

**Definition 7.2.1 (transversal)**

Given a set system \((V, \mathcal{V})\) as defined above, a set \(T \subseteq V\) is a transversal of \(\mathcal{V}\) if there is a bijection \(\pi : T \leftrightarrow I\) such that

\[
x \in V_{\pi(x)} \text{ for all } x \in T
\]

- Note that due to \(\pi : T \leftrightarrow I\) being a bijection, all of \(I\) and \(T\) are "covered" (so this makes things distinct automatically).
When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
- Given a set system \((V, \mathcal{V})\) with \(\mathcal{V} = (V_i : i \in I)\), and \(V_i \subseteq V\) for all \(i\). Then, for any \(J \subseteq I\), let

\[
V(J) = \bigcup_{j \in J} V_j
\]  

(7.1)

so \(|V(J)| : 2^I \to \mathbb{Z}_+\) is the set cover func. (we know is submodular).
- We have

**Theorem 7.2.1 (Hall’s theorem)**

*Given a set system \((V, \mathcal{V})\), the family of subsets \(\mathcal{V} = (V_i : i \in I)\) has a transversal \((v_i : i \in I)\) iff for all \(J \subseteq I\)*

\[
|V(J)| \geq |J|
\]  

(7.2)
When do transversals exist?

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- Hall’s theorem (\(\forall J \subseteq I, |V(J)| \geq |J|\)) as a bipartite graph.
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- Given a set system \((V, \mathcal{V})\) with \(\mathcal{V} = (V_i : i \in I)\), and \(V_i \subseteq V\) for all \(i\). Then, for any \(J \subseteq I\), let

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V(J) = \bigcup_{j \in J} V_j
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so \(|V(J)| : \mathcal{2}^I \to \mathbb{Z}_+\) is the set cover func. (we know is submodular).
- Hall’s theorem (\(\forall J \subseteq I, |V(J)| \geq |J|\)) as a bipartite graph.
When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
- Given a set system \((V, \mathcal{V})\) with \(\mathcal{V} = (V_i : i \in I)\), and \(V_i \subseteq V\) for all \(i\). Then, for any \(J \subseteq I\), let

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V(J) = \bigcup_{j \in J} V_j
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so \(|V(J)| : 2^I \to \mathbb{Z}_+\) is the set cover func. (we know is submodular).

Moreover, we have

**Theorem 7.2.2 (Rado’s theorem (1942))**

If \(M = (V, r)\) is a matroid on \(V\) with rank function \(r\), then the family of subsets \((V_i : i \in I)\) of \(V\) has a transversal \((v_i : i \in I)\) that is independent in \(M\) iff for all \(J \subseteq I\)

\[
r(V(J)) \geq |J|
\]

(7.3)

- Note, a transversal \(T\) independent in \(M\) means that \(r(T) = |T|\).
Consider a set of jobs $I$ and a set of applicants $V$ to the jobs. If an applicant $v \in V$ is qualified for job $i \in I$, we add edge $(v, i)$ to the bipartite graph $G = (V, I, E)$.
Application’s of Hall’s theorem

- Consider a set of jobs $I$ and a set of applicants $V$ to the jobs. If an applicant $v \in V$ is qualified for job $i \in I$, we add edge $(v, i)$ to the bipartite graph $G = (V, I, E)$.

- We wish all jobs to be filled, and hence Hall’s condition $(\forall J \subseteq I, |V(J)| \geq |J|)$ is a necessary and sufficient condition for this to be possible.
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We wish all jobs to be filled, and hence Hall’s condition $(\forall J \subseteq I, |V(J)| \geq |J|)$ is a necessary and sufficient condition for this to be possible.

Note if $|V| = |I|$, then Hall’s theorem is the Marriage Theorem (Frobenious 1917), where an edge $(v, i)$ in the graph indicate compatibility between two individuals $v \in V$ and $i \in I$ coming from two separate groups $V$ and $I$. 
Consider a set of jobs $I$ and a set of applicants $V$ to the jobs. If an applicant $v \in V$ is qualified for job $i \in I$, we add edge $(v, i)$ to the bipartite graph $G = (V, I, E)$.

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If $\forall J \subseteq I, |V(J)| \geq |J|$, then all individuals in each group can be matched with a compatible mate.
More general conditions for existence of transversals

Theorem 7.2.1 (Polymatroid transversal theorem)

If \( \mathcal{V} = (V_i : i \in I) \) is a finite family of non-empty subsets of \( V \), and \( f : 2^V \rightarrow \mathbb{Z}_+ \) is a non-negative, integral, monotone non-decreasing, and submodular function, then \( \mathcal{V} \) has a system of representatives \( (v_i : i \in I) \) such that

\[
f(\bigcup_{i \in J} \{v_i\}) \geq |J| \quad \text{for all } J \subseteq I \tag{7.1}
\]

if and only if

\[
f(\mathcal{V}(J)) \geq |J| \quad \text{for all } J \subseteq I \tag{7.2}
\]

- Given Theorem ??, we immediately get Theorem 7.2.1 by taking \( f(S) = |S| \) for \( S \subseteq V \).
- We get Theorem ?? by taking \( f(S) = r(S) \) for \( S \subseteq V \), the rank function of the matroid.
Transversals and Bipartite Matchings

- Transversals correspond exactly to matchings in bipartite graphs.
- Given a set system \((V, \mathcal{V})\), with \(\mathcal{V} = (V_i : i \in I)\), we can define a bipartite graph \(G = (V, I, E)\) associated with \(\mathcal{V}\) that has edge set \(\{(v, i) : v \in V, i \in I, v \in V_i\}\).
- A matching in this graph is a set of edges no two of which that have a common endpoint. In fact, we easily have:

**Lemma 7.2.4**

A subset \(T \subseteq V\) is a partial transversal of \(\mathcal{V}\) iff there is a matching in \((V, I, E)\) in which every edge has one endpoint in \(T\) \((T\) matched into \(I)\).
In fact, we have

**Theorem 7.2.4**

Let \((V, \mathcal{V})\) where \(\mathcal{V} = (V_1, V_2, \ldots, V_\ell)\) be a subset system. Let \(I = \{1, \ldots, \ell\}\). Let \(\mathcal{I}\) be the set of partial transversals of \(\mathcal{V}\). Then \((V, \mathcal{I})\) is a matroid.

**Proof.**

- We note that \(\emptyset \in \mathcal{I}\) since the empty set is a transversal of the empty subfamily of \(\mathcal{V}\), thus \((I1')\) holds.
- We already saw that if \(T\) is a partial transversal of \(\mathcal{V}\), and if \(T' \subseteq T\), then \(T'\) is also a partial transversal. So \((I2')\) holds.
- Suppose that \(T_1\) and \(T_2\) are partial transversals of \(\mathcal{V}\) such that \(|T_1| < |T_2|\). Exercise: show that \((I3')\) holds.
Definition 7.2.4 (Matroid isomorphism)

Two matroids $M_1$ and $M_2$ respectively on ground sets $V_1$ and $V_2$ are isomorphic if there is a bijection $\pi : V_1 \rightarrow V_2$ which preserves independence (equivalently, rank, circuits, and so on).

- Let $F$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $F$, such as a Galois field GF($p$) where $p$ is prime (such as GF(2)). Succinctly: A field is a set with $+$, $\times$, closure, associativity, commutativity, and additive and multiplicative identities and inverses.
- We can more generally define matroids on a field.

Definition 7.2.6 (representable (as a linear matroid))

Any matroid isomorphic to a linear matroid on a field is called representable over $F$. $F$-representable.
Logistics

Matroids, other definitions using matroid rank $r : 2^V \rightarrow \mathbb{Z}_+$

**Definition 7.2.1 (closed/flat/subspace)**

A subset $A \subseteq E$ is closed (equivalently, a flat or a subspace) of matroid $M$ if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

A hyperplane is a flat of rank $r(M) - 1$.

**Definition 7.2.2 (closure)**

Given $A \subseteq E$, the closure (or span) of $A$, is defined by

$$\text{span}(A) = \{ b \in E : r(A \cup \{b\}) = r(A) \}.$$  

Therefore, a closed set $A$ has $\text{span}(A) = A$.

**Definition 7.2.3 (circuit)**

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise-minimal dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).
Spanning Sets

- We have the following definitions:

**Definition 7.2.6 (spanning set of a set)**
Given a matroid \( \mathcal{M} = (V, \mathcal{I}) \), and a set \( Y \subseteq V \), then any set \( X \subseteq Y \) such that \( r(X) = r(Y) \) is called a **spanning set** of \( Y \).

**Definition 7.2.7 (spanning set of a matroid)**
Given a matroid \( \mathcal{M} = (V, \mathcal{I}) \), any set \( A \subseteq V \) such that \( r(A) = r(V) \) is called a **spanning set** of the matroid.

- A base of a matroid is a minimal spanning set (and it is independent) but supersets of a base are also spanning.
- \( V \) is always trivially spanning.
- Consider the terminology: “spanning tree in a graph”, comes from spanning in a matroid sense.
Dual of a Matroid

- Given a matroid $M = (V, \mathcal{I})$, a dual matroid $M^* = (V, \mathcal{I}^*)$ can be defined on the same ground set $V$, but using a very different set of independent sets $\mathcal{I}^*$.

- We define the set of sets $\mathcal{I}^*$ for $M^*$ as follows:

$$\mathcal{I}^* = \{ A \subseteq V : V \setminus A \text{ is a spanning set of } M \} \quad (7.21)$$

- That is, a set $A$ is independent in the dual matroid $M^*$ if removal of $A$ from $V$ does not decrease the rank in $M$:

$$\mathcal{I}^* = \{ A \subseteq V : \text{rank}_M(V \setminus A) = \text{rank}_M(V) \} \quad (7.22)$$

- In other words, a set $A \subseteq V$ is independent in the dual $M^*$ (i.e., $A \in \mathcal{I}^*$) if its complement is spanning in $M$ (residual $V \setminus A$ must contain a base in $M$).

- Dual of the dual: Note, we have that $(M^*)^* = M$.

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EE596b/Spring 2014/Submodularity - Lecture 7 - April 21st, 2014
F14/44 (pg.20/174)
Dual of a Matroid: Bases

Since the smallest spanning sets are bases, the bases of $M$ (when $V \setminus I$ is as small as possible while still spanning) are complements of the bases of $M^*$ (where $I$ is as large as possible while still being independent).
Dual of a Matroid: Bases

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- In fact, we have that

[Theorem 7.3.1 (Dual matroid bases)]

Let $M = (V, I)$ be a matroid and $B(M)$ be the set of bases of $M$. Then

$$B^*(M) = \{V \cap B : B \in B(M)\}.$$
Dual of a Matroid: Bases

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Theorem 7.3.1 (Dual matroid bases)

Let $M = (V, \mathcal{I})$ be a matroid and $\mathcal{B}(M)$ be the set of bases of $M$. Then define

$$\mathcal{B}^*(M) = \{V \setminus B : B \in \mathcal{B}(M)\}. \quad (7.1)$$

Then $\mathcal{B}^*(M)$ is the set of basis of $M^*$ (that is, $\mathcal{B}^*(M) = \mathcal{B}(M^*)$."

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Dual of a Matroid: Terminology

- $\mathcal{B}^*(M)$, the bases of $M^*$, are called cobases of $M$. 

Proposition 7.3.2 (from Oxley 2011)

Let $M = (V, I)$ be a matroid, and let $X \subseteq V$. Then

1. $X$ is independent in $M$ iff $V \cap X$ is cospanning in $M$ (spanning in $M^*$).
2. $X$ is spanning in $M$ iff $V \cap X$ is coindependent in $M$ (independent in $M^*$).
3. $X$ is a hyperplane in $M$ iff $V \cap X$ is a cocircuit in $M$ (circuit in $M^*$).
4. $X$ is a circuit in $M$ iff $V \cap X$ is a cohyperplane in $M$ (hyperplane in $M^*$).
Dual of a Matroid: Terminology

- $\mathcal{B}^*(M)$, the bases of $M^*$, are called cobases of $M$.
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- $B^*(M)$, the bases of $M^*$, are called **cobases** of $M$.
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- The independent sets of $M^*$ are called **coindependent** sets of $M$. 

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Let \( M = (V, I) \) be a matroid, and let \( X \subseteq V \). Then
\[ 1 \quad X \text{ is independent in } M \iff V \cap X \text{ is cospanning in } M^* \text{(spanning in } M^*) \]
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\[ 3 \quad X \text{ is a hyperplane in } M \iff V \cap X \text{ is a cocircuit in } M^* \text{(circuit in } M^*) \]
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Example duality: graphic matroid

Using a graphic/cycle matroid, we can already see how dual matroid concepts demonstrates the extraordinary flexibility and power that a matroid can have.
**Example duality: graphic matroid**

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- Recall, in cycle matroid, a spanning set of $G$ is any set of edges that are incident to all nodes (i.e., any superset of a spanning forest), a minimal spanning set is a spanning tree, and a circuit has a nice visual interpretation (a cycle in the graph).
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A cut in a graph $G$ is a set of edges, the removal of which increases the number of connected components. I.e., $X \subseteq E(G)$ is a cut in $G$ if $k(G) < k(G \setminus X)$. 

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- A cocycle (a cocircuit in a graphic matroid) is a “minimal cut” in the graph. Cocycle matroid sometimes called a “cut matroid”.

- All dependent sets in a cocycle matroid are cuts (i.e., a dependent set is a minimal cut or contains one).
Example: cocycle matroid (sometimes “cut matroid”)

The dual of the cycle matroid is called the cocycle matroid. Recall,
\[ \mathcal{I}^* = \{ A \subseteq V : V \setminus A \text{ is a spanning set of } M \} \]
Example: cocycle matroid (sometimes “cut matroid”)

- The dual of the cycle matroid is called the cocycle matroid. Recall, \( \mathcal{I}^* = \{ A \subseteq V : V \setminus A \text{ is a spanning set of } M \} \)
- It consists of all sets of edges the complement of which contains a spanning tree — i.e., an independent set can’t consist of edges that, if removed, would render the graph non-spanning.

A graph G
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Minimally spanning in \( M \) (and thus a base (maximally independent) in \( M \))

Maximally independent in \( M^* \) (thus a base, minimally spanning, in \( M^* \))
The dual of the cycle matroid is called the cocycle matroid. Recall, 
\[ I^* = \{ A \subseteq V : V \setminus A \text{ is a spanning set of } M \} \]

It consists of all sets of edges the complement of which contains a spanning tree — i.e., an independent set can’t consist of edges that, if removed, would render the graph non-spanning.

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Independent but not spanning in \( M \), and not closed in \( M \).

Dependent in \( M^* \) (contains a cocycle, is a nonminimal cut)
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- It consists of all sets of edges the complement of which contains a spanning tree — i.e., an independent set can’t consist of edges that, if removed, would render the graph non-spanning.

Spanning in $M$, but not a base, and not independent (has cycles)  
Independent in $M^*$ (does not contain a cut)
The dual of the cycle matroid is called the cocycle matroid. Recall, $\mathcal{I}^* = \{ A \subseteq V : V \setminus A \text{ is a spanning set of } M \}$

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A hyperplane in $M$, dependent but not spanning in $M$

A cycle in $M^*$ (minimally dependent in $M^*$, a cocycle, or a minimal cut)
Example: cocycle matroid (sometimes “cut matroid”)

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A hyperplane in $M$, dependent but not spanning in $M$

A cycle in $M^*$ (minimally dependent in $M^*$, a cocycle, or a minimal cut)
The dual of a matroid is (indeed) a matroid

**Theorem 7.3.3**

Let $M^* = (V, I^*)$ be previously defined. Then $M^*$ is a matroid.

**Proof.**

- Clearly $\emptyset \in I^*$, so ($I1'$) holds.
The dual of a matroid is (indeed) a matroid

Theorem 7.3.3

Let $M^* = (V, I^*)$ be previously defined. Then $M^*$ is a matroid.

Proof.

- Clearly $\emptyset \in I^*$, so (I1') holds.
- Also, if $I \subseteq J \in I^*$, then clearly also $I \in I^*$ since if $V \setminus J$ is spanning in $M$, so must $V \setminus I$. Therefore, (I2') holds.
The dual of a matroid is (indeed) a matroid

**Theorem 7.3.3**

Let \( M^* = (V, I^*) \) be previously defined. Then \( M^* \) is a matroid.

**Proof.**

Consider \( I, J \in I^* \) with \( |I| < |J| \). We need to show that there is some member \( v \in J \setminus I \) such that \( I + v \) is independent in \( M^* \), which means that \( V \setminus (I + v) = (V \setminus I) \setminus v \) is still spanning in \( M \). That is, removing \( v \) from \( V \setminus I \) doesn’t make \( (V \setminus I) \setminus v \) not spanning.

...
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Let $M^* = (V, \mathcal{I}^*)$ be previously defined. Then $M^*$ is a matroid.

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Consider $I, J \in \mathcal{I}^*$ with $|I| < |J|$. We need to show that there is some member $v \in J \setminus I$ such that $I + v$ is independent in $M^*$, which means that $V \setminus (I + v) = (V \setminus I) \setminus v$ is still spanning in $M$. That is, removing $v$ from $V \setminus I$ doesn’t make $(V \setminus I) \setminus v$ not spanning.

Since $V \setminus J$ is spanning in $M$, $V \setminus J$ contains some base (say $B \subseteq V \setminus J$) of $M$. Also, $V \setminus I$ contains a base of $M$, say $B' \subseteq V \setminus I$. 

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The dual of a matroid is (indeed) a matroid

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Consider $I, J \in \mathcal{I}^*$ with $|I| < |J|$. We need to show that there is some member $v \in J \setminus I$ such that $I + v$ is independent in $M^*$, which means that $V \setminus (I + v) = (V \setminus I) \setminus v$ is still spanning in $M$. That is, removing $v$ from $V \setminus I$ doesn’t make $(V \setminus I) \setminus v$ not spanning.

Since $V \setminus J$ is spanning in $M$, $V \setminus J$ contains some base (say $B \subseteq V \setminus J$) of $M$. Also, $V \setminus I$ contains a base of $M$, say $B' \subseteq V \setminus I$.

Since $B \setminus I \subseteq V \setminus I$, and $B \setminus I$ is independent in $M$, we can choose the base $B'$ of $M$ s.t. $B \setminus I \subseteq B' \subseteq V \setminus I$. 

...
The dual of a matroid is (indeed) a matroid

Theorem 7.3.3

Let \( M^* = (V, \mathcal{I}^*) \) be previously defined. Then \( M^* \) is a matroid.

Proof.

- Consider \( I, J \in \mathcal{I}^* \) with \( |I| < |J| \). We need to show that there is some member \( v \in J \setminus I \) such that \( I + v \) is independent in \( M^* \), which means that \( V \setminus (I + v) = (V \setminus I) \setminus v \) is still spanning in \( M \). That is, removing \( v \) from \( V \setminus I \) doesn’t make \( (V \setminus I) \setminus v \) not spanning.

- Since \( V \setminus J \) is spanning in \( M \), \( V \setminus J \) contains some base (say \( B \subseteq V \setminus J \)) of \( M \). Also, \( V \setminus I \) contains a base of \( M \), say \( B' \subseteq V \setminus I \).

- Since \( B \setminus I \subseteq V \setminus I \), and \( B \setminus I \) is independent in \( M \), we can choose the base \( B' \) of \( M \) s.t. \( B \setminus I \subseteq B' \subseteq V \setminus I \).

- Since \( B \) and \( J \) are disjoint, we have both: 1) \( B \setminus I \) and \( J \setminus I \) are disjoint; and 2) \( B \cap I \subseteq I \setminus J \). Also note, \( B' \) and \( I \) are disjoint.
The dual of a matroid is (indeed) a matroid

**Theorem 7.3.3**

Let $M^* = (V, \mathcal{I}^*)$ be previously defined. Then $M^*$ is a matroid.

**Proof.**

Now $J \setminus I \not\subseteq B'$, since otherwise (i.e., assuming $J \setminus I \subseteq B'$):

\[
|B| = |B \cap I| + |B \setminus I| 
\leq |I \setminus J| + |B \setminus I| 
\leq |J \setminus I| + |B \setminus I| \leq |B'|
\]

which is a contradiction. *The last inequality on the right follows since $J \setminus I \subseteq B'$ (by assumption) and $B \setminus I \subseteq B'$ implies that $(J \setminus I) \cup (B \setminus I) \subseteq B'$, but since $J$ and $B$ are disjoint, we have that $|J \setminus I| + |B \setminus I| \leq |B'|$.*
The dual of a matroid is (indeed) a matroid

**Theorem 7.3.3**

Let $M^* = (V, I^*)$ be previously defined. Then $M^*$ is a matroid.

**Proof.**

- Now $J \setminus I \not\subseteq B'$, since otherwise (i.e., assuming $J \setminus I \subseteq B'$):

\[
|B| = |B \cap I| + |B \setminus I| \leq |I \setminus J| + |B \setminus I| < |J \setminus I| + |B \setminus I| \leq |B'|
\]

(7.2) \hspace{1cm} (7.3) \hspace{1cm} (7.4)

which is a contradiction.

- Therefore, $J \setminus I \not\subseteq B'$, and there is a $v \in J \setminus I$ s.t. $v \notin B'$.
The dual of a matroid is (indeed) a matroid

**Theorem 7.3.3**

Let $M^* = (V, \mathcal{I}^*)$ be previously defined. Then $M^*$ is a matroid.

**Proof.**

- Now $J \setminus I \not\subseteq B'$, since otherwise (i.e., assuming $J \setminus I \subseteq B'$):

  \[
  |B| = |B \cap I| + |B \setminus I| \leq |I \setminus J| + |B \setminus I| \leq |J \setminus I| + |B \setminus I| \leq |B'|
  \]

  which is a contradiction.

- Therefore, $J \setminus I \not\subseteq B'$, and there is a $v \in J \setminus I$ s.t. $v \notin B'$.

- So $B'$ is disjoint with $I \cup \{v\}$, means $B' \subseteq V \setminus (I \cup \{v\})$, or $V \setminus (I \cup \{v\})$ is spanning in $M$, and therefore $I \cup \{v\} \in \mathcal{I}^*$.
Theorem 7.3.4

Let $M$ be an $\mathbb{F}$-representable matroid (i.e., one that can be represented by a finite sized matrix over field $\mathbb{F}$). Then $M^*$ is also $\mathbb{F}$-representable.
Theorem 7.3.4
Let $M$ be an $\mathbb{F}$-representable matroid (i.e., one that can be represented by a finite sized matrix over field $\mathbb{F}$). Then $M^*$ is also $\mathbb{F}$-representable.

Theorem 7.3.5
Let $M$ be a graphic matroid (i.e., one that can be represented by a graph $G = (V, E)$). Then $M^*$ is not necessarily also graphic.
Dual Matroid Rank

**Theorem 7.3.6**

The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified in terms of the rank $r_M$ in matroid $M$ as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \quad (7.5)$$

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2. *i.e.*, $|X|$ is modular, complement $f(V \setminus X)$ is submodular if $f$ is submodular, $r_M(V)$ is a constant, and summing submodular functions and a constant preserves submodularity.
The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified in terms of the rank $r_M$ in matroid $M$ as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \tag{7.5}$$

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2.
- Non-negativity integral follows since

  $$|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V).$$

  The right inequality follows since $r_M$ is submodular.
The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified in terms of the rank $r_M$ in matroid $M$ as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V)$$  \hspace{1cm} (7.5)

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2.
- Non-negativity integral follows since
  $$|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V).$$
- Monotone non-decreasing follows since, as $X$ increases by one, $|X|$ always increases by 1, while $r_M(V \setminus X)$ decreases by one or zero.
Dual Matroid Rank

Theorem 7.3.6

The rank function \( r_{M^*} \) of the dual matroid \( M^* \) may be specified in terms of the rank \( r_M \) in matroid \( M \) as follows. For \( X \subseteq V \):

\[
r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V)
\]  

(7.5)

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2.
- Non-negativity integral follows since
  \[
  |X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V).
  \]
- Monotone non-decreasing follows since, as \( X \) increases by one, \( |X| \) always increases by 1, while \( r_M(V \setminus X) \) decreases by one or zero.
- Therefore, \( r_{M^*} \) is the rank function of a matroid. That it is the dual matroid rank function is shown in the next proof.
Dual Matroid Rank

Theorem 7.3.6

The rank function \( r_{M^*} \) of the dual matroid \( M^* \) may be specified in terms of the rank \( r_M \) in matroid \( M \) as follows. For \( X \subseteq V \):

\[
r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V)
\]

(7.5)

Proof.

A set \( X \) is independent in \((V, r_{M^*})\) if and only if

\[
r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) = |X|
\]

(7.6)
Dual Matroid Rank

Theorem 7.3.6

The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified in terms of the rank $r_M$ in matroid $M$ as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \quad (7.5)$$

Proof.

A set $X$ is independent in $(V, r_{M^*})$ if and only if

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) = |X| \quad (7.6)$$

or

$$r_M(V \setminus X) = r_M(V) \quad (7.7)$$

...
**Theorem 7.3.6**

The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified in terms of the rank $r_M$ in matroid $M$ as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \quad (7.5)$$

**Proof.**

A set $X$ is independent in $(V, r_{M^*})$ if and only if

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) = |X| \quad (7.6)$$

or

$$r_M(V \setminus X) = r_M(V) \quad (7.7)$$

But a subset $X$ is independent in $M^*$ only if $V \setminus X$ is spanning in $M$ (by the definition of the dual matroid).
Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then

$$\mathcal{I}_Y = \{ Z : Z \subseteq Y, Z \in \mathcal{I} \}$$

(7.8)

is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid with rank $r(M_Y) = r(Y)$. 
Matroid restriction/deletion

Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then

$$\mathcal{I}_Y = \{ Z : Z \subseteq Y, Z \in \mathcal{I} \}$$  \hspace{1cm} (7.8)

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This is called the restriction of $M$ to $Y$, and is often written $M|Y$.  

Prof. Jeff Bilmes
Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then

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This is called the restriction of $M$ to $Y$, and is often written $M|_Y$.

If $Y = V \setminus X$, then we have

$$\mathcal{I}_Y = \{ Z : Z \cap X = \emptyset, Z \in \mathcal{I} \}$$

(7.9)

is considered a deletion of $X$ from $M$, and is often written $M \setminus Z$. 

$G \setminus A$
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is considered a deletion of $X$ from $M$, and is often written $M \setminus Z$.

Hence, $M|Y = M \setminus (V \setminus Y)$. 

Matroid restriction/deletion

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  \]
  is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid with rank $r(M_Y) = r(Y)$.
- This is called the restriction of $M$ to $Y$, and is often written $M|Y$.
- If $Y = V \setminus X$, then we have
  \[
  \mathcal{I}_Y = \{ Z : Z \cap X = \emptyset, Z \in \mathcal{I} \} \quad (7.9)
  \]
  is considered a deletion of $X$ from $M$, and is often written $M \setminus Z$.
- Hence, $M|Y = M \setminus (V \setminus Y)$.
- The rank function is of the same form. I.e., $r_Y : 2^Y \to \mathbb{Z}_+$, where
  $r_Y(Z) = r(Z)$ for $Z \subseteq Y$. 

**Matroid contraction**

- Contraction is dual to deletion, and is like a forced inclusion of contained base, but with a similar ground set removal. **Contracting** $Z$ is written $M/Z$. 

Let $Z \subseteq V$ and let $X$ be a base of $Z$. The subset $I$ of $V \cap Z$ is independent in $M/Z$ iff $I \cup X$ is independent in $M$. In fact, it is the case $M/Z = (M* \cap Z) * (\)$. Exercise: show why.

The rank function takes the form

$$r_{M/Z}(Y) = r(Y \cup Z) - r(Z) = r(Y|Z)$$

(7.10)

So given $I \subseteq V \cap Z$ and $X$ is a base of $Z$, $r_{M/Z}(I) = |I|$ is identical to $r(I \cup Z) = |I| + r(Z) = |I| + |X|$ but $r(I \cup Z) = r(I \cup X)$. This implies $r(I \cup X) = |I| + |X|$, or $I \cup X$ is independent in $M$. A minor of a matroid is any matroid obtained via a series of deletions and contractions of some matroid.
Matroid contraction

- Contraction is dual to deletion, and is like a forced inclusion of contained base, but with a similar ground set removal. Contracting $Z$ is written $M/Z$.

- Let $Z \subseteq V$ and let $X$ be a base of $Z$. Then a subset $I$ of $V \setminus Z$ is independent in $M/Z$ iff $I \cup X$ is independent in $M$. 

\[ r_{M/Z}(I) = |I| \text{ is identical to } r(I \cup Z) = |I| + r(Z) = |I| + |X| \]
\[ r(I \cup Z) = r(I \cup X) \]
This implies $r(I \cup X) = |I| + |X|$, or $I \cup X$ is independent in $M$.
Matroid contraction

- Contraction is dual to deletion, and is like a forced inclusion of contained base, but with a similar ground set removal. Contraction of $Z$ is written $M/Z$.

- Let $Z \subseteq V$ and let $X$ be a base of $Z$. Then a subset $I$ of $V \setminus Z$ is independent in $M/Z$ iff $I \cup X$ is independent in $M$.

- In fact, it is the case $M/Z = (M^* \setminus Z)^*$ (Exercise: show why).
Matroid contraction

- Contraction is dual to deletion, and is like a forced inclusion of contained base, but with a similar ground set removal. **Contracting** $Z$ is written $M/Z$.

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- In fact, it is the case $M/Z = (M^* \setminus Z)^*$ (**Exercise: show why**).

- The rank function takes the form

\[
    r_{M/Z}(Y) = r(Y \cup Z) - r(Z) = r(Y|Z) \quad (7.10)
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Matroid contraction

- Contraction is dual to deletion, and is like a forced inclusion of contained base, but with a similar ground set removal. Contracting $Z$ is written $M/Z$.
- Let $Z \subseteq V$ and let $X$ be a base of $Z$. Then a subset $I$ of $V \setminus Z$ is independent in $M/Z$ iff $I \cup X$ is independent in $M$.
- In fact, it is the case $M/Z = (M^* \setminus Z)^*$ (Exercise: show why).
- The rank function takes the form

$$r_{M/Z}(Y) = r(Y \cup Z) - r(Z) = r(Y \mid Z) \quad (7.10)$$

- So given $I \subseteq V \setminus Z$ and $X$ is a base of $Z$, $r_{M/Z}(I) = |I|$ is identical to $r(I \cup Z) = |I| + r(Z) = |I| + |X|$ but $r(I \cup Z) = r(I \cup X)$. This implies $r(I \cup X) = |I| + |X|$, or $I \cup X$ is independent in $M$. 
Matroid contraction

- Contraction is dual to deletion, and is like a forced inclusion of contained base, but with a similar ground set removal. **Contracting Z is written** $M/Z$.

- Let $Z \subseteq V$ and let $X$ be a base of $Z$. Then a subset $I$ of $V \setminus Z$ is independent in $M/Z$ iff $I \cup X$ is independent in $M$.

- In fact, it is the case $M/Z = (M^* \setminus Z)^*$ (Exercise: show why).

- The rank function takes the form

$$ r_{M/Z}(Y) = r(Y \cup Z) - r(Z) = r(Y \mid Z) \quad (7.10) $$

- So given $I \subseteq V \setminus Z$ and $X$ is a base of $Z$, $r_{M/Z}(I) = |I|$ is identical to $r(I \cup Z) = |I| + r(Z) = |I| + |X|$ but $r(I \cup Z) = r(I \cup X)$. This implies $r(I \cup X) = |I| + |X|$, or $I \cup X$ is independent in $M$.

- A **minor** of a matroid is any matroid obtained via a series of deletions and contractions of some matroid.
Matroid Intersection

Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$. 

Theorem 7.4.1
Let $M_1$ and $M_2$ be given as above, with rank functions $r_1$ and $r_2$. Then the size of the maximum size set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by $\left( r_1 \ast r_2 \right)(V) \equiv \min_{X \subseteq V} \left( r_1(X) + r_2(V \cap X) \right)$ (7.11)

This is an instance of the convolution of two submodular functions, $f_1$ and $f_2$ that, evaluated at $Y \subseteq V$, is written as:

$\left( f_1 \ast f_2 \right)(Y) = \min_{X \subseteq Y} \left( f_1(X) + f_2(Y \cap X) \right)$ (7.12)
Matroid Intersection

- Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$.

- While $(V, \mathcal{I}_1 \cap \mathcal{I}_2)$ is typically not a matroid (Exercise: show graphical example.), we might be interested in finding the maximum size common independent set. That is, find $\max |X|$ such that both $X \in \mathcal{I}_1$ and $X \in \mathcal{I}_2$. 

Theorem 7.4.1

Let $M_1$ and $M_2$ be given as above, with rank functions $r_1$ and $r_2$. Then the size of the maximum size set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by

$$\left( r_1 \ast r_2 \right)(V) \triangleq \min X \subseteq V (r_1(X) + r_2(V \cap X)) \hspace{1cm} (7.11)$$

This is an instance of the convolution of two submodular functions $f_1$ and $f_2$ evaluated at $Y \subseteq V$, written as:

$$\left( f_1 \ast f_2 \right)(Y) = \min X \subseteq Y (f_1(X) + f_2(Y \cap X)) \hspace{1cm} (7.12)$$
Matroid Intersection

- Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$.
- While $(V, \mathcal{I}_1 \cap \mathcal{I}_2)$ is typically not a matroid (Exercise: show graphical example.), we might be interested in finding the maximum size common independent set. That is, find $\max |X|$ such that both $X \in \mathcal{I}_1$ and $X \in \mathcal{I}_2$.

**Theorem 7.4.1**

Let $M_1$ and $M_2$ be given as above, with rank functions $r_1$ and $r_2$. Then the size of the maximum size set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by

$$(r_1 * r_2)(V) \triangleq \min_{X \subseteq V} \left( r_1(X) + r_2(V \setminus X) \right) \quad (7.11)$$
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This is an instance of the convolution of two submodular functions, $f_1$ and $f_2$ that, evaluated at $Y \subseteq V$, is written as:

$$ (f_1 * f_2)(Y) = \min_{X \subseteq Y} \left( f_1(X) + f_2(Y \setminus X) \right) \quad (7.12) $$
Recall Hall’s theorem, that a transversal exists iff for all $X \subseteq V$, we have $|\Gamma(X)| \geq |X|$.
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\( \Leftrightarrow \quad |\Gamma(X)| - |X| \geq 0, \forall X \)

\( \Leftrightarrow \quad \min_X |\Gamma(X)| - |X| \geq 0 \)

So Hall’s theorem can be expressed as convolution.

Note, in general, convolution of two submodular functions does not preserve submodularity (but in certain special cases it does).
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\[
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\[
\Leftrightarrow [\Gamma(\cdot) \ast |\cdot|](V) \geq |V|
\]

So Hall’s theorem can be expressed as convolution.

**Exercise:** \( f(\cdot) \ast |\cdot| \in \text{submodular} \)
Recall Hall’s theorem, that a transversal exists iff for all \( X \subseteq V \), we have \( |\Gamma(X)| \geq |X| \).

\[
\begin{align*}
\Leftrightarrow & \quad |\Gamma(X)| - |X| \geq 0, \forall X \\
\Leftrightarrow & \quad \min_X |\Gamma(X)| - |X| \geq 0 \\
\Leftrightarrow & \quad \min_X |\Gamma(X)| + |V| - |X| \geq |V| \\
\Leftrightarrow & \quad \min_X \left( |\Gamma(X)| + |V \setminus X| \right) \geq |V| \\
\Leftrightarrow & \quad [\Gamma(\cdot) * \cdot \mathbb{1}(V)](V) \geq |V|
\end{align*}
\]

So Hall’s theorem can be expressed as convolution.

Note, in general, convolution of two submodular functions does not preserve submodularity (but in certain special cases it does).
Matroid Union

Definition 7.4.2

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, \ldots, $M_k = (V_k, \mathcal{I}_k)$ be matroids. We define the union of matroids as

$$M_1 \vee M_2 \vee \cdots \vee M_k = (V_1 \cup V_2 \cup \cdots \cup V_k, \mathcal{I}_1 \vee \mathcal{I}_2 \vee \cdots \vee \mathcal{I}_k),$$

where

$$\mathcal{I}_1 \vee \mathcal{I}_2 \vee \cdots \vee \mathcal{I}_k = \{ I_1 \cup I_2 \cup \cdots \cup I_k | I_1 \in \mathcal{I}_1, \ldots, I_k \in \mathcal{I}_k \}. \tag{7.13}$$

Note $A \uplus B$ designates the disjoint union of $A$ and $B$.

$$\{1,2,3\} \uplus \{1,2\} \quad = \quad \frac{3}{2} \{ (1,1), (1,2), (1,3), (2,1), (2,3) \}$$
Matroid Union

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$$I_1 \vee I_2 \vee \cdots \vee I_k = \{I_1 \cup I_2 \cup \cdots \cup I_k | I_1 \in \mathcal{I}_1, \ldots, I_k \in \mathcal{I}_k\} \quad (7.13)$$

Note $A \cup B$ designates the disjoint union of $A$ and $B$.

Theorem 7.4.3

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, $\ldots$, $M_k = (V_k, \mathcal{I}_k)$ be matroids, with rank functions $r_1, \ldots, r_k$. Then the union of these matroids is still a matroid, having rank function

$$r(Y) = \min_{X \subseteq Y} \left(|Y \setminus X| + r_1(X \cap V_1) + \cdots + r_k(X \cap V_k)\right) \quad (7.14)$$

for any $Y \subseteq V_1 \cup \ldots V_k$. 
Exercise: Describe $M \lor M^*$. 
Matroids of three or fewer elements are graphic

- All matroids up to and including three elements are graphic.
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(a) The only matroid with zero elements.
(b) The two one-element matroids.
(c) The four two-element matroids.
(d) The eight three-element matroids.
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This is a nice way to show matroids with low ground set sizes. What about matroids that are low rank but with many elements?
Affine Matroids

Given an $n \times m$ matrix with entries over some field $\mathbb{F}$, we say that a subset $S \subseteq \{1, \ldots, m\}$ of indices (with corresponding column vectors $\{v_i : i \in S\}$, with $|S| = k$) is **affinely dependent** if $m \geq 1$ and there exists elements $\{a_1, \ldots, a_k\} \in \mathbb{F}$, not all zero with $\sum_{i=1}^{k} a_i = 0$, such that $\sum_{i=1}^{k} a_i v_i = 0$. Otherwise, the set is called **affinely independent**.
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- Otherwise, the set is called **affinely independent**.
- Concisely: points $\{v_1, v_2, \ldots, v_k\}$ are affinely independent if $v_2 - v_1, v_3 - v_1, \ldots, v_k - v_1$ are linearly independent.
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- **Example:** in 2D, three co-linear points are affinely dependent, and any 4 or more non co-linear points are affinely dependent.
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Proposition 7.4.4 (affine matroid)

Let ground set $E = \{1, \ldots, m\}$ index column vectors of a matrix, and let $\mathcal{I}$ be the set of subsets $X$ of $E$ such that $X$ indices affinely independent vectors. Then $(E, \mathcal{I})$ is a matroid.
Affine Matroids

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**Proposition 7.4.4 (affine matroid)**

Let ground set \( E = \{1, \ldots, m\} \) index column vectors of a matrix, and let \( \mathcal{I} \) be the set of subsets \( X \) of \( E \) such that \( X \) indices affinely independent vectors. Then \( (E, \mathcal{I}) \) is a matroid.

**Exercise:** prove this.
Consider the affine matroid with $n \times m = 2 \times 6$ matrix on the field $F = \mathbb{R}$, and let the elements be
\{(0, 0), (1, 0), (2, 0), (0, 1), (0, 2), (1, 1)\}. 
Consider the affine matroid with $n \times m = 2 \times 6$ matrix on the field $\mathbb{F} = \mathbb{R}$, and let the elements be:
\[ \{(0,0), (1,0), (2,0), (0,1), (0,2), (1,1)\}. \]

Hence, we can plot the points in $\mathbb{R}^2$ as follows:
Euclidean Representation of Low-rank Matroids

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- Hence, we can plot the points in $\mathbb{R}^2$ as follows:

- Dependent sets consist of all subsets with $\geq 4$ elements, or 3 collinear elements.
Euclidean Representation of Low-rank Matroids

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Hence, we can plot the points in $\mathbb{R}^2$ as follows:

Dependent sets consist of all subsets with $\geq 4$ elements, or 3 collinear elements.

In general, for a matroid $\mathcal{M}$ of rank $m + 1$ with $m \leq 3$, then a subset $X$ in a geometric representation in $\mathbb{R}^m$ is dependent if: 1) $|X| \geq 2$ and the points are identical; 2) $|X| \geq 3$ and the points are collinear; 3) $|X| \geq 4$ and the points are coplanar; or 4) $|X| \geq 5$ and the points are in space.
Theorem 7.4.5

Any matroid of rank $m \leq 4$ can be represented by an affine matroid in $\mathbb{R}^{m-1}$. 
Theorem 7.4.5

Any matroid of rank \( m \leq 4 \) can be represented by an affine matroid in \( \mathbb{R}^{m-1} \).

As another example on the right, a rank 4 matroid

All sets of 5 points are dependent. The only other sets of dependent points are coplanar ones of size 4. Namely:

- \{ (0,0,0), (0,0,1), (0,1,1), (0,1,0), (1,1,0) \}
- \{ (0,0,0), (0,0,1), (0,1,1), (0,1,0), (1,0,0) \}
- \{ (0,0,0), (0,0,1), (0,1,1), (0,1,0), (1,1,0) \}

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Euclidean Representation of Low-rank Matroids

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  - $\{(0, 0, 0), (0, 1, 0), (1, 1, 0), (1, 0, 0)\}$,
  - $\{(0, 0, 0), (0, 0, 1), (0, 1, 1), (0, 1, 0)\}$, and
  - $\{(0, 0, 1), (0, 1, 1), (1, 1, 0), (1, 0, 0)\}$.
Loops represented by a separate box indicating how many loops there are. Parallel elements indicated by a multiplicity next to a point.
Euclidean Representation of Low-rank Matroids

- Very useful for graphically depicting low-rank matrices but which still have rich structure. Also useful for answering questions.

Example: Is there a matroid that is not representable (i.e., not linear for some field)?

Yes, consider the matroid called the non-Pappus matroid. Has rank three, but any matric matroid with the above dependencies would require that \{7, 8, 9\} is dependent, hence requiring an additional line in the above.
Euclidean Representation of Low-rank Matroids

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```
1
   2
  /   \
7     8
  \
   9
```

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\[
\begin{align*}
1 & \quad 2 & \quad 3 \\
4 & \quad 5 & \quad 6 \\
7 & \quad 8 & \quad 9
\end{align*}
\]

- Called the non-Pappus matroid. Has rank three, but any matric matroid with the above dependencies would require that \{7, 8, 9\} is dependent, hence requiring an additional line in the above.
Check rank's submodularity: Let $X = \{1, 2, 3, 6, 7\}$, $Y = \{1, 4, 5, 6, 7\}$. So $r(X) = 3$, and $r(Y) = 3$, and $r(X \cup Y) = 4$, so we must have, by submodularity, that $r(\{1, 6, 7\}) = r(X \cap Y) \leq r(X) + r(Y) - r(X \cup Y) = 2$.

However, from the diagram, we have that since 1, 6, 7 are distinct non-collinear points, we have that $r(X \cap Y) = 3$.

If we extend the line from 6-7 to 1, then is it a matroid?

Hence, not all 2D or 3D graphs of points and lines are matroids.
Is this a matroid?

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Hence, not all 2D or 3D graphs of points and lines are matroids.
Other examples can be more complex, consider the following two matroids (from Oxley, 2011):
Euclidean Representation of Low-rank Matroids: Other Examples

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- Hence, lines (in 2D) may be curved and planes (in 3D) can be twisted.
Euclidean Rep. of Low-rank Matroids: Conditions

- rank-1 (resp. rank-2, rank-3) flats correspond to points (resp. lines, planes).

- A set of parallel points (could be size 1) does not touch another set of parallel points (could be size 1).

- Every line contains at least two points (not dependent unless more than two).

- Any two distinct points lie on a line (often not drawn when only two).

- Every plane contains at least three non-collinear points (not dependent unless more than three).

- Any three distinct non-collinear points lie on a plane.

- If the diagram has at most one plane, then any two distinct lines meet in at most one point.

- If the diagram has more than one plane, then:
  1) Any two distinct planes meeting in more than two points do so in a line.
  2) Any two distinct lines meeting in a point do so in at most one point and lie on a common plane.
  3) Any line not lying on a plane intersects it in at most one point.

Matroid of rank at most four (see Oxley 2011 for more details).
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- every plane contains at least three non-collinear points (not dependent unless > 3)
Euclidean Rep. of Low-rank Matroids: Conditions

- rank-1 (resp. rank-2, rank-3) flats correspond to points (resp. lines, planes).
- a set of parallel points (could be size 1) does not touch another set of parallel points (could be size 1).
- every line contains at least two points (not dependent unless \( \geq 2 \)).
- any two distinct points lie on a line (often not drawn when only two)
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- If diagram has at most one plane, then any two distinct lines meet in at most one point.
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- If diagram has at most one plane, then any two distinct lines meet in at most one point.
- If diagram has more than one plane, then: 1) any two distinct planes meeting in more than two points do so in a line; 2) any two distinct lines meeting in a point do so in at most one point and lie in on a common plane; 3) any line not lying on a plane intersects it in at most one point.
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- Matroid of rank at most four (see Oxley 2011 for more details).
Matroid Further Reading

- “The Coming of the Matroids”, William Cunningham, 2012 (a nice history)
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- Schrijver, “Combinatorial Optimization”, 2003
The greedy algorithm

- In combinatorial optimization, the greedy algorithm is often useful as a heuristic that can work quite well in practice.
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- Sometimes, this gives the optimal solution (we saw three greedy algorithms that can find the maximum weight spanning tree).
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- Greedy is good since it can be made to run very fast $O(n \log n)$. 

**Dual Matroid**

**Other Matroid Properties**

**Matroid and Greedy**
The greedy algorithm

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- The goal is to choose a good subset of items, and the fundamental tenet of the greedy algorithm is to choose next whatever currently looks best.
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The greedy algorithm

- In combinatorial optimization, the greedy algorithm is often useful as a heuristic that can work quite well in practice.
- The goal is to choose a good subset of items, and the fundamental tenet of the greedy algorithm is to **choose next whatever currently looks best**.
- Sometimes, this gives the optimal solution (we saw three greedy algorithms that can find the maximum weight spanning tree).
- Greedy is good since it can be made to run very fast $O(n \log n)$.
- Often, however, greedy is heuristic (it might work well in practice, but worst-case performance can be unboundedly poor).
- **We will next see that the greedy algorithm working is a defining property of a matroid, and is also a defining property of a polymatroid function.**
Matroid and the greedy algorithm

- Let $(E, I)$ be an independence system, and we are given a non-negative modular weight function $w : E \rightarrow \mathbb{R}_+$. 

Algorithm 1: The Matroid Greedy Algorithm

1. Set $X \leftarrow \emptyset$;
2. while $\exists v \in E \cap X$ s.t. $X \cup \{v\} \in I$ do
3. \begin{align*}
v & \in \text{argmax}\{w(v) : v \in E \cap X, X \cup \{v\} \in I\};
4. X & \leftarrow X \cup \{v\};
\end{align*}
5. Same as sorting items by decreasing weight $w$, and then choosing items in that order that retain independence.

Theorem 7.5.1

Let $(E, I)$ be an independence system. Then the pair $(E, I)$ is a matroid if and only if for each weight function $w \in \mathbb{R}_+^E$, Algorithm 1 leads to a set $I \in I$ of maximum weight $w(I)$. 

Prof. Jeff Bilmes
EE596b/Spring 2014/Submodularity - Lecture 7 - April 21st, 2014
Matroid and the greedy algorithm

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Algorithm 1: The Matroid Greedy Algorithm

1. Set \(X \leftarrow \emptyset\);
2. while \(\exists v \in E \setminus X\) s.t. \(X \cup \{v\} \in \mathcal{I}\) do
3. \(v \in \arg\max\ \{w(v) : v \in E \setminus X, X \cup \{v\} \in \mathcal{I}\}\);
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Matroid and the greedy algorithm

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- Same as sorting items by decreasing weight \(w\), and then choosing items in that order that retain independence.
Matroid and the greedy algorithm

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1. Set \(X \leftarrow \emptyset\);
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- Same as sorting items by decreasing weight \(w\), and then choosing items in that order that retain independence.

**Theorem 7.5.1**

Let \((E, \mathcal{I})\) be an independence system. Then the pair \((E, \mathcal{I})\) is a matroid if and only if for each weight function \(w \in \mathcal{R}^E_+\), Algorithm 1 leads to a set \(I \in \mathcal{I}\) of maximum weight \(w(I)\).
Matroids by bases

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

**Theorem 7.5.1 (Matroid (by bases))**

Let $E$ be a set and $\mathcal{B}$ be a nonempty collection of subsets of $E$. Then the following are equivalent.

1. $\mathcal{B}$ is the collection of bases of a matroid;
2. if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.
3. If $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.

Properties 2 and 3 are called “exchange properties.”

Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.
proof of Theorem 7.5.1.

- Assume \((E, \mathcal{I})\) is a matroid and \(w : E \to \mathbb{R}_+\) is given.

...
proof of Theorem 7.5.1.

- Assume \((E, \mathcal{I})\) is a matroid and \(w : E \rightarrow \mathbb{R}_+\) is given.
- Let \(A = (a_1, a_2, \ldots, a_r)\) be the solution returned by greedy, where \(r = r(M)\) the rank of the matroid, and we order the elements as they were chosen (so \(w(a_1) \geq w(a_2) \geq \cdots \geq w(a_r)\)).
proof of Theorem 7.5.1.

- Assume \((E, \mathcal{I})\) is a matroid and \(w : E \rightarrow \mathcal{R}_+\) is given.
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- \(A\) is a base of \(M\), and let \(B = (b_1, \ldots, b_r)\) be any another base of \(M\) with elements also ordered decreasing by weight.
proof of Theorem 7.5.1.

- Assume \((E, \mathcal{I})\) is a matroid and \(w : E \to \mathbb{R}_+\) is given.
- Let \(A = (a_1, a_2, \ldots, a_r)\) be the solution returned by greedy, where \(r = r(M)\) the rank of the matroid, and we order the elements as they were chosen (so \(w(a_1) \geq w(a_2) \geq \cdots \geq w(a_r)\)).
- \(A\) is a base of \(M\), and let \(B = (b_1, \ldots, b_r)\) be any another base of \(M\) with elements also ordered decreasing by weight.
- We next show that not only is \(w(A) \geq w(B)\) but that \(w(a_i) \geq w(b_i)\) for all \(i\).
proof of Theorem 7.5.1.

Assume otherwise, and let \( k \) be the first (smallest) integer such that \( w(a_k) < w(b_k) \). Hence \( w(a_j) \geq w(b_j) \) for \( j < k \).
proof of Theorem 7.5.1.

- Assume otherwise, and let $k$ be the first (smallest) integer such that $w(a_{k}) < w(b_{k})$. Hence $w(a_{j}) \geq w(b_{j})$ for $j < k$.
- Define independent sets $A_{k-1} = \{a_{1}, \ldots, a_{k-1}\}$ and $B_{k} = \{b_{1}, \ldots, b_{k}\}$. 

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proof of Theorem 7.5.1.

- Assume otherwise, and let $k$ be the first (smallest) integer such that $w(a_k) < w(b_k)$. Hence $w(a_j) \geq w(b_j)$ for $j < k$.
- Define independent sets $A_{k-1} = \{a_1, \ldots, a_{k-1}\}$ and $B_k = \{b_1, \ldots, b_k\}$.
- Since $|A_{k-1}| < |B_k|$, $A_{k-1} \cup \{b_i\} \in \mathcal{I}$ for some $1 \leq i \leq k$. 

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- Define independent sets $A_{k-1} = \{a_1, \ldots, a_{k-1}\}$ and $B_k = \{b_1, \ldots, b_k\}$.

- Since $|A_{k-1}| < |B_k|$, $A_{k-1} \cup \{b_i\} \in \mathcal{I}$ for some $1 \leq i \leq k$.

- But $w(b_i) \geq w(b_k) > w(a_k)$, and so the greedy algorithm would have chosen $b_i$ rather than $a_k$, contradicting what greedy does.
Matroid and the greedy algorithm

Converse proof of Theorem 7.5.1.

- Given an independence system \((E, I)\), suppose the greedy algorithm leads to an independent set of max weight for every non-negative weight function. We’ll show \((E, I)\) is a matroid.
converse proof of Theorem 7.5.1.

- Given an independence system $(E, \mathcal{I})$, suppose the greedy algorithm leads to an independent set of max weight for every non-negative weight function. We'll show $(E, \mathcal{I})$ is a matroid.
- Emptyset containing and down monotonicity already holds (since we've started with an independence system).
Given an independence system $(E, \mathcal{I})$, suppose the greedy algorithm leads to an independent set of max weight for every non-negative weight function. We'll show $(E, \mathcal{I})$ is a matroid.

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Let $I, J \in \mathcal{I}$ with $|I| < |J|$. Suppose to the contrary, that $I \cup \{z\} \notin \mathcal{I}$ for all $z \in J \setminus I$. 

Define the following modular weight function $w$ on $E$, and define $k = |I|$. 

$$w(v) = \begin{cases} k + 2 & \text{if } v \in I, \\ k + 1 & \text{if } v \in J \cap I, \\ 0 & \text{if } v \in E \cap (I \cup J) \end{cases}$$
converse proof of Theorem 7.5.1.

- Given an independence system \((E, \mathcal{I})\), suppose the greedy algorithm leads to an independent set of max weight for every non-negative weight function. We’ll show \((E, \mathcal{I})\) is a matroid.

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- Let \(I, J \in \mathcal{I}\) with \(|I| < |J|\). Suppose to the contrary, that \(I \cup \{z\} \notin \mathcal{I}\) for all \(z \in J \setminus I\).

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    w(v) = \begin{cases} 
    k + 2 & \text{if } v \in I, \\
    k + 1 & \text{if } v \in J \setminus I, \\
    0 & \text{if } v \in E \setminus (I \cup J) 
    \end{cases} \quad (7.15)
\]
Matroid and the greedy algorithm

Converse proof of Theorem 7.5.1.

Now greedy will, after \( \ell \) iterations, recover \( I \), but it cannot choose any element in \( J \setminus I \) by assumption. Thus, greedy chooses a set of weight \( \ell(\ell + 2) \).
Matroid and the greedy algorithm

Converse proof of Theorem 7.5.1.

- Now greedy will, after \( k \) iterations, recover \( I \), but it cannot choose any element in \( J \setminus I \) by assumption. Thus, greedy chooses a set of weight \( k(k + 2) \).

- On the other hand, \( J \) has weight

\[
    w(J) \geq |J|(k + 1) \geq (k + 1)(k + 1) > k(k + 2)
\]  

so \( J \) has strictly larger weight but is still independent, contradicting greedy’s optimality.
converse proof of Theorem 7.5.1.

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- On the other hand, \( J \) has weight

\[
\begin{align*}
\text{weight} \quad &\geq |J|(k + 1) \\
&\geq (k + 1)(k + 1) > k(k + 2)
\end{align*}
\]

so \( J \) has strictly larger weight but is still independent, contradicting greedy’s optimality.
- Therefore, \((E, I)\) must be a matroid.
As given, the theorem asked for a modular function \( w \in \mathbb{R}^E_+ \).
Matroid and greedy

- As given, the theorem asked for a modular function $w \in \mathbb{R}^E_+$. 
- This will not only return an independent set, but it will return a base if we keep going even if the weights are 0.
As given, the theorem asked for a modular function $w \in \mathbb{R}^E_+$. This will not only return an independent set, but it will return a base if we keep going even if the weights are 0. If we don’t want elements with weight 0, we can stop once (and if) the weight hits zero, thus giving us a maximum weight independent set.
As given, the theorem asked for a modular function $w \in \mathbb{R}^E_+$. This will not only return an independent set, but it will return a base if we keep going even if the weights are 0. If we don’t want elements with weight 0, we can stop once (and if) the weight hits zero, thus giving us a maximum weight independent set. We don’t need non-negativity, we can use any $w \in \mathbb{R}^E$ and keep going until we have a base.
Matroid and greedy

- As given, the theorem asked for a modular function $w \in \mathbb{R}^E$.
- This will not only return an independent set, but it will return a base if we keep going even if the weights are 0.
- If we don’t want elements with weight 0, we can stop once (and if) the weight hits zero, thus giving us a maximum weight independent set.
- We don’t need non-negativity, we can use any $w \in \mathbb{R}^E$ and keep going until we have a base.
- If we stop at a negative value, we’ll once again get a maximum weight independent set.
As given, the theorem asked for a modular function $w \in \mathbb{R}^E_+$.  

This will not only return an independent set, but it will return a base if we keep going even if the weights are 0.  

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We don’t need non-negativity, we can use any $w \in \mathbb{R}^E$ and keep going until we have a base.  

If we stop at a negative value, we’ll once again get a maximum weight independent set.  

We can instead do as small as possible thus giving us a minimum weight independent set/base.
Summary of Important (for us) Matroid Definitions

Given an independence system, matroids are defined equivalently by any of the following:

- All maximally independent sets have the same size.
- A monotone non-decreasing submodular integral rank function with unit increments.
- The greedy algorithm achieves the maximum weight independent set for all weight functions.