Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 5 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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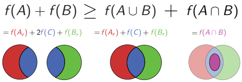
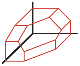
April 14th, 2014





Cumulative Outstanding Reading

- Read chapter 1 from Fujishige's book.

Announcements, Assignments, and Reminders

- our room (Mueller Hall Room 154) is changed!
- Please do use our discussion board (`https://canvas.uw.edu/courses/895956/discussion_topics`) for all questions, comments, so that all will benefit from them being answered.
- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids
- L6:
- L7:
- L8:
- L9:
- L10:
- L11:
- L12:
- L13:
- L14:
- L15:
- L16:
- L17:
- L18:
- L19:
- L20:

Finals Week: June 9th-13th, 2014.

Many (Equivalent) Definitions of Submodularity

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B), \quad \forall A, B \subseteq V \tag{5.6}$$

$$f(j|S) \geq f(j|T), \; \forall S \subseteq T \subseteq V, \text{ with } j \in V \setminus T \tag{5.7}$$

$$f(C|S) \geq f(C|T), \forall S \subseteq T \subseteq V, \text{ with } C \subseteq V \setminus T \tag{5.8}$$

$$f(j|S) \geq f(j|S \cup \{k\}), \; \forall S \subseteq V \text{ with } j \in V \setminus (S \cup \{k\}) \tag{5.9}$$

$$f(A \cup B | A \cap B) \leq f(A|A \cap B) + f(B|A \cap B), \quad \forall A, B \subseteq V \tag{5.10}$$

$$f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S) - \sum_{j \in S \setminus T} f(j|S \cup T - \{j\}), \; \forall S, T \subseteq V \tag{5.11}$$

$$f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S), \; \forall S \subseteq T \subseteq V \tag{5.12}$$

$$f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|S \setminus \{j\}) + \sum_{j \in T \setminus S} f(j|S \cap T) \; \forall S, T \subseteq V$$

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- We saw several different "greedy" algorithms that proced optimal spanning trees (Borůvka's, Jarník/Prim/Dijkstra's, and Kruskal's).
- We wish to more formally connect the above, and generalize further.

From Matrix Rank → Matroid

- So V is set of column vector indices of a matrix.
- Let $\mathcal{I}$ be a set of all subsets of V such that for any $I \in \mathcal{I}$, the vectors indexed by I are linearly independent.
- Given a set $B \in \mathcal{I}$ of linearly independent vectors, then any subset $A \subseteq B$ is also linearly independent. Hence, $\mathcal{I}$ is down-closed or "subclusive", under subsets. In other words,

$$A \subseteq B \text{ and } B \in \mathcal{I} \Rightarrow A \in \mathcal{I} \tag{5.32}$$

- maxInd: Inclusionwise maximal independent subsets (or bases) of any set $B \subseteq V$.

$$\text{maxInd}(B) \triangleq \{A \subseteq B : A \in \mathcal{I} \text{ and } \forall v \in B \setminus A, A \cup \{v\} \notin \mathcal{I}\} \tag{5.33}$$

- Given any set $B \subset V$ of vectors, all maximal (by set inclusion) subsets of linearly independent vectors are the same size. That is, for all $B \subseteq V$,

$$\forall A_1, A_2 \in \text{maxInd}(B), \quad |A_1| = |A_2| \tag{5.34}$$

From Matrix Rank → Matroid

- Thus, for all $I \in \mathcal{I}$, the matrix rank function has the property

$$r(I) = |I| \tag{5.32}$$

and for any $B \notin \mathcal{I}$,

$$r(B) = \max\{|A| : A \subseteq B \text{ and } A \in \mathcal{I}\} \leq |B| \tag{5.33}$$

Matroid

Independent set definition of a matroid is perhaps most natural. Note, if $J \in \mathcal{I}$, then J is said to be an independent set.

Definition 5.2.4 (Matroid)

A set system $(E, \mathcal{I})$ is a Matroid if

(I1) $\emptyset \in \mathcal{I}$

(I2) $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$

(I3) $\forall I, J \in \mathcal{I}$, with $|I| = |J| + 1$, then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$.

On Matroids

- Abstract properties of linear dependence (Hassler Whitney, 1935), but already then found instances of objects with those properties not based on a matrix.

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- Crapo & Rota preferred the term "combinatorial geometry", or more specifically a "pregeometry" and said that pregeometries are "often described by the ineffably cacaphonic [sic] term 'matroid', which we prefer to avoid in favor of the term 'pregeometry'."

Matroid

Slight modification (non unit increment) that is equivalent.

Definition 5.3.1 (Matroid-II)

A set system $(E, \mathcal{I})$ is a Matroid if

(I1') $\emptyset \in \mathcal{I}$

(I2') $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ (or "down-closed")

(I3') $\forall I, J \in \mathcal{I}$, with $|I| > |J|$, then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$

Note (I1)=(I1'), (I2)=(I2'), and we get (I3)≡(I3') using induction.

Matroids, independent sets, and bases

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- A base of $U \subseteq E$: For $U \subseteq E$, a subset $B \subseteq U$ is called a base of U if B is inclusionwise maximally independent subset of U. That is, $B \in \mathcal{I}$ and there is no $Z \in \mathcal{I}$ with $B \subset Z \subseteq U$.

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- A base of a matroid: If $U = E$, then a "base of E" is just called a base of the matroid M (this corresponds to a basis in a linear space).

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Proposition 5.3.2

In a matroid $M = (E, \mathcal{I})$, for any $U \subseteq E(M)$, any two bases of U have the same size.

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(I3') $\forall X \subseteq V$, and $I_1, I_2 \in \text{maxInd}(X)$, we have $|I_1| = |I_2|$ (all maximally independent subsets of X have the same size).

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Definition 5.3.4 (matroid rank function)

The rank of a matroid is a function $r : 2^E \to \mathbb{Z}_+$ defined by

$$r(A) = \max \{|X| : X \subseteq A, X \in \mathcal{I}\} = \max_{X \in \mathcal{I}} |A \cap X| \quad (5.1)$$

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- From the above, we immediately see that $r(A) \leq |A|$.
- Moreover, if $r(A) = |A|$, then $A \in \mathcal{I}$, meaning A is independent (in this case, A is a self base).

Matroids, other definitions using matroid rank $r : 2^V \to \mathbb{Z}_+$

Definition 5.3.5 (closed/flat/subspace)

A subset $A \subseteq E$ is closed (equivalently, a flat or a subspace) of matroid M if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

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Definition 5.3.6 (closure)

Given $A \subseteq E$, the closure (or span) of A, is defined by $\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}$.

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Therefore, a closed set A has $\text{span}(A) = A$.

$A \subseteq B \Rightarrow \text{span}(A) \subseteq \text{span}(B)$

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Definition 5.3.7 (circuit)

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise-minimal dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

Matroids by bases

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

Theorem 5.3.8 (Matroid (by bases))

Let E be a set and $\mathcal{B}$ be a nonempty collection of subsets of E. Then the following are equivalent.

1. *$\mathcal{B}$ is the collection of bases of a matroid;*
2. *if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.*
3. *If $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.*

Properties 2 and 3 are called "exchange properties."

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Properties 2 and 3 are called "exchange properties."
Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.

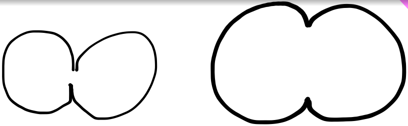
Matroids by circuits

A set is independent if and only if it contains no circuit. Therefore, it is not surprising that circuits can also characterize a matroid.

Theorem 5.3.9 (Matroid by circuits)

Let E be a set and $\mathcal{C}$ be a collection of subsets of E that satisfy the following three properties:

1. *(C1): $\emptyset \notin \mathcal{C}$*
2. *(C2): if $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.*
3. *(C3): if $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq C_2$, and $e \in C_1 \cap C_2$, then there exists a $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.*



Matroids by circuits

Several circuit definitions for matroids.

Theorem 5.3.10 (Matroid by circuits)

Let E be a set and $\mathcal{C}$ be a collection of nonempty subsets of E, such that no two sets in $\mathcal{C}$ are contained in each other. Then the following are equivalent.

1. *$\mathcal{C}$ is the collection of circuits of a matroid;*
2. *if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$;*
3. *if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, and $y \in C \setminus C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$ containing y;*

Matroids by circuits

$G=(V,E)$ $M=(E,\mathcal{I})$

$V(G)$ $E(G)$ $E(M)$ $\mathcal{I}(M)$

Several circuit definitions for matroids. $V(G_1)$

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Again, think about this for a moment in terms of linear spaces and matrices, and spanning trees.

Matroids by submodular functions

Theorem 5.3.11 (Matroid by submodular functions)

Let $f : 2^E \to \mathbb{Z}$ be a integer valued monotone non-decreasing submodular function. Define a set of sets as follows:

$$\mathcal{C}(f) = \Big\{ C \subseteq E : C \text{ is non-empty, is inclusionwise-minimal, and has } f(C) < |C| \Big\} \tag{5.2}$$

Then $\mathcal{C}(f)$ is the collection of circuits of a matroid on E.

Inclusionwise-minimal in this case means that if $C \in \mathcal{C}(f)$, then there exists no $C' \subset C$ with $C' \in \mathcal{C}(f)$ (i.e., $C' \subset C$ would either be empty or have $f(C') \geq |C'|$). Also, recall inclusionwise-minimal in Definition 5.3.7, the definition of a circuit.

Uniform Matroid

- Given E, consider $\mathcal{I}$ to be all subsets of E that are at most size k. That is $\mathcal{I} = \{A \subseteq E : |A| \leq k\}$.

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- Given E, consider $\mathcal{I}$ to be all subsets of E that are at most size k. That is $\mathcal{I} = \{A \subseteq E : |A| \leq k\}$.
- Then $(E, \mathcal{I})$ is a matroid called a k-uniform matroid.

Uniform Matroid

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- A "free" matroid sets $k = |E|$, so everything is independent.

Linear (or Matric) Matroid

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- Called both linear matroids and matric matroids.

Cycle Matroid of a graph: Graphic Matroids

- Let $G = (V, E)$ be a graph. Consider $(E, \mathcal{I})$ where the edges of the graph E are the ground set and $A \in \mathcal{I}$ if the edge-induced graph $G(V, A)$ by A does not contain any cycle.

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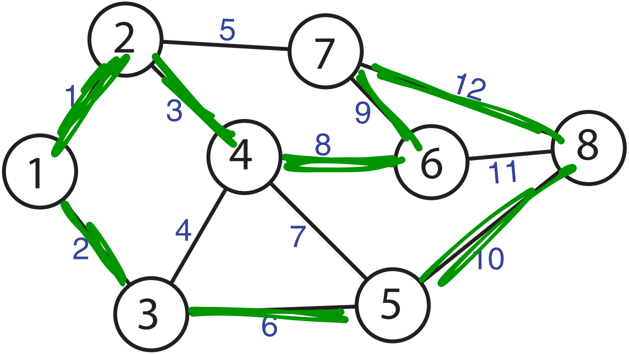
- Rank function $r(A)$ is the size of the largest spanning forest contained in $G(V, A)$.
- Closure function adds all edges between the vertices adjacent to any edge in A. Closure of a spanning forest is G.

Example: graphic matroid

- A graph defines a matroid on edge sets, independent sets are those without a cycle.



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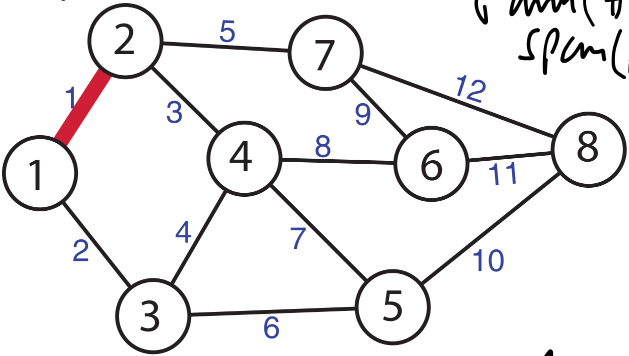
rank(A) = 1
span(A) = A



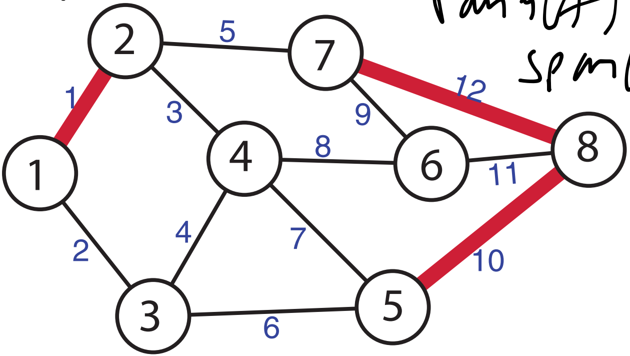
E = {1, 2, 3, ..., 12}

A = {1} ⊆ E

Example: graphic matroid

- A graph defines a matroid on edge sets, independent sets are those without a cycle.

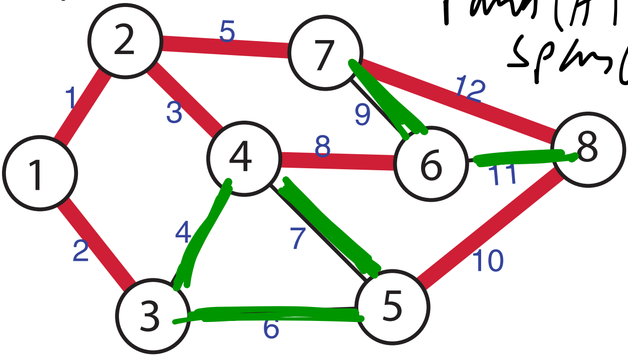
rank(A) = 3
span(A) = A



Example: graphic matroid

- A graph defines a matroid on edge sets, independent sets are those without a cycle.

rank(A) = 7
span(A) = E



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$\text{rank}(A) = 7$

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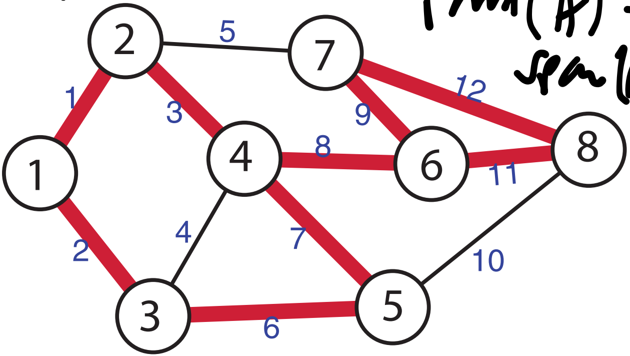
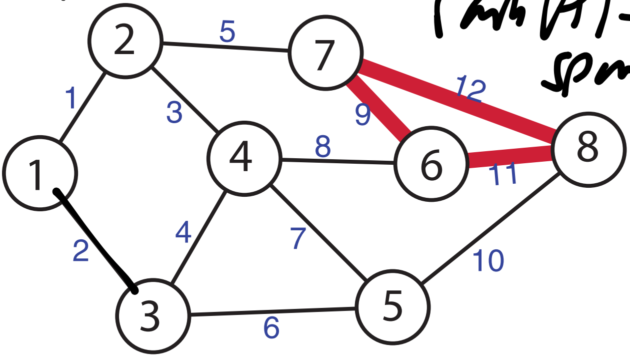


Example: graphic matroid

- A graph defines a matroid on edge sets, independent sets are those without a cycle.

rank(A) = 2
span(A) = A



Partition Matroid

- Let V be our ground set.

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- Let $V = V_1 \cup V_2 \cup \cdots \cup V_\ell$ be a partition of V into blocks or disjoint sets (disjoint union). Define a set of subsets of V as

$$\mathcal{I} = \{X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell\}. \tag{5.5}$$

where $k_1, \ldots, k_\ell$ are fixed parameters, $k_i \geq 0$. Then $M = (V, \mathcal{I})$ is a matroid.

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- Note that a k-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$.
- We'll show that property (I3') in Def 5.3.3 holds. If $X, Y \in \mathcal{I}$ with $|Y| > |X|$, then there must be at least one i with $|Y \cap V_i| > |X \cap V_i|$. Therefore, adding one element $e \in V_i \cap (Y \setminus X)$ to X won't break independence.

Partition Matroid

Ground set of objects, $V = \Bigg\{$

$\Bigg\}$

Partition Matroid

Partition of V into six blocks, $V_1, V_2, \ldots, V_6$

Partition Matroid

Limit associated with each block, $\{k_1, k_2, \ldots, k_6\}$

Partition Matroid

Independent subset but not maximally independent.

Partition Matroid

Maximally independent subset, what is called a base.

Partition Matroid

Not independent since over limit in set six.

Matroids - rank

Lemma 5.5.1

The rank function $r : 2^E \to \mathbb{Z}_+$ *of a matroid is submodular, that is* $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$

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$$\geq |X| + |Y| = r(A \cap B) + r(A \cup B) \tag{5.8}$$

$f(S) = |Y \cap S|$ $\quad f(A) + f(B) = f(A \cap B) + f(A \cup B)$

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Matroids

In fact, we can use the rank of a matroid for its definition.

Theorem 5.5.2 (Matroid from rank)

Let E be a set and let $r : 2^E \to \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with r being its rank function if and only if for all $A, B \subseteq E$:

(R1) $\forall A \subseteq E \;\; 0 \leq r(A) \leq |A|$ *(non-negative cardinality bounded)*

(R2) $r(A) \leq r(B)$ *whenever* $A \subseteq B \subseteq E$ *(monotone non-decreasing)*

(R3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ *for all* $A, B \subseteq E$ *(submodular)*

- So submodularity and non-negative monotone non-decreasing, and unit increase is necessary and sufficient to define the matroid.
- Given above, unit increment (if $r(A) = k$, then either $r(A \cup \{v\}) = k$ or $r(A \cup \{v\}) = k + 1$) holds.
- A matroid is sometimes given as (E, r) where E is ground set and r is rank function.

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- From above, $r(\emptyset) = 0$. Let $v \notin A$, then by monotonicity and submodularity, $r(A) \leq r(A \cup \{v\}) \leq r(A) + r(\{v\})$ which gives only two possible values to $r(A \cup \{v\})$.

$(+r(\emptyset)=0$

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Proof of Theorem 5.5.2 (matroid from rank).

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- Given a matroid $M = (E, \mathcal{I})$, we see its rank function as defined in Eq. 5.1 satisfies (R1), (R2), and, as we saw in Lemma 5.5.1, (R3) too.
- Next, assume we have (R1), (R2), and (R3). Define $\mathcal{I} = \{X \subseteq E : r(X) = |X|\}$. We will show that $(E, \mathcal{I})$ is a matroid.
- First, $\emptyset \in \mathcal{I}$.
- Also, if $Y \in \mathcal{I}$ and $X \subseteq Y$ then by submodularity,

$$r(X) \geq r(Y) - r(Y \setminus X) - r(\emptyset) \tag{5.9}$$

$$\geq |Y| - |Y \setminus X| \tag{5.10}$$

$$= |X| \tag{5.11}$$

$r(x) \leq |x|$, $r(x) \geq |x|$ $\Rightarrow$

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implying $r(X) = |X|$, and thus $X \in \mathcal{I}$.

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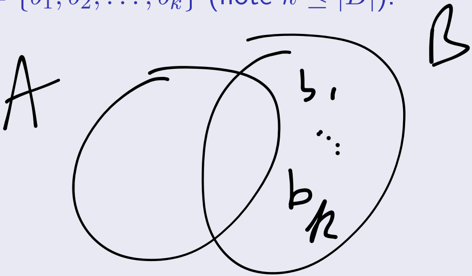
Matroids from rank

Proof of Theorem 5.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).



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- Suppose, to the contrary, that $\forall b \in B \setminus A$, $(A + b) \notin \mathcal{I}$, which means for all such b, $r(A + b) = r(A) = |A|$. Then

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Proof of Theorem 5.5.2 (matroid from rank) cont.

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$$r(B) \leq r(A \cup B) \tag{5.12}$$

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$$r(B) \leq r(A \cup B) \tag{5.12}$$

$$\leq r(A \cup (B \setminus \{b_1\})) + \underbrace{r(A \cup \{b_1\}) - r(A)}_{=0} \tag{5.13}$$

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Matroids from rank

Proof of Theorem 5.5.2 (matroid from rank) cont.

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r(B) &\le r(A \cup B) && (5.12)\\
&\le r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) && (5.13)\\
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&\le r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) && (5.15)
\end{aligned}
$$

□

Matroids from rank

Proof of Theorem 5.5.2 (matroid from rank) cont.

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&= r(A \cup (B \setminus \{b_1, b_2\})) && (5.16)
\end{aligned}
$$

□

Matroids from rank

Proof of Theorem 5.5.2 (matroid from rank) cont.

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$\square$

Matroids from rank

Proof of Theorem 5.5.2 (matroid from rank) cont.

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&\leq \dots \leq r(A) = |A| < |B| && (5.17)
\end{aligned}
$$

giving a contradiction since $B \in \mathcal{I}$. $\square$

Matroids from rank II

Another way of using function r to define a matroid.

Theorem 5.5.3 (Matroid from rank II)

Let E be a finite set and let $r : 2^E \to \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with r being its rank function if and only if for all $A \subseteq E$, and $x, y \in E$:

(R1') $r(\emptyset) = 0$;

(R2') $r(X) \leq r(X \cup \{y\}) \leq r(X) + 1$;

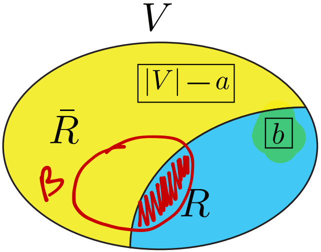
(R3') *If* $r(X \cup \{x\}) = r(X \cup \{y\}) = r(X)$, *then* $r(X \cup \{x, y\}) = r(X)$.

Matroid and Rank

- Thus, we can define a matroid as $M = (V, r)$ where r satisfies matroid rank axioms.
- Example: 2-partition matroid rank function: Given natural numbers $a, b \in \mathbb{Z}_+$ with $a > b$, and any set $R \subseteq V$ with $|R| = a$, two-block partition $V = (R, \bar{R})$, where $\bar{R} = V \setminus R$, define:

$$r(A) = \min(|A \cap R|, b) + \min(|A \cap \bar{R}|, |\bar{R}|) \quad (5.18)$$
$$= \min(|A \cap R|, b) + |A \cap \bar{R}| \quad (5.19)$$

- Partition matroid figure showing this:



$b < |R| = a$

Truncated Matroid Rank Function

- Can use this to define a truncated matroid rank function. With $r(A) = \min(|A \cap R|, b) + |A \cap \bar{R}|$, $b < a$, define:

$$f_R(A) = \min\{r(A), a\} \tag{5.20}$$
$$= \min\{\min(|A \cap \bar{R}| + |A \cap R|, |A \cap \bar{R}| + b), a\} \tag{5.21}$$
$$= \min\{|A|, b + |A \cap \bar{R}|, a\} \tag{5.22}$$

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- Defines a matroid $M = (V, f_R) = (V, \mathcal{I})$ (Goemans et. al.) with

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- For any B with $|B \cap R| = \ell$, with $b < \ell < a$, $f_R(B) = b + a - \ell$.
- R, the set with minimum valuation amongst size-a sets, is hidden within an exponentially larger set of size-a sets with larger valuation.

Summarizing: Many ways to define a Matroid

Summarizing what we've so far seen, we saw that it is possible to uniquely define a matroid based on any of:

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- Independence (define the independent sets).
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- Circuit axioms
- Closure axioms (we didn't see this, but it is possible)
- Rank axioms (normalized, monotone, cardinality bounded, submodular)

Maximization problems for matroids

- Given a matroid $M = (E, \mathcal{I})$ and a modular cost function $c : E \to \mathbb{R}$, the task is to find an $X \in \mathcal{I}$ such that $c(X) = \sum_{x \in X} c(x)$ is maximum.
- This seems remarkably similar to the max spanning tree problem.

Minimization problems for matroids

- Given a matroid $M = (E, \mathcal{I})$ and a modular cost function $c : E \to \mathbb{R}$, the task is to find a basis $B \in \mathcal{B}$ such that $c(B)$ is minimized.
- This sounds like a set cover problem (find the minimum cost covering set of sets).

Partition Matroid

- What is the partition matroid's rank function?

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  2. $\min(\text{submodular}(A), k_i)$ is submodular in A since $|A \cap V_i|$ is monotone.
  3. sums of submodular functions are submodular.

- $r(A)$ is also non-negative integral monotone non-decreasing, so it defines a matroid (the partition matroid).

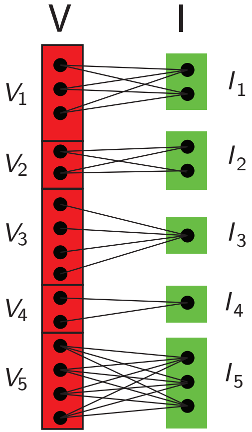
Partition Matroid, rank as matching

- A partition matroid can be viewed using a bipartite graph.
- Letting V denote the ground set, and $V_1, V_2, \ldots$ the partition, the graph is $G = (V, I, E)$ where V is the ground set, I is a set of "indices", and E is the set of edges.
- $I = (I_1, I_2, \ldots, I_\ell)$ is a set of $k = \sum_{i=1}^{\ell} k_i$ nodes, grouped into ℓ clusters, where there are k_i nodes in the i^{th} group I_i.
- $(v, i) \in E(G)$ iff $v \in V_j$ and $i \in I_j$.

$|I_i| = k_i$ $\quad\quad$ $|I| = k$

Partition Matroid, rank as matching

- Example where $\ell = 5$, $(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3)$.



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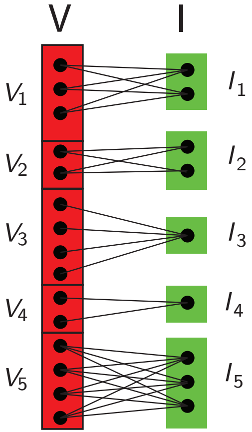

- Recall, $\Gamma : 2^V \to \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of X is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.

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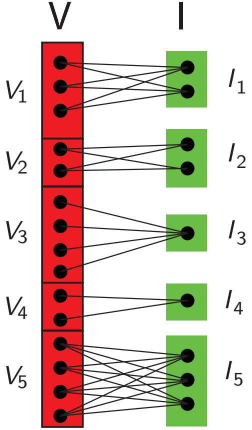

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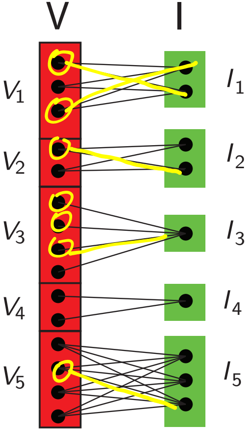

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- Here, for $X \subseteq V$, we have $\Gamma(X) = \{i \in I : (v, i) \in E(G) \text{ and } v \in X\}$.
- For such a constructed bipartite graph, the rank function of a partition matroid is $r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i) =$ the maximum matching involving X.

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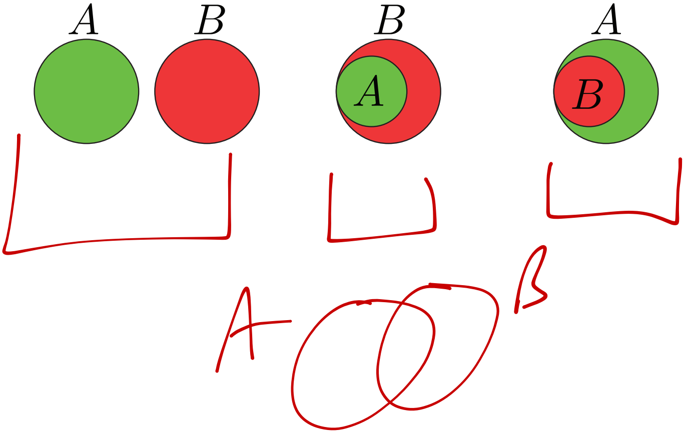
- A set system $(V, \mathcal{F})$ is called a laminar family if for any two sets $A, B \in \mathcal{F}$, at least one of the three sets $A \cap B$, $A \setminus B$, or $B \setminus A$ is empty.



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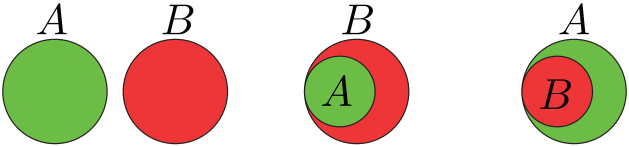

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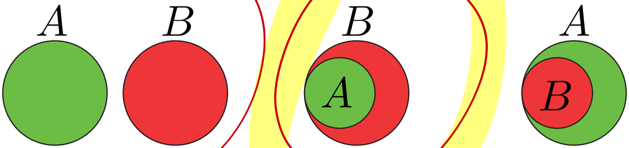

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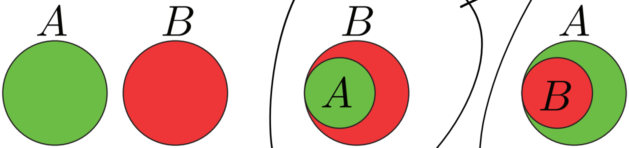

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- Suppose we have a laminar family $\mathcal{F}$ of subsets of V and an integer $k(A)$ for every set $A \in \mathcal{F}$.
- Then $(V, \mathcal{I})$ defines a matroid where

$$\mathcal{I} = \{I \subseteq E : |I \cap A| \leq k(A) \text{ for all } A \in \mathcal{F}\} \tag{5.25}$$

System of Representatives

- Let $(V, \mathcal{V})$ be a set system (i.e., $\mathcal{V} = (V_i : i \in I)$ where $\emptyset \subset V_i \subseteq V$ for all i), and I is an index set. Hence, $|I| = |\mathcal{V}|$.

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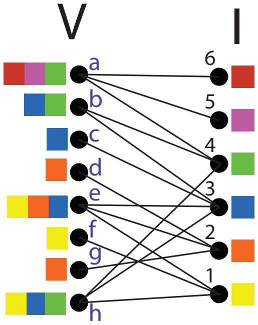
- Here, $\ell = 6$ groups, with $\mathcal{V} = (V_1, V_2, \dots, V_6)$ $= (\{e, f, h\}, \{d, e, g\}, \{b, c, e, h\}, \{a, b, h\}, \{a\}, \{a\})$.



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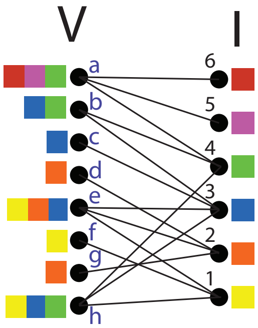

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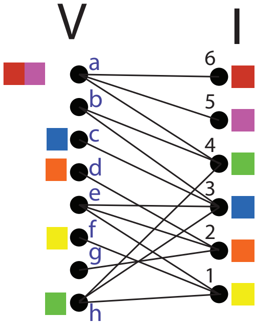

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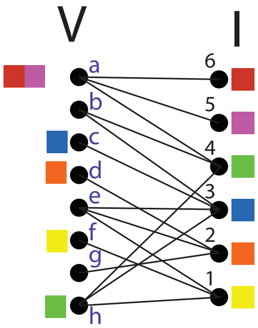

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- Here, the set of representatives is not distinct. In fact, due to the red and pink group, a distinct group of representatives is impossible (since there is only one common choice to represent both color groups).

System of Distinct Representatives

- Let $(V, \mathcal{V})$ be a set system (i.e., $\mathcal{V} = (V_k : i \in I)$ where $V_i \subseteq V$ for all i), and I is an index set. Hence, $|I| = |\mathcal{V}|$.

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Definition 5.7.1 (transversal)

Given a set system $(V, \mathcal{V})$ as defined above, a set $T \subseteq V$ is a transversal of $\mathcal{V}$ if there is a bijection $\pi : T \leftrightarrow I$ such that

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- Note that due to it being a bijection, all of I and T are "covered" (so this makes things distinct).

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- Thus, transversals are down closed (subclusive).

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- We have

Theorem 5.8.1 (Hall's theorem)

Given a set system $(V, \mathcal{V})$, the family of subsets $\mathcal{V} = (V_i : i \in I)$ has a transversal $(v_i : i \in I)$ iff for all $J \subseteq I$

$$|V(J)| \geq |J| \tag{5.28}$$

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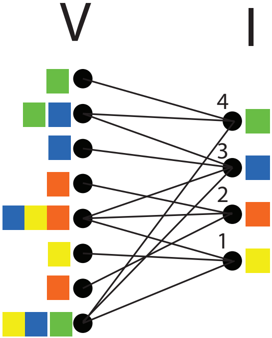

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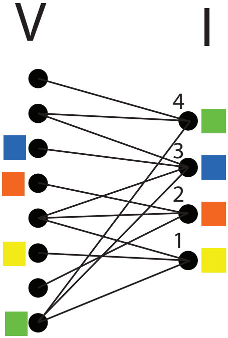

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- Moreover, we have

Theorem 5.8.2 (Rado's theorem)

If $M = (V, r)$ is a matroid on V with rank function r, then the family of subsets $(V_i : i \in I)$ of V has a transversal $(v_i : i \in I)$ that is independent in M iff for all $J \subseteq I$

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- Note, a transversal T independent in M means that $r(T) = |T|$.

More general conditions for existence of transversals

Theorem 5.8.3

If $\mathcal{V} = (V_i : I \in I)$ *is a finite family of non-empty subsets of* V*, and* $f : 2^V \to \mathbb{Z}_+$ *is a non-negative, integral, monotone non-decreasing, and submodular function, then* $\mathcal{V}$ *has a system of representatives* $(v_i : i \in I)$ *such that*

$$f(\cup_{i \in J}\{v_i\}) \geq |J| \text{ for all } J \subseteq I \tag{5.30}$$

if and only if

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- Given Theorem 5.8.3, we immediately get Theorem 5.8.1 by taking $f(S) = |S|$ for $S \subseteq V$. *In which case, Eq. 5.30 requires the system of representatives to be distinct.*

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- We get Theorem 5.8.2 by taking $f(S) = r(S)$ for $S \subseteq V$, the rank function of the matroid. *where, Eq. 5.30 insists the system of representatives is independent in M.*

More general conditions for existence of transversals

first part proof of Theorem 5.8.3.

- Suppose Eq. 5.30 is true. Then since f is monotone, and since $V(J) \supseteq \cup_{i \in J}\{v_i\}$ when $(v_i : i \in I)$ is a system of representatives, then Eq. 5.31 immediately follows.

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More general conditions for existence of transversals

Lemma 5.8.4

Suppose Eq. 5.31 ($f(V(J)) \geq |J|, \forall J \subseteq I$) is true for $\mathcal{V}$, and there exists an i such that $|V_i| \geq 2$ (w.l.o.g., say $i = 1$). Then there exists $\bar{v} \in V_1$ such that the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 5.31.

Proof.

- When Eq. 5.31 and the above holds, this means that for any subsets $J_1, J_2 \subseteq I \setminus \{1\}$, we have that

$$f(V_1 \cup V(J_1)) \geq |J_1| + 1 \tag{5.32}$$
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$$f((V_1 \setminus \{\bar{v}_1\}) \cup V(J_1)) < |J_1| + 1, \tag{5.34}$$

$$f((V_1 \setminus \{\bar{v}_2\}) \cup V(J_2)) < |J_2| + 1, \tag{5.35}$$

(note that either one or both of J_1, J_2 could be empty).

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Proof.

- Taking $X = (V_1 \setminus \{\bar{v}_1\}) \cup V(J_1)$ and $Y = (V_1 \setminus \{\bar{v}_2\}) \cup V(J_2)$, we have $f(X) \leq |J_1|$, $f(Y) \leq |J_2|$, and that:

$$X \cup Y = V_1 \cup V(J_1 \cup J_2), \tag{5.36}$$

$$X \cap Y \supseteq V(J_1 \cap J_2), \tag{5.37}$$

and

$$|J_1| + |J_2| \geq f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y). \tag{5.38}$$

More general conditions for existence of transversals

Lemma 5.8.4

Suppose Eq. 5.31 ($f(V(J)) \geq |J|, \forall J \subseteq I$) is true for $\mathcal{V}$, and there exists an i such that $|V_i| \geq 2$ (w.l.o.g., say $i = 1$). Then there exists $\bar{v} \in V_1$ such that the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \dots, V_{|I|})$ also satisfies Eq 5.31.

Proof.

- since f submodular monotone non-decreasing, & Eqs 5.32-5.35,

$$|J_1| + |J_2| \geq f(V_1 \cup V(J_1 \cup J_2)) + f(V(J_1 \cap J_2)) \tag{5.39}$$

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- Since $\mathcal{V}$ satisfies Eq. 5.31, $1 \notin J_1 \cup J_2$, & Eqs 5.32-5.33, this gives

$$|J_1| + |J_2| \geq |J_1 \cup J_2| + 1 + |J_1 \cap J_2| \tag{5.40}$$

which is a contradiction since cardinality is modular.

More general conditions for existence of transversals

converse proof of Theorem 5.8.3.

- Conversely, suppose Eq. 5.31 is true.

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This theorem can be used to produce a variety of other results quite easily, and shows how submodularity is the key ingredient in its truth.

Transversal Matroid

Transversals, themselves, define a matroid.

Theorem 5.9.1

If $\mathcal{V}$ is a family of finite subsets of a ground set V, then the collection of partial transversals of $\mathcal{V}$ is the set of independent sets of a matroid $M = (V, \mathcal{V})$ on V.

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- This means that the transversals of $\mathcal{V}$ are the bases of matroid M.
- Therefore, all maximal partial transversals of $\mathcal{V}$ have the same cardinality!

Transversals and Matchings

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- A matching in this graph is a set of edges no two of which that have a common endpoint.
- In fact, we easily have

Lemma 5.9.2

A subset $T \subseteq V$ is a partial transversal of $\mathcal{V}$ iff there is a matching in (V, I, E) in which every edge has one endpoint in T.

We say that T is matched into I.

Morphing Partition Matroid Rank

- Recall the partition matroid rank function. Note, $k_i = |I_i|$ in the bipartite graph representation, and since a matroid, w.l.o.g., $|V_i| \geq k_i$ (also, recall, $V(J) = \cup_{j \in J} V_j$).

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... Morphing Partition Matroid Rank

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$$r(A) = \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} \left(|V(J_i) \cap V(I_i) \cap A| - |I_i \cap J_i| + |I_i| \right) \tag{5.45}$$

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- In fact, this bottom (more general) expression is the expression for the rank of a transversal matroid.

Partial Transversals Are Matroids

In fact, we have

Theorem 5.9.3

Let $(V, \mathcal{V})$ where $\mathcal{V} = (V_1, V_2, \dots, V_\ell)$ be a subset system. Let $I = \{1, \dots, \ell\}$. Let $\mathcal{I}$ be the set of partial transversals of $\mathcal{V}$. Then $(V, \mathcal{I})$ is a matroid.

Proof.

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- Suppose that T_1 and T_2 are partial transversals of $\mathcal{V}$ such that $|T_1| < |T_2|$. Exercise: show that (I3') holds.

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- Transversal matroid has rank

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- Therefore, this function is submodular.
- Note that it is a minimum over a set of modular functions. Is this true in general? Exercise:

Matroid loops

- A circuit in a matroids is well defined, a subset $A \subseteq E$ is circuit if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

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- In a matric (i.e., linear) matroid, the only such loop is the value $\mathbf{0}$, as all non-zero vectors have rank 1. The $\mathbf{0}$ can appear > 1 time with different indices, as can a self loop in a graph appear on different nodes.

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- Note, we also say that two elements s, t are said to be parallel if $\{s, t\}$ is a circuit.

Representable

Definition 5.10.1 (Matroid isomorphism)

Two matroids M_1 and M_2 respectively on ground sets V_1 and V_2 are isomorphic if there is a bijection $\pi : V_1 \to V_2$ which preserves independence (equivalently, rank, circuits, and so on).

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- We can more generally define matroids on a field.

Definition 5.10.2 (linear matroids on a field)

Let $\mathbf{X}$ be an $n \times m$ matrix and $E = \{1, \ldots, m\}$, where $\mathbf{X}_{ij} \in \mathbb{F}$ for some field, and let $\mathcal{I}$ be the set of subsets of E such that the columns of $\mathbf{X}$ are linearly independent over $\mathbb{F}$.

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- We can more generally define matroids on a field.

Definition 5.10.3 (representable (as a linear matroid))

Any matroid isomorphic to a linear matroid on a field is called representable over $\mathbb{F}$

Representability of Transversal Matroids

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- In particular:

Theorem 5.10.4

Transversal matroids are representable over all finite fields of sufficiently large cardinality, and are representable over any infinite field.

Converse: Representability of Transversal Matroids

The converse is not true, however.

Example 5.10.5

Let $V = \{1,2,3,4,5,6\}$ be a ground set and let $M = (V, \mathcal{I})$ be a set system where $\mathcal{I}$ is all subsets of V of cardinality ≤ 2 except for the pairs $\{1,2\}$, $\{3,4\}$, $\{5,6\}$.

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- It can be shown that this is a matroid and is representable.
- However, this matroid is not isomorphic to any transversal matroid.