Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 2 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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Cumulative Outstanding Reading

- Read chapter 1 from Fujishige's book.

Announcements, Assignments, and Reminders

- our room (Mueller Hall Room 154) is changed!
- Please do use our discussion board (`https://canvas.uw.edu/courses/895956/discussion_topics`) for all questions, comments, so that all will benefit from them being answered.
- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3:
- L4:
- L5:
- L6:
- L7:
- L8:
- L9:
- L10:
- L11:
- L12:
- L13:
- L14:
- L15:
- L16:
- L17:
- L18:
- L19:
- L20:

Finals Week: June 9th-13th, 2014.

Submodular Definitions

Definition 2.2.2 (submodular concave)

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \tag{2.2}$$

An alternate and (as we will soon see) equivalent definition is:

Definition 2.2.3 (diminishing returns)

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A \subseteq B \subset V$, and $v \in V \setminus B$, we have that:

$$f(A \cup \{v\}) - f(A) \geq f(B \cup \{v\}) - f(B) \tag{2.3}$$

This means that the incremental "value", "gain", or "cost" of v decreases (diminishes) as the context in which v is considered grows from A to B.

Example Discrete Optimization Problems

- **Combinatorial Problems**: e.g., set cover, max k coverage, vertex cover, edge cover, graph cut problems.
- **Operations Research**: facility location (uncapacited)
- **Sensor placement**
- **Information:** Information gain and feature selection, information theory
- **Mathematics:** e.g., monge matrices
- **Networks**: Social networks, influence, viral marketing, information cascades, diffusion networks
- **Graphical models**: tree distributions, factors, and image segmentation
- **Diversity** and its models
- **NLP**: Natural language processing: document summarization, web search, information retrieval
- **ML**: Machine Learning: active/semi-supervised learning
- **Economics**: markets, economies of scale

Markets: Supply Side Economies of scale

- Economies of Scale: Many goods and services can be produced at a much lower per-unit cost only if they are produced in very large quantities.
- The profit margin for producing a unit of goods is improved as more of those goods are created.
- If you already make a good, making a similar good is easier than if you start from scratch (e.g., Apple making both iPod and iPhone).
- An argument in favor of free trade is that it opens up larger markets for firms (especially in otherwise small markets), thereby enabling better economies of scale, and hence greater efficiency (lower costs and resources per unit of good produced).

Supply Side Economies of scale

- What is a good model of the cost of manufacturing a set of items?
- Let V be a set of possible items that a company might possibly wish to manufacture, and let $f(S)$ for $S \subseteq V$ be the cost to that company to manufacture subset S.
- Ex: V might be colors of paint in a paint manufacturer: green, red, blue, yellow, white, etc.
- Producing green when you are already producing yellow and blue is probably cheaper than if you were only producing some other colors.

$$f(\text{green}, \text{blue}, \text{yellow}) - f(\text{blue}, \text{yellow}) <= f(\text{green}, \text{blue}) - f(\text{blue}) \tag{2.1}$$

- So a submodular function would be a good model.

Demand side Economies of Scale: Network Externalities

- consumers of a good derive positive value when size of the market increases.
- the value of a network to a user depends on the number of other users in that network. External use benefits internal use.
- This is called network externalities (Katz & Shapiro 1986), and is a form of "demand" economies of scale
- "value" in this case can be seen as a "willingness-to-pay" for the service (WTP)
- WTP tends to increase but then saturate (like a logistic function)
- Given network externalities, a consumer in today's market cares also about the future success of the product and competing products.
- If the good is durable (e.g., a car or phone) or there is human capital investment (e.g., education in a skill), the total benefits derived from a good will depend on the number of consumers who adopt compatible products in the future.

Positive Network Externalities

- railroad - standard rail format and shared access
- The telephone, who wants to talk by phone only to oneself?
- the internet, more valuable per person the more people use it.
- ebooks (the more people comment, the better it gets)
- social network sites: facebook more valuable with everyone online
- online education, Massive Open Online Courses (MOOCs) such as Coursera, edX, etc. – with many people simultaneously taking a class, all gain due to richer peer discussions due to greater pool of well matched study groups, more simultaneous similar questions/problems that are asked $\Rightarrow$ more efficient learning & training data for ML algorithms to learn how people learn.
- Software (e.g., Microsoft office, smartphone apps, etc.): more people means more bug reporting $\Rightarrow$ better & faster software evolution.
- gmail and web-based email (collaborative spam filtering).
- wikipedia, collaborative documents
- any widely used standard (job training now is useful in the future)
- the "tipping point", and "winner take all" (one platform prevails) markets.

Other Network Externalities

No Network Externalities

- food/drink - (should be) independent of how many others are eating the type of food.
- Music - your enjoyment should (ideally) be independent of others' enjoyment (but maybe not, see Salganik, Dodds, Watts'06).

Negative Network Externalities

- clothing
- (Halloween) costumes

Optimization Problem Involving Network Externalities

- (From Mirrokni, Roch, Sundararajan 2012): Let V be a set of users.
- Let $v_i(S)$ be the value that user i has for a good if $S \subseteq V$ already own the good — e.g. $v_i(S) = \omega_i + f_i(\sum_{j \in S} w_{ij})$ where ω_i is inherent value, and f_i might be a concave function, and w_{ij} is how important $j \in S$ is to i (e.g., a network). Weights might be random.
- Given price p for good, user i buys good if $v_i(S) \geq p$.
- We choose initial price p and initial set of users $A \subseteq V$ who get the good for free.
- Define $S_1 = \{i \notin A : v_i(A) \geq p\}$ initial set of buyers.
- $S_2 = \{i \notin A \cup S_1 : v_i(A \cup S_1) \geq p\}$.
- This starts a cascade. Let $S_k = \{i \notin \cup_{j<k} S_j \cup A : v_j(\cup_{j<k} S_j \cup A) \geq p\}$,
- and let S_{k^*} be the saturation point, lowest value of k such that $S_k = S_{k+1}$
- Goal: find A and p to maximize $f_p(A) = \mathbb{E}[p \times |S_{k^*}|]$.

Shared Fixed Costs

- It is often inaccurate to consider individual costs in isolation, without accounting for the various interactions that might exist between them.
- Ex: Let $V = \{v_1, v_2\}$ be items with v_1 being the action "buy milk at the store" and v_2 being the action "buy honey at the store.".
- For $A \subseteq V$, let $f(A)$ be the cost of set of items A.
- $f(\{v_1\}) =$ cost to drive to/from store and cost to purchase milk, say $c_d + c_m$.
- $f(\{v_2\}) =$ cost to drive to/from store and cost to purchase honey, say $c_d + c_h$.
- But $f(\{v_1, v_2\}) = c_d + c_m + c_h < 2c_d + c_m + c_h$ since c_d is a shared fixed cost.
- Shared fixed costs are submodular.

Anecdote

From David Brooks, NYTs column, March 28th, 2011 on "Tools for Thinking". In response to Steven Pinker (Harvard) asking a number of people "What scientific concept would improve everybody's cognitive toolkit?"

> *Emergent systems are ones in which many different elements interact. The pattern of interaction then produces a new element that is greater than the sum of the parts, which then exercises a top-down influence on the constituent elements.*

Submodular Motivation Recap

- Given a set of objects $V = \{v_1, \ldots, v_n\}$ and a function $f : 2^V \to \mathbb{R}$ that returns a real value for any subset $S \subseteq V$.
- Suppose we are interested in finding the subset that either maximizes or minimizes the function, e.g., $\operatorname{argmax}_{S \subseteq V} f(S)$, possibly subject to some constraints.
- In general, this problem has exponential time complexity.
- Example: f might correspond to the value (e.g., information gain) of a set of sensor locations in an environment, and we wish to find the best set $S \subseteq V$ of sensors locations given a fixed upper limit on the number of sensors $|S|$.
- In many cases (such as above) f has properties that make its optimization tractable to either exactly or approximately compute.
- One such property is *submodularity*.

Submodular Definitions

Definition 2.4.2 (submodular concave)

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \tag{2.2}$$

An alternate and (as we will soon see) equivalent definition is:

Definition 2.4.3 (diminishing returns)

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A \subseteq B \subset V$, and $v \in V \setminus B$, we have that:

$$f(A \cup \{v\}) - f(A) \geq f(B \cup \{v\}) - f(B) \tag{2.3}$$

This means that the incremental "value", "gain", or "cost" of v decreases (diminishes) as the context in which v is considered grows from A to B.

Subadditive Definitions

Definition 2.4.1 (subadditive)

A function $f : 2^V \to \mathbb{R}$ is subadditive if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \geq f(A \cup B) \tag{2.2}$$

This means that the "whole" is less than the sum of the parts.

Supermodular Definitions

Definition 2.4.2 (supermodular convex)

A function $f : 2^V \to \mathbb{R}$ is supermodular if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \leq f(A \cup B) + f(A \cap B) \tag{2.3}$$

An alternate and equivalent definition is:

Definition 2.4.3 (increasing returns)

A function $f : 2^V \to \mathbb{R}$ is supermodular if for any $A \subseteq B \subset V$, and $v \in V \setminus B$, we have that:

$$f(A \cup \{v\}) - f(A) \leq f(B \cup \{v\}) - f(B) \tag{2.4}$$

The incremental "value", "gain", or "cost" of v increases as the context in which v is considered grows from A to B.

Submodular vs. Supermodular

- Submodular and supermodular functions are closely related.
- In fact, f is submodular iff $-f$ is supermodular.

Superadditive Definitions

Definition 2.4.4 (superadditive)

A function $f : 2^V \to \mathbb{R}$ is superadditive if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \leq f(A \cup B) \tag{2.5}$$

- This means that the "whole" is greater than the sum of the parts.
- In general, submodular and subadditive (and supermodular and superadditive) are different properties.

Modular Definitions

Definition 2.4.5 (modular)

A function that is both submodular and supermodular is called **modular**

If f is a modular function, than for any $A, B \subseteq V$, we have

$$f(A) + f(B) = f(A \cap B) + f(A \cup B) \tag{2.6}$$

In modular functions, elements do not interact (or cooperate, or compete, or influence each other), and have value based only on singleton values.

Proposition 2.4.6

If f is modular, it may be written as

$$f(A) = f(\emptyset) + \sum_{a \in A} \Big(f(\{a\}) - f(\emptyset) \Big) \tag{2.7}$$

Modular Definitions

Proof.

We inductively construct the value for $A = \{a_1, a_2, \ldots, a_k\}$.
For $k = 2$,

$$f(a_1) + f(a_2) = f(a_1, a_2) + f(\emptyset) \tag{2.8}$$

$$\text{implies } f(a_1, a_2) = f(a_1) - f(\emptyset) + f(a_2) - f(\emptyset) + f(\emptyset) \tag{2.9}$$

then for $k = 3$,

$$f(a_1, a_2) + f(a_3) = f(a_1, a_2, a_3) + f(\emptyset) \tag{2.10}$$

$$\text{implies } f(a_1, a_2, a_3) = f(a_1, a_2) - f(\emptyset) + f(a_3) - f(\emptyset) + f(\emptyset) \tag{2.11}$$

$$= f(\emptyset) + \sum_{i=1}^{3} \big(f(a_i) - f(\emptyset) \big) \tag{2.12}$$

and so on . . . □

Complement function

Given a function $f : 2^V \to \mathbb{R}$, we can find a complement function $\bar{f} : 2^V \to \mathbb{R}$ as $\bar{f}(A) = f(V \setminus A)$ for any A.

Proposition 2.4.7

$\bar{f}$ is submodular if f is submodular.

Proof.

$$\bar{f}(A) + \bar{f}(B) \geq \bar{f}(A \cup B) + \bar{f}(A \cap B) \tag{2.13}$$

follows from

$$f(V \setminus A) + f(V \setminus B) \geq f(V \setminus (A \cup B)) + f(V \setminus (A \cap B)) \tag{2.14}$$

which is true because $V \setminus (A \cup B) = (V \setminus A) \cap (V \setminus B)$ and $V \setminus (A \cap B) = (V \setminus A) \cup (V \setminus B)$. □

Submodularity

- Submodular functions have a long history in economics, game theory, combinatorial optimization, electrical networks, and operations research.
- They are gaining importance in machine learning as well (one of our main motivations for offering this course).
- Arbitrary set functions are hopelessly difficult to optimize, while the minimum of submodular functions can be found in polynomial time, and the maximum can be constant-factor approximated in low-order polynomial time.
- Submodular functions share properties in common with both convex and concave functions, but they are quite different.

Attractions of Convex Functions

Why do we like Convex Functions? (Quoting Lovász 1983):

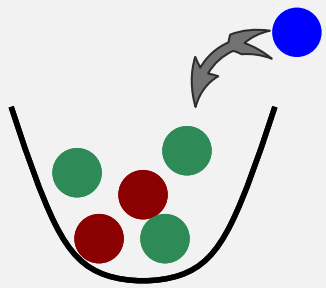
1. *Convex functions occur in many mathematical models in economy, engineering, and other sciences. Convexity is a very natural property of various functions and domains occurring in such models; quite often the only non-trivial property which can be stated in general.*
2. *Convexity is preserved under many natural operations and transformations, and thereby the effective range of results can be extended, elegant proof techniques can be developed as well as unforeseen applications of certain results can be given.*
3. *Convex functions and domains exhibit sufficient structure so that a mathematically beautiful and practically useful theory can be developed.*
4. *There are theoretically and practically (reasonably) efficient methods to find the minimum of a convex function.*

Attractions of Submodular Functions

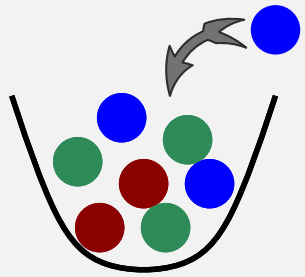
In this course, we wish to demonstrate that submodular functions also possess attractions of these four sorts as well.

Example Submodular: Number of Colors of Balls in Urns

- Consider an urn containing colored balls. Given a set S of balls, $f(S)$ counts the number of distinct colors.



Initial value: 2 (colors in urn).
New value with added blue ball: 3

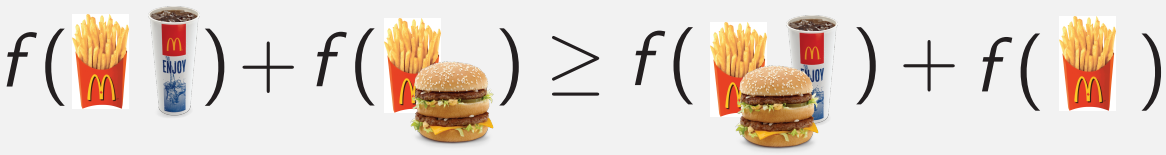


Initial value: 3 (colors in urn).
New value with added blue ball: 3

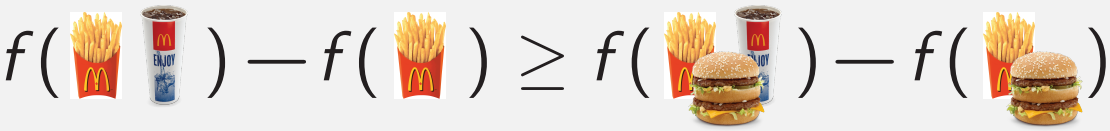
- Submodularity: Incremental Value of Object Diminishes in a Larger Context (diminishing returns).
- Thus, f is submodular.

Ex. Submodular: Consumer Costs of Living

- Consumer costs are very often submodular. For example:



- Rearranging terms, we can see this as diminishing returns:

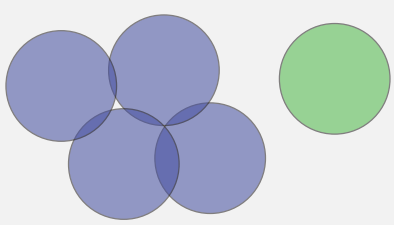


- This is very common: The additional cost of a coke is, say, free if you add it to fries and a hamburger, but when added just to an order of fries, the coke is not free.

Area of the union of areas indexed by A

- Let V be a set of indices, and each $v \in V$ indexes a given sub-area of some region. Let area(v) be the area corresponding to item v.
- Let $f(S) = \bigcup_{s \in S} \text{area}(s)$ be the union of the areas indexed by elements in S.
- Then $f(S)$ is submodular.

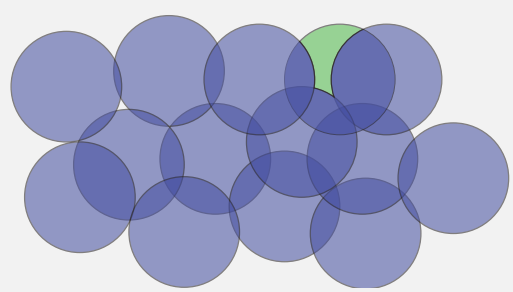
Area of the union of areas indexed by A



Gain (value) of v in context of A:

$$f(A \cup \{v\}) - f(A) = f(\{v\})$$

We get full value $f(\{v\})$ in this case since the area of v has no overlap with that of A.



Incremental value of v in the context of $B \supset A$.

$$\begin{aligned} f(B \cup \{v\}) - f(B) &< f(\{v\}) \\ &= f(A \cup \{v\}) - f(A) \end{aligned}$$

So benefit of v in the context of A is greater than the benefit of v in the context of $B \supseteq A$.

Example Submodular: Entropy from Information Theory

- Entropy is submodular. Let V be the index set of a set of random variables, then the function

$$f(A) = H(X_A) = -\sum_{x_A} p(x_A) \log p(x_A) \tag{2.15}$$

is submodular.

- Proof: conditioning reduces entropy. With $A \subseteq B$ and $v \notin B$,

$$H(X_v|X_B) = H(X_{B+v}) - H(X_B) \tag{2.16}$$
$$\leq H(X_{A+v}) - H(X_A) = H(X_v|X_A) \tag{2.17}$$

Example Submodular: Entropy from Information Theory

- Alternate Proof: Conditional mutual Information is always non-negative.
- Given $A, B, C \subseteq V$, consider conditional mutual information quantity:

$$\begin{aligned} I(X_{A\setminus B}; X_{B\setminus A}|X_{A\cap B}) &= \sum_{x_{A\cup B}} p(x_{A\cup B}) \log \frac{p(x_{A\setminus B}, x_{B\setminus A}|x_{A\cap B})}{p(x_{A\setminus B}|x_{A\cap B})p(x_{B\setminus A}|x_{A\cap B})} \\ &= \sum_{x_{A\cup B}} p(x_{A\cup B}) \log \frac{p(x_{A\cup B})p(x_{A\cap B})}{p(x_A)p(x_B)} \geq 0 \end{aligned} \tag{2.18}$$

then

$$\begin{aligned} &I(X_{A\setminus B}; X_{B\setminus A}|X_{A\cap B}) \\ &\quad = H(X_A) + H(X_B) - H(X_{A\cup B}) - H(X_{A\cap B}) \geq 0 \end{aligned} \tag{2.19}$$

so entropy satisfies

$$H(X_A) + H(X_B) \geq H(X_{A\cup B}) + H(X_{A\cap B}) \tag{2.20}$$

Example Submodular: Mutual Information

- Also, symmetric mutual information is submodular,

$$f(A) = I(X_A; X_{V \setminus A}) = H(X_A) + H(X_{V \setminus A}) - H(X_V) \qquad (2.21)$$

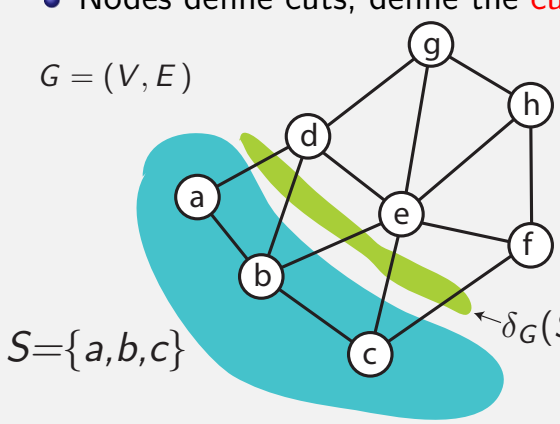
Note that $f(A) = H(X_A)$ and $\bar{f}(A) = H(X_{V \setminus A})$, and adding submodular functions preserves submodularity (which we will see quite soon).

Undirected Graphs

- Let $G = (V, E)$ be a graph with vertices $V = V(G)$ and edges $E = E(G) \subseteq V \times V$.
- If G is undirected, define

$$E(X, Y) = \{\{x, y\} \in E(G) : x \in X \setminus Y, y \in Y \setminus X\} \qquad (2.22)$$

as the edges between X and Y.
- Nodes define cuts, define the cut function $\delta(X) = E(X, V \setminus X)$.



$\delta_G(S) = \{\{u, v\} \in E : u \in S, v \in V \setminus S\}.$
$= \{\{a,d\},\{b,d\},\{b,e\},\{c,e\},\{c,f\}\}$

Directed graphs, and cuts and flows

- If G is directed, define

$$E^+(X,Y) \triangleq \{(x,y) \in E(G) : x \in X \setminus Y, y \in Y \setminus X\} \tag{2.23}$$

as the edges directed from X towards Y.

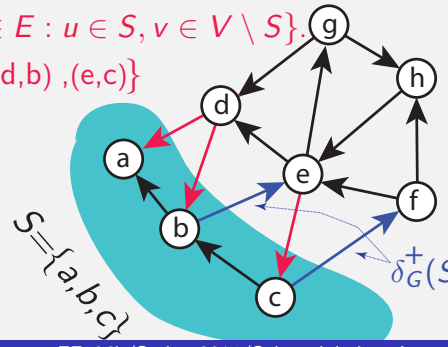
- Nodes define cuts and flows. Define edges leaving X (out-flow) as

$$\delta^+(X) \triangleq E^+(X, V \setminus X) \tag{2.24}$$

and edges entering X (in-flow) as

$$\delta^-(X) \triangleq E^+(V \setminus X, X) \tag{2.25}$$

$\delta^-_G(S) = \{(v,u) \in E : u \in S, v \in V \setminus S\}.$
$= \{$(d,a) ,(d,b) ,(e,c)$\}$



$\delta^+_G(S) = \{(u,v) \in E : u \in S, v \in V \setminus S\}.$
$= \{$(b,e) ,(c,f)$\}$

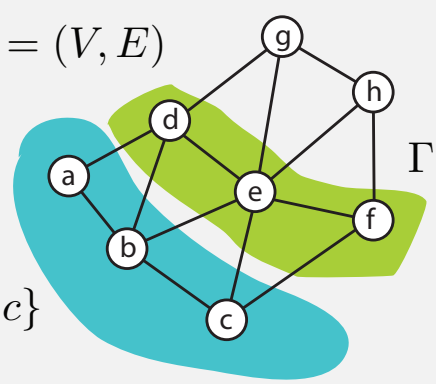
The Neighbor function in undirected graphs

- Given a set $X \subseteq V$, the neighbors function of X is defined as

$$\Gamma(X) \triangleq \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\} \tag{2.26}$$

- Example:

$G = (V, E)$



$\Gamma(S) = \{d, e, f\}$

$S = \{a, b, c\}$

Directed Cut function: property

Lemma 2.6.1

For a digraph $G = (V, E)$ and any $X, Y \subseteq V$: we have

$$|\delta^+(X)| + |\delta^+(Y)| = |\delta^+(X \cap Y)| + |\delta^+(X \cup Y)| + |E^+(X, Y)| + |E^+(Y, X)| \tag{2.27}$$

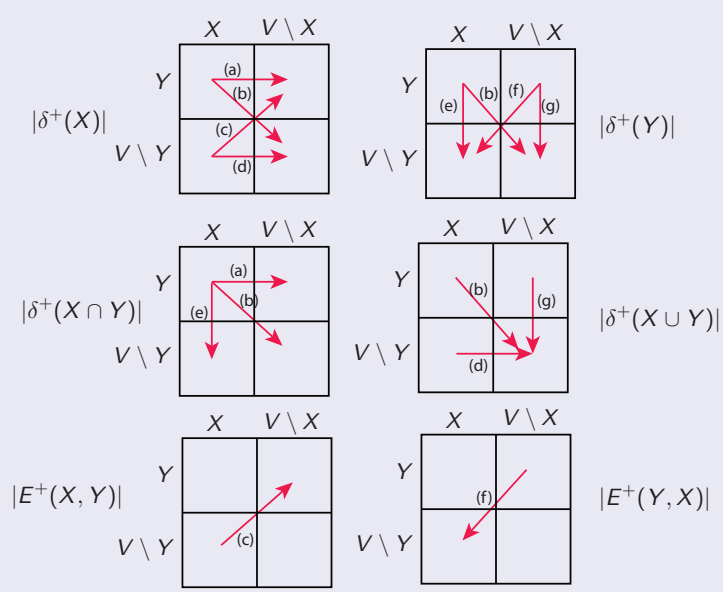
and

$$|\delta^-(X)| + |\delta^-(Y)| = |\delta^-(X \cap Y)| + |\delta^-(X \cup Y)| + |E^-(X, Y)| + |E^-(Y, X)| \tag{2.28}$$

Directed Cut function: proof of property

Proof.

We can prove this using a simple geometric counting argument ($\delta^-(X)$ is similar)



□

Directed cut/flow functions: submodular

Lemma 2.6.2

For a digraph $G = (V, E)$ and any $X, Y \subseteq V$: both functions $|\delta^+(X)|$ and $|\delta^-(X)|$ are submodular.

Proof.

$|E^+(X, Y)| \geq 0$ and $|E^-(X, Y)| \geq 0$. □

More generally, in the non-negative weighted case, both in-flow and out-flow are submodular on subsets of the vertices.

Undirected Cut/Flow & the Neighbor function: submodular

Lemma 2.6.3

For an undirected graph $G = (V, E)$ and any $X, Y \subseteq V$: we have that both the undirected cut (or flow) function $|\delta(X)|$ and the neighbor function $|\Gamma(X)|$ are submodular. I.e.,

$$|\delta(X)| + |\delta(Y)| = |\delta(X \cap Y)| + |\delta(X \cup Y)| + 2|E(X, Y)| \tag{2.29}$$

and

$$|\Gamma(X)| + |\Gamma(Y)| \geq |\Gamma(X \cap Y)| + |\Gamma(X \cup Y)| \tag{2.30}$$

Proof.

- Eq. (2.29) follows from Eq. (2.27): we replace each undirected edge $\{u, v\}$ with two oppositely-directed directed edges (u, v) and (v, u). Then we use same counting argument.
- Eq. (2.30) follows as shown in the following page.

...

Undirected cut/flow is submodular: alternate proof

- Another simple proof shows that $|\delta(X)|$ is submodular.
- Define a graph $G_{uv} = (\{u, v\}, \{e\}, w)$ with two nodes u, v and one edge $e = \{u, v\}$ with non-negative weight $w(e) \in \mathbb{R}_+$.
- Cut weight function over those two nodes: $w(\delta_{u,v}(\cdot))$ has valuation:

$$w(\delta_{u,v}(\emptyset)) = w(\delta_{u,v}(\{u, v\})) = 0 \tag{2.31}$$

and

$$w(\delta_{u,v}(\{u\})) = w(\delta_{u,v}(\{v\})) = w \geq 0 \tag{2.32}$$

- Thus, $w(\delta_{u,v}(\cdot))$ is submodular since

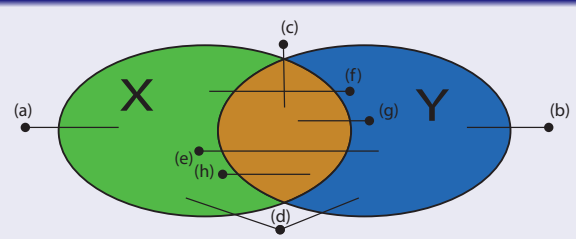
$$w(\delta_{u,v}(\{u\})) + w(\delta_{u,v}(\{v\})) \geq w(\delta_{u,v}(\{u, v\})) + w(\delta_{u,v}(\emptyset)) \tag{2.33}$$

- General non-negative weighted graph $G = (V, E, w)$, define $w(\delta(\cdot))$:

$$f(A) = w(\delta(A)) = \sum_{(u,v) \in E(G)} w(\delta_{u,v}(A \cap \{u, v\})) \tag{2.34}$$

- This is easily shown to be submodular using properties we will soon see (namely, submodularity closed under summation and restriction).

cont.



Graphically, we can count and see that

$$\Gamma(X) = (a) + (c) + (f) + (g) + (d) \tag{2.35}$$
$$\Gamma(Y) = (b) + (c) + (e) + (h) + (d) \tag{2.36}$$
$$\Gamma(X \cup Y) = (a) + (b) + (c) + (d) \tag{2.37}$$
$$\Gamma(X \cap Y) = (c) + (g) + (h) \tag{2.38}$$

so

$$\begin{aligned} |\Gamma(X)| + |\Gamma(Y)| &= (a) + (b) + 2(c) + 2(d) + (e) + (f) + (g) + (h) \\ &\geq (a) + (b) + 2(c) + (d) + (g) + (h) = |\Gamma(X \cup Y)| + |\Gamma(X \cap Y)| \end{aligned} \tag{2.39}$$

Undirected Neighbor functions

Therefore, the undirected cut function $|\delta(A)|$ and the neighbor function $|\Gamma(A)|$ of a graph G are both submodular.

Other graph functions that are submodular/supermodular

These come from Narayanan's book 1997. Let G be an undirected graph.

- Let $V(X)$ be the vertices adjacent to some edge in $X \subseteq E(G)$, then $|V(X)|$ (the vertex function) is submodular.
- Let $E(S)$ be the edges with both vertices in $S \subseteq V(G)$. Then $|E(S)|$ (the interior edge function) is supermodular.
- Let $I(S)$ be the edges with at least one vertex in $S \subseteq V(G)$. Then $|I(S)|$ (the incidence function) is submodular.
- Recall $|\delta(S)|$, is the set size of edges with exactly one vertex in $S \subseteq V(G)$ is submodular (cut size function). Thus, we have $I(S) = E(S) \cup \delta(S)$ and $E(S) \cap \delta(S) = \emptyset$, and thus that $|I(S)| = |E(S)| + |\delta(S)|$. So we can get a submodular function by summing a submodular and a supermodular function. If you had to guess, is this always the case?
- Consider $f(A) = |\delta^+(A)| - |\delta^+(V \setminus A)|$. Guess, submodular, supermodular, modular, or neither? Exercise: determine which one and prove it.

Other graph functions that are submodular/supermodular

- Recall, $f : 2^V \to \mathbb{R}$ is submodular, then so is $\bar{f} : 2^V \to \mathbb{R}$ defined as $\bar{f}(S) = f(V \setminus S)$.
- Hence, if $f : 2^V \to \mathbb{R}$ is supermodular, then so is $\bar{f} : 2^V \to \mathbb{R}$ defined as $\bar{f}(S) = f(V \setminus S)$.
- Given a graph $G = (V, E)$, for each $A \subseteq E(G)$, let $c(A)$ denote the number of connected components of the (spanning) subgraph $(V(G), A)$, with $c : 2^E \to \mathbb{R}_+$.
- $c(A)$ is monotone non-increasing, $c(A + a) - c(A) \leq 0$.
- Then $c(A)$ is supermodular. Intuition why: an edge is "more" (no less) able to bridge separate components (and thus reduce the number of conected components) when the edge is added in a smaller context than when added in a larger context.
- I.e., $c(A + a) - c(A) \leq c(B + a) - c(B)$ with $A \subseteq B \subseteq E \setminus \{a\}$.
- $\bar{c}(A) = c(E \setminus A)$ is the number of connected components in G when we remove A, and hence is also supermodular.

Graph Strength

- So $\bar{c}(A) = c(E \setminus A)$ is the number of connected components in G when we remove A, is supermodular.
- Maximizing $\bar{c}(A)$ might seem as a goal for a network attacker — many connected components means that many points in the network have lost connectivity to many other points (unprotected network).
- If we can remove a small set A and shatter the graph into many connected components, then the graph is weak.
- An attacker wishes to choose a small number of edges (since it is cheap) to shatter the graph into as many components as possible.
- Let $G = (V, E, w)$ with $w : E \to \mathbb{R}+$ be a weighted graph with non-negative weights.
- For $(u, v) = e \in E$, let $w(e)$ be a measure of the strength of the connection between vertices u and v (strength meaning the difficulty of cutting the edge e).

Graph Strength

- Then $w(A)$ for $A \subseteq E$ is a modular function

$$w(A) = \sum_{e \in A} w_e \tag{2.40}$$

  so that $w(E(G[S]))$ is the "internal strength" of the vertex set S.
- Suppose removing A shatters G into a graph with $\bar{c}(A) > 1$ components — then $w(A)/(\bar{c}(A) - 1)$ is like the "effort per achieved component" for a network attacker.
- A form of graph strength can then be defined as the following:

$$\textit{strength}(G, w) = \min_{A \subseteq E(G): \bar{c}(A) > 1} \frac{w(A)}{\bar{c}(A) - 1} \tag{2.41}$$

- Graph strength is like the minimum effort per component. An attacker would use the argument of the min to choose which edges to attack. A network designer would maximize, over G and/or w, the graph strength, *strength*(G, w).
- Since submodularity, problems have strongly-poly-time solutions.

Matrix Rank functions

- Let V, with $|V| = m$ be an index set of a set of vectors in $\mathbb{R}^n$ for some n (unrelated to m).
- For a given set $\{v, v_1, v_2, \ldots, v_k\}$, it might or might not be possible to find $(\alpha_i)_i$ such that:

$$x_v = \sum_{i=1}^{k} \alpha_i x_{v_i} \tag{2.42}$$

  If not, then x_v is linearly independent of $x_{v_1}, \ldots, x_{v_k}$.
- Let $r(S)$ for $S \subseteq V$ be the rank of the set of vectors S. Then $r(\cdot)$ is a submodular function, and in fact is called a matric matroid rank function.

Example: Rank function of a matrix

- Given $n \times m$ matrix $\mathbf{X} = (x_1, x_2, \ldots, x_m)$ with $x_i \in \mathbb{R}^n$ for all i. There are m length-n column vectors $\{x_i\}_i$
- Let $V = \{1, 2, \ldots, m\}$ be the set of column vector indices.
- For any $A \subseteq V$, let $r(A)$ be the rank of the column vectors indexed by A.
- $r(A)$ is the dimensionality of the vector space spanned by the set of vectors $\{x_a\}_{a \in A}$.
- Thus, $r(V)$ is the rank of the matrix $\mathbf{X}$.

Skip matrix rank example

Example: Rank function of a matrix

Consider the following 4×8 matrix, so $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

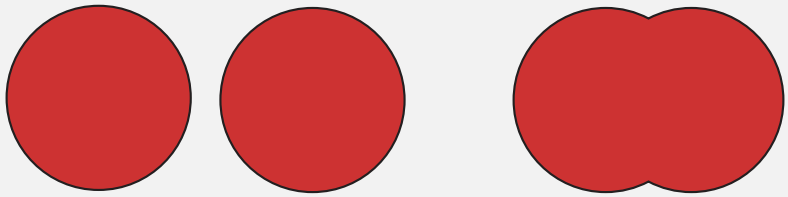
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 0 & 2 & 2 & 3 & 0 & 1 & 3 & 1 \\ 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} | & | & | & | & | & | & | & | \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ | & | & | & | & | & | & | & | \end{pmatrix} \end{array}$$

- Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{6, 7\}$, $A_r = \{1\}$, $B_r = \{5\}$.
- Then $r(A) = 3$, $r(B) = 3$, $r(C) = 2$.
- $r(A \cup C) = 3$, $r(B \cup C) = 3$.
- $r(A \cup A_r) = 3$, $r(B \cup B_r) = 3$, $r(A \cup B_r) = 4$, $r(B \cup A_r) = 4$.
- $r(A \cup B) = 4$, $r(A \cap B) = 1 < r(C) = 2$.
- $6 = r(A) + r(B) > r(A \cup B) + r(A \cap B) = 5$

Rank function of a matrix

- Let $A, B \subseteq V$ be two subsets of column indices.
- The rank of the two sets unioned together $A \cup B$ is no more than the sum of the two individual ranks.
- In Venn diagram, Let area correspond to dimensions spanned by vectors indexed by a set. Hence, $r(A)$ can be viewed as an area.

$$r(A) \quad + \quad r(B) \qquad \geq \qquad r(A \cup B)$$



- If some of the dimensions spanned by A overlap some of the dimensions spanned by B (i.e., if $\exists$ common span), then that area is counted twice in $r(A) + r(B)$, so the inequality will be strict.
- Any function where the above inequality is true for all $A, B \subseteq V$ is called subadditive.

Rank functions of a matrix

- Vectors A and B have a (possibly empty) common span and two (possibly empty) non-common residual spans.
- Let C index vectors spanning dimensions common to A and B.
- Let A_r index vectors spanning dimensions spanned by A but not B.
- Let B_r index vectors spanning dimensions spanned by B but not A.
- Then, $r(A) = r(C) + r(A_r)$
- Similarly, $r(B) = r(C) + r(B_r)$.
- Then $r(A) + r(B)$ counts the dimensions spanned by C twice, i.e.,

$$r(A) + r(B) = r(A_r) + 2r(C) + r(B_r). \tag{2.43}$$

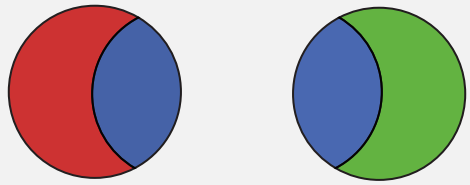
- But $r(A \cup B)$ counts the dimensions spanned by C only once.

$$r(A \cup B) = r(A_r) + r(C) + r(B_r) \tag{2.44}$$

Rank functions of a matrix

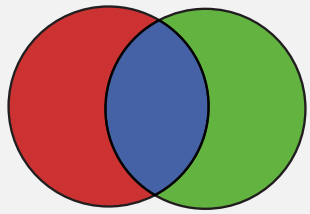
- Then $r(A) + r(B)$ counts the dimensions spanned by C twice, i.e.,

$$r(A) + r(B) = r(A_r) + 2r(C) + r(B_r)$$



- But $r(A \cup B)$ counts the dimensions spanned by C only once.

$$r(A \cup B) = r(A_r) + r(C) + r(B_r)$$

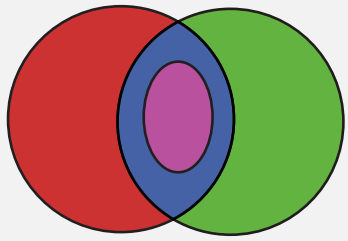


- Thus, we have subadditivity: $r(A) + r(B) \geq r(A \cup B)$. Can we add more to the r.h.s. and still have an inequality? Yes.

Rank function of a matrix

- Note, $r(A \cap B) \leq r(C)$. Why? Vectors indexed by $A \cap B$ (i.e., the common index set) span no more than the dimensions commonly spanned by A and B (namely, those spanned by the professed C).

$$r(C) \geq r(A \cap B)$$



In short:

- Common span (blue) is "more" (no less) than span of common index (magenta).
- More generally, common information (blue) is "more" (no less) than information within common index (magenta).

The Venn and Art of Submodularity

$$\underbrace{r(A) + r(B)}_{= r(A_r) + 2r(C) + r(B_r)} \geq \underbrace{r(A \cup B)}_{= r(A_r) + r(C) + r(B_r)} + \underbrace{r(A \cap B)}_{= r(A \cap B)}$$