Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 7 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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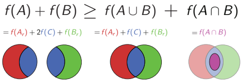
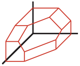
April 21st, 2014





Cumulative Outstanding Reading

- Read chapters 1 and 2, and sections 3.1-3.2 from Fujishige's book.

Announcements, Assignments, and Reminders

- Homework 1 is out, due Wednesday April 23rd, 11:45pm, electronically via our assignment dropbox (`https://canvas.uw.edu/courses/895956/assignments`).
- All homeworks must be done electronically, only PDF file format accepted.
- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, matroids and greedy
- L8:
- L9:
- L10:
- L11:
- L12:
- L13:
- L14:
- L15:
- L16:
- L17:
- L18:
- L19:
- L20:

Finals Week: June 9th-13th, 2014.

System of Distinct Representatives

- Let $(V, \mathcal{V})$ be a set system (i.e., $\mathcal{V} = (V_k : i \in I)$ where $V_i \subseteq V$ for all i), and I is an index set. Hence, $|I| = |\mathcal{V}|$.
- A family $(v_i : i \in I)$ with $v_i \in V$ is said to be a system of distinct representatives of $\mathcal{V}$ if $\exists$ a bijection $\pi : I \leftrightarrow I$ such that $v_i \in V_{\pi(i)}$ and $v_i \neq v_j$ for all $i \neq j$.
- In a system of distinct representatives, there is a requirement for the representatives to be distinct. Lets re-state (and rename) this as a:

Definition 7.2.1 (transversal)

Given a set system $(V, \mathcal{V})$ as defined above, a set $T \subseteq V$ is a transversal of $\mathcal{V}$ if there is a bijection $\pi : T \leftrightarrow I$ such that

$$x \in V_{\pi(x)} \text{ for all } x \in T \tag{7.1}$$

- Note that due to $\pi : T \leftrightarrow I$ being a bijection, all of I and T are "covered" (so this makes things distinct automatically).

When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
- Given a set system $(V, \mathcal{V})$ with $\mathcal{V} = (V_i : i \in I)$, and $V_i \subseteq V$ for all i. Then, for any $J \subseteq I$, let

$$V(J) = \cup_{j \in J} V_j \tag{7.1}$$

so $|V(J)| : 2^I \to \mathbb{Z}_+$ is the set cover func. (we know is submodular).
- We have

Theorem 7.2.1 (Hall's theorem)

Given a set system $(V, \mathcal{V})$, the family of subsets $\mathcal{V} = (V_i : i \in I)$ has a transversal $(v_i : i \in I)$ iff for all $J \subseteq I$

$$|V(J)| \geq |J| \tag{7.2}$$

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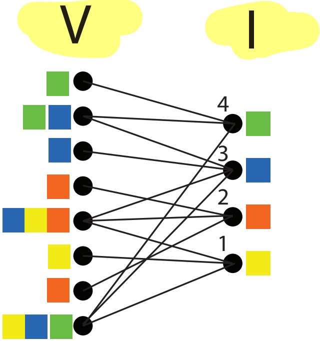

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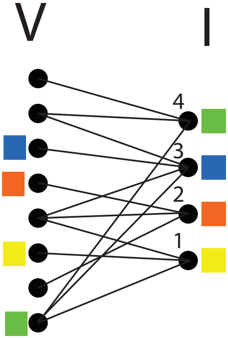

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- Moreover, we have

Theorem 7.2.2 (Rado's theorem (1942))

If $M = (V, r)$ is a matroid on V with rank function r, then the family of subsets $(V_i : i \in I)$ of V has a transversal $(v_i : i \in I)$ that is independent in M iff for all $J \subseteq I$

$$|V(J)| \geq r(V(J)) \geq |J| \tag{7.3}$$

- Note, a transversal T independent in M means that $r(T) = |T|$.

Application's of Hall's theorem

- Consider a set of jobs I and a set of applicants V to the jobs. If an applicant $v \in V$ is qualified for job $i \in I$, we add edge (v, i) to the bipartite graph $G = (V, I, E)$.

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- Note if $|V| = |I|$, then Hall's theorem is the Marriage Theorem (Frobenious 1917), where an edge (v, i) in the graph indicate compatibility between two individuals $v \in V$ and $i \in I$ coming from two separate groups V and I.

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- If $\forall J \subseteq I, |V(J)| \geq |J|$, then all individuals in each group can be matched with a compatible mate.

More general conditions for existence of transversals

Theorem 7.2.1 (Polymatroid transversal theorem)

If $\mathcal{V} = (V_i : i \in I)$ is a finite family of non-empty subsets of V, and $f : 2^V \to \mathbb{Z}_+$ is a non-negative, integral, monotone non-decreasing, and submodular function, then $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that

$$f(\cup_{i \in J}\{v_i\}) \geq |J| \text{ for all } J \subseteq I \tag{7.1}$$

if and only if

$$f(V(J)) \geq |J| \text{ for all } J \subseteq I \tag{7.2}$$

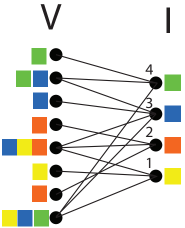
- Given Theorem **??**, we immediately get Theorem 7.2.1 by taking $f(S) = |S|$ for $S \subseteq V$.
- We get Theorem **??** by taking $f(S) = r(S)$ for $S \subseteq V$, the rank function of the matroid.

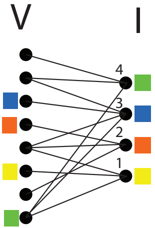
Transversals and Bipartite Matchings

- Transversals correspond exactly to matchings in bipartite graphs.
- Given a set system $(V, \mathcal{V})$, with $\mathcal{V} = (V_i : i \in I)$, we can define a bipartite graph $G = (V, I, E)$ associated with $\mathcal{V}$ that has edge set $\{(v, i) : v \in V, i \in I, v \in V_i\}$.
- A matching in this graph is a set of edges no two of which that have a common endpoint. In fact, we easily have:

Lemma 7.2.4

A subset $T \subseteq V$ is a partial transversal of $\mathcal{V}$ iff there is a matching in (V, I, E) in which every edge has one endpoint in T (T matched into I).





Partial Transversals Are Independent Sets in a Matroid

In fact, we have

Theorem 7.2.4

Let $(V, \mathcal{V})$ where $\mathcal{V} = (V_1, V_2, \dots, V_\ell)$ be a subset system. Let $I = \{1, \dots, \ell\}$. Let $\mathcal{I}$ be the set of partial transversals of $\mathcal{V}$. Then $(V, \mathcal{I})$ is a matroid.

Proof.

- We note that $\emptyset \in \mathcal{I}$ since the empty set is a transversal of the empty subfamily of $\mathcal{V}$, thus (I1') holds.
- We already saw that if T is a partial transversal of $\mathcal{V}$, and if $T' \subseteq T$, then T' is also a partial transversal. So (I2') holds.
- Suppose that T_1 and T_2 are partial transversals of $\mathcal{V}$ such that $|T_1| < |T_2|$. Exercise: show that (I3') holds.

$\square$

Representable

Definition 7.2.4 (Matroid isomorphism)

Two matroids M_1 and M_2 respectively on ground sets V_1 and V_2 are isomorphic if there is a bijection $\pi : V_1 \to V_2$ which preserves independence (equivalently, rank, circuits, and so on).

- Let $\mathbb{F}$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $\mathbb{F}$, such as a Galois field GF(p) where p is prime (such as GF(2)). Succinctly: A field is a set with $+$, $*$, closure, associativity, commutativity, and additive and multiplictaive identities and inverses.
- We can more generally define matroids on a field.

Definition 7.2.6 (representable (as a linear matroid))

Any matroid isomorphic to a linear matroid on a field is called representable over $\mathbb{F}$

F - representable.

Matroids, other definitions using matroid rank $r : 2^V \to \mathbb{Z}_+$

Definition 7.2.1 (closed/flat/subspace)

A subset $A \subseteq E$ is closed (equivalently, a flat or a subspace) of matroid M if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

A hyperplane is a flat of rank $r(M) - 1$.

Definition 7.2.2 (closure)

Given $A \subseteq E$, the closure (or span) of A, is defined by $\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}$.

Therefore, a closed set A has $\text{span}(A) = A$.

Definition 7.2.3 (circuit)

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise-minimal dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

Spanning Sets

- We have the following definitions:

Definition 7.2.6 (spanning set of a set)

Given a matroid $\mathcal{M} = (V, \mathcal{I})$, and a set $Y \subseteq V$, then any set $X \subseteq Y$ such that $r(X) = r(Y)$ is called a spanning set of Y.

Definition 7.2.7 (spanning set of a matroid)

Given a matroid $\mathcal{M} = (V, \mathcal{I})$, any set $A \subseteq V$ such that $r(A) = r(V)$ is called a spanning set of the matroid.

- A base of a matroid is a minimal spanning set (and it is independent) but supersets of a base are also spanning.
- V is always trivially spanning.
- Consider the terminology: "spanning tree in a graph", comes from spanning in a matroid sense.

Dual of a Matroid

- Given a matroid $M = (V, \mathcal{I})$, a dual matroid $M^* = (V, \mathcal{I}^*)$ ~~can be~~ is defined on the same ground set V, but using a very different set of independent sets $\mathcal{I}^*$.
- We define the set of sets $\mathcal{I}^*$ for M^* as follows:

$$\mathcal{I}^* = \{A \subseteq V : V \setminus A \text{ is a spanning set of } M\} \tag{7.21}$$

- That is, a set A is independent in the dual matroid M^* if removal of A from V does not decrease the rank in M:

$$\mathcal{I}^* = \{A \subseteq V : \text{rank}_M(V \setminus A) = \text{rank}_M(V)\} \tag{7.22}$$

- In other words, a set $A \subseteq V$ is independent in the dual M^* (i.e., $A \in \mathcal{I}^*$) if its complement is spanning in M (residual $V \setminus A$ must contain a base in M).
- Dual of the dual: Note, we have that $(M^*)^* = M$.

Dual of a Matroid: Bases

- Since the smallest spanning sets are bases, the bases of M (when $V \setminus I$ is as small as possible while still spanning) are complements of the bases of M^* (where I is as large as possible while still being independent).

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Theorem 7.3.1 (Dual matroid bases)

Let $M = (V, \mathcal{I})$ be a matroid and $\mathcal{B}(M)$ be the set of bases of M. Then define

$$\mathcal{B}^*(M) = \{V \setminus B : B \in \mathcal{B}(M)\}. \tag{7.1}$$

Then $\mathcal{B}^*(M)$ is the set of basis of M^* (that is, $\mathcal{B}^*(M) = \mathcal{B}(M^*)$.

Dual of a Matroid: Terminology

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4. *X is a circuit in M iff $V \setminus X$ is a cohyperplane in M (hyperplane in M^*).*

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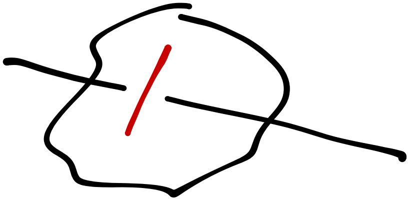

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- All dependent sets in a cocycle matroid are cuts (i.e., a dependent set is a minimal cut or contains one).

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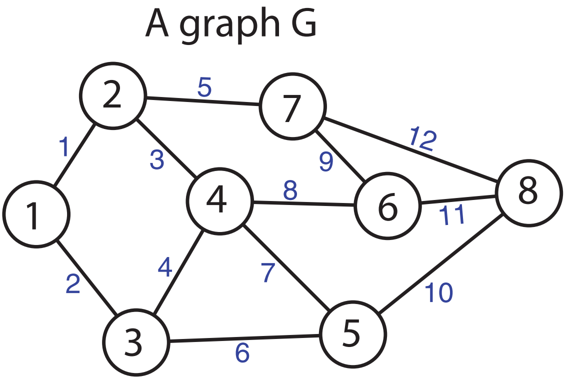

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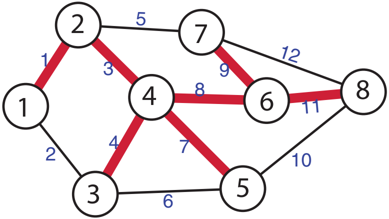
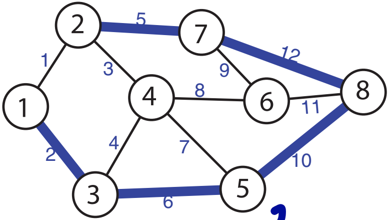


Maximally independent in M* (thus a base, minimally spanning, in M*)



$A \in \mathcal{I}^*$

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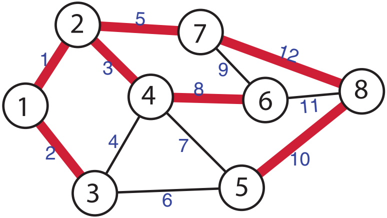
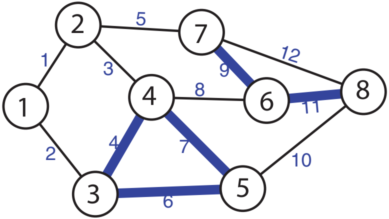
- It consists of all sets of edges the complement of which contains a spanning tree — i.e., an independent set can't consist of edges that, if removed, would render the graph non-spanning.

Minimally spanning in M (and thus a base (maximally independent) in M)



Maximally independent in M* (thus a base, minimally spanning, in M*)



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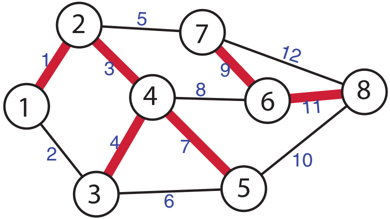
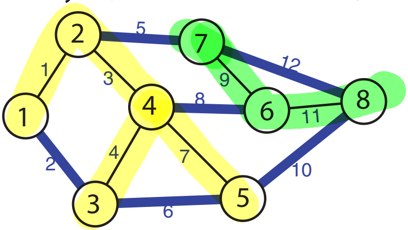
Independent but not spanning in M, and not closed in M.



Dependent in M* (contains a cocycle, is a nonminimal cut)



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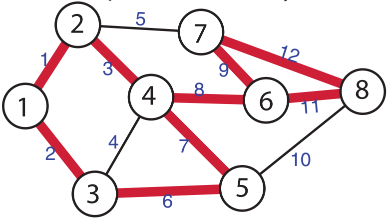
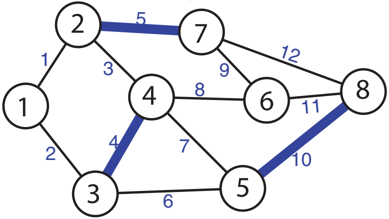
Spanning in M, but not a base, and not independent (has cycles)



Independent in M* (does not contain a cut)



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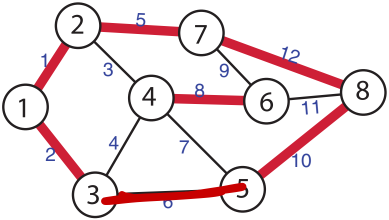

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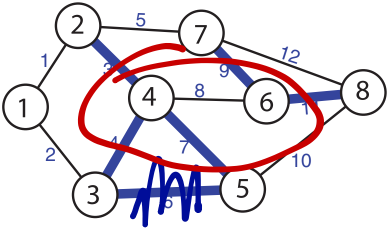

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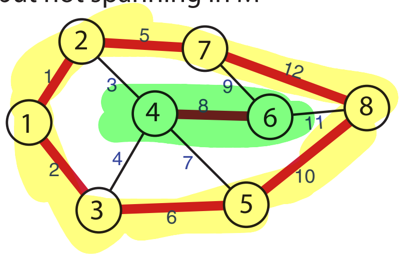
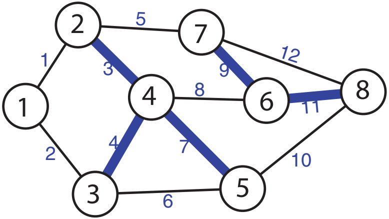
A hyperplane in M, dependent but not spanning in M



A cycle in M* (minimally dependent in M*, a cocycle, or a minimal cut)



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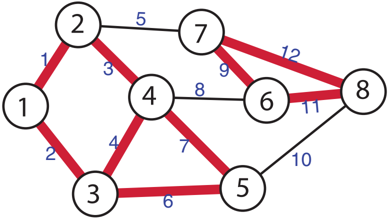

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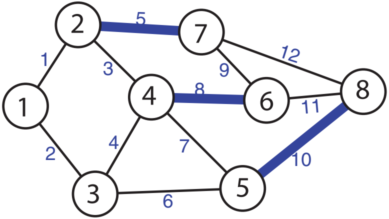

The dual of a matroid is (indeed) a matroid

Theorem 7.3.3

Let $M^* = (V, \mathcal{I}^*)$ be previously defined. Then M^* is a matroid.

Proof.

- Clearly $\emptyset \in \mathcal{I}^*$, so (I1') holds.

...

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Let $M^* = (V, \mathcal{I}^*)$ be previously defined. Then M^* is a matroid.

Proof.

- Clearly $\emptyset \in I^*$, so (I1') holds.
- Also, if $I \subseteq J \in \mathcal{I}^*$, then clearly also $I \in \mathcal{I}^*$ since if $V \setminus J$ is spanning in M, so must $V \setminus I$. Therefore, (I2') holds.

. . .

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Proof.

- Consider $I, J \in \mathcal{I}^*$ with $|I| < |J|$. We need to show that there is some member $v \in J \setminus I$ such that $I + v$ is independent in M^*, which means that $V \setminus (I + v) = (V \setminus I) \setminus v$ is still spanning in M. That is, removing v from $V \setminus I$ doesn't make $(V \setminus I) \setminus v$ not spanning.

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- Since $V \setminus J$ is spanning in M, $V \setminus J$ contains some base (say $B \subseteq V \setminus J$) of M. Also, $V \setminus I$ contains a base of M, say $B' \subseteq V \setminus I$.

Base   spanning set in M.

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- Since $B \setminus I \subseteq V \setminus I$, and $B \setminus I$ is independent in M, we can choose the base B' of M s.t. $B \setminus I \subseteq B' \subseteq V \setminus I$.

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- Since $B \setminus I \subseteq V \setminus I$, and $B \setminus I$ is independent in M, we can choose the base B' of M s.t $B \setminus I \subseteq B' \subseteq V \setminus I$.
- Since B and J are disjoint, we have both: 1) $B \setminus I$ and $J \setminus I$ are disjoint; and 2) $B \cap I \subseteq I \setminus J$. Also note, B' and I are disjoint.

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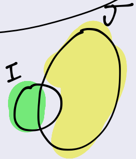
Proof.

- Now $J \setminus I \not\subseteq B'$, since otherwise (i.e., assuming $J \setminus I \subseteq B'$):

$|J| > |I|$

$$|B| = |B \cap I| + |B \setminus I| \tag{7.2}$$
$$\leq |I \setminus J| + |B \setminus I| \tag{7.3}$$
$$< |J \setminus I| + |B \setminus I| \leq |B'| \tag{7.4}$$



which is a contradiction. *The last inequality on the right follows since $J \setminus I \subseteq B'$ (by assumption) and $B \setminus I \subseteq B'$ implies that $(J \setminus I) \cup (B \setminus I) \subseteq B'$, but since J and B are disjoint, we have that $|J \setminus I| + |B \setminus I| \leq |B'|$.*

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  which is a contradiction.

- Therefore, $J \setminus I \not\subseteq B'$, and there is a $v \in J \setminus I$ s.t. $v \notin B'$.
- So B' is disjoint with $I \cup \{v\}$, means $B' \subseteq V \setminus (I \cup \{v\})$, or $V \setminus (I \cup \{v\})$ is spanning in M, and therefore $I \cup \{v\} \in \mathcal{I}^*$.

□

Matroid Duals and Representability

Theorem 7.3.4

Let M be an $\mathbb{F}$-representable matroid (i.e., one that can be represented by a finite sized matrix over field $\mathbb{F}$). Then M^* is also $\mathbb{F}$-representable.

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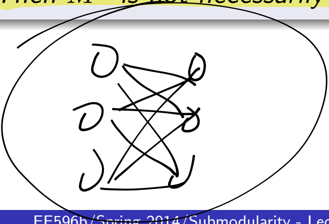
Theorem 7.3.5

Let M be a graphic matroid (i.e., one that can be represented by a graph $G = (V, E)$). Then M^* is not necessarily also graphic.



Dual Matroid Rank

Theorem 7.3.6

The rank function r_{M^*} of the dual matroid M^* may be specified in terms of the rank r_M in matroid M as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \tag{7.5}$$

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2. *I.e., $|X|$ is modular, complement $f(V \setminus X)$ is submodular if f is submodular, $r_M(V)$ is a constant, and summing submodular functions and a constant preserves submodularity.*

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- Therefore, r_{M^*} is the rank function of a matroid. That it is the dual matroid rank function is shown in the next proof.

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Proof.

A set X is independent in (V, r_{M^*}) if and only if

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) = |X| \tag{7.6}$$

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or

$$r_M(V \setminus X) = r_M(V) \tag{7.7}$$

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But a subset X is independent in M^* only if $V \setminus X$ is spanning in M (by the definition of the dual matroid). □

Matroid restriction/deletion

- Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then

$$\mathcal{I}_Y = \{Z : Z \subseteq Y, Z \in \mathcal{I}\} \tag{7.8}$$

is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid with rank $r(M_Y) = r(Y)$.

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$G \setminus A$

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- Hence, $M|Y = M \setminus (V \setminus Y)$.
- The rank function is of the same form. I.e., $r_Y : 2^Y \to \mathbb{Z}_+$, where $r_Y(Z) = r(Z)$ for $Z \subseteq Y$.

Matroid contraction

- Contraction is dual to deletion, and is like a forced inclusion of contained base, but with a similar ground set removal. Contracting Z is written M/Z.

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$r(Z) = r(X)$

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- A minor of a matroid is any matroid obtained via a series of deletions and contractions of some matroid.

Matroid Intersection

- Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$.

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- While $(V, \mathcal{I}_1 \cap \mathcal{I}_2)$ is typically not a matroid (Exercise: show graphical example.), we might be interested in finding the maximum size common independent set. That is, find $\max |X|$ such that both $X \in \mathcal{I}_1$ and $X \in \mathcal{I}_2$.

Matroid Intersection

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Let M_1 and M_2 be given as above, with rank functions r_1 and r_2. Then the size of the maximum size set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by

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This is an instance of the convolution of two submodular functions, f_1 and f_2 that, evaluated at $Y \subseteq V$, is written as:

$$(f_1 * f_2)(Y) = \min_{X \subseteq Y} \Big(f_1(X) + f_2(Y \setminus X) \Big) \tag{7.12}$$

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Exercise: $f(\cdot) * |\cdot| \in$ submodular

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- $\Leftrightarrow \quad [\Gamma(\cdot) * |\cdot|](V) \geq |V|$
- So Hall's theorem can be expressed as convolution.
- Note, in general, convolution of two submodular functions does not preserve submodularity (but in certain special cases it does).

Matroid Union

Definition 7.4.2

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, $\dots$, $M_k = (V_k, \mathcal{I}_k)$ be matroids. We define the union of matroids as
$M_1 \vee M_2 \vee \dots \vee M_k = (V_1 \uplus V_2 \uplus \dots \uplus V_k, \mathcal{I}_1 \vee \mathcal{I}_2 \vee \dots \vee \mathcal{I}_k)$, where

$$I_1 \vee \mathcal{I}_2 \vee \dots \vee \mathcal{I}_k = \{I_1 \uplus I_2 \uplus \dots \uplus I_k | I_1 \in \mathcal{I}_1, \dots, I_k \in \mathcal{I}_k\} \quad (7.13)$$

Note $A \uplus B$ designates the disjoint union of A and B.

$\{1,2,3\} \uplus \{1,2\}$

$= \{(1,1), (1,2), (1,3), (2,1), (2,3)\}$

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Theorem 7.4.3

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, $\dots$, $M_k = (V_k, \mathcal{I}_k)$ be matroids, with rank functions $r_1, \dots, r_k$. Then the union of these matroids is still a matroid, having rank function

$$r(Y) = \min_{X \subseteq Y} \Big(|Y \setminus X| + r_1(X \cap V_1) + \dots + r_k(X \cap V_k) \Big) \quad (7.14)$$

for any $Y \subseteq V_1 \cup \dots V_k$.

Exercise: Matroid Union, and Matroid duality

Exercise: Describe $M \vee M^*$.

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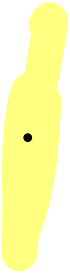
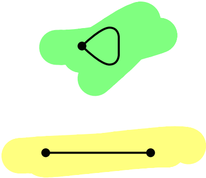
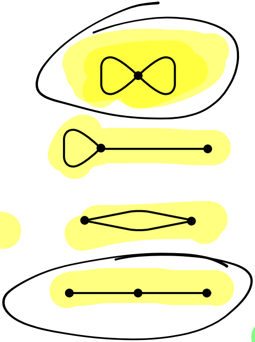


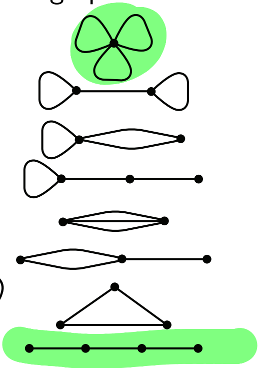
(a) The only matroid with zero elements.



(b) The two one-element matroids.



(c) The four two-element matroids.



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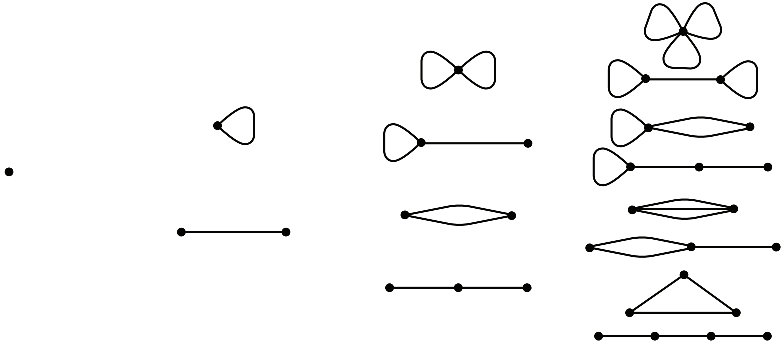

(a) The only matroid with zero elements.

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(c) The four two-element matroids.

(d) The eight three-element matroids.

- This is a nice way to show matroids with low ground set sizes. What about matroids that are low rank but with many elements?

Affine Matroids

- Given an $n \times m$ matrix with entries over some field $\mathbb{F}$, we say that a subset $S \subseteq \{1, \dots, m\}$ of indices (with corresponding column vectors $\{v_i : i \in S\}$, with $|S| = k$) is affinely dependent if $m \geq 1$ and there exists elements $\{a_1, \dots, a_k\} \in \mathbb{F}$, not all zero with $\sum_{i=1}^k a_i = 0$, such that $\sum_{i=1}^k a_i v_i = 0$.

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Exercise: prove this.

Euclidean Representation of Low-rank Matroids

- Consider the affine matroid with $n \times m = 2 \times 6$ matrix on the field $\mathbb{F} = \mathbb{R}$, and let the elements be $\{(0,0),(1,0),(2,0),(0,1),(0,2),(1,1)\}$.

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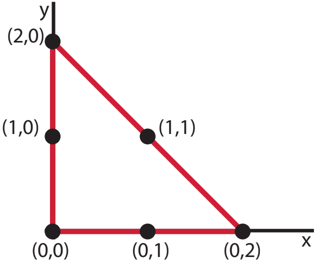
- Hence, we can plot the points in $\mathbb{R}^2$ as follows:



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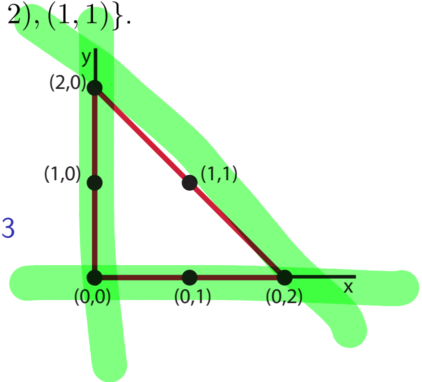

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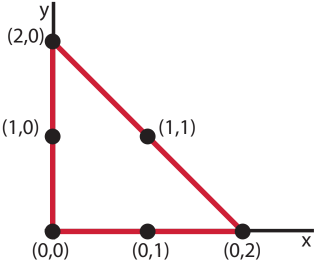

- In general, for a matroid $\mathcal{M}$ of rank $m+1$ with $m \leq 3$, then a subset X in a geometric representation in $\mathbb{R}^m$ is dependent if: 1) $|X| \geq 2$ and the points are identical; 2) $|X| \geq 3$ and the points are collinear; 3) $|X| \geq 4$ and the points are coplanar; or 4) $|X| \geq 5$ and the points are in space.

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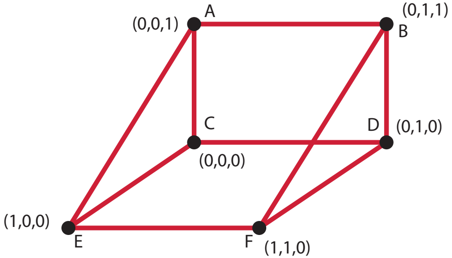

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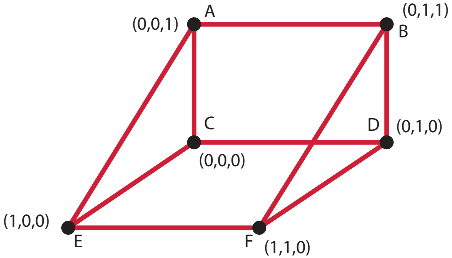

- All sets of 5 points are dependent. The only other sets of dependent points are coplanar ones of size 4. Namely:
  $\{(0,0,0),(0,1,0),(1,1,0),(1,0,0)\}$,
  $\{(0,0,0),(0,0,1),(0,1,1),(0,1,0)\}$, and
  $\{(0,0,1),(0,1,1),(1,1,0),(1,0,0)\}$.

Euclidean Representation of Low-rank Matroids: A test

- Loops represented by a separate box indicating how many loops there are. Parallel elements indicated by a multiplicity next to a point.

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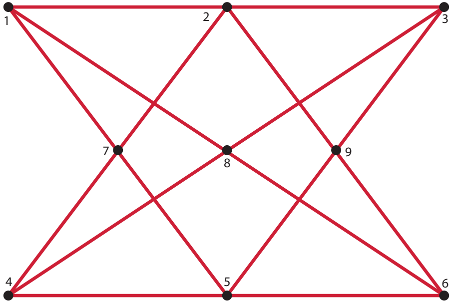

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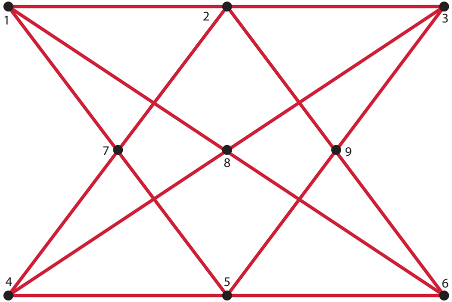

- Called the non-Pappus matroid. Has rank three, but any matric matroid with the above dependencies would require that $\{7, 8, 9\}$ is dependent, hence requiring an additional line in the above.

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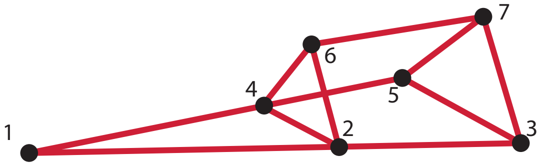

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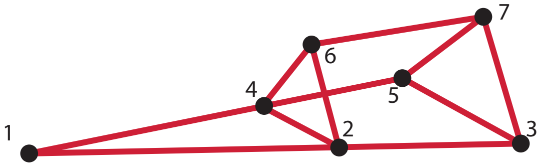

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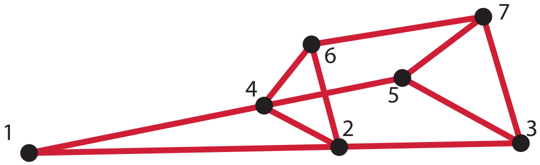

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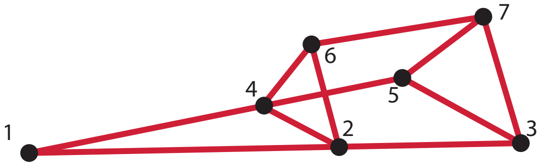

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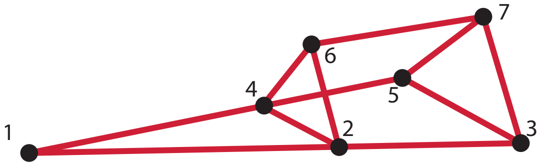

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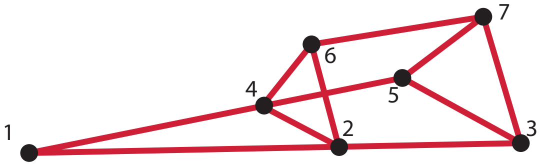

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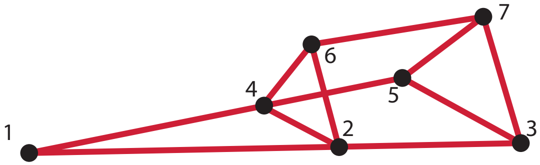

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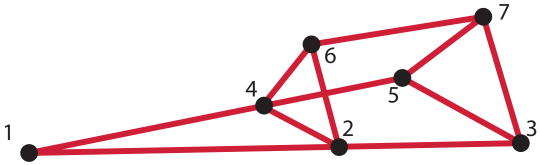

- Check rank's submodularity: Let $X = \{1, 2, 3, 6, 7\}$, $Y = \{1, 4, 5, 6, 7\}$. So $r(X) = 3$, and $r(Y) = 3$, and $r(X \cup Y) = 4$, so we must have, by submodularity, that $r(\{1, 6, 7\}) = r(X \cap Y) \leq r(X) + r(Y) - r(X \cup Y) = 2$.

Euclidean Representation of Low-rank Matroids: A test

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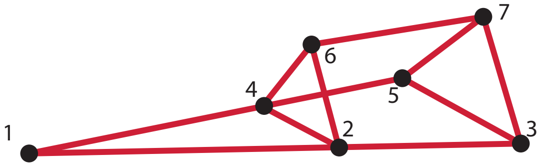

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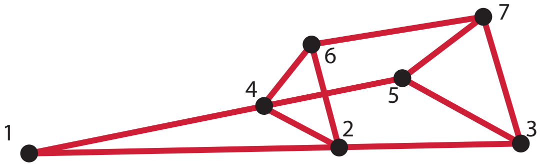

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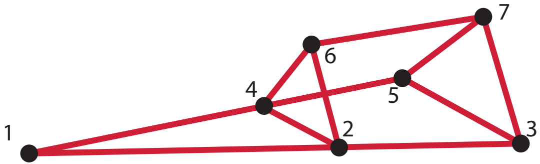

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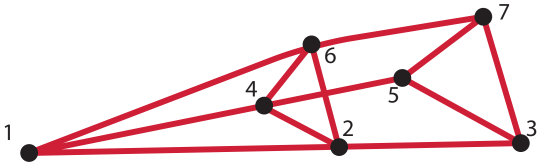
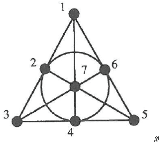
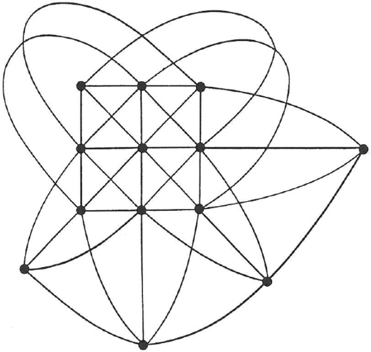


- If we extend the line from 6-7 to 1, then is it a matroid?
- Hence, not all 2D or 3D graphs of points and lines are matroids.

Euclidean Representation of Low-rank Matroids: Other Examples

- Other examples can be more complex, consider the following two matroids (from Oxley, 2011):





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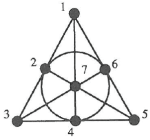

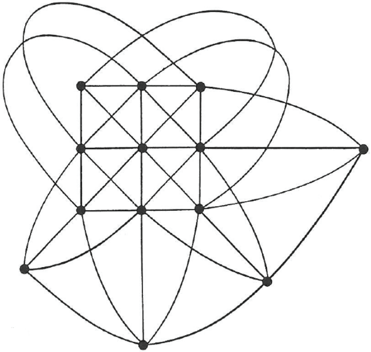

- Hence, lines (in 2D) may be curved and planes (in 3D) can be twisted.

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- Matroid of rank at most four (see Oxley 2011 for more details).

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- Greedy is good since it can be made to run very fast $O(n \log n)$.
- Often, however, greedy is heuristic (it might work well in practice, but worst-case performance can be unboundedly poor).
- We will next see that the greedy algorithm working is a defining property of a matroid, and is also a defining property of a polymatroid function.

Matroid and the greedy algorithm

- Let $(E, \mathcal{I})$ be an independence system, and we are given a non-negative modular weight function $w : E \to \mathbb{R}_+$.

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while $\exists v \in E \setminus X$ *s.t.* $X \cup \{v\} \in \mathcal{I}$ **do**
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Theorem 7.5.1

Let $(E, \mathcal{I})$ be an independence system. Then the pair $(E, \mathcal{I})$ is a matroid if and only if for each weight function $w \in \mathcal{R}_+^E$, Algorithm 1 leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.

Matroids by bases

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

Theorem 7.5.1 (Matroid (by bases))

Let E be a set and $\mathcal{B}$ be a nonempty collection of subsets of E. Then the following are equivalent.

1. *$\mathcal{B}$ is the collection of bases of a matroid;*
2. *if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.*
3. *If $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.*

Properties 2 and 3 are called "exchange properties."
Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.

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proof of Theorem 7.5.1.

- Assume $(E, \mathcal{I})$ is a matroid and $w : E \to \mathcal{R}_+$ is given.

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- A is a base of M, and let $B = (b_1, \dots, b_r)$ be any another base of M with elements also ordered decreasing by weight.
- We next show that not only is $w(A) \geq w(B)$ but that $w(a_i) \geq w(b_i)$ for all i.

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- But $w(b_i) \geq w(b_k) > w(a_k)$, and so the greedy algorithm would have chosen b_i rather than a_k, contradicting what greedy does.

$\square$

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converse proof of Theorem 7.5.1.

- Given an independence system $(E, \mathcal{I})$, suppose the greedy algorithm leads to an independent set of max weight for every non-negative weight function. We'll show $(E, \mathcal{I})$ is a matroid.

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- Let $I, J \in \mathcal{I}$ with $|I| < |J|$. Suppose to the contrary, that $I \cup \{z\} \notin \mathcal{I}$ for all $z \in J \setminus I$.
- Define the following modular weight function w on E, and define $k = |I|$.

$$w(v) = \begin{cases} k+2 & \text{if } v \in I, \\ k+1 & \text{if } v \in J \setminus I, \\ 0 & \text{if } v \in E \setminus (I \cup J) \end{cases} \tag{7.15}$$

Matroid and the greedy algorithm

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- Now greedy will, after k iterations, recover I, but it cannot choose any element in $J \setminus I$ by assumption. Thus, greedy chooses a set of weight $k(k+2)$.

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  so J has strictly larger weight but is still independent, contradicting greedy's optimality.
- Therefore, $(E, \mathcal{I})$ must be a matroid.

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- If we stop at a negative value, we'll once again get a maximum weight independent set.
- We can instead do as small as possible thus giving us a minimum weight independent set/base.

Summary of Important (for us) Matroid Definitions

Given an independence system, matroids are defined equivalently by any of the following:

- All maximally independent sets have the same size.
- A monotone non-decreasing submodular integral rank function with unit increments.
- The greedy algorithm achieves the maximum weight independent set for all weight functions.