Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 5 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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\[
\begin{align*}
 f(A) + f(B) & \geq f(A \cup B) + f(A \cap B) \\
 -f(A) + 2f(C) + f(B) & \leq -f(A) + f(C) + f(B) + f(A \cap B)
\end{align*}
\]
Read chapter 1 from Fujishige’s book.
Announcements, Assignments, and Reminders

- our room (Mueller Hall Room 154) is changed!
- Please do use our discussion board (https://canvas.uw.edu/courses/895956/discussion_topics) for all questions, comments, so that all will benefit from them being answered.
- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).
Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, Combinatorial Geometries
- L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
- L9: From Matroid Polytopes to Polymatroids.
- L10: Polymatroids and Submodularity
- L11: More properties of polymatroids, SFM special cases
- L12: polymatroid properties, extreme points polymatroids,
- L13: sat, dep, supp, exchange capacity, examples
- L14: Lattice theory: partially ordered sets; lattices; distributive, modular, submodular, and boolean lattices; ideals and join irreducibles.
- L15: Supp, Base polytope, polymatroids and entropic Venn diagrams, exchange capacity,
- L16: proof that minimum norm point yields min of submodular function, and the lattice of minimizers of a submodular function, Lovasz extension
- L17: Lovasz extension, Choquet Integration, more properties/examples of Lovasz extension, convex minimization and SFM.
- L18: Lovasz extension examples and structured convex norms, The Min-Norm Point Algorithm detailed.
- L19: symmetric submodular function minimization, maximizing monotone submodular function w. card constraints.
- L20: maximizing monotone submodular function w. other constraints, non-monotone maximization.

Finals Week: June 9th-13th, 2014.

Prof. Jeff Bilmes
Many (Equivalent) Definitions of Submodularity

\[ f(A) + f(B) \geq f(A \cup B) + f(A \cap B), \ \forall A, B \subseteq V \]  \hspace{1cm} (5.6)

\[ f(j|S) \geq f(j|T), \ \forall S \subseteq T \subseteq V, \text{ with } j \in V \setminus T \]  \hspace{1cm} (5.7)

\[ f(C|S) \geq f(C|T), \forall S \subseteq T \subseteq V, \text{ with } C \subseteq V \setminus T \]  \hspace{1cm} (5.8)

\[ f(j|S) \geq f(j|S \cup \{k\}), \ \forall S \subseteq V \text{ with } j \in V \setminus (S \cup \{k\}) \]  \hspace{1cm} (5.9)

\[ f(A \cup B|A \cap B) \leq f(A|A \cap B) + f(B|A \cap B), \ \forall A, B \subseteq V \]  \hspace{1cm} (5.10)

\[ f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S) - \sum_{j \in S \setminus T} f(j|S \cup T - \{j\}), \ \forall S, T \subseteq V \]  \hspace{1cm} (5.11)

\[ f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S), \ \forall S \subseteq T \subseteq V \]  \hspace{1cm} (5.12)

\[ f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|S \setminus \{j\}) + \sum_{j \in T \setminus S} f(j|S \cap T) \ \forall S, T \subseteq V \]  \hspace{1cm} (5.13)
We saw: column space of a matrix, dimensionality of span of subset of columns as rank function.
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Incidence matrix of (arbitrarily oriented version of) graph $G = (V, E)$, rank of matrix columns $F$ corresponded to spanning tree of edge-induced graph $G' = (V', F)$ where $v'$ are vertices incident to edges in $F$. 
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We saw several different “greedy” algorithms that proceed optimal spanning trees (Borůvka’s, Jarník/Prim/Dijkstra’s, and Kruskal’s).
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We wish to more formally connect the above, and generalize further.
From Matrix Rank → Matroid

- So $V$ is set of column vector indices of a matrix.
- Let $\mathcal{I}$ be a set of all subsets of $V$ such that for any $I \in \mathcal{I}$, the vectors indexed by $I$ are linearly independent.
- Given a set $B \in \mathcal{I}$ of linearly independent vectors, then any subset $A \subseteq B$ is also linearly independent. Hence, $\mathcal{I}$ is down-closed or "subclusive", under subsets. In other words,

$$A \subseteq B \text{ and } B \in \mathcal{I} \Rightarrow A \in \mathcal{I} \tag{5.32}$$

- **maxInd**: Inclusionwise maximal independent subsets (or bases) of any set $B \subseteq V$.

$$\text{maxInd}(B) \triangleq \{A \subseteq B : A \in \mathcal{I} \text{ and } \forall v \in B \setminus A, A \cup \{v\} \notin \mathcal{I}\} \tag{5.33}$$

- Given any set $B \subset V$ of vectors, all maximal (by set inclusion) subsets of linearly independent vectors are the same size. That is, for all $B \subseteq V$,

$$\forall A_1, A_2 \in \text{maxInd}(B), \mid A_1 \mid = \mid A_2 \mid \tag{5.34}$$
Thus, for all $I \in \mathcal{I}$, the matrix rank function has the property

$$r(I) = |I|$$  \hspace{1cm} (5.32)

and for any $B \notin \mathcal{I}$,

$$r(B) = \max \{|A| : A \subseteq B \text{ and } A \in \mathcal{I}\} \leq |B|$$  \hspace{1cm} (5.33)
Matroid

Independent set definition of a matroid is perhaps most natural. Note, if \( J \in \mathcal{I} \), then \( J \) is said to be an independent set.

**Definition 5.2.4 (Matroid)**

A set system \((E, \mathcal{I})\) is a Matroid if

- (I1) \( \emptyset \in \mathcal{I} \)
- (I2) \( \forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I} \)
- (I3) \( \forall I, J \in \mathcal{I} \), with \( |I| = |J| + 1 \), then there exists \( x \in I \setminus J \) such that \( J \cup \{x\} \in \mathcal{I} \).

Why is (I1) is not redundant given (I2)? Because without (I1) could have a non-matroid where \( \mathcal{I} = \{\} \).
On Matroids

- Abstract properties of linear dependence (Hassler Whitney, 1935), but already then found instances of objects with those properties not based on a matrix.
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- Matroid independent sets (i.e., $A$ s.t. $r(A) = |A|$) are useful constraint set, and fast algorithms for submodular optimization subject to one (or more) matroid independence constraints exist.
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- Matroid independent sets (i.e., $A$ s.t. $r(A) = |A|$) are useful constraint set, and fast algorithms for submodular optimization subject to one (or more) matroid independence constraints exist.
- Crapo & Rota preferred the term “combinatorial geometry”, or more specifically a “pregeometry” and said that pregeometries are “often described by the ineffably cacaphonic [sic] term ’matroid’, which we prefer to avoid in favor of the term ’pregeometry’.”
Slight modification (non unit increment) that is equivalent.

**Definition 5.3.1 (Matroid-II)**

A set system \((E, \mathcal{I})\) is a **Matroid** if

(I1') \(\emptyset \in \mathcal{I}\)

(I2') \(\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}\) (or “down-closed”)

(I3') \(\forall I, J \in \mathcal{I}, \text{ with } |I| > |J|, \text{ then there exists } x \in I \setminus J \text{ such that } J \cup \{x\} \in \mathcal{I}\)

Note (I1)=\(\text{(I1')}\), (I2)=\(\text{(I2')}\), and we get (I3)≡\(\text{(I3')}\) using induction.
Matroids, independent sets, and bases

- **Independent sets**: Given a matroid $M = (E, \mathcal{I})$, a subset $A \subseteq E$ is called independent if $A \in \mathcal{I}$ and otherwise $A$ is called dependent.
Matroids, independent sets, and bases

- **Independent sets**: Given a matroid $M = (E, \mathcal{I})$, a subset $A \subseteq E$ is called **independent** if $A \in \mathcal{I}$ and otherwise $A$ is called **dependent**.

- **A base of $U \subseteq E$**: For $U \subseteq E$, a subset $B \subseteq U$ is called a base of $U$ if $B$ is inclusionwise maximally independent subset of $U$. That is, $B \in \mathcal{I}$ and there is no $Z \in \mathcal{I}$ with $B \subset Z \subseteq U$. 
Matroids, independent sets, and bases

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- **A base of a matroid**: If $U = E$, then a “base of $E$” is just called a base of the matroid $M$ (this corresponds to a basis in a linear space).
Proposition 5.3.2

In a matroid $M = (E, I)$, for any $U \subseteq E(M)$, any two bases of $U$ have the same size.
Matroids - important property

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In a matroid $M = (E, \mathcal{I})$, for any $U \subseteq E(M)$, any two bases of $U$ have the same size.

- In matrix terms, given a set of vectors $U$, all sets of independent vectors that span the space spanned by $U$ have the same size.
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- In fact, under (I1),(I2), this condition is equivalent to (I3). Exercise: show the following is equivalent to the above.
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**Definition 5.3.3 (Matroid)**

A set system $(V, I)$ is a Matroid if
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Definition 5.3.3 (Matroid)

A set system $(V, I)$ is a Matroid if

(I1') $\emptyset \in I$ (emptyset containing)
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(I2’) $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ (down-closed or subclusive)

(I3’) $\forall X \subseteq V$, and $I_1, I_2 \in \maxInd(X)$, we have $|I_1| = |I_2|$ (all maximally independent subsets of $X$ have the same size).
Matroids - rank

Thus, in any matroid $M = (E, \mathcal{I})$, $\forall U \subseteq E(M)$, any two bases of $U$ have the same size.
Thus, in any matroid $M = (E, I)$, $\forall U \subseteq E(M)$, any two bases of $U$ have the same size.

The common size of all the bases of $U$ is called the rank of $U$, denoted $r_M(U)$ or just $r(U)$ when the matroid in equation is unambiguous.
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$r(E) = r_{(E, \mathcal{I})}$ is the rank of the matroid, and is the common size of all the bases of the matroid.
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- We can a bit more formally define the rank function this way.
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**Definition 5.3.4 (matroid rank function)**

The rank of a matroid is a function $r : 2^E \to \mathbb{Z}_+$ defined by

$$r(A) = \max \{|X| : X \subseteq A, X \in \mathcal{I}\} = \max_{X \in \mathcal{I}} |A \cap X|$$  \hspace{1cm} (5.1)
Matroids - rank

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- From the above, we immediately see that $r(A) \leq |A|$.
Matroids - rank

- Thus, in any matroid \( M = (E, \mathcal{I}) \), \( \forall U \subseteq E(M) \), any two bases of \( U \) have the same size.
- The common size of all the bases of \( U \) is called the rank of \( U \), denoted \( r_M(U) \) or just \( r(U) \) when the matroid in equation is unambiguous.
- \( r(E) = r(E, \mathcal{I}) \) is the rank of the matroid, and is the common size of all the bases of the matroid.
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\]  

(5.1)

- From the above, we immediately see that \( r(A) \leq |A| \).
- Moreover, if \( r(A) = |A| \), then \( A \in \mathcal{I} \), meaning \( A \) is independent (in this case, \( A \) is a self base).

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EE596b/Spring 2014/Submodularity - Lecture 5 - April 14th, 2014
F14/39 (pg.38/144)
Matroids, other definitions using matroid rank $r : 2^V \to \mathbb{Z}_+$

**Definition 5.3.5 (closed/flat/subspace)**

A subset $A \subseteq E$ is closed (equivalently, a flat or a subspace) of matroid $M$ if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

A hyperplane is a flat of rank $r(M) - 1$. 

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F15/39 (pg.39/144)
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A hyperplane is a flat of rank $r(M) - 1$.

Definition 5.3.6 (closure)

Given $A \subseteq E$, the closure (or span) of $A$, is defined by $\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}$. 

Therefore, a closed set $A$ has $\text{span}(A) = A$. 

Definition 5.3.7 (circuit)

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise-minimal dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \{a\}) = |A| - 1$).
Matroids, other definitions using matroid rank $r : 2^V \to \mathbb{Z}_+$

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Matroids by bases

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

Theorem 5.3.8 (Matroid (by bases))

Let $E$ be a set and $\mathcal{B}$ be a nonempty collection of subsets of $E$. Then the following are equivalent.

1. $\mathcal{B}$ is the collection of bases of a matroid;
2. if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.
3. if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.

Properties 2 and 3 are called “exchange properties.”
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Properties 2 and 3 are called “exchange properties.”
Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.
A set is independent if and only if it contains no circuit. Therefore, it is not surprising that circuits can also characterize a matroid.

**Theorem 5.3.9 (Matroid by circuits)**

Let $E$ be a set and $\mathcal{C}$ be a collection of subsets of $E$ that satisfy the following three properties:

1. **(C1):** $\emptyset \notin \mathcal{C}$
2. **(C2):** if $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.
3. **(C3):** if $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq C_2$, and $e \in C_1 \cap C_2$, then there exists a $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\}$.
Several circuit definitions for matroids.

**Theorem 5.3.10 (Matroid by circuits)**

Let $E$ be a set and $C$ be a collection of nonempty subsets of $E$, such that no two sets in $C$ are contained in each other. Then the following are equivalent.

1. $C$ is the collection of circuits of a matroid;
2. if $C, C' \in C$, and $x \in C \cap C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $C$;
3. if $C, C' \in C$, and $x \in C \cap C'$, and $y \in C \setminus C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $C$ containing $y$;
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Let $E$ be a set and $C$ be a collection of nonempty subsets of $E$, such that no two sets in $C$ are contained in each other. Then the following are equivalent.

1. $C$ is the collection of circuits of a matroid;
2. if $C, C' \in C$, and $x \in C \cap C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $C$;
3. if $C, C' \in C$, and $x \in C \cap C'$, and $y \in C \setminus C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $C$ containing $y$;

Again, think about this for a moment in terms of linear spaces and matrices, and spanning trees.
Theorem 5.3.11 (Matroid by submodular functions)

Let $f : 2^E \rightarrow \mathbb{Z}$ be a integer valued monotone non-decreasing submodular function. Define a set of sets as follows:

$$C(f) = \left\{ C \subseteq E : C \text{ is non-empty, is inclusionwise-minimal, and has } f(C) < |C| \right\}$$

(5.2)

Then $C(f)$ is the collection of circuits of a matroid on $E$.

Inclusionwise-minimal in this case means that if $C \in C(f)$, then there exists no $C' \subset C$ with $C' \in C(f)$ (i.e., $C' \subset C$ would either be empty or have $f(C') \geq |C'|$). Also, recall inclusionwise-minimal in Definition 5.3.7, the definition of a circuit.
Uniform Matroid

- Given \( E \), consider \( \mathcal{I} \) to be all subsets of \( E \) that are at most size \( k \). That is \( \mathcal{I} = \{A \subseteq E : |A| \leq k\} \).
Uniform Matroid

- Given $E$, consider $I$ to be all subsets of $E$ that are at most size $k$. That is $I = \{A \subseteq E : |A| \leq k\}$.
- Then $(E, I)$ is a matroid called a $k$-uniform matroid.
Uniform Matroid

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- Note, if $I, J \in \mathcal{I}$, and $|I| < |J| \leq k$, and $j \in J$ such that $j \not\in I$, then $j$ is such that $|I + j| \leq k$ and so $I + j \in \mathcal{I}$. 

Rank function

$\text{r}(A) = \begin{cases} 
|A| & \text{if } |A| \leq k \\
 k & \text{if } |A| > k 
\end{cases}$

Note, this function is submodular. Not surprising since $\text{r}(A) = \text{min}(|A|, k)$ which is a non-decreasing concave function applied to a modular function.

Closure function

$\text{span}(A) = \begin{cases} 
A & \text{if } |A| < k, \\
E & \text{if } |A| \geq k, 
\end{cases}$

A "free" matroid sets $k = |E|$, so everything is independent.
Uniform Matroid

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- Rank function

$$r(A) = \begin{cases} |A| & \text{if } |A| \leq k \\ k & \text{if } |A| > k \end{cases}$$ (5.3)
Uniform Matroid

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Linear (or Matric) Matroid

Let $X$ be an $n \times m$ matrix and $E = \{1, \ldots, m\}$.
Linear (or Matric) Matroid

- Let $\mathbf{X}$ be an $n \times m$ matrix and $E = \{1, \ldots, m\}$
- Let $\mathcal{I}$ consists of subsets of $E$ such that if $A \in \mathcal{I}$, and $A = \{a_1, a_2, \ldots, a_k\}$ then the vectors $x_{a_1}, x_{a_2}, \ldots, x_{a_k}$ are linearly independent.
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- the rank function is just the rank of the space spanned by the corresponding set of vectors.
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- The rank function is just the rank of the space spanned by the corresponding set of vectors.
- Rank is submodular, it is intuitive that it satisfies the diminishing returns property (a given vector can only become linearly dependent in a greater context, thereby no longer contributing to rank).
- Called both linear matroids and matric matroids.
Let $G = (V, E)$ be a graph. Consider $(E, \mathcal{I})$ where the edges of the graph $E$ are the ground set and $A \in \mathcal{I}$ if the edge-induced graph $G(V, A)$ by $A$ does not contain any cycle.
Let $G = (V, E)$ be a graph. Consider $(E, I)$ where the edges of the graph $E$ are the ground set and $A \in I$ if the edge-induced graph $G(V, A)$ by $A$ does not contain any cycle.

Then $M = (E, I)$ is a matroid.
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Bases are spanning forests (spanning trees if $G$ is connected).
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Rank function $r(A)$ is the size of the largest spanning forest contained in $G(V, A)$.

Closure function adds all edges between the vertices adjacent to any edge in $A$. Closure of a spanning forest is $G$. 
Example: graphic matroid

- A graph defines a matroid on edge sets, independent sets are those without a cycle.
Example: graphic matroid

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Partition Matroid

- Let $V$ be our ground set.
Partition Matroid

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- Let $V = V_1 \cup V_2 \cup \cdots \cup V_\ell$ be a partition of $V$ into $\ell$ blocks (i.e., disjoint sets). Define a set of subsets of $V$ as

$$I = \{ X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell \}. \quad (5.5)$$

where $k_1, \ldots, k_\ell$ are fixed parameters, $k_i \geq 0$. Then $M = (V, I)$ is a matroid.
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- Parameters associated with a partition matroid: $\ell$ and $k_1, k_2, \ldots, k_\ell$ although often the $k_i$’s are all the same.
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Parameters associated with a partition matroid: $\ell$ and $k_1, k_2, \ldots, k_\ell$ although often the $k_i$’s are all the same.

We’ll show that property (I3’) in Def 5.3.3 holds. If $X, Y \in I$ with $|Y| > |X|$, then there must be at least one $i$ with $|Y \cap V_i| > |X \cap V_i|$. Therefore, adding one element $e \in V_i \cap (Y \setminus X)$ to $X$ won’t break independence.
Partition Matroid

Ground set of objects, $V = \{ \}$

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Partition Matroid

Partition of $V$ into six blocks, $V_1, V_2, \ldots, V_6$
Partition Matroid

Limit associated with each block, \(\{k_1, k_2, \ldots, k_6\}\)
Partition Matroid

Independent subset but not maximally independent.
Partition Matroid

Maximally independent subset, what is called a base.
Partition Matroid

Not independent since over limit in set six.
Lemma 5.5.1

The rank function \( r : 2^E \to \mathbb{Z}_+ \) of a matroid is submodular, that is
\[
r(A) + r(B) \geq r(A \cup B) + r(A \cap B)
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Proof.

1. Let $X \in \mathcal{I}$ be an inclusionwise maximal set with $X \subseteq A \cap B$
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Proof.

1. Let $X \in \mathcal{I}$ be an inclusionwise maximal set with $X \subseteq A \cap B$

2. Let $Y \in \mathcal{I}$ be inclusionwise maximal set with $X \subseteq Y \subseteq A \cup B$. We can find such a $Y \supseteq X$ because the following. Let $Y' \in \mathcal{I}$ be any inclusionwise maximal set with $Y' \subseteq A \cup B$, which might not have $X \subseteq Y'$. Starting from $X \subseteq A \cup B$, since $|Y'| \geq |X|$, there exists a $y \in Y' \setminus X \subseteq A \cup B$ such that $X + y \in \mathcal{I}$ but since $y \in A \cup B$, also $X + y \in A \cup B$ — we then add $y$ to $X$. We can keep doing this while $|Y'| > |X|$ since this is a matroid. We end up with an inclusionwise maximal set $Y$ (so $|Y| = |Y'|$) with $Y \in \mathcal{I}$ and $X \subseteq Y$. 
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3. Since \( M \) is a matroid, we know that \( r(A \cap B) = r(X) = |X| \), and \( r(A \cup B) = r(Y) = |Y| \). Also, for any \( U \in \mathcal{I} \), \( r(A) \geq |A \cap U| \).
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(5.6) (5.7)
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\]

\[
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\]

\[
\geq |X| + |Y| = r(A \cap B) + r(A \cup B) \tag{5.8}
\]
In fact, we can use the rank of a matroid for its definition.

**Theorem 5.5.2 (Matroid from rank)**

Let $E$ be a set and let $r : 2^E \rightarrow \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with $r$ being its rank function if and only if for all $A, B \subseteq E$:

1. **(R1)** $\forall A \subseteq E \ 0 \leq r(A) \leq |A|$ (non-negative cardinality bounded)
2. **(R2)** $r(A) \leq r(B)$ whenever $A \subseteq B \subseteq E$ (monotone non-decreasing)
3. **(R3)** $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ for all $A, B \subseteq E$ (submodular)

- So submodularity and non-negative monotone non-decreasing, and unit increase is necessary and sufficient to define the matroid.
- Given above, unit increment (if $r(A) = k$, then either $r(A \cup \{v\}) = k$ or $r(A \cup \{v\}) = k + 1$) holds.
- A matroid is sometimes given as $(E, r)$ where $E$ is ground set and $r$ is rank function.
Matroids

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3. (R3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ for all $A, B \subseteq E$ (submodular)

From above, $r(\emptyset) = 0$. Let $v \notin A$, then by monotonicity and submodularity, $r(A) \leq r(A \cup \{v\}) \leq r(A) + r(\{v\})$ which gives only two possible values to $r(A \cup \{v\})$. 

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Proof of Theorem 5.5.2 (matroid from rank).

- Given a matroid $M = (E, I)$, we see its rank function as defined in Eq. 5.1 satisfies (R1), (R2), and, as we saw in Lemma 5.5.1, (R3) too.
Matroids from rank

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- Next, assume we have (R1), (R2), and (R3). Define $\mathcal{I} = \{ X \subseteq E : r(X) = |X| \}$. We will show that $(E, \mathcal{I})$ is a matroid.

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r(X) \geq r(Y) - r(Y \setminus X) \tag{5.9}
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Matroids from rank

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First, $\emptyset \in \mathcal{I}$.

Also, if $Y \in \mathcal{I}$ and $X \subseteq Y$ then by submodularity,

$$r(X) \geq r(Y) - r(Y \setminus X) - r(\emptyset) \quad (5.9)$$
Proof of Theorem 5.5.2 (matroid from rank).

Given a matroid $M = (E, I)$, we see its rank function as defined in Eq. 5.1 satisfies (R1), (R2), and, as we saw in Lemma 5.5.1, (R3) too.

Next, assume we have (R1), (R2), and (R3). Define $I = \{X \subseteq E : r(X) = |X|\}$. We will show that $(E, I)$ is a matroid.

First, $\emptyset \in I$.

Also, if $Y \in I$ and $X \subseteq Y$ then by submodularity,

$$r(X) \geq r(Y) - r(Y \setminus X) - r(\emptyset) \geq |Y| - |Y \setminus X|$$

(5.9)

(5.10)
Matroids from rank

Proof of Theorem 5.5.2 (matroid from rank).

- Given a matroid $M = (E, \mathcal{I})$, we see its rank function as defined in Eq. 5.1 satisfies (R1), (R2), and, as we saw in Lemma 5.5.1, (R3) too.

- Next, assume we have (R1), (R2), and (R3). Define $\mathcal{I} = \{ X \subseteq E : r(X) = |X| \}$. We will show that $(E, \mathcal{I})$ is a matroid.

- First, $\emptyset \in \mathcal{I}$.

- Also, if $Y \in \mathcal{I}$ and $X \subseteq Y$ then by submodularity,

$$
\begin{align*}
    r(X) &\geq r(Y) - r(Y \setminus X) - r(\emptyset) \\
    &\geq |Y| - |Y \setminus X| \\
    &= |X|
\end{align*}
$$

...
Given a matroid $M = (E, I)$, we see its rank function as defined in Eq. 5.1 satisfies (R1), (R2), and, as we saw in Lemma 5.5.1, (R3) too.

Next, assume we have (R1), (R2), and (R3). Define $I = \{X \subseteq E : r(X) = |X|\}$. We will show that $(E, I)$ is a matroid.

First, $\emptyset \in I$.

Also, if $Y \in I$ and $X \subseteq Y$ then by submodularity,

$$r(X) \geq r(Y) - r(Y \setminus X) - r(\emptyset) \quad (5.9)$$

$$\geq |Y| - |Y \setminus X| \quad (5.10)$$

$$= |X| \quad (5.11)$$

implying $r(X) = |X|$, and thus $X \in I$. 

...
Proof of Theorem 5.5.2 (matroid from rank) cont.

Let \( A, B \in \mathcal{I} \), with \(|A| < |B|\), so \( r(A) = |A| < r(B) = |B| \). Let \( B \setminus A = \{b_1, b_2, \ldots, b_k\} \) (note \( k \leq |B| \)).
Proof of Theorem 5.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).

- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A|$. Then
Proof of Theorem 5.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).

- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A|$. Then

$$r(B) \leq r(A \cup B) \quad (5.12)$$
Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).

Suppose, to the contrary, that $\forall b \in B \setminus A, A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A|$. Then

\[
r(B) \leq r(A \cup B) \leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A)
\]
Proof of Theorem 5.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).

- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A|$. Then

  \[
  r(B) \leq r(A \cup B) \leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) = r(A \cup (B \setminus \{b_1\}))
  \]

  giving a contradiction since $B \in \mathcal{I}$. 

Proof of Theorem 5.5.2 (matroid from rank) cont.

Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).

Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A|$. Then

\[
    r(B) \leq r(A \cup B)
\]

\[
    \leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A)
\]

\[
    = r(A \cup (B \setminus \{b_1\}))
\]

\[
    \leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A)
\]

\[
\text{giving a contradiction since } B \in \mathcal{I}.
\]
Proof of Theorem 5.5.2 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).

- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A|$. Then

\[
\begin{align*}
r(B) &\leq r(A \cup B) \quad \text{(5.12)} \\
&\leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) \quad \text{(5.13)} \\
&= r(A \cup (B \setminus \{b_1\})) \quad \text{(5.14)} \\
&\leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) \quad \text{(5.15)} \\
&= r(A \cup (B \setminus \{b_1, b_2\})) \quad \text{(5.16)}
\end{align*}
\]

giving a contradiction since $B \in \mathcal{I}$.
Proof of Theorem 5.5.2 (matroid from rank) cont.

Let \( A, B \in \mathcal{I} \), with \( |A| < |B| \), so \( r(A) = |A| < r(B) = |B| \). Let \( B \setminus A = \{b_1, b_2, \ldots, b_k\} \) (note \( k \leq |B| \)).

Suppose, to the contrary, that \( \forall b \in B \setminus A, A + b \notin \mathcal{I} \), which means for all such \( b \), \( r(A + b) = r(A) = |A| \). Then

\[
 r(B) \leq r(A \cup B) \\
\leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) \\
= r(A \cup (B \setminus \{b_1\})) \\
\leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) \\
= r(A \cup (B \setminus \{b_1, b_2\})) \\
\leq \ldots \leq r(A) = |A| < |B| 
\]

(giving a contradiction since \( B \in \mathcal{I} \)).
Proof of Theorem 5.5.2 (matroid from rank) cont.

Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $k \leq |B|$).

Suppose, to the contrary, that $\forall b \in B \setminus A, A + b \notin \mathcal{I}$, which means for all such $b$, $r(A + b) = r(A) = |A|$. Then

$$r(B) \leq r(A \cup B)$$

$$\leq r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) \quad \text{(5.13)}$$

$$= r(A \cup (B \setminus \{b_1\})) \quad \text{(5.14)}$$

$$\leq r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) \quad \text{(5.15)}$$

$$= r(A \cup (B \setminus \{b_1, b_2\})) \quad \text{(5.16)}$$

$$\leq \ldots \leq r(A) = |A| < |B|$$

(giving a contradiction since $B \in \mathcal{I}$.)
Another way of using function $r$ to define a matroid.

**Theorem 5.5.3 (Matroid from rank II)**

Let $E$ be a finite set and let $r : 2^E \to \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with $r$ being its rank function if and only if for all $A \subseteq E$, and $x, y \in E$:

(R1') $r(\emptyset) = 0$;

(R2') $r(X) \leq r(X \cup \{y\}) \leq r(X) + 1$;

(R3') If $r(X \cup \{x\}) = r(X \cup \{y\}) = r(X)$, then $r(X \cup \{x, y\}) = r(X)$.
Summarizing what we’ve so far seen, we saw that it is possible to uniquely define a matroid based on any of:

- Independence (define the independent sets).
Summarizing: Many ways to define a Matroid

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Summarizing what we’ve so far seen, we saw that it is possible to uniquely define a matroid based on any of:

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Summarizing: Many ways to define a Matroid

Summarizing what we’ve so far seen, we saw that it is possible to uniquely define a matroid based on any of:

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- Circuit axioms
- Closure axioms (we didn’t see this, but it is possible)
Summarizing: Many ways to define a Matroid

Summarizing what we’ve so far seen, we saw that it is possible to uniquely define a matroid based on any of:

- Independence (define the independent sets).
- Base axioms (exchangeability)
- Circuit axioms
- Closure axioms (we didn’t see this, but it is possible)
- Rank axioms (normalized, monotone, cardinality bounded, submodular)
Maximization problems for matroids

- Given a matroid $M = (E, \mathcal{I})$ and a modular cost function $c : E \rightarrow \mathbb{R}$, the task is to find an $X \in \mathcal{I}$ such that $c(X) = \sum_{x \in X} c(x)$ is maximum.
- This seems remarkably similar to the max spanning tree problem.
Minimization problems for matroids

- Given a matroid $M = (E, \mathcal{I})$ and a modular cost function $c : E \rightarrow \mathbb{R}$, the task is to find a basis $B \in \mathcal{B}$ such that $c(B)$ is minimized.

- This sounds like a set cover problem (find the minimum cost covering set of sets).
What is the partition matroid’s rank function?

\[ r(A) = \ell \sum_{i=1}^{\min(|A \cap V_i|, k_i)} \] (5.18)

which we also immediately see is submodular using properties we spoke about last week. That is:

1. \(|A \cap V_i|\) is submodular (in fact modular) in
2. \(\min(\text{submodular}(A), k_i)\) is submodular in
3. sums of submodular functions are submodular.

\[ r(A) \] is also non-negative integral monotone non-decreasing, so it defines a matroid (the partition matroid).
Partition Matroid

- What is the partition matroid’s rank function?
- A partition matroids rank function:

\[ r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i) \]  

(5.18)

which we also immediately see is submodular using properties we spoke about last week. That is:

1. \( |A \cap V_i| \) is submodular (in fact modular) in \( A \)
2. \( \min(\text{submodular}(A), k_i) \) is submodular in \( A \) since \( |A \cap V_i| \) is monotone.
3. Sums of submodular functions are submodular.

\( r(A) \) is also non-negative integral monotone non-decreasing, so it defines a matroid (the partition matroid).
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Partition Matroid

- What is the partition matroid’s rank function?
- A partition matroids rank function:

$$r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i)$$ (5.18)

which we also immediately see is submodular using properties we spoke about last week. That is:

1. $|A \cap V_i|$ is submodular (in fact modular) in $A$
2. $\min(\text{submodular}(A), k_i)$ is submodular in $A$ since $|A \cap V_i|$ is monotone.
3. sums of submodular functions are submodular.

- $r(A)$ is also non-negative integral monotone non-decreasing, so it defines a matroid (the partition matroid).
Thus, we can define a matroid as $M = (V, r)$ where $r$ satisfies matroid rank axioms.

Example: 2-partition matroid rank function: Given natural numbers $a, b \in \mathbb{Z}_+$ with $a > b$, and any set $R \subseteq V$ with $|R| = a$, two-block partition $V = (R, \bar{R})$, where $\bar{R} = V \setminus R$, define:

$$r(A) = \min(|A \cap R|, b) + \min(|A \cap \bar{R}|, |\bar{R}|)$$ 

$$= \min(|A \cap R|, b) + |A \cap \bar{R}|$$  

Partition matroid figure showing this:
Can use this to define a truncated matroid rank function. With 
\( r(A) = \min(|A \cap R|, b) + |A \cap \bar{R}|, \ b < a \), define:

\[
f_R(A) = \min \{ r(A), a \} \tag{5.21}
\]

\[
= \min \{ \min(|A \cap \bar{R}| + |A \cap R|, |A \cap \bar{R}| + b), a \} \tag{5.22}
\]

\[
= \min \{ |A|, b + |A \cap \bar{R}|, a \} \tag{5.23}
\]
Can use this to define a truncated matroid rank function. With \( r(A) = \min(|A \cap R|, b) + |A \cap \bar{R}|, \ b < a \), define:

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f_R(A) = \min \{ r(A), a \} = \min \{ \min(|A \cap \bar{R}| + |A \cap R|, |A \cap \bar{R}| + b), a \} = \min \{|A|, b + |A \cap \bar{R}|, a\}
\]

Defines a matroid \( M = (V, f_R) = (V, \mathcal{I}) \) (Goemans et. al.) with

\[
\mathcal{I} = \{ I \subseteq V : |I| \leq a \text{ and } |I \cap R| \leq b \},
\]
Truncated Matroid Rank Function

- Can use this to define a truncated matroid rank function. With \( r(A) = \min(|A \cap R|, b) + |A \cap \bar{R}|, \ b < a \), define:

  \[
  f_R(A) = \min \{ r(A), a \} \\
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  = \min \{ |A|, b + |A \cap \bar{R}|, a \}
  \]

(5.21) (5.22) (5.23)

- Defines a matroid \( M = (V, f_R) = (V, \mathcal{I}) \) (Goemans et. al.) with

\[
\mathcal{I} = \{ I \subseteq V : |I| \leq a \text{ and } |I \cap R| \leq b \},
\]

(5.24)

- Useful for showing hardness of constrained submodular minimization. Consider sets \( B \subseteq V \) with \( |B| = a \).
Can use this to define a **truncated matroid rank** function. With 

\[ r(A) = \min(|A \cap R|, b) + |A \cap \bar{R}|, \quad b < a, \]

define:

\[
f_R(A) = \min \{ r(A), a \} = \min \left\{ \min(|A \cap \bar{R}| + |A \cap R|, |A \cap \bar{R}| + b), a \right\}
\]

\[ = \min \{ |A|, b + |A \cap \bar{R}|, a \} \]

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\]

Useful for showing hardness of constrained submodular minimization. Consider sets \( B \subseteq V \) with \( |B| = a \).

For \( R \), we have \( f_R(R) = b < a \).
Can use this to define a truncated matroid rank function. With $r(A) = \min(|A \cap R|, b) + |A \cap \bar{R}|$, $b < a$, define:

$$f_R(A) = \min \{r(A), a\} \quad (5.21)$$

$$= \min \{\min(|A \cap \bar{R}| + |A \cap R|, |A \cap \bar{R}| + b), a\} \quad (5.22)$$

$$= \min \{|A|, b + |A \cap \bar{R}|, a\} \quad (5.23)$$

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$$\mathcal{I} = \{I \subseteq V : |I| \leq a \text{ and } |I \cap R| \leq b\}, \quad (5.24)$$

Useful for showing hardness of constrained submodular minimization. Consider sets $B \subseteq V$ with $|B| = a$.

- For $R$, we have $f_R(R) = b < a$.
- For any $B$ with $|B \cap R| \leq b$, $f_R(B) = a$. 
Truncated Matroid Rank Function

- Can use this to define a truncated matroid rank function. With \( r(A) = \min(|A \cap R|, b) + |A \cap \bar{R}|, \ b < a \), define:

\[
    f_R(A) = \min \{r(A), a\} \tag{5.21}
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\[
    = \min \{\min(|A \cap \bar{R}| + |A \cap R|, |A \cap \bar{R}| + b), a\} \tag{5.22}
\]

\[
    = \min \{|A|, b + |A \cap \bar{R}|, a\} \tag{5.23}
\]

- Defines a matroid \( M = (V, f_R) = (V, \mathcal{I}) \) (Goemans et. al.) with

\[
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- Useful for showing hardness of constrained submodular minimization. Consider sets \( B \subseteq V \) with \( |B| = a \).
  - For \( R \), we have \( f_R(R) = b < a \).
  - For any \( B \) with \( |B \cap R| \leq b \), \( f_R(B) = a \).
  - For any \( B \) with \( |B \cap R| = \ell \), with \( b < \ell < a \), \( f_R(B) = b + a - \ell \).
Truncated Matroid Rank Function

- Can use this to define a truncated matroid rank function. With \( r(A) = \min(|A \cap R|, b) + |A \cap \bar{R}|, \ b < a, \) define:
  \[
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  \]

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  \]

- Useful for showing hardness of constrained submodular minimization. Consider sets \( B \subseteq V \) with \( |B| = a \).
  - For \( R \), we have \( f_R(R) = b < a \).
  - For any \( B \) with \( |B \cap R| \leq b \), \( f_R(B) = a \).
  - For any \( B \) with \( |B \cap R| = \ell \), with \( b < \ell < a \), \( f_R(B) = b + a - \ell \).
  - \( R \), the set with minimum valuation amongst size-\( a \) sets, is hidden within an exponentially larger set of size-\( a \) sets with larger valuation.
A partition matroid can be viewed using a bipartite graph.

Letting $V$ denote the ground set, and $V_1, V_2, \ldots$ the partition, the graph is $G = (V, I, E)$ where $V$ is the ground set, $I$ is a set of “indices”, and $E$ is the set of edges.

$I = (I_1, I_2, \ldots, I_\ell)$ is a set of $k = \sum_{i=1}^\ell k_i$ nodes, grouped into $\ell$ clusters, where there are $k_i$ nodes in the $i^{th}$ group $I_i$.

$(v, i) \in E(G)$ iff $v \in V_j$ and $i \in I_j$. 
Example where $\ell = 5$, 
$(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3).$
Example where $\ell = 5$, 
$(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3)$.

Recall, $\Gamma : 2^V \rightarrow \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of $X$ is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.
Example where $\ell = 5$, 
$(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3)$.

- Recall, $\Gamma : 2^V \to \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of $X$ is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.
- Here, for $X \subseteq V$, we have $\Gamma(X) = \{i \in I : (v, i) \in E(G) \text{ and } v \in X\}$.

![Partition Matroid Diagram]
Example where $\ell = 5$,

$$(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3).$$

Recall, $\Gamma : 2^V \rightarrow \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of $X$ is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.

Here, for $X \subseteq V$, we have $\Gamma(X) = \{i \in I : (v, i) \in E(G)$ and $v \in X\}$.

For such a constructed bipartite graph, the rank function of a partition matroid is $r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i) =$ the maximum matching matching involving $X$. 
Laminar Family and Laminar Matroid

- We can define a matroid with structures richer than just partitions.
Laminar Family and Laminar Matroid

- We can define a matroid with structures richer than just partitions.
- A set system \((V, \mathcal{F})\) is called a \textit{laminar} family if for any two sets \(A, B \in \mathcal{F}\), at least one of the three sets \(A \cap B\), \(A \setminus B\), or \(B \setminus A\) is empty.
Laminar Family and Laminar Matroid

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- A set system \((V, \mathcal{F})\) is called a **laminar** family if for any two sets \(A, B \in \mathcal{F}\), at least one of the three sets \(A \cap B, A \setminus B,\) or \(B \setminus A\) is empty.

![Laminar Family Diagram]

- Family is laminar \(\exists\) no two **properly** intersecting members: \(\forall A, B \in \mathcal{F},\) either \(A, B\) disjoint \((A \cap B = \emptyset)\) or comparable \((A \subseteq B\) or \(B \subseteq A)\).
Laminar Family and Laminar Matroid

- We can define a matroid with structures richer than just partitions.
- A set system \((V, F)\) is called a \textit{laminar} family if for any two sets \(A, B \in F\), at least one of the three sets \(A \cap B\), \(A \setminus B\), or \(B \setminus A\) is empty.

Family is laminar \(\exists\) no two properly intersecting members: \(\forall A, B \in F\), either \(A, B\) disjoint \((A \cap B = \emptyset)\) or comparable \((A \subseteq B\) or \(B \subseteq A)\).

Suppose we have a laminar family \(F\) of subsets of \(V\) and an integer \(k_A\) for every set \(A \in F\).
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- Suppose we have a laminar family \(\mathcal{F}\) of subsets of \(V\) and an integer \(k_A\) for every set \(A \in \mathcal{F}\). Then \((V, \mathcal{I})\) defines a matroid where

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\mathcal{I} = \{I \subseteq E : |I \cap A| \leq k_A \text{ for all } A \in \mathcal{F}\}
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Exercise: what is the rank function here?
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![Laminar Family Diagram]  

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(5.25)

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