Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 3 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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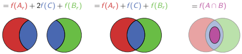

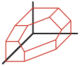

Cumulative Outstanding Reading

- Read chapter 1 from Fujishige's book.

Announcements, Assignments, and Reminders

- our room (Mueller Hall Room 154) is changed!
- Please do use our discussion board (`https://canvas.uw.edu/courses/895956/discussion_topics`) for all questions, comments, so that all will benefit from them being answered.
- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3:
- L4:
- L5:
- L6:
- L7:
- L8:
- L9:
- L10:
- L11:
- L12:
- L13:
- L14:
- L15:
- L16:
- L17:
- L18:
- L19:
- L20:

Finals Week: June 9th-13th, 2014.

Submodular Definitions

Definition 3.2.2 (submodular concave)

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \tag{3.2}$$

An alternate and (as we will soon see) equivalent definition is:

Definition 3.2.3 (diminishing returns)

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A \subseteq B \subset V$, and $v \in V \setminus B$, we have that:

$$f(A \cup \{v\}) - f(A) \geq f(B \cup \{v\}) - f(B) \tag{3.3}$$

This means that the incremental "value", "gain", or "cost" of v decreases (diminishes) as the context in which v is considered grows from A to B.

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- We began to see that there were many functions that were submodular, and operations on sets of submodular functions that preserved submodularity.

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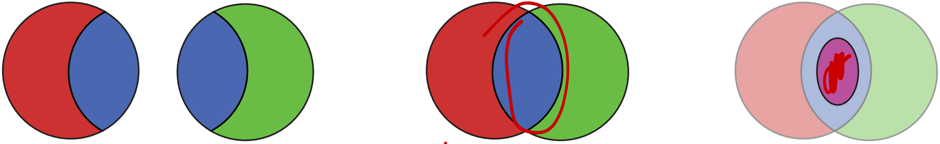
- Social Network Influence
- Information and Summarization - document summarization via sentence selection

The Venn and Art of Submodularity

$$\underbrace{r(A) + r(B)}_{= r(A_r) + 2r(C) + r(B_r)} \geq \underbrace{r(A \cup B)}_{= r(A_r) + r(C) + r(B_r)} + \underbrace{r(A \cap B)}_{= r(A \cap B)}$$



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- We can think of S as a set of sets of vectors from the matrix rank example, and for each $s \in S$, let X_s being a set of vector indices.
- Then, defining $f : 2^S \to \mathbb{R}_+$ as follows,

$$f(X) = r(\cup_{s \in S} X_s) \tag{3.1}$$

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  we have that f is submodular, and is known to be a polymatroid rank function.
- In general (as we will see) polymatroid rank functions are submodular, normalized $f(\emptyset) = 0$, and monotone non-decreasing ($f(A) \leq f(B)$ whenever $A \subseteq B$).

Spanning trees

- Let E be a set of edges of some graph $G = (V, E)$, and let $r(S)$ for $S \subseteq E$ be the maximum size (in terms of number of edges) spanning forest in the vertex-induced graph, induced by vertices incident to edges S.

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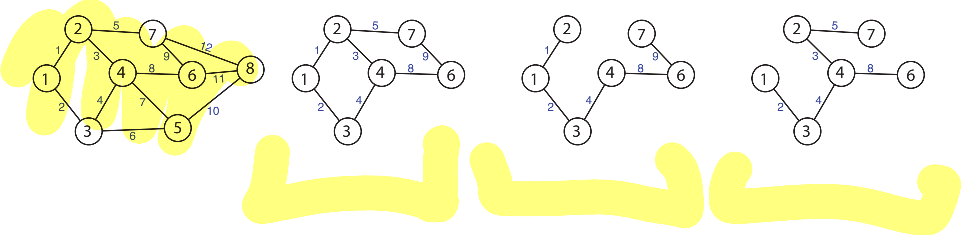
- Example: Given $G = (V, E)$, $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $E = \{1, 2, \ldots, 12\}$. $S = \{1, 2, 3, 4, 5, 8, 9\} \subset E$. Two spanning trees have the same edge count (the rank of S).



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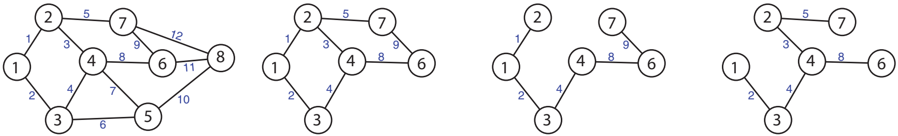

- Then $r(S)$ is submodular, and is another matrix rank function corresponding to the incidence matrix of the graph.

Supply Side Economies of scale

- What is a good model of the cost of manufacturing a set of items?
- Let V be a set of possible items that a company might possibly wish to manufacture, and let $f(S)$ for $S \subseteq V$ be the cost to that company to manufacture subset S.
- Ex: V might be colors of paint in a paint manufacturer: green, red, blue, yellow, white, etc.
- Producing green when you are already producing yellow and blue is probably cheaper than if you were only producing some other colors.

$$f(\text{green}, \text{blue}, \text{yellow}) - f(\text{blue}, \text{yellow}) <= f(\text{green}, \text{blue}) - f(\text{blue}) \tag{3.1}$$

- So diminishing returns (a submodular function) would be a good model.

A model of Influence in Social Networks

- Given a graph $G = (V, E)$, each $v \in V$ corresponds to a person, to each v we have an activation function $f_v : 2^V \to [0, 1]$ dependent only on its neighbors. I.e., $f_v(A) = f_v(A \cap \Gamma(v))$.
- Goal - Viral Marketing: find a small subset $S \subseteq V$ of individuals to directly influence, and thus indirectly influence the greatest number of possible other individuals (via the social network G).
- We define a function $f : 2^V \to \mathbb{Z}^+$ that models the ultimate influence of an initial set S of nodes based on the following iterative process: At each step, a given set of nodes S are activated, and we activate new nodes $v \in V \setminus S$ if $f_v(S) \geq U[0, 1]$ (where $U[0, 1]$ is a uniform random number between 0 and 1).
- It can be shown that for many f_v (including simple linear functions, and where f_v is submodular itself) that f is submodular.

The value of a friend

- Let V be a group of individuals. How valuable to you is a given friend $v \in V$?
- It depends on how many friends you have.
- Given a group of friends $S \subseteq V$, can you valuate them with a function $f(S)$ an how?
- Let $f(S)$ be the value of the set of friends S. Is submodular or supermodular a good model?

Information and Summarization

- Let V be a set of information containing elements (V might say be either words, sentences, documents, web pages, or blogs, each $v \in V$ is one element, so v might be a word, a sentence, a document, etc.). The total amount of information in V is measure by a function $f(V)$, and any given subset $S \subseteq V$ measures the amount of information in S, given by $f(S)$.
- How informative is any given item v in different sized contexts? Any such real-world information function would exhibit diminishing returns, i.e., the value of v decreases when it is considered in a larger context.
- So a submodular function would likely be a good model.

Submodular Polyhedra

- Submodular functions have associated polyhedra with nice properties: when a set of constraints in a linear program is a submodular polyhedron, a simple greedy algorithm can find the optimal solution even though the polyhedron is formed via an exponential number of constraints.

$$P_f = \{x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E\} \quad (3.2)$$

$$P_f^+ = P_f \cap \{x \in \mathbb{R}^E : x \geq 0\} \quad (3.3)$$

$$B_f = P_f \cap \{x \in \mathbb{R}^E : x(E) = f(E)\} \quad (3.4)$$

- The linear programming problem is to, given $c \in \mathbb{R}^E$, compute:

$$\tilde{f}(c) \triangleq \max \{c^T x : x \in P_f\} \quad (3.5)$$

- This can be solved using the greedy algorithm! Moreover, $\tilde{f}(c)$ computed using greedy is convex if and only of f is submodular (we will go into this in some detail this quarter).

Ground set: E or V?

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- It is common in the literature to use either E or V as the ground set.
- We will follow this inconsistency in the literature and will inconsistently use either E or V as our ground set (hopefully not in the same equation, if so, please point this out).

Notation $\mathbb{R}^E$

What does $x \in \mathbb{R}^E$ mean?

$m = |E|$

$$\mathbb{R}^E = \{x = (x_j \in \mathbb{R} : j \in E)\} \tag{3.6}$$

$$\mathbb{R}_+^E = \{x = (x_j : j \in E) : x \geq 0\} \tag{3.7}$$

Any vector $x \in \mathbb{R}^E$ can be treated as a normalized modular function, and vice verse. That is

$$x(A) = \sum_{a \in A} x_a \tag{3.8}$$

Note that x is said to be normalized since $x(\emptyset) = 0$.

characteristic vectors of sets & modular functions

- Given an $A \subseteq E$, define the vector $\mathbf{1}_A \in \mathbb{R}_+^E$ to be

$$\mathbf{1}_A(j) = \begin{cases} 1 & \text{if } j \in A; \\ 0 & \text{if } j \notin A \end{cases} \tag{3.9}$$

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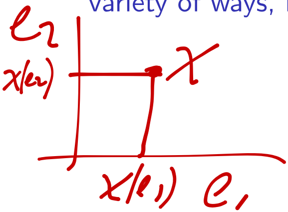
- Thus, given modular function $x \in \mathbb{R}^E$, we can write $x(A)$ in a variety of ways, i.e.,



$$x(A) = x \cdot \mathbf{1}_A = \sum_{i \in A} x(i) \tag{3.10}$$

Other Notation: singletons and sets

When A is a set and k is a singleton (i.e., a single item), the union is properly written as $A \cup \{k\}$, but sometimes I will write just $A + k$.

General notation: what does S^T mean when S and T are arbitrary sets

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- Hence, given a finite set E, $\mathbb{R}^E$ is the set of all functions that map from elements of E to the reals $\mathbb{R}$, and such functions are identical to a vector in a vector space with axes labeled as elements of E (i.e., if $m \in \mathbb{R}^E$, then for all $e \in E$, $m(e) \in \mathbb{R}$).

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- Similarly, 2^E is the set of all functions from E to "two" — in this case, we really mean $2 \equiv \{0, 1\}$, so 2^E is shorthand for $\{0, 1\}^V$

$S \subseteq V$ $\quad f : 2^V \to \mathbb{R}$

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Summing Submodular Functions

Given E, let $f_1, f_2 : 2^E \to \mathbb{R}$ be two submodular functions. Then

$$f : 2^E \to \mathbb{R} \text{ with } f(A) = f_1(A) + f_2(A) \tag{3.11}$$

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is submodular. This follows easily since

$$f(A) + f(B) = f_1(A) + f_2(A) + f_1(B) + f_2(B) \tag{3.12}$$

$$\geq f_1(A \cup B) + f_2(A \cup B) + f_1(A \cap B) + f_2(A \cap B) \tag{3.13}$$

$$= f(A \cup B) + f(A \cap B). \tag{3.14}$$

I.e., it holds for each component of f in each term in the inequality.

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$$\begin{aligned} f(A) + f(B) &= f_1(A) + f_2(A) + f_1(B) + f_2(B) && (3.12)\\ &\geq f_1(A \cup B) + f_2(A \cup B) + f_1(A \cap B) + f_2(A \cap B) && (3.13)\\ &= f(A \cup B) + f(A \cap B). && (3.14) \end{aligned}$$

I.e., it holds for each component of f in each term in the inequality. In fact, any conic combination (i.e., non-negative linear combination) of submodular functions is submodular, as in $f(A) = \alpha_1 f_1(A) + \alpha_2 f_2(A)$ for $\alpha_1, \alpha_2 \geq 0$.

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is submodular (as is $f(A) = f_1(A) + m(A)$). This follows easily since

$$f(A) + f(B) = f_1(A) - m(A) + f_1(B) - m(B) \tag{3.16}$$

$$\geq f_1(A \cup B) - m(A \cup B) + f_1(A \cap B) - m(A \cap B) \tag{3.17}$$

$$= f(A \cup B) + f(A \cap B). \tag{3.18}$$

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$$= f(A \cup B) + f(A \cap B). \tag{3.18}$$

That is, the modular component with $m(A) + m(B) = m(A \cup B) + m(A \cap B)$ never destroys the inequality. Note of course that if m is modular than so is $-m$.

Restricting Submodular Functions

Given E, let $f : 2^E \to \mathbb{R}$ be a submodular functions. And let $S \subseteq E$ be an arbitrary fixed set. Then

$$f' : 2^E \to \mathbb{R} \text{ with } f'(A) = f(A \cap S) \tag{3.19}$$

is submodular.

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Proof.

$\square$

Restricting Submodular Functions

Given E, let $f : 2^E \to \mathbb{R}$ be a submodular functions. And let $S \subseteq E$ be an arbitrary fixed set. Then

$$f' : 2^E \to \mathbb{R} \text{ with } f'(A) = f(A \cap S) \tag{3.19}$$

is submodular.

Proof.

Given $A \subseteq B \subseteq E \setminus v$, consider

$$f((A+v) \cap S) - f(A \cap S) \geq f((B+v) \cap S) - f(B \cap S) \tag{3.20}$$

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If $v \notin S$, then both differences on each size are zero. If $v \in S$, then we can consider this

$$f(A' + v) - f(A') \geq f(B' + v) - f(B') \tag{3.21}$$

with $A' = A \cap S$ and $B' = B \cap S$. Since $A' \subseteq B'$, this holds due to submodularity of f. $\square$

Summing Restricted Submodular Functions

Given V, let $f_1, f_2 : 2^V \to \mathbb{R}$ be two submodular functions and let S_1, S_2 be two arbitrary fixed sets. Then

$$f : 2^V \to \mathbb{R} \text{ with } f(A) = f_1(A \cap S_1) + f_2(A \cap S_2) \tag{3.22}$$

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Given V, let $\mathcal{C} = \{C_1, C_2, \dots, C_k\}$ be a set of subsets of V, and for each $C \in \mathcal{C}$, let $f_C : 2^V \to \mathbb{R}$ be a submodular function. Then

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is submodular. This property is critical for image processing and graphical models. For example, let $\mathcal{C}$ be all pairs of the form $\{\{u, v\} : u, v \in V\}$, or let it be all pairs corresponding to the edges of some undirected graphical model. We plan to revisit this topic later in the term.

Max - normalized

Given V, let $c \in \mathbb{R}_+^V$ be a given fixed vector. Then $f : 2^V \to \mathbb{R}_+$, where

$$f(A) = \max_{j \in A} c_j \tag{3.24}$$

is submodular and normalized (we take $f(\emptyset) = 0$).

Proof.

Consider

$$\max_{j \in A} c_j + \max_{j \in B} c_j \geq \max_{j \in A \cup B} c_j + \max_{j \in A \cap B} c_j \tag{3.25}$$

which follows since we have that

$$\max(\max_{j \in A} c_j, \max_{j \in B} c_j) = \max_{j \in A \cup B} c_j \tag{3.26}$$

and

$$\min(\max_{j \in A} c_j, \max_{j \in B} c_j) \geq \max_{j \in A \cap B} c_j \tag{3.27}$$

□

Max

Given V, let $c \in \mathbb{R}^V$ be a given fixed vector (not necessarily non-negative). Then $f : 2^V \to \mathbb{R}$, where

$$f(A) = \max_{j \in A} c_j \tag{3.28}$$

is submodular, where we take $f(\emptyset) \leq \min_j c_j$ (so the function is not normalized).

Proof.

The proof is identical to the normalized case. □

Facility/Plant Location (uncapacitated)

- Let $F = \{1, \dots, f\}$ be a set of possible factory/plant locations for facilities to be built.
- $S = \{1, \dots, s\}$ is a set of sites (e.g., cities, clients) needing service.
- Let c_{ij} be the "benefit" (e.g., $1/c_{ij}$ is the cost) of servicing site i with facility location j.
- Let m_j be the benefit (e.g., either $1/m_j$ is the cost or $-m_j$ is the cost) to build a plant at location j.
- Each site should be serviced by only one plant but no less than one.
- Define $f(A)$ as the "delivery benefit" plus "construction benefit" when the locations $A \subseteq F$ are to be constructed.
- We can define the (uncapacitated) facility location function

$$f(A) = \sum_{j \in A} m_j + \sum_{i \in F} \max_{j \in A} c_{ij}. \tag{3.4}$$

- Goal is to find a set A that maximizes $f(A)$ (the benefit) placing a bound on the number of plants A (e.g., $|A| \le k$).

Facility Location

Given V, E, let $c \in \mathbb{R}^{V \times E}$ be a given $|V| \times |E|$ matrix. Then

$$f : 2^E \to \mathbb{R}, \text{ where } f(A) = \sum_{i \in V} \max_{j \in A} c_{ij} \tag{3.29}$$

is submodular.

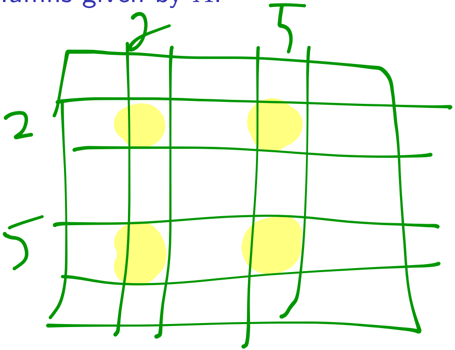
Proof.

We can write $f(A)$ as $f(A) = \sum_{i \in V} f_i(A)$ where $f_i(A) = \max_{j \in A} c_{ij}$ is submodular (max of a i^{th} row vector), so f can be written as a sum of submodular functions. □

Thus, the facility location function (which only adds a modular function to the above) is submodular.

Log Determinant

- Let $\mathbf{\Sigma}$ be an $n \times n$ positive definite matrix. Let $V = \{1, 2, \dots, n\} \equiv [n]$ be an index set, and for $A \subseteq V$, let $\mathbf{\Sigma}_A$ be the (square) submatrix of $\mathbf{\Sigma}$ obtained by including only entries in the rows/columns given by A.



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- We have that:

$$f(A) = \log\det(\mathbf{\Sigma}_A) \text{ is submodular.} \tag{3.30}$$

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Proof of submodularity of the logdet function.

Suppose $X \in \mathbf{R}^n$ is multivariate Gaussian random variable, that is

$$x \in p(x) = \frac{1}{\sqrt{|2\pi\mathbf{\Sigma}|}} \exp\left(-\frac{1}{2}(x-\mu)^T \mathbf{\Sigma}^{-1}(x-\mu)\right) \tag{3.31}$$

...

Log Determinant

...cont.

Then the (differential) entropy of the r.v. X is given by

$$h(X) = \log \sqrt{|2\pi e \mathbf{\Sigma}|} = \log \sqrt{(2\pi e)^n |\mathbf{\Sigma}|} \tag{3.32}$$

and in particular, for a variable subset A,

$$f(A) = h(X_A) = \log \sqrt{(2\pi e)^{|A|} |\mathbf{\Sigma}_A|} \tag{3.33}$$

Entropy is submodular (conditioning reduces entropy), and moreover

$$f(A) = h(X_A) = m(A) + \frac{1}{2} \log |\mathbf{\Sigma}_A| \tag{3.34}$$

where $m(A)$ is a modular function. □

Note: still submodular in the semi-definite case as well.

Summary so far

- Summing: if $\alpha_i \geq 0$ and $f_i : 2^V \to \mathbb{R}$ is submodular, then so is $\sum_i \alpha_i f_i$.

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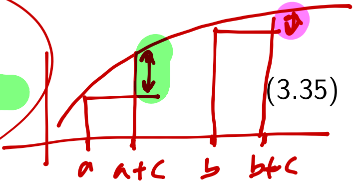
- Summing: if $\alpha_i \geq 0$ and $f_i : 2^V \to \mathbb{R}$ is submodular, then so is $\sum_i \alpha_i f_i$.
- Restrictions: $f'(A) = f(A \cap S)$
- max: $f(A) = \max_{j \in A} c_j$ and facility location.
- Log determinant $f(A) = \log \det(\mathbf{\Sigma}_A)$

Concave over non-negative modular

Let $m \in \mathbb{R}_+^E$ be a modular function, and g a concave function over $\mathbb{R}$. Define $f : 2^E \to \mathbb{R}$ as

$$f(A) = g(m(A)) \tag{3.35}$$



then f is submodular.

Proof.

Given $A \subseteq B \subseteq E \setminus v$, we have $0 \leq a = m(A) \leq b = m(B)$, and $0 \leq c = m(v)$. For g concave, we have $g(a+c) - g(a) \geq g(b+c) - g(b)$, and thus

$$g(m(A) + m(v)) - g(m(A)) \geq g(m(B) + m(v)) - g(m(B)) \tag{3.36}$$

□

A form of converse is true as well.

Concave composed with non-negative modular

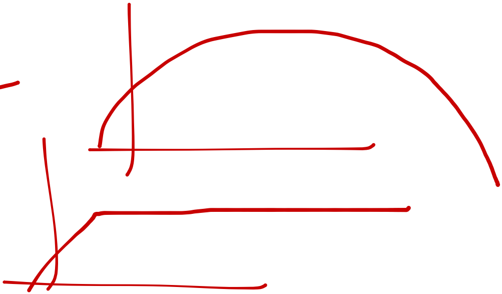
Theorem 3.5.1

Given a ground set V. The following two are equivalent:

1. *For all modular functions $m : 2^V \to \mathbb{R}_+$, then $f : 2^V \to \mathbb{R}$ defined as $f(A) = g(m(A))$ is submodular*
2. *$g : \mathbb{R}_+ \to \mathbb{R}$ is concave.*

- If g is non-decreasing concave, then f is polymatroidal.

$g(0) = 0$



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- If g is non-decreasing concave, then f is polymatroidal.
- Sums of concave over modular functions are submodular

$$f(A) = \sum_{i=1}^{K} g_i(m_i(A)) \tag{3.37}$$

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- Very large class of functions, including graph cut, bipartite neighborhoods, set cover (Stobbe & Krause).
- However, Vondrak showed that a graphic matroid rank function over K_4 (we'll define this after we define matroids) are not members.

Monotonicity

Definition 3.6.1

A function $f : 2^V \to \mathbb{R}$ is monotone nondecreasing (resp. monotone increasing) if for all $A \subset B$, we have $f(A) \leq f(B)$ (resp. $f(A) < f(B)$).

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Definition 3.6.2

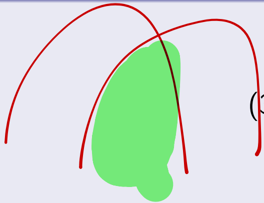
A function $f : 2^V \to \mathbb{R}$ is monotone nonincreasing (resp. monotone decreasing) if for all $A \subset B$, we have $f(A) \geq f(B)$ (resp. $f(A) > f(B)$).

Composition of submodular an concave

Theorem 3.6.3

Given two functions, one defined on sets

$$f : 2^V \to \mathbb{R} \tag{3.38}$$



and another continuous valued one:

$$g : \mathbb{R} \to \mathbb{R} \tag{3.39}$$

the composition formed as $h = g \circ f : 2^V \to \mathbb{R}$ *(defined as* $h(S) = g(f(S))$*) is nondecreasing submodular, if* g *is non-decreasing concave and* f *is nondecreasing submodular.*

Monotone difference of two functions

Let f and g both be submodular functions on subsets of V and let $(f-g)(\cdot)$ be either monotone increasing or monotone decreasing. Then $h : 2^V \to R$ defined by

$$h(A) = \min(f(A), g(A)) \tag{3.40}$$

is submodular.

Proof.

If $h(A)$ agrees with either f or g on both X and Y, and since

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \tag{3.41}$$

$$g(X) + g(Y) \geq g(X \cup Y) + g(X \cap Y), \tag{3.42}$$

the result (Equation 3.40) follows since

$$\begin{matrix} f(X) + f(Y) \\ g(X) + g(Y) \end{matrix} \geq \min(f(X \cup Y), g(X \cup Y)) + \min(f(X \cap Y), g(X \cap Y)) \tag{3.43}$$

Monotone difference of two functions

...cont.

Otherwise, w.l.o.g., $h(X) = f(X)$ and $h(Y) = g(Y)$, giving

$$h(X) + h(Y) = f(X) + g(Y) \geq f(X \cup Y) + f(X \cap Y) + g(Y) - f(Y) \tag{3.44}$$

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Assume the case where $f - g$ is monotone increasing. Hence, $f(X \cup Y) + g(Y) - f(Y) \geq g(X \cup Y)$ giving

$$h(X) + h(Y) \geq g(X \cup Y) + f(X \cap Y) \geq h(X \cup Y) + h(X \cap Y) \tag{3.45}$$

□

What is an easy way to prove the case where $f - g$ is monotone decreasing?

Saturation via the $\min(\cdot)$ function

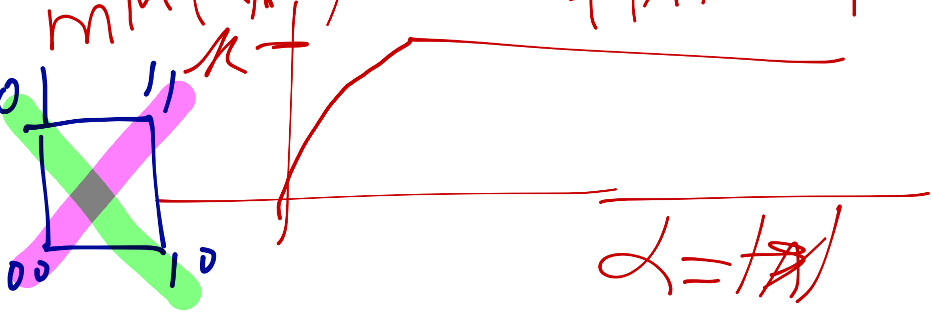
Let $f : 2^V \to \mathbb{R}$ be an monotone increasing or decreasing submodular function and let k be a constant. Then the function $h : 2^V \to \mathbb{R}$ defined by

$$h(A) = \min(k, f(A)) \tag{3.46}$$

$f(A) = \sqrt{|A|}$

is submodular.

min(h,k)



$\alpha = |B|$

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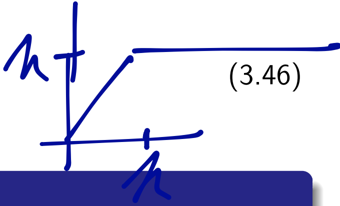

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Proof.

For constant k, we have that $(f - k)$ is increasing (or decreasing) so this follows from the previous result. □

Note also, $g(a) = \min(k, a)$ for constant k is a non-decreasing concave function, so when f is monotone nondecreasing submodular, we can use the earlier result about composing a monotone concave function with a monotone submodular function to get a version of this.

More on Min - the saturate trick

- In general, the minimum of two submodular functions is not submodular (unlike concave functions).

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- However, when wishing to maximize two monotone non-decreasing submodular functions, we can define function $h : 2^V \to \mathbb{R}$ as

$$h(A) = \frac{1}{2}(\min(k, f) + \min(k, g)) \tag{3.47}$$

then h is submodular, and $h(A) \geq k$ if and only if both $f(A) \geq k$ and $g(A) \geq k$.

More on Min - the saturate trick

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  then h is submodular, and $h(A) \geq k$ if and only if both $f(A) \geq k$ and $g(A) \geq k$.
- This can be useful in many applications. Moreover, this is an instance of a submodular surrogate (where we take a non-submodular problem and find a submodular one that can tell us something). We hope to revisit this again later in the quarter.

Arbitrary functions as difference between submodular funcs.

Given an arbitrary set function f, it can be expressed as a difference between two submodular functions: $f = g - h$ where both g and h are submodular.

Proof.

Let f be given and arbitrary, and define:

$$\alpha \triangleq \min_{X,Y} \Big(f(X) + f(Y) - f(X \cup Y) - f(X \cap Y) \Big) \tag{3.48}$$

If $\alpha \geq 0$ then f is submodular, so by assumption $\alpha < 0$.

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If $\alpha \geq 0$ then f is submodular, so by assumption $\alpha < 0$. Now let h be an arbitrary strict submodular function and define

$$\beta \triangleq \min_{X,Y} \Big(h(X) + h(Y) - h(X \cup Y) - h(X \cap Y) \Big). \tag{3.49}$$

Strict means that $\beta > 0$. . . .

Arbitrary functions as difference between submodular funcs.

...cont.

Define $f' : 2^V \to \mathbb{R}$ as

$$f'(A) = f(A) + \frac{|\alpha|}{\beta} h(A) \tag{3.50}$$

Then f' is submodular (why?), and $f = f'(A) - \frac{|\alpha|}{\beta} h(A)$, a difference between two submodular functions as desired.

□

Arbitrary function as difference between two polymatroids

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- Given submodular $g : 2^V \to \mathbb{R}$, construct $\bar{g} : 2^V \to \mathbb{R}$ as $\bar{g}(A) = g(A) - \sum_{a \in A} g(a|V \setminus \{a\})$. Let $m_g(A) \triangleq \sum_{a \in A} g(a|V \setminus \{a\})$

$\bar{g}(j|A) = g(j|A) - g(j|V \setminus j) \geq 0$

$f(j|A) \geq f(j|B) \geq f(j|V \setminus \{j\})$

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- Then, given arbitrary $f = g - h$ where g and h are submodular,

$$\begin{aligned} f = g - h &= \bar{g} + m_g - \bar{h} - m_h && (3.51) \\ &= \bar{g} - \bar{h} + (m_g - m_h) && (3.52) \\ &= \bar{g} - \bar{h} + m_{g-h} && (3.53) \\ &= \bar{g} + m^+_{g-h} - (\bar{h} + (-m_{g-h})^+) && (3.54) \end{aligned}$$

where m^+ is the positive part of modular function m. That is, $m^+(A) = \sum_{a \in A} m(a)\mathbf{1}(m(a) > 0)$.

Arbitrary function as difference between two polymatroids

- Any submodular function g can be represented as a sum of a polymatroid (normalized monotone non-decreasing submodular) function $\bar{g}$ and a modular function m_g.
- Given submodular $g : 2^V \to \mathbb{R}$, construct $\bar{g} : 2^V \to \mathbb{R}$ as $\bar{g}(A) = g(A) - \sum_{a \in A} g(a|V \setminus \{a\})$. Let $m_g(A) \triangleq \sum_{a \in A} g(a|V \setminus \{a\})$
- Then, given arbitrary $f = g - h$ where g and h are submodular,

$$f = g - h = \bar{g} + m_g - \bar{h} - m_h \tag{3.51}$$
$$= \bar{g} - \bar{h} + (m_g - m_h) \tag{3.52}$$
$$= \bar{g} - \bar{h} + m_{g-h} \tag{3.53}$$
$$= \bar{g} + m^+_{g-h} - (\bar{h} + (-m_{g-h})^+) \tag{3.54}$$

  where m^+ is the positive part of modular function m. That is, $m^+(A) = \sum_{a \in A} m(a)\mathbf{1}(m(a) > 0)$.
- But both $g + m^+_{g-h}$ and $\bar{h} + (-m_{g-h})^+$ are polymatroid functions.

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- But both $g + m^+_{g-h}$ and $\bar{h} + (-m_{g-h})^+$ are polymatroid functions.
- Thus, any function can be expressed as a difference between two polymatroid functions.

Applications

- Sensor placement with submodular costs. I.e., let V be a set of possible sensor locations, $f(A) = I(X_A; X_{V \setminus A})$ measures the quality of a subset A of placed sensors, and $c(A)$ the submodular cost. We have $\cancel{\min}_A f(A) - \lambda c(A)$.

max

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- Feature selection: a problem of maximizing $I(X_A; C) - \lambda c(A) = H(X_A) - [H(X_A|C) + \lambda c(A)]$, the difference between two submodular functions, where H is the entropy and c is a feature cost function.

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- Graphical Model Inference. Finding x that maximizes $p(x) \propto \exp(-v(x))$ where $x \in \{0, 1\}^n$ and v is a pseudo-Boolean function. When v is non-submodular, it can be represented as a difference between submodular functions.

Submodular Definitions

Definition 3.7.2 (submodular concave)

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \tag{3.2}$$

An alternate and (as we will soon see) equivalent definition is:

Definition 3.7.3 (diminishing returns)

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A \subseteq B \subset V$, and $v \in V \setminus B$, we have that:

$$f(A \cup \{v\}) - f(A) \geq f(B \cup \{v\}) - f(B) \tag{3.3}$$

This means that the incremental "value", "gain", or "cost" of v decreases (diminishes) as the context in which v is considered grows from A to B.

Submodular Definition: Group Diminishing Returns

An alternate and equivalent definition is:

Definition 3.7.1 (group diminishing returns)

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A \subseteq B \subset V$, and $C \subseteq V \setminus B$, we have that:

$$f(A \cup C) - f(A) \geq f(B \cup C) - f(B) \tag{3.55}$$

This means that the incremental "value" or "gain" of set C decreases as the context in which C is considered grows from A to B (diminishing returns)

Gain

- We often wish to express the gain of an item $j \in V$ in context A, namely $f(A \cup \{j\}) - f(A)$.

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- This is called the gain and is used so often, there are equally as many ways to notate this. I.e., you might see:

$$f(A \cup \{j\}) - f(A) \triangleq \rho_j(A) \tag{3.56}$$
$$\triangleq \rho_A(j) \tag{3.57}$$
$$\triangleq \nabla_j f(A) \tag{3.58}$$
$$\triangleq f(\{j\}|A) \tag{3.59}$$
$$\triangleq f(j|A) \tag{3.60}$$

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- We'll use $f(j|A)$.
- Submodularity's diminishing returns definition can be stated as saying that $f(j|A)$ is a monotone non-increasing function of A, since $f(j|A) \geq f(j|B)$ whenever $A \subseteq B$ (conditioning reduces valuation).

Gain Notation

It will also be useful to extend this to sets.
Let A, B be any two sets. Then

$$f(A|B) \triangleq f(A \cup B) - f(B) \tag{3.61}$$

So when j is any singleton

$$f(j|B) = f(\{j\}|B) = f(\{j\} \cup B) - f(B) \tag{3.62}$$

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Note that this is inspired from information theory and the notation used for conditional entropy $H(X_A|X_B) = H(X_A, X_B) - H(X_B)$.

Submodular Definition Basic Equivalencies

We want to show that Submodular Concave (Definition 3.7.2), Diminishing Returns (Definition 3.7.3), and Group Diminishing Returns (Definition 3.7.1) are identical.

Submodular Definition Basic Equivalencies

We want to show that Submodular Concave (Definition 3.7.2), Diminishing Returns (Definition 3.7.3), and Group Diminishing Returns (Definition 3.7.1) are identical. We will show that:

- Submodular Concave $\Rightarrow$ Diminishing Returns
- Diminishing Returns $\Rightarrow$ Group Diminishing Returns
- Group Diminishing Returns $\Rightarrow$ Submodular Concave

Submodular Concave $\Rightarrow$ Diminishing Returns

$f(S)+f(T) \geq f(S \cup T)+f(S \cap T) \Rightarrow f(v|A) \geq f(v|B), A \subseteq B \subseteq V \setminus v.$

- Assume Submodular concave, so $\forall S, T$ we have $f(S)+f(T) \geq f(S \cup T)+f(S \cap T)$.

$\square$

Submodular Concave ⇒ Diminishing Returns

$f(S)+f(T) \geq f(S \cup T)+f(S \cap T) \Rightarrow f(v|A) \geq f(v|B), A \subseteq B \subseteq V \setminus v.$

- Assume Submodular concave, so $\forall S, T$ we have $f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$.
- Given A, B and $v \in V$ such that: $A \subseteq B \subseteq V \setminus \{v\}$, we have from submodular concave that:

$$f(A + v) + f(B) \geq f(B + v) + f(A) \tag{3.63}$$

□

Submodular Concave ⇒ Diminishing Returns

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$$f(A + v) + f(B) \geq f(B + v) + f(A) \tag{3.63}$$

- Rearranging, we have

$$f(A + v) - f(A) \geq f(B + v) - f(B) \tag{3.64}$$

□

Diminishing Returns ⇒ Group Diminishing Returns

$f(v|S) \geq f(v|T), S \subseteq T \subseteq V \setminus v \Rightarrow f(C|A) \geq f(C|B), A \subseteq B \subseteq V \setminus C.$

Let $C = \{c_1, c_2, \dots, c_k\}$. Then diminishing returns implies

$$f(A \cup C) - f(A) \tag{3.65}$$

$$= f(A \cup C) - \sum_{i=1}^{k-1} \Big(f(A \cup \{c_1, \dots, c_i\}) - f(A \cup \{c_1, \dots, c_i\}) \Big) - f(A) \tag{3.66}$$

$$= \sum_{i=1}^{k} f(A \cup \{c_1 \dots c_i\}) - f(A \cup \{c_1 \dots c_{i-1}\}) \tag{3.67}$$

$$\geq \sum_{i=1}^{k} f(B \cup \{c_1 \dots c_i\}) - f(B \cup \{c_1 \dots c_{i-1}\}) \tag{3.68}$$

$$= f(B \cup C) - \sum_{i=1}^{k-1} \Big(f(B \cup \{c_1, \dots, c_i\}) - f(B \cup \{c_1, \dots, c_i\}) \Big) - f(B) \tag{3.69}$$

$$= f(B \cup C) - f(B) \tag{3.70}$$

□

Group Diminishing Returns ⇒ Submodular Concave

$f(U|S) \geq f(U|T), S \subseteq T \subseteq V \setminus U \Rightarrow f(A)+f(B) \geq f(A \cup B)+f(A \cap B).$

Assume group diminishing returns. Assume $A \neq B$ otherwise trivial. Define $A' = A \cap B$, $C = A \setminus B$, and $B' = B$. Then since $A' \subseteq B'$,

$$f(A' + C) - f(A') \geq f(B' + C) - f(B') \tag{3.71}$$

giving

$$f(A' + C) + f(B') \geq f(B' + C) + f(A') \tag{3.72}$$

or

$$f(A \cap B + A \setminus B) + f(B) \geq f(B + A \setminus B) + f(A \cap B) \tag{3.73}$$

which is the same as the submodular concave condition

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \tag{3.74}$$

Submodular Definition: Four Points

Definition 3.7.2 ("singleton", or "four points")

A function $f : 2^V \to \mathbb{R}$ is submodular iff for any $A \subset V$, and any $a, b \in V \setminus A$, we have that:

$$f(A \cup \{a\}) + f(A \cup \{b\}) \geq f(A \cup \{a, b\}) + f(A) \tag{3.75}$$

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This follows immediately from diminishing returns. To achieve diminishing returns, assume $A \subset B$ with $B \setminus A = \{b_1, b_2, \dots, b_k\}$. Then

$$\begin{aligned}
f(A + a) - f(A) &\geq f(A + b_1 + a) - f(A + b_1) && (3.76)\\
&\geq f(A + b_1 + b_2 + a) - f(A + b_1 + b_2) && (3.77)\\
&\geq \dots && (3.78)\\
&\geq f(A + b_1 + \dots + b_k + a) - f(A + b_1 + \dots + b_k) && (3.79)\\
&= f(B + a) - f(B) && (3.80)
\end{aligned}$$

Submodular Definitions

Theorem 3.7.3

Given function $f : 2^V \to \mathbb{R}$, then

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \text{ for all } A, B \subseteq V \qquad \text{(SC)}$$

if and only if

$$f(v|X) \geq f(v|Y) \text{ for all } X \subseteq Y \subseteq V \text{ and } v \notin B \qquad \text{(DR)}$$

Submodular Definitions

Theorem 3.7.3

Given function $f : 2^V \to \mathbb{R}$, *then*

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$$f(v|X) \geq f(v|Y) \text{ for all } X \subseteq Y \subseteq V \text{ and } v \notin B \quad \text{(DR)}$$

Proof.

(SC)⇒(DR): Set $A \leftarrow X \cup \{v\}$, $B \leftarrow Y$. Then $A \cup B = B \cup \{v\}$ and $A \cap B = X$ and $f(A) - f(A \cap B) \geq f(A \cup B) - f(B)$ implies (DR).

(DR)⇒(SC): Order $A \setminus B = \{v_1, v_2, \ldots, v_r\}$ arbitrarily. Then
$f(v_i|A \cap B \cup \{v_1, v_2, \ldots, v_{i-1}\}) \geq f(v_i|B \cup \{v_1, v_2, \ldots, v_{i-1}\}),\ i \in [r-1]$
Applying telescoping summation to both sides, we get:

$$\sum_{i=0}^{r} f(v_i|A \cap B \cup \{v_1, v_2, \ldots, v_{i-1}\}) \geq \sum_{i=0}^{r} f(v_i|B \cup \{v_1, v_2, \ldots, v_{i-1}\})$$

or

$$f(A) - f(A \cap B) \geq f(A \cup B) - f(B)$$

Use of gain: submodular bounds of a difference

- Given submodular f, and given you have $C, D \subseteq E$ with either $D \supseteq C$ or $D \subseteq C$, and have an expression of the form:

$$f(C) - f(D) \tag{3.81}$$

Use of gain: submodular bounds of a difference

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- If $D \supseteq C$, then for any X with $D = C \cup X$ then

$$f(C) - f(D) = f(C) - f(C \cup X) \geq f(C \cap X) - f(X)$$

(3.83)

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$$f(C \cup X | C) \leq f(X | C \cap X) \tag{3.83}$$

- Alternatively, if $D \subseteq C$, given any Y such that $D = C \cap Y$ then

$$f(C) - f(D) = f(C) - f(C \cap Y) \geq f(C \cup Y) - f(Y)$$

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- Equations (3.83) and (3.85) have same form.

Many (Equivalent) Definitions of Submodularity

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$$f(j|S) \geq f(j|S \cup \{k\}), \ \forall S \subseteq V \text{ with } j \in V \setminus (S \cup \{k\}) \tag{3.89}$$

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$$f(A \cup B | A \cap B) \leq f(A|A \cap B) + f(B|A \cap B), \quad \forall A, B \subseteq V \tag{3.90}$$

$$f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S) - \sum_{j \in S \setminus T} f(j|S \cup T - \{j\}), \; \forall S, T \subseteq V \tag{3.91}$$

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$$f(C|S) \geq f(C|T), \forall S \subseteq T \subseteq V, \; \text{with } C \subseteq V \setminus T \tag{3.88}$$

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$$f(A \cup B|A \cap B) \leq f(A|A \cap B) + f(B|A \cap B), \;\; \forall A, B \subseteq V \tag{3.90}$$

$$f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S) - \sum_{j \in S \setminus T} f(j|S \cup T - \{j\}), \; \forall S, T \subseteq V \tag{3.91}$$

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Many (Equivalent) Definitions of Submodularity

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B), \;\; \forall A, B \subseteq V \tag{3.86}$$

$$f(j|S) \geq f(j|T), \; \forall S \subseteq T \subseteq V, \text{ with } j \in V \setminus T \tag{3.87}$$

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$$f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|S \setminus \{j\}), \; \forall T \subseteq S \subseteq V \tag{3.94}$$

Equivalent Definitions of Submodularity

We've already seen that Eq. 3.86 $\equiv$ Eq. 3.87 $\equiv$ Eq. 3.88 $\equiv$ Eq. 3.89 $\equiv$ Eq. 3.90.

Equivalent Definitions of Submodularity

We've already seen that Eq. 3.86 $\equiv$ Eq. 3.87 $\equiv$ Eq. 3.88 $\equiv$ Eq. 3.89 $\equiv$ Eq. 3.90.

We next show that Eq. 3.89 $\Rightarrow$ Eq. 3.91 $\Rightarrow$ Eq. 3.92 $\Rightarrow$ Eq. 3.89.

Approach

To show these next results, we essentially first use:

$$f(S \cup T) = f(S) + f(T|S) \leq f(S) + \text{upper bond} \tag{3.95}$$

and

$$f(T) + \text{lower bound} \leq f(T) + f(S|T) = f(S \cup T) \tag{3.96}$$

leading to

$$f(T) + \text{lower bound} \leq f(S) + \text{upper bound} \tag{3.97}$$

or

$$f(T) \leq f(S) + \text{upper bound} - \text{lower bound} \tag{3.98}$$

Eq. 3.89 $\Rightarrow$ Eq. 3.91

Let $T \setminus S = \{j_1, \ldots, j_r\}$ and $S \setminus T = \{k_1, \ldots, k_q\}$.
First, we upper bound the gain of T in the context of S:

$$f(S \cup T) - f(S) = \sum_{t=1}^{r} \Big(f(S \cup \{j_1, \ldots, j_t\}) - f(S \cup \{j_1, \ldots, j_{t-1}\}) \Big) \tag{3.99}$$

$$= \sum_{t=1}^{r} f(j_t | S \cup \{j_1, \ldots, j_{t-1}\}) \leq \sum_{t=1}^{r} f(j_t|S) \tag{3.100}$$

$$= \sum_{j \in T \setminus S} f(j|S) \tag{3.101}$$

or

$$f(T|S) \leq \sum_{j \in T \setminus S} f(j|S) \tag{3.102}$$

Eq. 3.89 $\Rightarrow$ Eq. 3.91

Let $T \setminus S = \{j_1, \ldots, j_r\}$ and $S \setminus T = \{k_1, \ldots, k_q\}$.
Next, lower bound S in the context of T:

$$f(S \cup T) - f(T) = \sum_{t=1}^{q} [f(T \cup \{k_1, \ldots, k_t\}) - f(T \cup \{k_1, \ldots, k_{t-1}\})] \tag{3.103}$$

$$= \sum_{t=1}^{q} f(k_t | T \cup \{k_1, \ldots, k_t\} \setminus \{k_t\}) \geq \sum_{t=1}^{q} f(k_t | T \cup S \setminus \{k_t\}) \tag{3.104}$$

$$= \sum_{j \in S \setminus T} f(j | S \cup T \setminus \{j\}) \tag{3.105}$$

Eq. 3.89 $\Rightarrow$ Eq. 3.91

Let $T \setminus S = \{j_1, \ldots, j_r\}$ and $S \setminus T = \{k_1, \ldots, k_q\}$.
So we have the upper bound

$$f(T|S) = f(S \cup T) - f(S) \leq \sum_{j \in T \setminus S} f(j|S) \tag{3.106}$$

and the lower bound

$$f(S|T) = f(S \cup T) - f(T) \geq \sum_{j \in S \setminus T} f(j|S \cup T \setminus \{j\}) \tag{3.107}$$

This gives upper and lower bounds of the form

$$f(T) + \text{lower bound} \leq f(S \cup T) \leq f(S) + \text{upper bound}, \tag{3.108}$$

and combining directly the left and right hand side gives the desired inequality.

Eq. 3.91 $\Rightarrow$ Eq. 3.92

This follows immediately since if $S \subseteq T$, then $S \setminus T = \emptyset$, and the last term of Eq. 3.91 vanishes.

Eq. 3.92 $\Rightarrow$ Eq. 3.89

Here, we set $T = S \cup \{j, k\}$, $j \notin S \cup \{k\}$ into Eq. 3.92 to obtain

$$f(S \cup \{j,k\}) \leq f(S) + f(j|S) + f(k|S) \tag{3.109}$$
$$= f(S) + f(S + \{j\}) - f(S) + f(S + \{k\}) - f(S) \tag{3.110}$$
$$= f(S + \{j\}) + f(S + \{k\}) - f(S) \tag{3.111}$$
$$= f(j|S) + f(S + \{k\}) \tag{3.112}$$

giving

$$f(j|S \cup \{k\}) = f(S \cup \{j,k\}) - f(S \cup \{k\}) \tag{3.113}$$
$$\leq f(j|S) \tag{3.114}$$

Example: Rank function of a matrix

Consider the following 4×8 matrix, so $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 0 & 2 & 2 & 3 & 0 & 1 & 3 & 1 \\ 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} | & | & | & | & | & | & | & | \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ | & | & | & | & | & | & | & | \end{pmatrix} \end{array}$$

- Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{6, 7\}$, $A_r = \{1\}$, $B_r = \{5\}$.
- Then $r(A) = 3$, $r(B) = 3$, $r(C) = 2$.
- $r(A \cup C) = 3$, $r(B \cup C) = 3$.
- $r(A \cup A_r) = 3$, $r(B \cup B_r) = 3$, $r(A \cup B_r) = 4$, $r(B \cup A_r) = 4$.
- $r(A \cup B) = 4$, $r(A \cap B) = 1 < r(C) = 2$.
- $6 = r(A) + r(B) > r(A \cup B) + r(A \cap B) = 5$

On Rank

- Let rank : $2^V \to \mathbb{Z}_+$ be the rank function.

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- If A, B are such that rank$(A) = |A|$ and rank$(B) = |B|$, with $|A| < |B|$, then the space spanned by B is greater, and we can find a vector in B that is linearly independent of the space spanned by vectors in A.

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- To stress this point, note that the above condition is $|A| < |B|$, **not** $A \subseteq B$ which is sufficient (to be able to find an independent vector) but not necessary.

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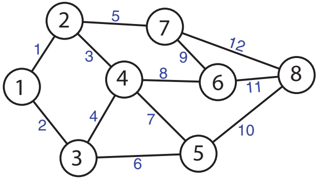
- To stress this point, note that the above condition is $|A| < |B|$, **not** $A \subseteq B$ which is sufficient (to be able to find an independent vector) but not necessary.
- In other words, given A, B with $\text{rank}(A) = |A|$ & $\text{rank}(B) = B$, then $|A| < |B| \Leftrightarrow \exists$ an $b \in B$ such that $\text{rank}(A \cup \{b\}) = |A| + 1$.

Spanning trees/forests

- We are given a graph $G = (V, E)$, and consider the edges $E = E(G)$ as an index set.
- Consider the $|V| \times |E|$ incidence matrix of undirected graph G, which is the matrix $\mathbf{X}_G = (x_{v,e})_{v \in V(G), e \in E(G)}$ where

$$x_{v,e} = \begin{cases} 1 & \text{if } v \in e \\ 0 & \text{if } v \notin e \end{cases} \tag{3.115}$$



$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array} \begin{array}{c} \begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array} \tag{3.116}$$

Spanning trees/forests & incidence matrices

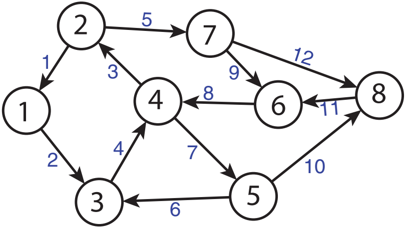
- We are given a graph $G = (V, E)$, we can arbitrarily orient the graph (make it directed) consider again the edges $E = E(G)$ as an index set.
- Consider instead the $|V| \times |E|$ incidence matrix of undirected graph G, which is the matrix $\mathbf{X}_G = (x_{v,e})_{v \in V(G), e \in E(G)}$ where

$$x_{v,e} = \begin{cases} 1 & \text{if } v \in e^+ \\ -1 & \text{if } v \in e^- \\ 0 & \text{if } v \notin e \end{cases} \tag{3.117}$$

and where e^+ is the tail and e^- is the head of (now) directed edge e.

Spanning trees/forests & incidence matrices

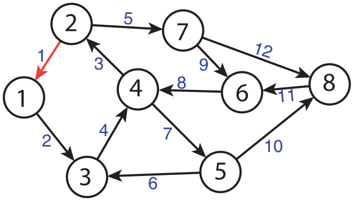
- A directed version of the graph (right) and its adjacency matrix (below).
- Orientation can be arbitrary.
- Note, rank of this matrix is 7.



$$
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12
\end{array} \\
\begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1
\end{pmatrix}
\end{array}
$$

Spanning trees

- We can consider edge-induced subgraphs and the corresponding matrix columns.

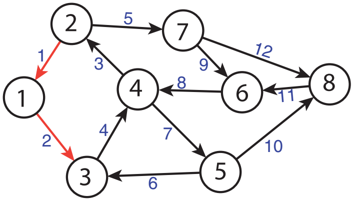


$$\begin{array}{c c} & 1 \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} & \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{array} \tag{3.118}$$

Here, $\text{rank}(\{x_1\}) = 1$.

Spanning trees

- We can consider edge-induced subgraphs and the corresponding matrix columns.



$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \end{array} \tag{3.118}$$

Here, $\text{rank}(\{x_1, x_2\}) = 2$.

Spanning trees

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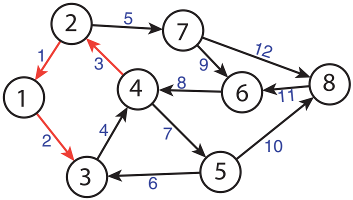

$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} \tag{3.118}$$

Here, rank$(\{x_1, x_2, x_3\}) = 3$.

Spanning trees

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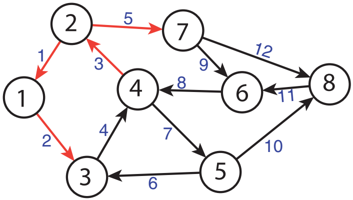

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 5 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \tag{3.118}$$

Here, rank$(\{x_1, x_2, x_3, x_5\}) = 4$.

Spanning trees

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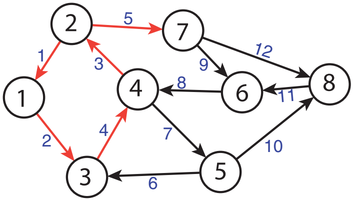

$$\begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \tag{3.118}$$

Here, $\text{rank}(\{x_1, x_2, x_3, x_4, x_5\}) = 4$.

Spanning trees

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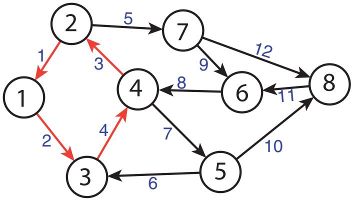

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Here, $\text{rank}(\{x_1, x_2, x_3, x_4\}) = 3$ since $x_4 = -x_1 - x_2 - x_3$.

Spanning trees

- In general, whenever the edges specify a cycle, there will be a linear dependence between the corresponding set of vectors in the matrix.
- This means that all forests in the graph correspond to a set of linearly independent column vectors in the matrix.
- Consider a "rank" function defined as follows: given a set of edges $A \subseteq E(G)$, the $\text{rank}(A)$ is the size of the largest forest in the A-edge induced subgraph of G.
- The rank of the entire graph then is then a spanning forest of the graph (spanning tree if the graph is connected).
- The rank of the graph is $\text{rank}(G) = |V| - k$ where k is the number of connected components of G (recall, we saw that $k_G(A)$ is a supermodular function in previous lectures).

Spanning Tree Algorithms

- We are now given a positive edge-weighted connected graph $G = (V, E, w)$ where $w : E \to \mathbb{R}_+$ is a modular function the edges of the graph. The goal is to find the minimum spanning tree (MST) of the graph.
- Given a tree T, the cost of the tree is $\text{cost}(T) = \sum_{e \in T} w(e)$, the sum of the weights of the edges.
- There are several algorithms for MST:

Algorithm 1: Borůvka's Algorithm

```
F ← ∅  /* We build up the edges of a forest in F */
while G(V, F) is disconnected do
    forall the components C_i of F do
        F ← F ∪ {e_i} for e_i = the min-weight edge out of C_i;
```

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Algorithm 2: Jarník/Prim/Dijkstra Algorithm

1 $T \leftarrow \emptyset$;
2 **while** T *is not a spanning tree* **do**
3 $\quad T \leftarrow T \cup \{e\}$ for $e =$ the minimum weight edge extending the tree T to a new vertex ;

Spanning Tree Algorithms

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- There are several algorithms for MST:

Algorithm 3: Kruskal's Algorithm

Sort the edges so that $w(e_1) \leq w(e_2) \leq \cdots \leq w(e_m)$;
$T \leftarrow (V(G), \emptyset) = (V, E)$;
for $i = 1$ **to** m **do**
  if $E(T) \cup \{e_i\}$ *does not create a cycle in* T **then**
    $E(T) \leftarrow E(T) \cup \{e\}$;

Spanning Tree Algorithms

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- These three algorithms are all guaranteed to find the optimal minimum spanning tree in (low order) polynomial time.

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- There are several algorithms for MST:
- These three algorithms are all guaranteed to find the optimal minimum spanning tree in (low order) polynomial time.
- All these algorithms are related to the "greedy" algorithm. I.e., "add next whatever looks best".
- These algorithms will also always find a basis (a set of linearly independent vectors that span the underlying space) in the matrix example we saw earlier.

Spanning Tree Algorithms

- We are now given a positive edge-weighted connected graph $G = (V, E, w)$ where $w : E \to \mathbb{R}_+$ is a modular function the edges of the graph. The goal is to find the minimum spanning tree (MST) of the graph.
- Given a tree T, the cost of the tree is $\text{cost}(T) = \sum_{e \in T} w(e)$, the sum of the weights of the edges.
- There are several algorithms for MST:
- These three algorithms are all guaranteed to find the optimal minimum spanning tree in (low order) polynomial time.
- All these algorithms are related to the "greedy" algorithm. I.e., "add next whatever looks best".
- These algorithms will also always find a basis (a set of linearly independent vectors that span the underlying space) in the matrix example we saw earlier.
- The above are all examples of a matroid, which is the fundamental reason why the greedy algorithms work.