Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 6 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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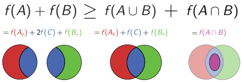
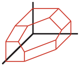
April 16th, 2014





Cumulative Outstanding Reading

- Read chapters 1 and 2, and sections 3.1-3.2 from Fujishige's book.

Announcements, Assignments, and Reminders

- Homework 1 is out, due Wednesday April 23rd, 11:45pm, electronically via our assignment dropbox (`https://canvas.uw.edu/courses/895956/assignments`).
- All homeworks must be done electronically, only PDF file format accepted.
- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation, Dual Matroid
- L7:
- L8:
- L9:
- L10:
- L11:
- L12:
- L13:
- L14:
- L15:
- L16:
- L17:
- L18:
- L19:
- L20:

Finals Week: June 9th-13th, 2014.

Matroid

Slight modification (non unit increment) that is equivalent.

Definition 6.2.1 (Matroid-II)

A set system $(E, \mathcal{I})$ is a Matroid if

(I1') $\emptyset \in \mathcal{I}$

(I2') $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ (or "down-closed")

(I3') $\forall I, J \in \mathcal{I}$, with $|I| > |J|$, then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$

Note (I1)=(I1'), (I2)=(I2'), and we get (I3)$\equiv$(I3') using induction.

Matroids - important property

Proposition 6.2.1

In a matroid $M = (E, \mathcal{I})$, for any $U \subseteq E(M)$, any two bases of U have the same size.

- In matrix terms, given a set of vectors U, all sets of independent vectors that span the space spanned by U have the same size.
- In fact, under (I1),(I2), this condition is equivalent to (I3). Exercise: show the following is equivalent to the above.

Definition 6.2.2 (Matroid)

A set system $(V, \mathcal{I})$ is a Matroid if

(I1') $\emptyset \in \mathcal{I}$ (emptyset containing)

(I2') $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ (down-closed or subclusive)

(I3') $\forall X \subseteq V$, and $I_1, I_2 \in \text{maxInd}(X)$, we have $|I_1| = |I_2|$ (all maximally independent subsets of X have the same size).

Matroids - rank

- Thus, in any matroid $M = (E, \mathcal{I})$, $\forall U \subseteq E(M)$, any two bases of U have the same size.
- The common size of all the bases of U is called the rank of U, denoted $r_M(U)$ or just $r(U)$ when the matroid in equation is unambiguous.
- $r(E) = r_{(E,\mathcal{I})}$ is the rank of the matroid, and is the common size of all the bases of the matroid.
- We can a bit more formally define the rank function this way.

Definition 6.2.1 (matroid rank function)

The rank of a matroid is a function $r : 2^E \to \mathbb{Z}_+$ defined by

$$r(A) = \max \{|X| : X \subseteq A, X \in \mathcal{I}\} = \max_{X \in \mathcal{I}} |A \cap X| \tag{6.1}$$

- From the above, we immediately see that $r(A) \leq |A|$.
- Moreover, if $r(A) = |A|$, then $A \in \mathcal{I}$, meaning A is independent (in this case, A is a self base).

Matroids - rank

Lemma 6.2.1

The rank function $r : 2^E \to \mathbb{Z}_+$ of a matroid is submodular, that is $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$

Proof.

1. Let $X \in \mathcal{I}$ be an inclusionwise maximal set with $X \subseteq A \cap B$
2. Let $Y \in \mathcal{I}$ be inclusionwise maximal set with $X \subseteq Y \subseteq A \cup B$.
3. Since M is a matroid, we know that $r(A \cap B) = r(X) = |X|$, and $r(A \cup B) = r(Y) = |Y|$. Also, for any $U \in \mathcal{I}$, $r(A) \geq |A \cap U|$.
4. Then we have

$$r(A) + r(B) \geq |Y \cap A| + |Y \cap B| \tag{6.3}$$

$$= |Y \cap (A \cap B)| + |Y \cap (A \cup B)| \tag{6.4}$$

$$\geq |X| + |Y| = r(A \cap B) + r(A \cup B) \tag{6.5}$$

□

Partition Matroid

- Let V be our ground set.
- Let $V = V_1 \cup V_2 \cup \cdots \cup V_\ell$ be a partition of V into blocks or disjoint sets (disjoint union). Define a set of subsets of V as

$$\mathcal{I} = \{X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell\}. \tag{6.3}$$

  where $k_1, \ldots, k_\ell$ are fixed parameters, $k_i \geq 0$. Then $M = (V, \mathcal{I})$ is a matroid.
- Note that a k-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$.
- We'll show that property (I3') in Def **??** holds. If $X, Y \in \mathcal{I}$ with $|Y| > |X|$, then there must be at least one i with $|Y \cap V_i| > |X \cap V_i|$. Therefore, adding one element $e \in V_i \cap (Y \setminus X)$ to X won't break independence.

Partition Matroid

- What is the partition matroid's rank function?
- A partition matroids rank function:

$$r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i) \tag{6.12}$$

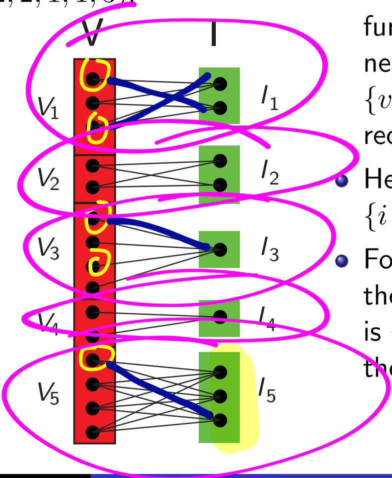
which we also immediately see is submodular using properties we spoke about last week. That is:

1. $|A \cap V_i|$ is submodular (in fact modular) in A
2. $\min(\text{submodular}(A), k_i)$ is submodular in A since $|A \cap V_i|$ is monotone.
3. sums of submodular functions are submodular.

- $r(A)$ is also non-negative integral monotone non-decreasing, so it defines a matroid (the partition matroid).

Partition Matroid, rank as matching

- Example where $\ell = 5$, $(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3)$.



- Recall, $\Gamma : 2^V \to \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of X is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.
- Here, for $X \subseteq V$, we have $\Gamma(X) = \{i \in I : (v, i) \in E(G) \text{ and } v \in X\}$.
- For such a constructed bipartite graph, the rank function of a partition matroid is $r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i) =$ the maximum matching involving X.

System of Representatives

- Let $(V, \mathcal{V})$ be a set system (i.e., $\mathcal{V} = (V_i : i \in I)$ where $\emptyset \subset V_i \subseteq V$ for all i), and I is an index set. Hence, $|I| = |\mathcal{V}|$.

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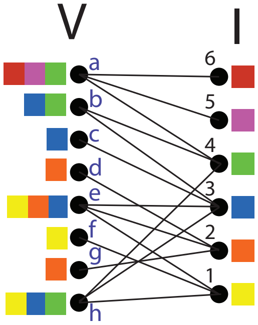
- Here, $\ell = 6$ groups, with $\mathcal{V} = (V_1, V_2, \dots, V_6)$ $= (\{e, f, h\}, \{d, e, g\}, \{b, c, e, h\}, \{a, b, h\}, \{a\}, \{a\})$.



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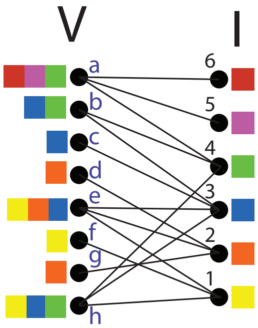

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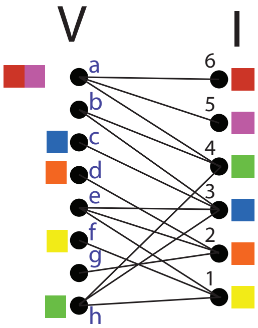

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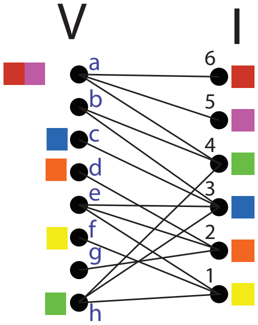

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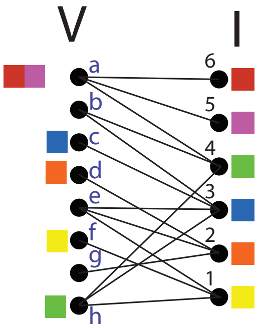

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- Here, the set of representatives is not distinct. Why? In fact, due to the red and pink group, a distinct group of representatives is impossible (since there is only one common choice to represent both color groups).

System of Distinct Representatives

- Let $(V, \mathcal{V})$ be a set system (i.e., $\mathcal{V} = (V_k : i \in I)$ where $V_i \subseteq V$ for all i), and I is an index set. Hence, $|I| = |\mathcal{V}|$.

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Definition 6.3.1 (transversal)

Given a set system $(V, \mathcal{V})$ as defined above, a set $T \subseteq V$ is a transversal of $\mathcal{V}$ if there is a bijection $\pi : T \leftrightarrow I$ such that

$$x \in V_{\pi(x)} \text{ for all } x \in T \tag{6.1}$$

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- Note that due to $\pi : T \leftrightarrow I$ being a bijection, all of I and T are "covered" (so this makes things distinct automatically).

Transversals are Subclusive

- A set $X \subseteq V$ is a partial transversal if X is a transversal of some subfamily $\mathcal{V}' = (V_i : i \in I')$ where $I' \subseteq I$.

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- Therefore, for any transversal T, any subset $T' \subseteq T$ is a partial transversal.
- Thus, transversals are down closed (subclusive).

When do transversals exist?

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- Given a set system $(V, \mathcal{V})$ with $\mathcal{V} = (V_i : i \in I)$, and $V_i \subseteq V$ for all i. Then, for any $J \subseteq I$, let

$$\boxed{V(J) = \cup_{j \in J} V_j} \tag{6.2}$$

so $|V(J)|$ is the set cover function (which we know is submodular).

$$|V(J)| : 2^I \to \mathbb{Z}_+$$

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- We have

Theorem 6.4.1 (Hall's theorem)

Given a set system $(V, \mathcal{V})$, the family of subsets $\mathcal{V} = (V_i : i \in I)$ has a transversal $(v_i : i \in I)$ iff for all $J \subseteq I$

$$|V(J)| \geq |J| \tag{6.3}$$

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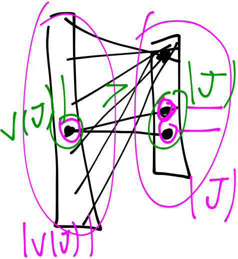
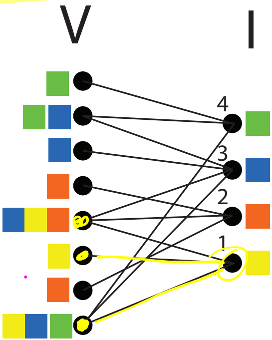
so $|V(J)|$ is the set cover function (which we know is submodular).
- Hall's theorem $(\forall J, |V(J)| \geq |J|)$ as a bipartite graph.





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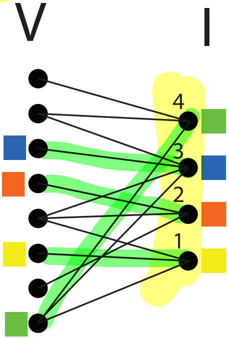

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- Moreover, we have

Theorem 6.4.2 (Rado's theorem)

If $M = (V, r)$ is a matroid on V with rank function r, then the family of subsets $(V_i : i \in I)$ of V has a transversal $(v_i : i \in I)$ that is independent in M iff for all $J \subseteq I$

$$r(V(J)) \geq |J| \tag{6.4}$$

When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
- Given a set system $(V, \mathcal{V})$ with $\mathcal{V} = (V_i : i \in I)$, and $V_i \subseteq V$ for all i. Then, for any $J \subseteq I$, let

$$V(J) = \cup_{j \in J} V_j \tag{6.2}$$

  so $|V(J)|$ is the set cover function (which we know is submodular).
- Moreover, we have

Theorem 6.4.2 (Rado's theorem)

If $M = (V, r)$ is a matroid on V with rank function r, then the family of subsets $(V_i : i \in I)$ of V has a transversal $(v_i : i \in I)$ that is independent in M iff for all $J \subseteq I$

$$r(V(J)) \geq |J| \tag{6.4}$$

- Note, a transversal T independent in M means that $r(T) = |T|$.

More general conditions for existence of transversals

Theorem 6.4.3 (Polymatroid transversal theorem)

If $\mathcal{V} = (V_i : i \in I)$ is a finite family of non-empty subsets of V, and $f : 2^V \to \mathbb{Z}_+$ is a non-negative, integral, monotone non-decreasing, and submodular function, then $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that

$$f(\cup_{i \in J}\{v_i\}) \geq |J| \text{ for all } J \subseteq I \tag{6.5}$$

if and only if

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- Given Theorem 6.4.3, we immediately get Theorem 6.4.1 by taking $f(S) = |S|$ for $S \subseteq V$. *In which case, Eq. 6.5 requires the system of representatives to be distinct.*

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$\subseteq V(J)$

$$f(\cup_{i \in J}\{v_i\}) \geq |J| \text{ for all } J \subseteq I \tag{6.5}$$

$V(J) \supseteq \bigcup_{i \in J} \{v_i$

if and only if

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- Given Theorem 6.4.3, we immediately get Theorem 6.4.1 by taking $f(S) = |S|$ for $S \subseteq V$.
- We get Theorem 6.4.2 by taking $f(S) = r(S)$ for $S \subseteq V$, the rank function of the matroid. *where, Eq. 6.5 insists the system of representatives is independent in M.*

Submodular Composition with Set-to-Set functions

- Note the condition in Theorem 6.4.3 is $f(V(J)) \geq |J|$ for all $J \subseteq I$, where $f : 2^V \to \mathbb{Z}_+$ is non-negative, integral, monotone non-decreasing and submodular, and $V(J) = \cup_{j \in J} V_j$ with $V_i \subseteq V$.

$V(J) : 2^I \to 2^V$

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- Define $g : 2^I \to \mathbb{Z}$ with $g(J) = f(V(J)) - |J|$, then the condition for the existence of a system of representatives, with quality Equation 6.5, becomes:

$$\min_{J \subseteq I} g(J) \geq 0 \tag{6.7}$$

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Proposition 6.4.4

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- Hence, the condition for existence can be solved by (a special case of) submodular function minimization, or vice verse!

More general conditions for existence of transversals

first part proof of Theorem 6.4.3.

- Suppose $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that Eq. 6.5 is true.

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first part proof of Theorem 6.4.3.

- Suppose $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that Eq. 6.5 is true.
- Then since f is monotone, and since $V(J) \supseteq \cup_{i \in J}\{v_i\}$ when $(v_i : i \in I)$ is a system of representatives, then Eq. 6.6 immediately follows.

...

More general conditions for existence of transversals

Lemma 6.4.5 (contraction lemma)

Suppose Eq. 6.6 ($f(V(J)) \geq |J|, \forall J \subseteq I$) is true for $\mathcal{V} = (V_i : i \in I)$, and there exists an i such that $|V_i| \geq 2$ (w.l.o.g., say $i = 1$). Then there exists $\bar{v} \in V_1$ such that the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 6.6.

Proof.

- When Eq. 6.6 holds, this means that for any subsets $J_1, J_2 \subseteq I \setminus \{1\}$, we have that, for $J \in \{J_1, J_2\}$,

$$f(V(J \cup \{1\})) \geq |J \cup \{1\}| \tag{6.8}$$

and hence

$$f(V_1 \cup V(J_1)) \geq |J_1| + 1 \tag{6.9}$$
$$f(V_1 \cup V(J_2)) \geq |J_2| + 1 \tag{6.10}$$

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- Suppose, to the contrary, the consequent is false. Then we may take any $\bar{v}_1, \bar{v}_2 \in V_1$ as two distinct elements in V_1 ...

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- Suppose, to the contrary, the consequent is false. Then we may take any $\bar{v}_1, \bar{v}_2 \in V_1$ as two distinct elements in V_1 ...
- ...and there must exist subsets J_1, J_2 of $I \setminus \{1\}$ such that

$$f(X) = f(\overbrace{(V_1 \setminus \{\bar{v}_1\}) \cup V(J_1)}^{X}) < |J_1| + 1, \qquad (6.11)$$

$$f(Y) = f(\underbrace{(V_1 \setminus \{\bar{v}_2\}) \cup V(J_2)}_{Y}) < |J_2| + 1, \qquad (6.12)$$

(note that either one or both of J_1, J_2 could be empty).

...

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Proof.

- Taking $X = (V_1 \setminus \{\bar{v}_1\}) \cup V(J_1)$ and $Y = (V_1 \setminus \{\bar{v}_2\}) \cup V(J_2)$, we have $f(X) \leq |J_1|$, $f(Y) \leq |J_2|$, and that:

$$X \cup Y = V_1 \cup V(J_1 \cup J_2), \tag{6.13}$$

$$X \cap Y \supseteq V(J_1 \cap J_2), \tag{6.14}$$

$|J_1| + |J_2| \geq f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$

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Proof.

- since f submodular monotone non-decreasing, & Eqs 6.13-6.15,

$$|J_1| + |J_2| \geq f(V_1 \cup V(J_1 \cup J_2)) + f(V(J_1 \cap J_2)) \tag{6.16}$$

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- Since $\mathcal{V}$ satisfies Eq. 6.6, $1 \notin J_1 \cup J_2$, & Eqs 6.9-6.10, this gives

$$\boxed{|J_1| + |J_2| \geq |J_1 \cup J_2| + 1 + |J_1 \cap J_2|} \tag{6.17}$$

which is a contradiction since cardinality is modular.

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- Given Theorem 6.4.3, we immediately get Theorem 6.4.1 by taking $f(S) = |S|$ for $S \subseteq V$.
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- W.l.o.g., let $|V_1| \geq 2$, then by Lemma 6.4.5, the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 6.6 for the right $\bar{v}$.

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- We can continue to reduce the family, deleting elements from V_i for some i while $|V_i| \geq 2$, until we arrive at a family of singleton sets.

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- This family will be the required system of representatives.

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This theorem can be used to produce a variety of other results quite easily, and shows how submodularity is the key ingredient in its truth.

Transversal Matroid

Transversals, themselves, define a matroid.

Theorem 6.5.1

If $\mathcal{V}$ is a family of finite subsets of a ground set V, then the collection of partial transversals of $\mathcal{V}$ is the set of independent sets of a matroid $M = (V, \mathcal{V})$ on V.

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- This means that the transversals of $\mathcal{V}$ are the bases of matroid M.
- Therefore, all maximal partial transversals of $\mathcal{V}$ have the same cardinality!

Transversals and Bipartite Matchings

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- A matching in this graph is a set of edges no two of which that have a common endpoint.
- In fact, we easily have

Lemma 6.5.2

A subset $T \subseteq V$ is a partial transversal of $\mathcal{V}$ iff there is a matching in (V, I, E) in which every edge has one endpoint in T.

We say that T is matched into I.

Arbitrary Matchings and Matroids?

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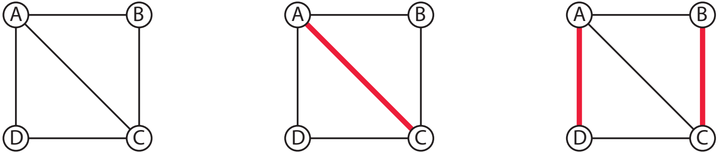
- Consider the following graph (left), and two max-matchings (two right instances)



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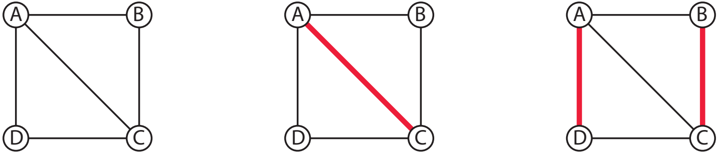
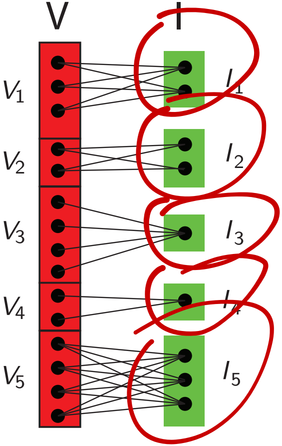


- $\{AC\}$ is a maximum matching, as is $\{AD, BC\}$, but they are not the same size.

Partition Matroid, rank as matching

- Example where $\ell = 5$, $(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3)$.



- Recall, $\Gamma : 2^V \to \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of X is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.
- Here, for $X \subseteq V$, we have $\Gamma(X) = \{i \in I : (v, i) \in E(G) \text{ and } v \in X\}$.
- For such a constructed bipartite graph, the rank function of a partition matroid is $r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i) =$ the maximum matching involving X.

Morphing Partition Matroid Rank

- Recall the partition matroid rank function. Note, $k_i = |I_i|$ in the bipartite graph representation, and since a matroid, w.l.o.g., $|V_i| \geq k_i$ (also, recall, $V(J) = \cup_{j \in J} V_j$).

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- We start with partition matroid rank function in the subsequent equations.

$$r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i) \tag{6.18}$$

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$$= \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} \left(\begin{Bmatrix} |A \cap V(I_i)| & \text{if } J_i \neq \emptyset \\ 0 & \text{if } J_i = \emptyset \end{Bmatrix} + |I_i \setminus J_i| \right) \tag{6.20}$$

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Exercise:

$$= \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} \left(|V(J_i) \cap A| + |I_i \setminus J_i| \right) \tag{6.21}$$

... Morphing Partition Matroid Rank

- Continuing,

$$r(A) = \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} \left(|V(J_i) \cap V(I_i) \cap A| - |I_i \cap J_i| + |I_i| \right) \tag{6.22}$$

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$$= \min_{J \subseteq I} \left(|V(J) \cap V(I) \cap A| - |J| + |I| \right) \tag{6.24}$$

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$$r(A) = \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} \left(|V(J_i) \cap V(I_i) \cap A| - |I_i \cap J_i| + |I_i| \right) \tag{6.22}$$

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... Morphing Partition Matroid Rank

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- In fact, this bottom (more general) expression is the expression for the rank of a transversal matroid.

Partial Transversals Are Matroids

In fact, we have

Theorem 6.5.3

Let $(V, \mathcal{V})$ *where* $\mathcal{V} = (V_1, V_2, \dots, V_\ell)$ *be a subset system. Let* $I = \{1, \dots, \ell\}$. *Let* $\mathcal{I}$ *be the set of partial transversals of* $\mathcal{V}$. *Then* $(V, \mathcal{I})$ *is a matroid.*

Proof.

□

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- We already saw that if T is a partial transversal of $\mathcal{V}$, and if $T' \subseteq T$, then T' is also a partial transversal. So (I2') holds.

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- We already saw that if T is a partial transversal of $\mathcal{V}$, and if $T' \subseteq T$, then T' is also a partial transversal. So (I2') holds.
- Suppose that T_1 and T_2 are partial transversals of $\mathcal{V}$ such that $|T_1| < |T_2|$. Exercise: show that (I3') holds.

□

Transversal Matroid Rank

- Transversal matroid has rank

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- Therefore, this function is submodular.
- Note that it is a minimum over a set of modular functions. Is this true in general? Exercise:

Matroid loops

- A circuit in a matroids is well defined, a subset $A \subseteq E$ is circuit if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

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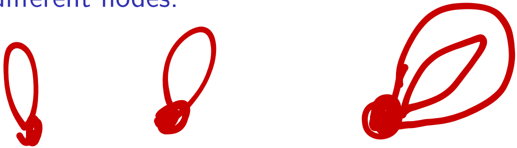
- In a matric (i.e., linear) matroid, the only such loop is the value $\mathbf{0}$, as all non-zero vectors have rank 1. The $\mathbf{0}$ can appear > 1 time with different indices, as can a self loop in a graph appear on different nodes.



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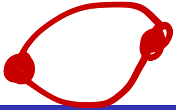
- Note, we also say that two elements s, t are said to be parallel if $\{s, t\}$ is a circuit.



Representable

Definition 6.6.1 (Matroid isomorphism)

Two matroids M_1 and M_2 respectively on ground sets V_1 and V_2 are isomorphic if there is a bijection $\pi : V_1 \to V_2$ which preserves independence (equivalently, rank, circuits, and so on).

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- Let $\mathbb{F}$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $\mathbb{F}$, such as a Galois field GF(p) where p is prime (such as GF(2)). Succinctly: A field is a set with $+$, $*$, closure, associativity, commutativity, and additive and multiplictaive identities and inverses.

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- We can more generally define matroids on a field.

Definition 6.6.2 (linear matroids on a field)

Let $\mathbf{X}$ be an $n \times m$ matrix and $E = \{1, \dots, m\}$, where $\mathbf{X}_{ij} \in \mathbb{F}$ for some field, and let $\mathcal{I}$ be the set of subsets of E such that the columns of $\mathbf{X}$ are linearly independent over $\mathbb{F}$.

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- We can more generally define matroids on a field.

Definition 6.6.3 (representable (as a linear matroid))

Any matroid isomorphic to a linear matroid on a field is called representable over $\mathbb{F}$

Representability of Transversal Matroids

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Representability of Transversal Matroids

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- In particular:

Theorem 6.6.4

Transversal matroids are representable over all finite fields of sufficiently large cardinality, and are representable over any infinite field.

Converse: Representability of Transversal Matroids

The converse is not true, however.

Example 6.6.5

Let $V = \{1, 2, 3, 4, 5, 6\}$ be a ground set and let $M = (V, \mathcal{I})$ be a set system where $\mathcal{I}$ is all subsets of V of cardinality ≤ 2 except for the pairs $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$.

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- It can be shown that this is a matroid and is representable.
- However, this matroid is not isomorphic to any transversal matroid.

Matroids, other definitions using matroid rank $r : 2^V \to \mathbb{Z}_+$

Definition 6.7.1 (closed/flat/subspace)

A subset $A \subseteq E$ is closed (equivalently, a flat or a subspace) of matroid M if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

Definition 6.7.2 (closure)

Given $A \subseteq E$, the closure (or span) of A, is defined by $\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}$.

Therefore, a closed set A has $\text{span}(A) = A$.

Definition 6.7.3 (circuit)

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise-minimal dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

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Definition 6.7.1 (spanning set of a set)

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- A base of a matroid is a minimal spanning set (and it is independent) but supersets of a base are also spanning.
- V is always trivially spanning.
- Consider the terminology: "spanning tree in a graph", comes from spanning in a matroid sense.

Dual of a Matroid

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- Given a matroid $M = (V, \mathcal{I})$, a dual matroid M^* $= (V, \mathcal{I}^*)$ can be defined in a way such that $(M^*)^* = M$.
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- Recall, in cycle matroid of a graph, a spanning set of G is any set of edges that are incident to all nodes (i.e., any superset of a spanning forest).
- Since the smallest spanning sets are bases, the bases of M (when $V \setminus I$ is as small as possible while still spanning) are complements of the bases of M^* (where I is as large as possible).

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Theorem 6.7.3

Let M^* be defined as on previous slide. Then M^* is a matroid.

Proof.

- Clearly $\emptyset \in I^*$, so (I1') holds.

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Proof.

- Clearly $\emptyset \in I^*$, so (I1') holds.
- Also, if $I \subseteq J \in \mathcal{I}^*$, then clearly also $I \in \mathcal{I}^*$ since if $V \setminus J$ is spanning in M, so must $V \setminus I$. Therefore, (I2') holds.

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Theorem 6.7.3

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Proof.

- Consider $I, J \in \mathcal{I}^*$ with $|I| < |J|$. We need to show that there is some member $v \in J \setminus I$ such that $I + v$ is a base in M^*, which means that $V \setminus (I + v) = (V \setminus I) \setminus v$ is still spanning in M. That is, removing v from $V \setminus I$ doesn't make $(V \setminus I) \setminus v$ not spanning.

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- Since $V \setminus J$ is spanning in M, $V \setminus J$ contain some base (say $B \subseteq V \setminus J$) of M. Also, $V \setminus I$ contains a base of M, say $B' \subseteq V \setminus I$.

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- Since $B \setminus I \subseteq V \setminus I$, and $B \setminus I$ is independent in M, we can choose the base B' of M s.t. $B \setminus I \subseteq B' \subseteq V \setminus I$.

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- Since $B \setminus I \subseteq V \setminus I$, and $B \setminus I$ is independent in M, we can choose the base B' of M s.t. $B \setminus I \subseteq B' \subseteq V \setminus I$.
- Since B and J are disjoint, we have both: 1) $B \setminus I$ and $J \setminus I$ are disjoint; and 2) $B \cap I \subseteq I \setminus J$. Also note, B' and I are disjoint.

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Proof.

- Now $J \setminus I \not\subseteq B'$, since otherwise (i.e., assuming $J \setminus I \subseteq B'$):

$$|B| = |B \cap I| + |B \setminus I| \tag{6.28}$$
$$\leq |I \setminus J| + |B \setminus I| \tag{6.29}$$
$$< |J \setminus I| + |B \setminus I| \leq |B'| \tag{6.30}$$

which is a contradiction. *The last inequality on the right follows since $J \setminus I \subseteq B'$ (by assumption) and $B \setminus I \subseteq B'$ implies that $(J \setminus I) \cup (B \setminus I) \subseteq B'$, but since J and B are disjoint, we have that $|J \setminus I| + |B \setminus I| \leq B'$.*

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  which is a contradiction.
- Therefore, $J \setminus I \not\subseteq B'$, and there is a $v \in J \setminus I$ s.t. $v \notin B'$.
- So B' is disjoint with $I \cup \{v\}$, meaning $B' \subseteq V \setminus (I \cup \{v\})$, or $V \setminus (I \cup \{v\})$ is spanning in M, and therefore $I \cup \{v\} \in \mathcal{I}^*$.

□

Dual Matroid Rank

Theorem 6.7.4

The rank function r_{M^*} of the dual matroid M^* may be specified in terms of the rank r_M in matroid M as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \tag{6.31}$$

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2. *I.e., $|X|$ is modular, complement $f(V \setminus X)$ is submodular if f is submodular, $r_M(V)$ is a constant, and summing submodular functions and a constant preserves submodularity.*

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- Non-negativity integral follows since $|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V)$. *The right inequality follows since r_M is submodular.*

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- Non-negativity integral follows since $|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V)$.
- Monotone non-decreasing follows since, as X increases by one, $|X|$ always increases by 1, while $r_M(V \setminus X)$ decreases by one or zero.

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- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2.
- Non-negativity integral follows since $|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V)$.
- Monotone non-decreasing follows since, as X increases by one, $|X|$ always increases by 1, while $r_M(V \setminus X)$ decreases by one or zero.
- Therefore, r_{M^*} is the rank function of a matroid. That it is the dual matroid rank function is shown in the next proof.

Dual Matroid Rank

Theorem 6.7.4

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Proof.

A set X is independent in (V, r_{M^*}) if and only if

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But a subset X is independent in M^* only if $V \setminus X$ is spanning in M (by the definition of the dual matroid). □

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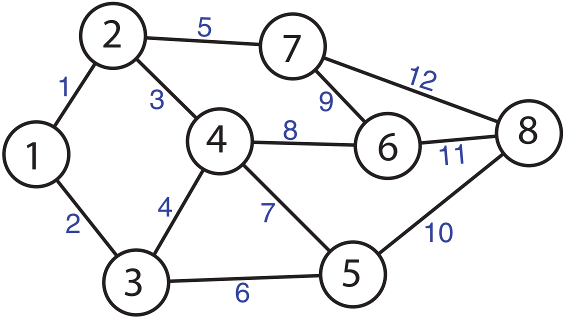
A graph G



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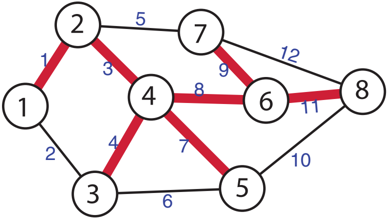
Minimally spanning in M (and thus a base in M)



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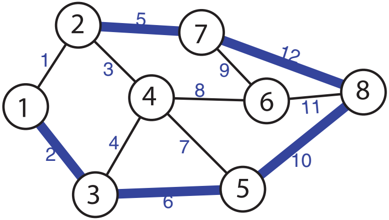

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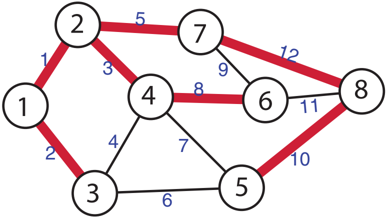

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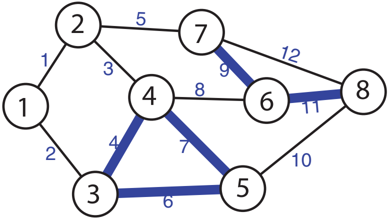

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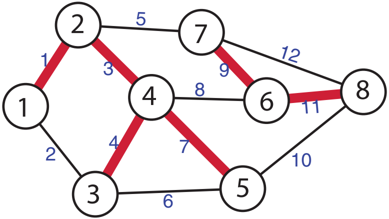
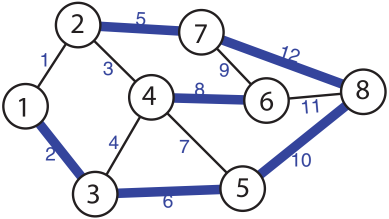


Dependent in M* (contains a cocycle, is a nonminimal cut)



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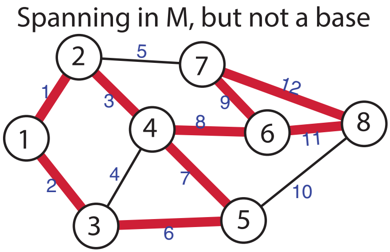

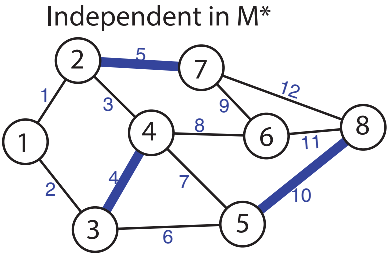

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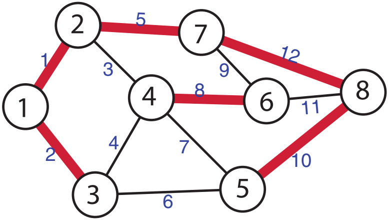

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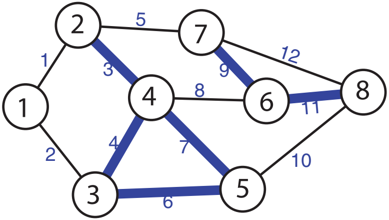

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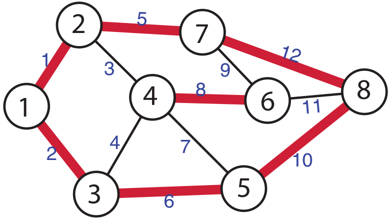
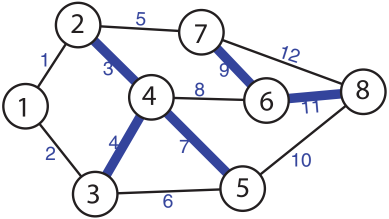


A cycle in M* (a cocycle, or a minimal cut)



Matroid and the greedy algorithm

- Let $\mathcal{I}$ be a set of subsets of E that is down-closed. Consider a non-negative modular weight function $w : E \to \mathbb{R}_+$, and we want to find the $A \in \mathcal{I}$ that maximizes $w(A)$.
- Consider the greedy algorithm: Set $A = \emptyset$, and repeatedly choose $y \in E \setminus A$ such that: 1) $A \cup \{y\} \in \mathcal{I}$, and 2) $w(y)$ is as large as possible. We stop when no such y exists.

Theorem 6.8.1

Let $\mathcal{I}$ be a non-empty collection of subsets of a set E, down-closed (i.e., an independence system). Then the pair $(E, \mathcal{I})$ is a matroid if and only if for each weight function $w \in \mathcal{R}_+^E$, the greedy algorithm leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.

Matroids by bases

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

Theorem 6.8.1 (Matroid (by bases))

Let E be a set and $\mathcal{B}$ be a nonempty collection of subsets of E. Then the following are equivalent.

1. *$\mathcal{B}$ is the collection of bases of a matroid;*
2. *if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.*
3. *If $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.*

Properties 2 and 3 are called "exchange properties."
Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.

Matroid and the greedy algorithm

proof of Theorem 6.8.1.

- Assume $(E, \mathcal{I})$ is a matroid and $w : E \to \mathcal{R}_+$ is given.

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- Assume $(E, \mathcal{I})$ is a matroid and $w : E \to \mathcal{R}_+$ is given.
- Let $A = (a_1, a_2, \dots, a_r)$ be the solution returned by greedy, where $r = r(M)$ the rank of the matroid, and we order the elements as they were chosen (so $w(a_1) \geq w(a_2) \geq \cdots \geq w(a_r)$).

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- A is a base of M, and let $B = (b_1, \dots, b_r)$ be any another base of M with elements also ordered decreasing by weight.
- We next show that not only is $w(A) \geq w(B)$ but that $w(a_i) \geq w(b_i)$ for all i.

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- Since $|A_{k-1}| < |B_k|$, $A_{k-1} \cup \{b_i\} \in \mathcal{I}$ for some $1 \leq i \leq k$.

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- But $w(b_i) \geq w(b_k) > w(a_k)$, and so the greedy algorithm would have chosen b_i rather than a_k, contradicting what greedy does.

$\square$

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converse proof of Theorem 6.8.1.

- Given an independence system $(E, \mathcal{I})$, suppose the greedy algorithm leads to an independent set of max weight for every such weight function. We'll show $(E, \mathcal{I})$ is a matroid.

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- Define the following modular weight function w on V, and define $k = |I|$.

$$w(v) = \begin{cases} k+2 & \text{if } v \in I, \\ k+1 & \text{if } v \in J \setminus I, \\ 0 & \text{if } v \in S \setminus (I \cup J) \end{cases} \tag{6.34}$$

Matroid and the greedy algorithm

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- Now greedy will clearly, after k iterations recover I, but can not choose any element in $J \setminus I$ by assumption. Thus, greedy chooses a set of weight $k(k+2)$.

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  so J has strictly larger weight but is still independent, contradicting greedy's optimality.
- Therefore, $(E, \mathcal{I})$ must be a matroid.

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- If we stop at a negative value, we'll once again get a maximum weight independent set.
- We can instead do as small as possible thus giving us a minimum weight independent set/base.

Matroid restriction/deletion

- Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then

$$\mathcal{I}_Y = \{Z : Z \subseteq Y, Z \in \mathcal{I}\} \tag{6.36}$$

is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid with rank $r(M_Y) = r(Y)$.

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- Hence, $M|Y = M \setminus (V \setminus Y)$.
- The rank function is of the same form. I.e., $r_Y : 2^Y \to \mathbb{Z}_+$, where $r_Y(Z) = r(Z)$ for $Z \subseteq Y$.

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- A minor of a matroid is any matroid obtained via a series of deletions and contractions of some matroid.

Matroid Intersection

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This is an instance of the convolution of two submodular functions, f_1 and f_2 that, evaluated at $Y \subseteq V$, is written as:

$$(f_1 * f_2)(Y) = \min_{X \subseteq Y} \Big(f_1(X) + f_2(Y \setminus X) \Big) \tag{6.40}$$

Convolution and Hall's Theorem

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- So Hall's theorem can be expressed as convolution.
- Note, in general, convolution of two submodular functions does not preserve submodularity (but in certain special cases it does).

Matroid Union

Definition 6.9.2

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, $\dots$, $M_k = (V_k, \mathcal{I}_k)$ be matroids. We define the union of matroids as $M_1 \vee M_2 \vee \dots \vee M_k = (V_1 \uplus V_2 \uplus \dots \uplus V_k, \mathcal{I}_1 \vee \mathcal{I}_2 \vee \dots \vee \mathcal{I}_k)$, where

$$I_1 \vee \mathcal{I}_2 \vee \dots \vee \mathcal{I}_k = \{I_1 \uplus I_2 \uplus \dots \uplus I_k | I_1 \in \mathcal{I}_1, \dots, I_k \in \mathcal{I}_k\} \quad (6.41)$$

Note $A \uplus B$ designates the disjoint union of A and B.

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Note $A \uplus B$ designates the disjoint union of A and B.

Theorem 6.9.3

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, $\dots$, $M_k = (V_k, \mathcal{I}_k)$ be matroids, with rank functions $r_1, \dots, r_k$. Then the union of these matroids is still a matroid, having rank function

$$r(Y) = \min_{X \subseteq Y} \Big(|Y \setminus X| + r_1(X \cap V_1) + \dots + r_k(X \cap V_k) \Big) \tag{6.42}$$

for any $Y \subseteq V_1 \cup \dots V_k$.

Exercise: Matroid Union, and Matroid duality

Exercise: Describe $M \vee M^*$.

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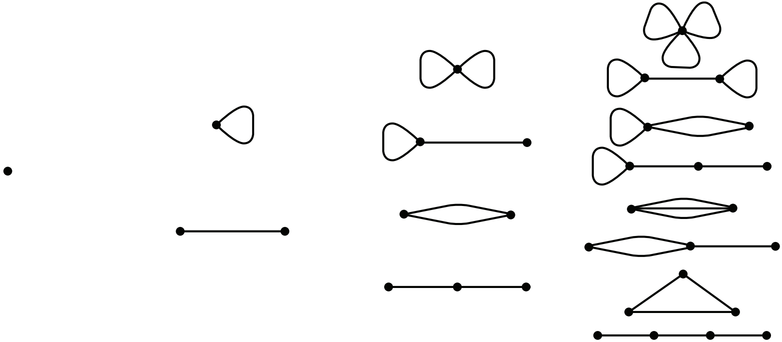

(a) The only matroid with zero elements.

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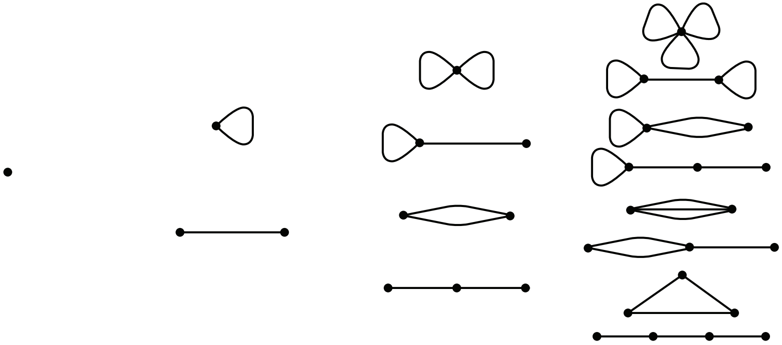

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- Nice way to show low element size matroids. What about matroids that are low rank but with many elements?

Affine Matroids

- Given an $n \times m$ matrix with entries over some field $\mathbb{F}$, we say that a subset $S \subseteq \{1, \ldots, m\}$ of indices (with corresponding column vectors $\{v_i : i \in S\}$, with $|S| = k$ is affinely dependent if $m \geq 1$ and there exists elements $\{a_1, \ldots, a_k\} \in \mathbb{F}$, not all zero, such that $\sum_{i=1}^{k} a_i v_i = 0$ and $\sum_{i=1}^{k} a_i = 0$, and otherwise affinely independent.

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- Concisely: points $\{v_1, v_2, \ldots, v_k\}$ are affinely independent if $v_2 - v_1, v_3 - v_1, \ldots, v_k - v_1$ are linearly independent.

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Proposition 6.9.4 (affine matroid)

Let ground set $E = \{1, \ldots, m\}$ index column vectors of a matrix, and let $\mathcal{I}$ be the set of subsets X of E such that X indices affinely independent vectors. Then $(E, \mathcal{I})$ is a matroid.

Proof.

Exercise: $\square$

Euclidean Representation of Low-rank Matroids

- Consider the affine matroid with $n \times m = 2 \times 6$ matrix on the field $\mathbb{F} = \mathbb{R}$, and let the elements be $\{(0,0),(1,0),(2,0),(0,1),(0,2),(1,1)\}$.

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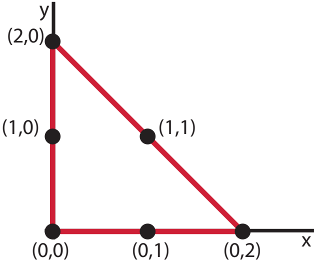
- Hence, we can plot the points in $\mathbb{R}^2$ as follows:



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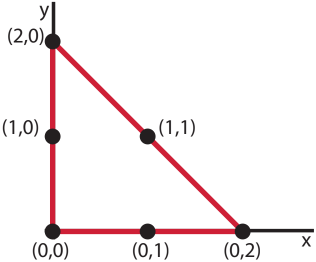
- Dependent sets consist of all subsets with ≥ 4 elements, or 3 collinear elements.



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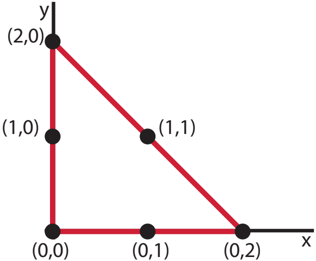

- In general, for a matroid $\mathcal{M}$ of rank $m+1$ with $m \leq 3$, then a subset X in a geometric representation in $\mathbb{R}^m$ is dependent if: 1) $|X| \geq 2$ and the points are identical; 2) $|X| \geq 3$ and the points are collinear; 3) $|X| \geq 4$ and the points are coplanar; or 4) $|X| \geq 5$ and the points are in space.

Euclidean Representation of Low-rank Matroids

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Any matroid of rank $m \leq 4$ can be represented by an affine matroid in $\mathcal{R}^{m-1}$.

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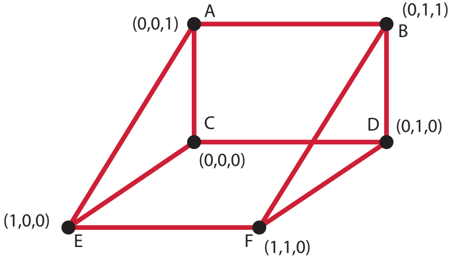
- As another example on the right, a rank 4 matroid



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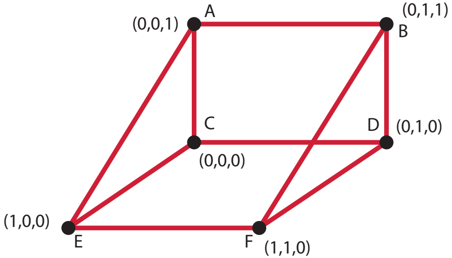

- All sets of 5 points are dependent. The only other sets of dependent points are coplanar ones of size 4. Namely: $\{(0,0,0),(0,1,0),(1,1,0),(1,0,0)\}$, $\{(0,0,0),(0,0,1),(0,1,1),(0,1,0)\}$, and $\{(0,0,1),(0,1,1),(1,1,0),(1,0,0)\}$.

Euclidean Representation of Low-rank Matroids: A test

- Loops represented by a separate box indicating how many loops there are. Parallel elements indicated by a multiplicity next to a point.

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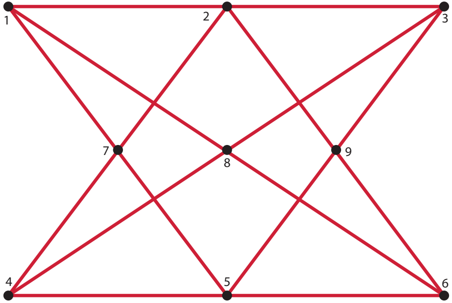

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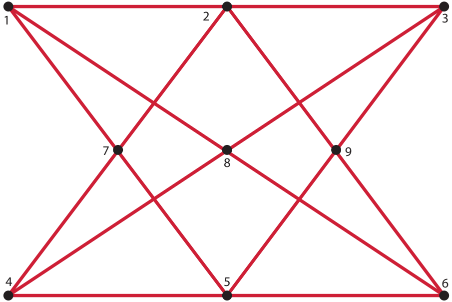

- Called the non-Pappus matroid. Has rank three, but any matric matroid with the above dependencies would require that $\{7, 8, 9\}$ is dependent, hence requiring an additional line in the above.

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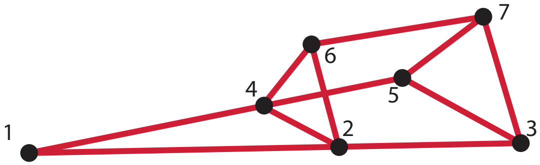

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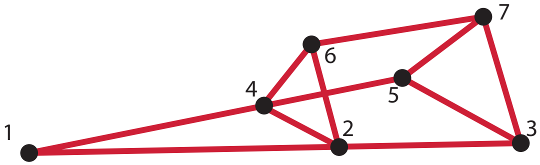

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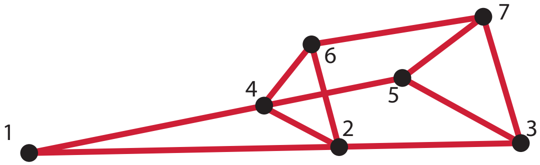

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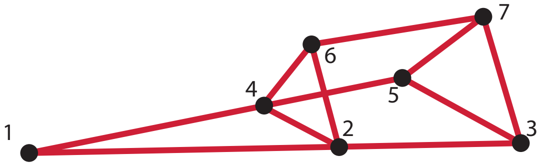

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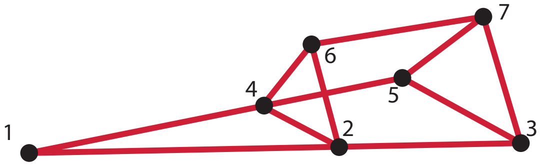

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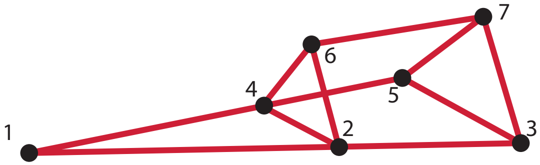

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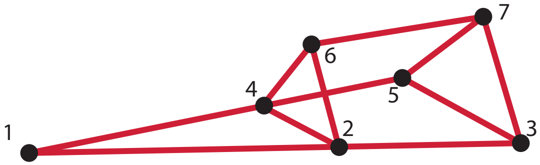

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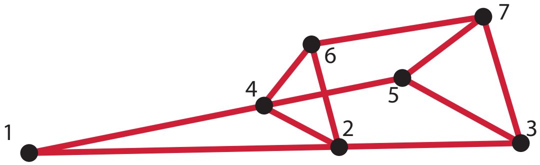

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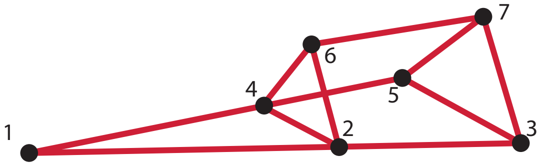

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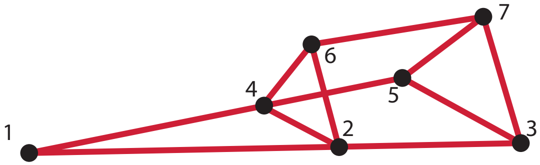

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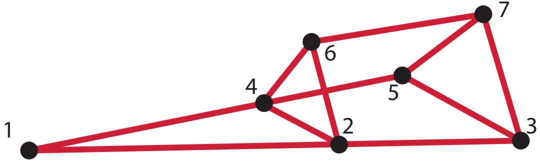
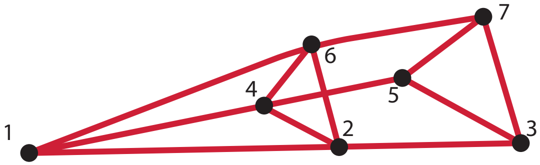


- Check rank's submodularity: Let $X = \{1, 2, 3, 6, 7\}$, $Y = \{1, 4, 5, 6, 7\}$. So $r(X) = 3$, and $r(Y) = 3$, and $r(X \cup Y) = 4$, so we must have, by submodularity, that $r(\{1, 6, 7\}) = r(X \cap Y) \leq r(X) + r(Y) - r(X \cup Y) = 2$.
- However, from the diagram, we have that since 1, 6, 7 are distinct non-collinear points, we have that $r(X \cap Y) = 3$

Euclidean Representation of Low-rank Matroids: A test

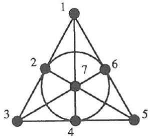
- Is this a matroid?

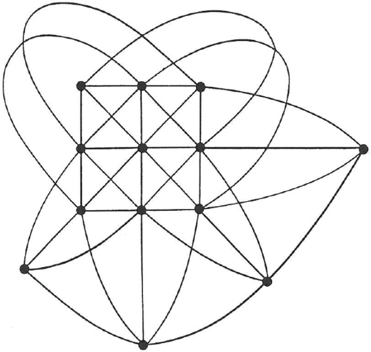


- If we extend the line from 6-7 to 1, then is it a matroid?
- Hence, not all 2D or 3D graphs of points and lines are matroids.

Euclidean Representation of Low-rank Matroids: Other Examples

- Other examples can be more complex, consider the following two matroids (from Oxley, 2011):





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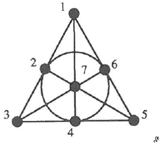

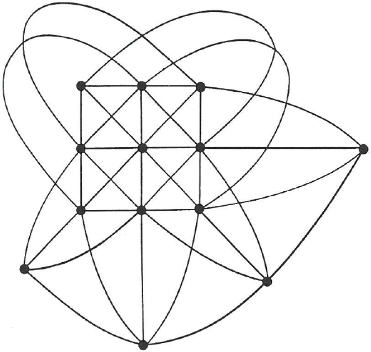

- Hence, lines (in 2D) may be curved and planes (in 3D) can be twisted.

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- Matroid of rank at most four (see Oxley 2011 for more details).