Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 3 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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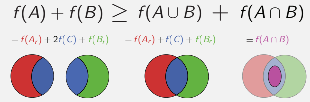
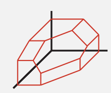
$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$





Cumulative Outstanding Reading

- Read chapter 1 from Fujishige's book.

Announcements, Assignments, and Reminders

- our room (Mueller Hall Room 154) is changed!
- Please do use our discussion board (https://canvas.uw.edu/courses/895956/discussion_topics) for all questions, comments, so that all will benefit from them being answered.
- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5:
- L6:
- L7:
- L8:
- L9:
- L10:
- L11:
- L12:
- L13:
- L14:
- L15:
- L16:
- L17:
- L18:
- L19:
- L20:

Finals Week: June 9th-13th, 2014.

Submodular Definitions

Definition 3.2.2 (submodular concave)

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \quad (3.2)$$

An alternate and (as we will soon see) equivalent definition is:

Definition 3.2.3 (diminishing returns)

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A \subseteq B \subset V$, and $v \in V \setminus B$, we have that:

$$f(A \cup \{v\}) - f(A) \geq f(B \cup \{v\}) - f(B) \quad (3.3)$$

This means that the incremental "value", "gain", or "cost" of v decreases (diminishes) as the context in which v is considered grows from A to B.

Many Properties

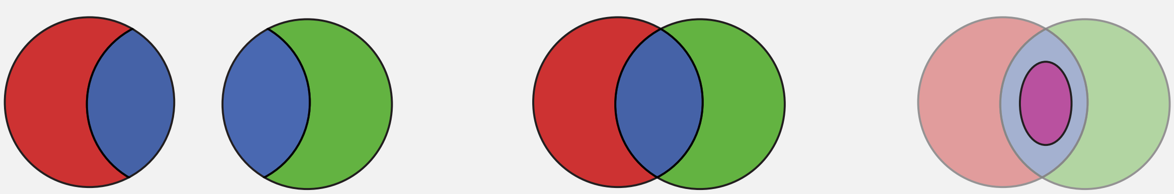
- In the last lecture, we started looking at properties of and gaining intuition about submodular functions.
- We began to see that there were many functions that were submodular, and operations on sets of submodular functions that preserved submodularity.

Some examples form last time

- Coverage functions (either via sets, or via regions in n-D space).
- Entropy function (as a function of sets of random variables), symmetric mutual information.
- Many functions based on graphs are either submodular or supermodular, and other functions might not be (e.g., graph strength) but involve submodularity in a critical way.
- Matrix rank - rank of a set of vectors from a set of vector indices.
- Geometric interpretation of $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$.
- Cost of manufacturing – supply side economies of scale
- Network Externalities – Demand side Economies of Scale
- Social Network Influence
- Information and Summarization - document summarization via sentence selection

The Venn and Art of Submodularity

$$\underbrace{r(A) + r(B)}_{= r(A_r) + 2r(C) + r(B_r)} \geq \underbrace{r(A \cup B)}_{= r(A_r) + r(C) + r(B_r)} + \underbrace{r(A \cap B)}_{= r(A \cap B)}$$



Polymatroid rank function

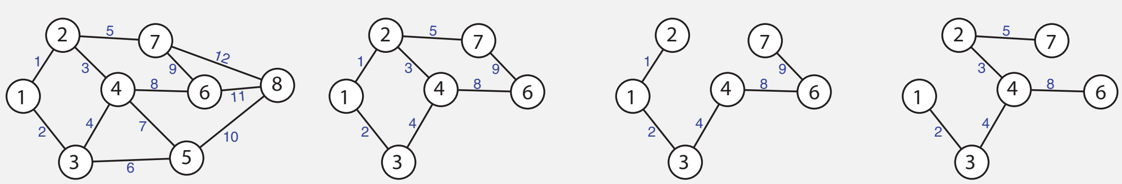
- Let S be a set of subspaces of a linear space (i.e., each $s \in S$ is a subspace of dimension ≥ 1).
- For each $X \subseteq S$, let $f(X)$ denote the dimensionality of the linear subspace spanned by the subspaces in X.
- We can think of S as a set of sets of vectors from the matrix rank example, and for each $s \in S$, let X_s being a set of vector indices.
- Then, defining $f : 2^S \to \mathbb{R}_+$ as follows,

$$f(X) = r(\cup_{s \in S} X_s) \tag{3.1}$$

  we have that f is submodular, and is known to be a polymatroid rank function.
- In general (as we will see) polymatroid rank functions are submodular, normalized $f(\emptyset) = 0$, and monotone non-decreasing ($f(A) \leq f(B)$ whenever $A \subseteq B$).

Spanning trees

- Let E be a set of edges of some graph $G = (V, E)$, and let $r(S)$ for $S \subseteq E$ be the maximum size (in terms of number of edges) spanning forest in the vertex-induced graph, induced by vertices incident to edges S.
- Example: Given $G = (V, E)$, $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $E = \{1, 2, \ldots, 12\}$. $S = \{1, 2, 3, 4, 5, 8, 9\} \subset E$. Two spanning trees have the same edge count (the rank of S).



- Then $r(S)$ is submodular, and is another matrix rank function corresponding to the incidence matrix of the graph.

Information and Summarization

- Let V be a set of information containing elements (V might say be either words, sentences, documents, web pages, or blogs, each $v \in V$ is one element, so v might be a word, a sentence, a document, etc.). The total amount of information in V is measure by a function $f(V)$, and any given subset $S \subseteq V$ measures the amount of information in S, given by $f(S)$.
- How informative is any given item v in different sized contexts? Any such real-world information function would exhibit diminishing returns, i.e., the value of v decreases when it is considered in a larger context.
- So a submodular function would likely be a good model.

Submodular Polyhedra

- Submodular functions have associated polyhedra with nice properties: when a set of constraints in a linear program is a submodular polyhedron, a simple greedy algorithm can find the optimal solution even though the polyhedron is formed via an exponential number of constraints.

$$P_f = \left\{x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E\right\} \tag{3.2}$$

$$P_f^+ = P_f \cap \left\{x \in \mathbb{R}^E : x \geq 0\right\} \tag{3.3}$$

$$B_f = P_f \cap \left\{x \in \mathbb{R}^E : x(E) = f(E)\right\} \tag{3.4}$$

- The linear programming problem is to, given $c \in \mathbb{R}^E$, compute:

$$\tilde{f}(c) \triangleq \max\left\{c^T x : x \in P_f\right\} \tag{3.5}$$

- This can be solved using the greedy algorithm! Moreover, $\tilde{f}(c)$ computed using greedy is convex if and only of f is submodular (we will go into this in some detail this quarter).

Ground set: E or V?

Submodular functions are functions defined on subsets of some finite set, called the ground set .

- It is common in the literature to use either E or V as the ground set.
- We will follow this inconsistency in the literature and will inconsistently use either E or V as our ground set (hopefully not in the same equation, if so, please point this out).

Notation $\mathbb{R}^E$

What does $x \in \mathbb{R}^E$ mean?

$$\mathbb{R}^E = \{x = (x_j \in \mathbb{R} : j \in E)\} \tag{3.6}$$

$$\mathbb{R}^E_+ = \{x = (x_j : j \in E) : x \geq 0\} \tag{3.7}$$

Any vector $x \in \mathbb{R}^E$ can be treated as a normalized modular function, and vice verse. That is

$$x(A) = \sum_{a \in A} x_a \tag{3.8}$$

Note that x is said to be normalized since $x(\emptyset) = 0$.

characteristic vectors of sets & modular functions

- Given an $A \subseteq E$, define the vector $\mathbf{1}_A \in \mathbb{R}_+^E$ to be

$$\mathbf{1}_A(j) = \begin{cases} 1 & \text{if } j \in A; \\ 0 & \text{if } j \notin A \end{cases} \tag{3.9}$$

- Sometimes this will be written as $\chi_A \equiv \mathbf{1}_A$.
- Thus, given modular function $x \in \mathbb{R}^E$, we can write $x(A)$ in a variety of ways, i.e.,

$$x(A) = x \cdot \mathbf{1}_A = \sum_{i \in A} x(i) \tag{3.10}$$

Other Notation: singletons and sets

When A is a set and k is a singleton (i.e., a single item), the union is properly written as $A \cup \{k\}$, but sometimes I will write just $A + k$.

General notation: what does S^T mean when S and T are arbitrary sets

- Let S and T be two arbitrary sets (either of which could be countable, or uncountable).
- We define the notation S^T to be the set of all functions that map from T to S. That is, if $f \in S^T$, then $f : T \to S$.
- Hence, given a finite set E, $\mathbb{R}^E$ is the set of all functions that map from elements of E to the reals $\mathbb{R}$, and such functions are identical to a vector in a vector space with axes labeled as elements of E (i.e., if $m \in \mathbb{R}^E$, then for all $e \in E$, $m(e) \in \mathbb{R}$).
- Similarly, 2^E is the set of all functions from E to "two" — in this case, we really mean $2 \equiv \{0,1\}$, so 2^E is shorthand for $\{0,1\}^V$ — hence, 2^E is the set of all functions that map from elements of E to $\{0,1\}$, equivalent to all binary vectors with elements indexed by elements of E, equivalent to subsets of E. Hence, if $A \in 2^E$ then $A \subseteq E$. What might 3^E mean?

Summing Submodular Functions

Given E, let $f_1, f_2 : 2^E \to \mathbb{R}$ be two submodular functions. Then

$$f : 2^E \to \mathbb{R} \text{ with } f(A) = f_1(A) + f_2(A) \tag{3.11}$$

is submodular.This follows easily since

$$f(A) + f(B) = f_1(A) + f_2(A) + f_1(B) + f_2(B) \tag{3.12}$$
$$\geq f_1(A \cup B) + f_2(A \cup B) + f_1(A \cap B) + f_2(A \cap B) \tag{3.13}$$
$$= f(A \cup B) + f(A \cap B). \tag{3.14}$$

I.e., it holds for each component of f in each term in the inequality. In fact, any conic combination (i.e., non-negative linear combination) of submodular functions is submodular, as in $f(A) = \alpha_1 f_1(A) + \alpha_2 f_2(A)$ for $\alpha_1, \alpha_2 \geq 0$.

Summing Submodular and Modular Functions

Given E, let $f_1, m : 2^E \to \mathbb{R}$ be a submodular and a modular function. Then

$$f : 2^E \to \mathbb{R} \text{ with } f(A) = f_1(A) - m(A) \tag{3.15}$$

is submodular (as is $f(A) = f_1(A) + m(A)$). This follows easily since

$$f(A) + f(B) = f_1(A) - m(A) + f_1(B) - m(B) \tag{3.16}$$

$$\geq f_1(A \cup B) - m(A \cup B) + f_1(A \cap B) - m(A \cap B) \tag{3.17}$$

$$= f(A \cup B) + f(A \cap B). \tag{3.18}$$

That is, the modular component with $m(A) + m(B) = m(A \cup B) + m(A \cap B)$ never destroys the inequality. Note of course that if m is modular than so is $-m$.

Restricting Submodular Functions

Given E, let $f : 2^E \to \mathbb{R}$ be a submodular functions. And let $S \subseteq E$ be an arbitrary fixed set. Then

$$f' : 2^E \to \mathbb{R} \text{ with } f'(A) = f(A \cap S) \tag{3.19}$$

is submodular.

Proof.

Given $A \subseteq B \subseteq E \setminus v$, consider

$$f((A + v) \cap S) - f(A \cap S) \geq f((B + v) \cap S) - f(B \cap S) \tag{3.20}$$

If $v \notin S$, then both differences on each size are zero. If $v \in S$, then we can consider this

$$f(A' + v) - f(A') \geq f(B' + v) - f(B') \tag{3.21}$$

with $A' = A \cap S$ and $B' = B \cap S$. Since $A' \subseteq B'$, this holds due to submodularity of f. □

Summing Restricted Submodular Functions

Given V, let $f_1, f_2 : 2^V \to \mathbb{R}$ be two submodular functions and let S_1, S_2 be two arbitrary fixed sets. Then

$$f : 2^V \to \mathbb{R} \text{ with } f(A) = f_1(A \cap S_1) + f_2(A \cap S_2) \tag{3.22}$$

is submodular. This follows easily from the preceding two results.
Given V, let $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ be a set of subsets of V, and for each $C \in \mathcal{C}$, let $f_C : 2^V \to \mathbb{R}$ be a submodular function. Then

$$f : 2^V \to \mathbb{R} \text{ with } f(A) = \sum_{C \in \mathcal{C}} f_C(A \cap C) \tag{3.23}$$

is submodular. This property is critical for image processing and graphical models. For example, let $\mathcal{C}$ be all pairs of the form $\{\{u, v\} : u, v \in V\}$, or let it be all pairs corresponding to the edges of some undirected graphical model. We plan to revisit this topic later in the term.

Max - normalized

Given V, let $c \in \mathbb{R}_+^V$ be a given fixed vector. Then $f : 2^V \to \mathbb{R}_+$, where

$$f(A) = \max_{j \in A} c_j \tag{3.24}$$

is submodular and normalized (we take $f(\emptyset) = 0$).

Proof.

Consider

$$\max_{j \in A} c_j + \max_{j \in B} c_j \geq \max_{j \in A \cup B} c_j + \max_{j \in A \cap B} c_j \tag{3.25}$$

which follows since we have that

$$\max(\max_{j \in A} c_j, \max_{j \in B} c_j) = \max_{j \in A \cup B} c_j \tag{3.26}$$

and

$$\min(\max_{j \in A} c_j, \max_{j \in B} c_j) \geq \max_{j \in A \cap B} c_j \tag{3.27}$$

$\square$

Max

Given V, let $c \in \mathbb{R}^V$ be a given fixed vector (not necessarily non-negative). Then $f : 2^V \to \mathbb{R}$, where

$$f(A) = \max_{j \in A} c_j \tag{3.28}$$

is submodular, where we take $f(\emptyset) \leq \min_j c_j$ (so the function is not normalized).

Proof.

The proof is identical to the normalized case. □

Facility/Plant Location (uncapacitated)

- Let $F = \{1, \ldots, f\}$ be a set of possible factory/plant locations for facilities to be built.
- $S = \{1, \ldots, s\}$ is a set of sites (e.g., cities, clients) needing service.
- Let c_{ij} be the "benefit" (e.g., $1/c_{ij}$ is the cost) of servicing site i with facility location j.
- Let m_j be the benefit (e.g., either $1/m_j$ is the cost or $-m_j$ is the cost) to build a plant at location j.
- Each site should be serviced by only one plant but no less than one.
- Define $f(A)$ as the "delivery benefit" plus "construction benefit" when the locations $A \subseteq F$ are to be constructed.
- We can define the (uncapacitated) facility location function

$$f(A) = \sum_{j \in A} m_j + \sum_{i \in F} \max_{j \in A} c_{ij}. \tag{3.4}$$

- Goal is to find a set A that maximizes $f(A)$ (the benefit) placing a bound on the number of plants A (e.g., $|A| \leq k$).

Facility Location

Given V, E, let $c \in \mathbb{R}^{V \times E}$ be a given $|V| \times |E|$ matrix. Then

$$f : 2^E \to \mathbb{R}, \text{ where } f(A) = \sum_{i \in V} \max_{j \in A} c_{ij} \tag{3.29}$$

is submodular.

Proof.

We can write $f(A)$ as $f(A) = \sum_{i \in V} f_i(A)$ where $f_i(A) = \max_{j \in A} c_{ij}$ is submodular (max of a i^{th} row vector), so f can be written as a sum of submodular functions. □

Thus, the facility location function (which only adds a modular function to the above) is submodular.

Log Determinant

- Let $\mathbf{\Sigma}$ be an $n \times n$ positive definite matrix. Let $V = \{1, 2, \ldots, n\} \equiv [n]$ be an index set, and for $A \subseteq V$, let $\mathbf{\Sigma}_A$ be the (square) submatrix of $\mathbf{\Sigma}$ obtained by including only entries in the rows/columns given by A.
- We have that:

$$f(A) = \log \det(\mathbf{\Sigma}_A) \text{ is submodular.} \tag{3.30}$$

- The submodularity of the log determinant is crucial for determinantal point processes (DPPs) (defined later in the class).

Proof of submodularity of the logdet function.

Suppose $X \in \mathbf{R}^n$ is multivariate Gaussian random variable, that is

$$x \in p(x) = \frac{1}{\sqrt{|2\pi\mathbf{\Sigma}|}} \exp\left(-\frac{1}{2}(x-\mu)^T \mathbf{\Sigma}^{-1}(x-\mu)\right) \tag{3.31}$$

...

Log Determinant

...cont.

Then the (differential) entropy of the r.v. X is given by

$$h(X) = \log \sqrt{|2\pi e \mathbf{\Sigma}|} = \log \sqrt{(2\pi e)^n |\mathbf{\Sigma}|} \tag{3.32}$$

and in particular, for a variable subset A,

$$f(A) = h(X_A) = \log \sqrt{(2\pi e)^{|A|} |\mathbf{\Sigma}_A|} \tag{3.33}$$

Entropy is submodular (conditioning reduces entropy), and moreover

$$f(A) = h(X_A) = m(A) + \frac{1}{2} \log |\mathbf{\Sigma}_A| \tag{3.34}$$

where $m(A)$ is a modular function. $\square$

Note: still submodular in the semi-definite case as well.

Summary so far

- Summing: if $\alpha_i \geq 0$ and $f_i : 2^V \to \mathbb{R}$ is submodular, then so is $\sum_i \alpha_i f_i$.
- Restrictions: $f'(A) = f(A \cap S)$
- max: $f(A) = \max_{j \in A} c_j$ and facility location.
- Log determinant $f(A) = \log \det(\mathbf{\Sigma}_A)$

Concave over non-negative modular

Let $m \in \mathbb{R}_+^E$ be a modular function, and g a concave function over $\mathbb{R}$. Define $f : 2^E \to \mathbb{R}$ as

$$f(A) = g(m(A)) \tag{3.35}$$

then f is submodular.

Proof.

Given $A \subseteq B \subseteq E \setminus v$, we have $0 \leq a = m(A) \leq b = m(B)$, and $0 \leq c = m(v)$. For g concave, we have $g(a+c) - g(a) \geq g(b+c) - g(b)$, and thus

$$g(m(A) + m(v)) - g(m(A)) \geq g(m(B) + m(v)) - g(m(B)) \tag{3.36}$$

□

A form of converse is true as well.

Concave composed with non-negative modular

Theorem 3.5.1

Given a ground set V. The following two are equivalent:

1. *For all modular functions $m : 2^V \to \mathbb{R}_+$, then $f : 2^V \to \mathbb{R}$ defined as $f(A) = g(m(A))$ is submodular*
2. *$g : \mathbb{R}_+ \to \mathbb{R}$ is concave.*

- If g is non-decreasing concave, then f is polymatroidal.
- Sums of concave over modular functions are submodular

$$f(A) = \sum_{i=1}^{K} g_i(m_i(A)) \tag{3.37}$$

- Very large class of functions, including graph cut, bipartite neighborhoods, set cover (Stobbe & Krause).
- However, Vondrak showed that a graphic matroid rank function over K_4 (we'll define this after we define matroids) are not members.

Monotonicity

Definition 3.6.1

A function $f : 2^V \to \mathbb{R}$ is monotone nondecreasing (resp. monotone increasing) if for all $A \subset B$, we have $f(A) \leq f(B)$ (resp. $f(A) < f(B)$).

Definition 3.6.2

A function $f : 2^V \to \mathbb{R}$ is monotone nonincreasing (resp. monotone decreasing) if for all $A \subset B$, we have $f(A) \geq f(B)$ (resp. $f(A) > f(B)$).

Composition of non-decreasting submodular and non-decreasing concave

Theorem 3.6.3

Given two functions, one defined on sets

$$f : 2^V \to \mathbb{R} \tag{3.38}$$

and another continuous valued one:

$$g : \mathbb{R} \to \mathbb{R} \tag{3.39}$$

the composition formed as $h = g \circ f : 2^V \to \mathbb{R}$ (defined as $h(S) = g(f(S))$) is nondecreasing submodular, if g is non-decreasing concave and f is nondecreasing submodular.

Monotone difference of two functions

Let f and g both be submodular functions on subsets of V and let $(f - g)(\cdot)$ be either monotone increasing or monotone decreasing. Then $h : 2^V \to R$ defined by

$$h(A) = \min(f(A), g(A)) \tag{3.40}$$

is submodular.

Proof.

If $h(A)$ agrees with either f or g on both X and Y, and since

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \tag{3.41}$$
$$g(X) + g(Y) \geq g(X \cup Y) + g(X \cap Y), \tag{3.42}$$

the result (Equation 3.40) follows since

$$\begin{matrix} f(X) + f(Y) \\ g(X) + g(Y) \end{matrix} \geq \min(f(X \cup Y), g(X \cup Y)) + \min(f(X \cap Y), g(X \cap Y)) \tag{3.43}$$

...

Monotone difference of two functions

...cont.

Otherwise, w.l.o.g., $h(X) = f(X)$ and $h(Y) = g(Y)$, giving

$$h(X) + h(Y) = f(X) + g(Y) \geq f(X \cup Y) + f(X \cap Y) + g(Y) - f(Y) \tag{3.44}$$

Assume the case where $f - g$ is monotone increasing. Hence, $f(X \cup Y) + g(Y) - f(Y) \geq g(X \cup Y)$ giving

$$h(X) + h(Y) \geq g(X \cup Y) + f(X \cap Y) \geq h(X \cup Y) + h(X \cap Y) \tag{3.45}$$

$\square$

What is an easy way to prove the case where $f - g$ is monotone decreasing?

Saturation via the $\min(\cdot)$ function

Let $f : 2^V \rightarrow \mathbb{R}$ be an monotone increasing or decreasing submodular function and let k be a constant. Then the function $h : 2^V \rightarrow \mathbb{R}$ defined by

$$h(A) = \min(k, f(A)) \tag{3.46}$$

is submodular.

Proof.

For constant k, we have that $(f - k)$ is increasing (or decreasing) so this follows from the previous result. □

Note also, $g(a) = \min(k, a)$ for constant k is a non-decreasing concave function, so when f is monotone nondecreasing submodular, we can use the earlier result about composing a monotone concave function with a monotone submodular function to get a version of this.

More on Min - the saturate trick

- In general, the minimum of two submodular functions is not submodular (unlike concave functions).
- However, when wishing to maximize two monotone non-decreasing submodular functions, we can define function $h : 2^V \rightarrow \mathbb{R}$ as

$$h(A) = \frac{1}{2}(\min(k, f) + \min(k, g)) \tag{3.47}$$

  then h is submodular, and $h(A) \geq k$ if and only if both $f(A) \geq k$ and $g(A) \geq k$.
- This can be useful in many applications. Moreover, this is an instance of a submodular surrogate (where we take a non-submodular problem and find a submodular one that can tell us something). We hope to revisit this again later in the quarter.

Arbitrary functions as difference between submodular funcs.

Given an arbitrary set function f, it can be expressed as a difference between two submodular functions: $f = g - h$ where both g and h are submodular.

Proof.

Let f be given and arbitrary, and define:

$$\alpha \triangleq \min_{X,Y} \Big(f(X) + f(Y) - f(X \cup Y) - f(X \cap Y) \Big) \tag{3.48}$$

If $\alpha \geq 0$ then f is submodular, so by assumption $\alpha < 0$. Now let h be an arbitrary strict submodular function and define

$$\beta \triangleq \min_{X,Y} \Big(h(X) + h(Y) - h(X \cup Y) - h(X \cap Y) \Big). \tag{3.49}$$

Strict means that $\beta > 0$. ...

Arbitrary functions as difference between submodular funcs.

...cont.

Define $f' : 2^V \to \mathbb{R}$ as

$$f'(A) = f(A) + \frac{|\alpha|}{\beta} h(A) \tag{3.50}$$

Then f' is submodular (why?), and $f = f'(A) - \frac{|\alpha|}{\beta} h(A)$, a difference between two submodular functions as desired.

□

Gain

- We often wish to express the gain of an item $j \in V$ in context A, namely $f(A \cup \{j\}) - f(A)$.
- This is called the gain and is used so often, there are equally as many ways to notate this. I.e., you might see:

$$f(A \cup \{j\}) - f(A) \triangleq \rho_j(A) \tag{3.51}$$
$$\triangleq \rho_A(j) \tag{3.52}$$
$$\triangleq \nabla_j f(A) \tag{3.53}$$
$$\triangleq f(\{j\}|A) \tag{3.54}$$
$$\triangleq f(j|A) \tag{3.55}$$

- We'll use $f(j|A)$.
- Submodularity's diminishing returns definition can be stated as saying that $f(j|A)$ is a monotone non-increasing function of A, since $f(j|A) \geq f(j|B)$ whenever $A \subseteq B$ (conditioning reduces valuation).

Gain Notation

It will also be useful to extend this to sets.
Let A, B be any two sets. Then

$$f(A|B) \triangleq f(A \cup B) - f(B) \tag{3.56}$$

So when j is any singleton

$$f(j|B) = f(\{j\}|B) = f(\{j\} \cup B) - f(B) \tag{3.57}$$

Note that this is inspired from information theory and the notation used for conditional entropy $H(X_A|X_B) = H(X_A, X_B) - H(X_B)$.

Arbitrary function as difference between two polymatroids

- Any submodular function g can be represented as a sum of a polymatroid (normalized monotone non-decreasing submodular) function $\bar{g}$ and a modular function m_g.
- Given submodular $g : 2^V \to \mathbb{R}$, construct $\bar{g} : 2^V \to \mathbb{R}$ as $\bar{g}(A) = g(A) - \sum_{a \in A} g(a|V \setminus \{a\})$. Let $m_g(A) \triangleq \sum_{a \in A} g(a|V \setminus \{a\})$
- Then, given arbitrary $f = g - h$ where g and h are submodular,

$$f = g - h = \bar{g} + m_g - \bar{h} - m_h \tag{3.58}$$

$$= \bar{g} - \bar{h} + (m_g - m_h) \tag{3.59}$$

$$= \bar{g} - \bar{h} + m_{g-h} \tag{3.60}$$

$$= \bar{g} + m^+_{g-h} - (\bar{h} + (-m_{g-h})^+) \tag{3.61}$$

  where m^+ is the positive part of modular function m. That is, $m^+(A) = \sum_{a \in A} m(a)\mathbf{1}(m(a) > 0)$.
- But both $g + m^+_{g-h}$ and $\bar{h} + (-m_{g-h})^+$ are polymatroid functions.
- Thus, any function can be expressed as a difference between two polymatroid functions.

Applications

- Sensor placement with submodular costs. I.e., let V be a set of possible sensor locations, $f(A) = I(X_A; X_{V \setminus A})$ measures the quality of a subset A of placed sensors, and $c(A)$ the submodular cost. We have $f(A) - \lambda c(A)$ as the overall objective.
- Discriminatively structured graphical models, EAR measure $I(X_A; X_{V \setminus A}) - I(X_A; X_{V \setminus A}|C)$, and synergy in neuroscience.
- Feature selection: a problem of maximizing $I(X_A; C) - \lambda c(A) = H(X_A) - [H(X_A|C) + \lambda c(A)]$, the difference between two submodular functions, where H is the entropy and c is a feature cost function.
- Graphical Model Inference. Finding x that maximizes $p(x) \propto \exp(-v(x))$ where $x \in \{0, 1\}^n$ and v is a pseudo-Boolean function. When v is non-submodular, it can be represented as a difference between submodular functions.