Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 2 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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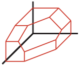
$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

$= f(A_r) + 2f(C) + f(B_r)$ $\quad = f(A_r) + f(C) + f(B_r)$ $\quad = f(A \cap B)$



Cumulative Outstanding Reading

- Read chapter 1 from Fujishige's book.

Announcements, Assignments, and Reminders

- our room (Mueller Hall Room 154) is changed!
- Please do use our discussion board (`https://canvas.uw.edu/courses/895956/discussion_topics`) for all questions, comments, so that all will benefit from them being answered.
- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3:
- L4:
- L5:
- L6:
- L7:
- L8:
- L9:
- L10:
- L11:
- L12:
- L13:
- L14:
- L15:
- L16:
- L17:
- L18:
- L19:
- L20:

Finals Week: June 9th-13th, 2014.

Submodular Definitions

Definition 2.2.2 (submodular concave)

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \tag{2.2}$$

An alternate and (as we will soon see) equivalent definition is:

Definition 2.2.3 (diminishing returns)

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A \subseteq B \subset V$, and $v \in V \setminus B$, we have that:

$$f(A \cup \{v\}) - f(A) \geq f(B \cup \{v\}) - f(B) \tag{2.3}$$

This means that the incremental "value", "gain", or "cost" of v decreases (diminishes) as the context in which v is considered grows from A to B.

Example Discrete Optimization Problems

- **Combinatorial Problems**: e.g., set cover, max k coverage, vertex cover, edge cover, graph cut problems.
- **Operations Research**: facility location (uncapacited)
- **Sensor placement**
- **Information:** Information gain and feature selection, information theory
- **Mathematics:** e.g., monge matrices
- **Networks**: Social networks, influence, viral marketing, information cascades, diffusion networks
- **Graphical models**: tree distributions, factors, and image segmentation
- **Diversity** and its models
- **NLP**: Natural language processing: document summarization, web search, information retrieval
- **ML**: Machine Learning: active/semi-supervised learning
- **Economics**: markets, economies of scale

Markets: Supply Side Economies of scale

- Economies of Scale: Many goods and services can be produced at a much lower per-unit cost only if they are produced in very large quantities.
- The profit margin for producing a unit of goods is improved as more of those goods are created.
- If you already make a good, making a similar good is easier than if you start from scratch (e.g., Apple making both iPod and iPhone).
- An argument in favor of free trade is that it opens up larger markets for firms (especially in otherwise small markets), thereby enabling better economies of scale, and hence greater efficiency (lower costs and resources per unit of good produced).

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- Ex: V might be colors of paint in a paint manufacturer: green, red, blue, yellow, white, etc.
- Producing green when you are already producing yellow and blue is probably cheaper than if you were only producing some other colors.

$$f(\text{green}, \text{blue}, \text{yellow}) - f(\text{blue}, \text{yellow}) <= f(\text{green}, \text{blue}) - f(\text{blue}) \tag{2.1}$$

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- Given network externalities, a consumer in today's market cares also about the future success of the product and competing products.
- If the good is durable (e.g., a car or phone) or there is human capital investment (e.g., education in a skill), the total benefits derived from a good will depend on the number of consumers who adopt compatible products in the future.

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- online education, Massive Open Online Courses (MOOCs) such as Coursera, edX, etc. – with many people simultaneously taking a class, all gain due to richer peer discussions due to greater pool of well matched study groups, more simultaneous similar questions/problems that are asked $\Rightarrow$ more efficient learning & training data for ML algorithms to learn how people learn.

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- any widely used standard (job training now is useful in the future)
- the "tipping point", and "winner take all" (one platform prevails)

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Optimization Problem Involving Network Externalities

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- and let S_{k^*} be the saturation point, lowest value of k such that $S_k = S_{k+1}$
- Goal: find A and p to maximize $f_p(A) = \mathbb{E}[p \times |S_{k^*}|]$.

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- But $f(\{v_1, v_2\}) = c_d + c_m + c_h < 2c_d + c_m + c_h$ since c_d is a shared fixed cost.

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- But $f(\{v_1, v_2\}) = c_d + c_m + c_h < 2c_d + c_m + c_h$ since c_d is a shared fixed cost.
- Shared fixed costs are submodular.

Anecdote

From David Brooks, NYTs column, March 28th, 2011 on "Tools for Thinking". In response to Steven Pinker (Harvard) asking a number of people "What scientific concept would improve everybody's cognitive toolkit?"

> *Emergent systems are ones in which many different elements interact. The pattern of interaction then produces a new element that is greater than the sum of the parts, which then exercises a top-down influence on the constituent elements.*

Submodular Motivation Recap

- Given a set of objects $V = \{v_1, \ldots, v_n\}$ and a function $f : 2^V \to \mathbb{R}$ that returns a real value for any subset $S \subseteq V$.
- Suppose we are interested in finding the subset that either maximizes or minimizes the function, e.g., $\operatorname{argmax}_{S \subseteq V} f(S)$, possibly subject to some constraints.
- In general, this problem has exponential time complexity.
- Example: f might correspond to the value (e.g., information gain) of a set of sensor locations in an environment, and we wish to find the best set $S \subseteq V$ of sensors locations given a fixed upper limit on the number of sensors $|S|$.
- In many cases (such as above) f has properties that make its optimization tractable to either exactly or approximately compute.
- One such property is *submodularity*.

Submodular Definitions

Definition 2.4.2 (submodular concave)

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \quad (2.2)$$

An alternate and (as we will soon see) equivalent definition is:

Definition 2.4.3 (diminishing returns)

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A \subseteq B \subset V$, and $v \in V \setminus B$, we have that:

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This means that the incremental "value", "gain", or "cost" of v decreases (diminishes) as the context in which v is considered grows from A to B.

Subadditive Definitions

Definition 2.4.1 (subadditive)

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$$f(A) + f(B) \geq f(A \cup B) \tag{2.2}$$

This means that the "whole" is less than the sum of the parts.

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An alternate and equivalent definition is:

Definition 2.4.3 (increasing returns)

A function $f : 2^V \to \mathbb{R}$ is supermodular if for any $A \subseteq B \subset V$, and $v \in V \setminus B$, we have that:

$$f(A \cup \{v\}) - f(A) \leq f(B \cup \{v\}) - f(B) \tag{2.4}$$

The incremental "value", "gain", or "cost" of v increases as the context in which v is considered grows from A to B.

Submodular vs. Supermodular

- Submodular and supermodular functions are closely related.

Submodular vs. Supermodular

- Submodular and supermodular functions are closely related.
- In fact, f is submodular iff $-f$ is supermodular.

Superadditive Definitions

Definition 2.4.4 (superadditive)

A function $f : 2^V \to \mathbb{R}$ is superadditive if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \leq f(A \cup B) \tag{2.5}$$

- This means that the “whole” is greater than the sum of the parts.

Superadditive Definitions

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- This means that the "whole" is greater than the sum of the parts.
- In general, submodular and subadditive (and supermodular and superadditive) are different properties.

Modular Definitions

Definition 2.4.5 (modular)

A function that is both submodular and supermodular is called **modular**

If f is a modular function, than for any $A, B \subseteq V$, we have

$$f(A) + f(B) = f(A \cap B) + f(A \cup B) \tag{2.6}$$

In modular functions, elements do not interact (or cooperate, or compete, or influence each other), and have value based only on singleton values.

Proposition 2.4.6

If f is modular, it may be written as

$$f(A) = f(\emptyset) + \sum_{a \in A} \Big(f(\{a\}) - f(\emptyset) \Big) \tag{2.7}$$

Modular Definitions

Proof.

We inductively construct the value for $A = \{a_1, a_2, \dots, a_k\}$.
For $k = 2$,

$$f(a_1) + f(a_2) = f(a_1, a_2) + f(\emptyset) \tag{2.8}$$

$$\text{implies } f(a_1, a_2) = f(a_1) - f(\emptyset) + f(a_2) - f(\emptyset) + f(\emptyset) \tag{2.9}$$

then for $k = 3$,

$$f(a_1, a_2) + f(a_3) = f(a_1, a_2, a_3) + f(\emptyset) \tag{2.10}$$

$$\text{implies } f(a_1, a_2, a_3) = f(a_1, a_2) - f(\emptyset) + f(a_3) - f(\emptyset) + f(\emptyset) \tag{2.11}$$

$$= f(\emptyset) + \sum_{i=1}^{3} \big(f(a_i) - f(\emptyset)\big) \tag{2.12}$$

and so on ... $\square$

Complement function

Given a function $f : 2^V \to \mathbb{R}$, we can find a complement function $\bar{f} : 2^V \to \mathbb{R}$ as $\bar{f}(A) = f(V \setminus A)$ for any A.

Proposition 2.4.7

$\bar{f}$ is submodular if f is submodular.

Proof.

$$\bar{f}(A) + \bar{f}(B) \geq \bar{f}(A \cup B) + \bar{f}(A \cap B) \tag{2.13}$$

follows from

$$f(V \setminus A) + f(V \setminus B) \geq f(V \setminus (A \cup B)) + f(V \setminus (A \cap B)) \tag{2.14}$$

which is true because $V \setminus (A \cup B) = (V \setminus A) \cap (V \setminus B)$ and $V \setminus (A \cap B) = (V \setminus A) \cup (V \setminus B)$. □

Submodularity

- Submodular functions have a long history in economics, game theory, combinatorial optimization, electrical networks, and operations research.
- They are gaining importance in machine learning as well (one of our main motivations for offering this course).
- Arbitrary set functions are hopelessly difficult to optimize, while the minimum of submodular functions can be found in polynomial time, and the maximum can be constant-factor approximated in low-order polynomial time.
- Submodular functions share properties in common with both convex and concave functions, but they are quite different.

Attractions of Convex Functions

Why do we like Convex Functions? (Quoting Lovász 1983):

1. *Convex functions occur in many mathematical models in economy, engineering, and other sciences. Convexity is a very natural property of various functions and domains occurring in such models; quite often the only non-trivial property which can be stated in general.*

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3. *Convex functions and domains exhibit sufficient structure so that a mathematically beautiful and practically useful theory can be developed.*
4. *There are theoretically and practically (reasonably) efficient methods to find the minimum of a convex function.*

Attractions of Submodular Functions

In this course, we wish to demonstrate that submodular functions also possess attractions of these four sorts as well.

Example Submodular: Number of Colors of Balls in Urns

- Consider an urn containing colored balls. Given a set S of balls, $f(S)$ counts the number of distinct colors.

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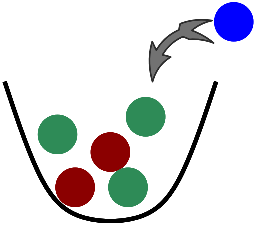
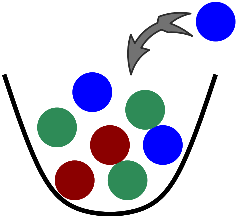


Initial value: 2 (colors in urn).
New value with added blue ball: 3



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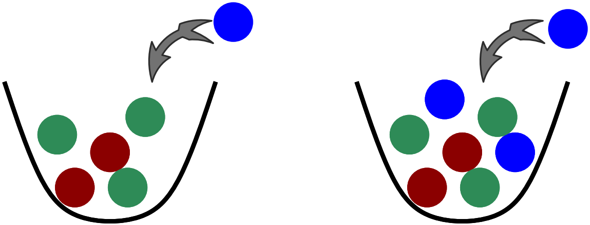

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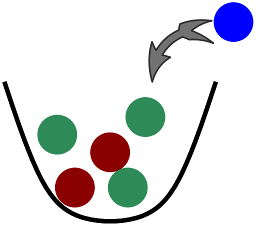

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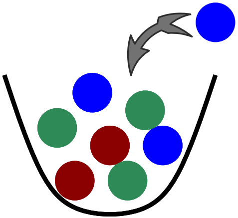

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- Submodularity: Incremental Value of Object Diminishes in a Larger Context (diminishing returns).
- Thus, f is submodular.

Ex. Submodular: Consumer Costs of Living

- Consumer costs are very often submodular.

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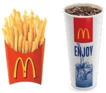
$f($ 

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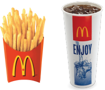
$f($ 

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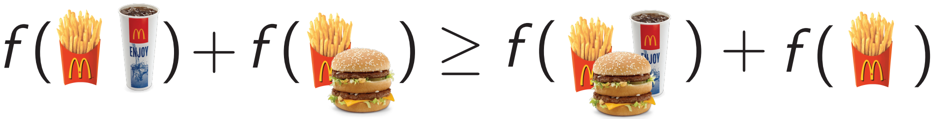

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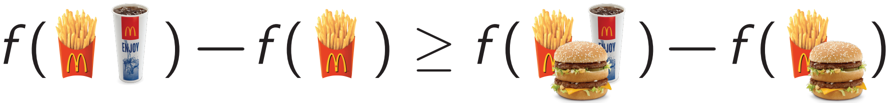

Ex. Submodular: Consumer Costs of Living

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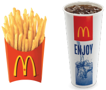
$f($  $) + f($  $) \geq f($  $) + f($  $)$

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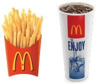
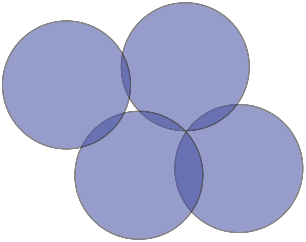
$f($  $) - f($  $) \geq f($  $) - f($  $)$

- This is very common: The additional cost of a coke is, say, free if you add it to fries and a hamburger, but when added just to an order of fries, the coke is not free.

Area of the union of areas indexed by A

- Let V be a set of indices, and each $v \in V$ indexes a given sub-area of some region. Let $\text{area}(v)$ be the area corresponding to item v.
- Let $f(S) = \bigcup_{s \in S} \text{area}(s)$ be the union of the areas indexed by elements in S.
- Then $f(S)$ is submodular.

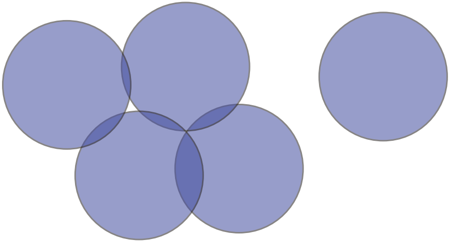
Area of the union of areas indexed by A



Union of areas of elements of A is given by:

$$f(A) = f(\{a_1, a_2, a_3, a_4\})$$

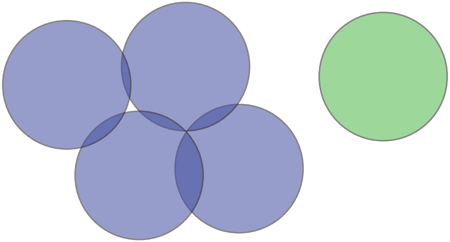
Area of the union of areas indexed by A



Area of A along with with v:

$$f(A \cup \{v\}) = f(\{a_1, a_2, a_3, a_4\} \cup \{v\})$$

Area of the union of areas indexed by A

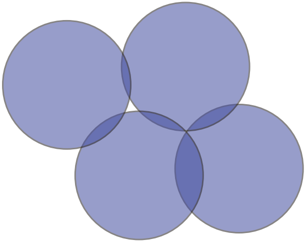


Gain (value) of v in context of A:

$$f(A \cup \{v\}) - f(A) = f(\{v\})$$

We get full value $f(\{v\})$ in this case since the area of v has no overlap with that of A.

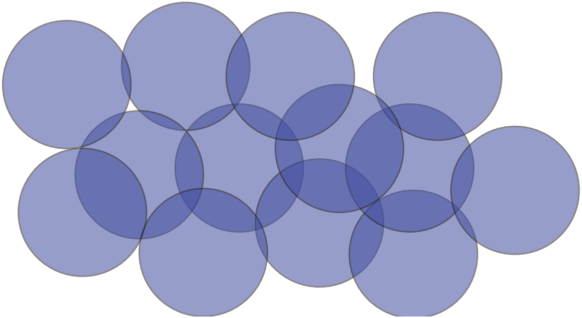
Area of the union of areas indexed by A



Area of A once again.

$$f(A) = f(\{a_1, a_2, a_3, a_4\})$$

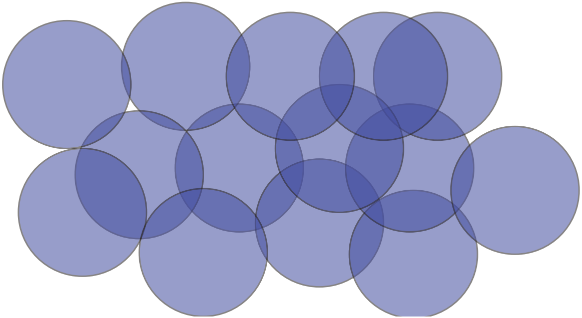
Area of the union of areas indexed by A



Union of areas of elements of $B \supset A$, where v is not included:

$$f(B) \text{ where } v \notin B \text{ and where } A \subseteq B$$

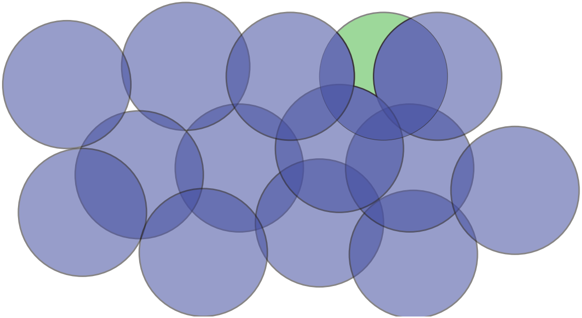
Area of the union of areas indexed by A



Area of B now also including v:

$$f(B \cup \{v\})$$

Area of the union of areas indexed by A



Incremental value of v in the context of $B \supset A$.

$$f(B \cup \{v\}) - f(B) < f(\{v\}) = f(A \cup \{v\}) - f(A)$$

So benefit of v in the context of A is greater than the benefit of v in the context of $B \supseteq A$.

Example Submodular: Entropy from Information Theory

- Entropy is submodular. Let V be the index set of a set of random variables, then the function

$$f(A) = H(X_A) = -\sum_{x_A} p(x_A) \log p(x_A) \tag{2.15}$$

  is submodular.

- Proof: conditioning reduces entropy. With $A \subseteq B$ and $v \notin B$,

$$\begin{aligned} H(X_v|X_B) &= H(X_{B+v}) - H(X_B) && (2.16) \\ &\leq H(X_{A+v}) - H(X_A) = H(X_v|X_A) && (2.17) \end{aligned}$$

Example Submodular: Entropy from Information Theory

- Alternate Proof: Conditional mutual Information is always non-negative.
- Given $A, B, C \subseteq V$, consider conditional mutual information quantity:

$$I(X_{A\setminus B}; X_{B\setminus A}|X_{A\cap B}) = \sum_{x_{A\cup B}} p(x_{A\cup B}) \log \frac{p(x_{A\setminus B}, x_{B\setminus A}|x_{A\cap B})}{p(x_{A\setminus B}|x_{A\cap B})p(x_{B\setminus A}|x_{A\cap B})}$$
$$= \sum_{x_{A\cup B}} p(x_{A\cup B}) \log \frac{p(x_{A\cup B})p(x_{A\cap B})}{p(x_A)p(x_B)} \geq 0 \tag{2.18}$$

then

$$I(X_{A\setminus B}; X_{B\setminus A}|X_{A\cap B})$$
$$= H(X_A) + H(X_B) - H(X_{A\cup B}) - H(X_{A\cap B}) \geq 0 \tag{2.19}$$

so entropy satisfies

$$H(X_A) + H(X_B) \geq H(X_{A\cup B}) + H(X_{A\cap B}) \tag{2.20}$$

Example Submodular: Mutual Information

- Also, symmetric mutual information is submodular,

$$f(A) = I(X_A; X_{V \setminus A}) = H(X_A) + H(X_{V \setminus A}) - H(X_V) \quad (2.21)$$

Note that $f(A) = H(X_A)$ and $\bar{f}(A) = H(X_{V \setminus A})$, and adding submodular functions preserves submodularity (which we will see quite soon).

Undirected Graphs

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$$E(X, Y) = \{\{x, y\} \in E(G) : x \in X \setminus Y, y \in Y \setminus X\} \tag{2.22}$$

as the edges between X and Y.

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- Nodes define cuts, define the cut function $\delta(X) = E(X, V \setminus X)$.

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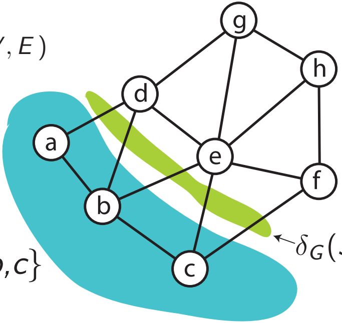
$G = (V, E)$



$S = \{a, b, c\}$

$$\delta_G(S) = \{\{u, v\} \in E : u \in S, v \in V \setminus S\}.$$
$$= \{\{a,d\},\{b,d\},\{b,e\},\{c,e\},\{c,f\}\}$$

Directed graphs, and cuts and flows

- If G is directed, define

$$E^+(X,Y) \triangleq \{(x,y) \in E(G) : x \in X \setminus Y, y \in Y \setminus X\} \tag{2.23}$$

as the edges directed from X towards Y.

Directed graphs, and cuts and flows

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- Nodes define cuts and flows. Define edges leaving X (out-flow) as

$$\delta^+(X) \triangleq E^+(X, V \setminus X) \qquad (2.24)$$

and edges entering X (in-flow) as

$$\delta^-(X) \triangleq E^+(V \setminus X, X) \qquad (2.25)$$

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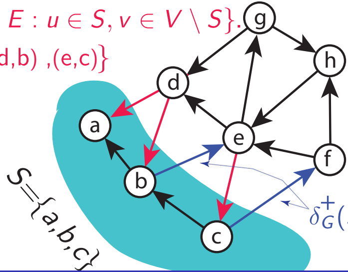
$\delta^-_G(S) = \{(v,u) \in E : u \in S, v \in V \setminus S\}.$
$= \{(d,a), (d,b), (e,c)\}$



$\delta^+_G(S) = \{(u,v) \in E : u \in S, v \in V \setminus S\}.$
$= \{(b,e), (c,f)\}$

The Neighbor function in undirected graphs

- Given a set $X \subseteq V$, the neighbors function of X is defined as

$$\Gamma(X) \triangleq \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\} \tag{2.26}$$

The Neighbor function in undirected graphs

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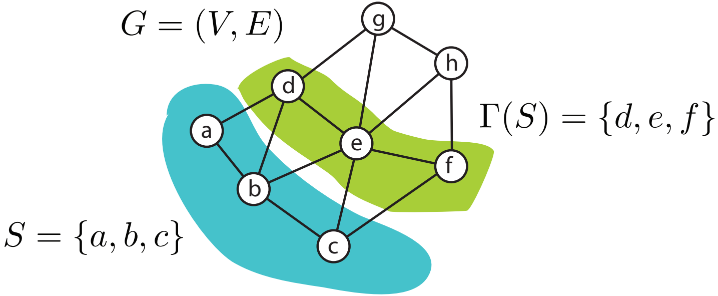
- Example:



Directed Cut function: property

Lemma 2.6.1

For a digraph $G = (V, E)$ and any $X, Y \subseteq V$: we have

$$|\delta^+(X)| + |\delta^+(Y)| = |\delta^+(X \cap Y)| + |\delta^+(X \cup Y)| + |E^+(X, Y)| + |E^+(Y, X)| \tag{2.27}$$

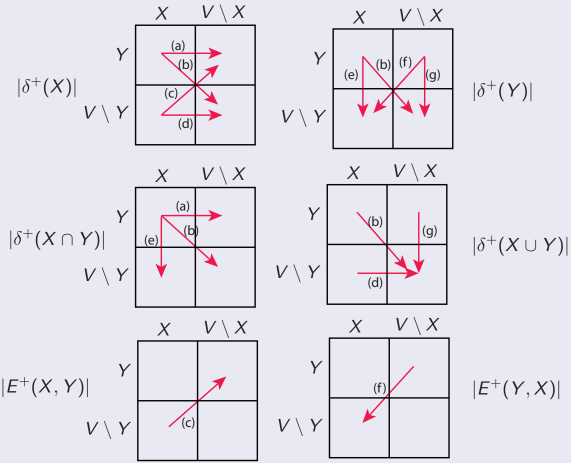
and

$$|\delta^-(X)| + |\delta^-(Y)| = |\delta^-(X \cap Y)| + |\delta^-(X \cup Y)| + |E^-(X, Y)| + |E^-(Y, X)| \tag{2.28}$$

Directed Cut function: proof of property

Proof.

We can prove this using a simple geometric counting argument ($\delta^-(X)$ is similar)



Directed cut/flow functions: submodular

Lemma 2.6.2

For a digraph $G = (V, E)$ and any $X, Y \subseteq V$: both functions $|\delta^+(X)|$ and $|\delta^-(X)|$ are submodular.

Proof.

$|E^+(X,Y)| \geq 0$ and $|E^-(X,Y)| \geq 0$. $\square$

More generally, in the non-negative weighted case, both in-flow and out-flow are submodular on subsets of the vertices.

Undirected Cut/Flow & the Neighbor function: submodular

Lemma 2.6.3

For an undirected graph $G = (V, E)$ and any $X, Y \subseteq V$: we have that both the undirected cut (or flow) function $|\delta(X)|$ and the neighbor function $|\Gamma(X)|$ are submodular. I.e.,

$$|\delta(X)| + |\delta(Y)| = |\delta(X \cap Y)| + |\delta(X \cup Y)| + 2|E(X, Y)| \tag{2.29}$$

and

$$|\Gamma(X)| + |\Gamma(Y)| \geq |\Gamma(X \cap Y)| + |\Gamma(X \cup Y)| \tag{2.30}$$

Proof.

- Eq. (2.29) follows from Eq. (2.27): we replace each undirected edge $\{u, v\}$ with two oppositely-directed directed edges (u, v) and (v, u). Then we use same counting argument.

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Proof.

- Eq. (2.29) follows from Eq. (2.27): we replace each undirected edge $\{u, v\}$ with two oppositely-directed directed edges (u, v) and (v, u). Then we use same counting argument.
- Eq. (2.30) follows as shown in the following page.

...

Undirected cut/flow is submodular: alternate proof

- Another simple proof shows that $|\delta(X)|$ is submodular.

Undirected cut/flow is submodular: alternate proof

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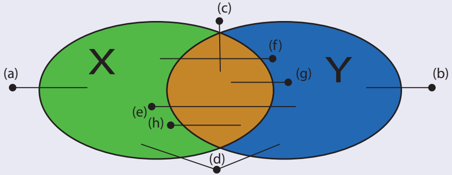
- This is easily shown to be submodular using properties we will soon

cont.



Graphically, we can count and see that

$$\Gamma(X) = (a) + (c) + (f) + (g) + (d) \tag{2.35}$$
$$\Gamma(Y) = (b) + (c) + (e) + (h) + (d) \tag{2.36}$$
$$\Gamma(X \cup Y) = (a) + (b) + (c) + (d) \tag{2.37}$$
$$\Gamma(X \cap Y) = (c) + (g) + (h) \tag{2.38}$$

so

$$\begin{aligned}|\Gamma(X)| + |\Gamma(Y)| &= (a) + (b) + 2(c) + 2(d) + (e) + (f) + (g) + (h) \\ &\geq (a) + (b) + 2(c) + (d) + (g) + (h) = |\Gamma(X \cup Y)| + |\Gamma(X \cap Y)|\end{aligned} \tag{2.39}$$

Undirected Neighbor functions

Therefore, the undirected cut function $|\delta(A)|$ and the neighbor function $|\Gamma(A)|$ of a graph G are both submodular.

Other graph functions that are submodular/supermodular

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- Consider $f(A) = |\delta^+(A)| - |\delta^+(V \setminus A)|$. Guess, submodular, supermodular, modular, or neither? Exercise: determine which one and prove it.

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$$w(A) = \sum_{e \in A} w_e \tag{2.40}$$

so that $w(E(G[S]))$ is the "internal strength" of the vertex set S. *Notation: S is a set of nodes, $G[S]$ is the vertex-induced subgraph of G induced by vertices S, $E(G[S])$ are the edges contained within this induced subgraph, and $w(E(G[S]))$ is the weight of these edges.*

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- Since submodularity, problems have strongly-poly-time solutions.

Matrix Rank functions

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$$x_v = \sum_{i=1}^{k} \alpha_i x_{v_i} \tag{2.42}$$

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If not, then x_v is linearly independent of $x_{v_1}, \ldots, x_{v_k}$.
- Let $r(S)$ for $S \subseteq V$ be the rank of the set of vectors S. Then $r(\cdot)$ is a submodular function, and in fact is called a matric matroid rank function.

Example: Rank function of a matrix

- Given $n \times m$ matrix $\mathbf{X} = (x_1, x_2, \ldots, x_m)$ with $x_i \in \mathbb{R}^n$ for all i. There are m length-n column vectors $\{x_i\}_i$

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- Thus, $r(V)$ is the rank of the matrix $\mathbf{X}$.

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Example: Rank function of a matrix

Consider the following 4×8 matrix, so $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

$$\begin{array}{c c} & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 2 & 2 & 3 & 0 & 1 & 3 & 1 \\ 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} | & | & | & | & | & | & | & | \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ | & | & | & | & | & | & | & | \end{pmatrix} \end{array}$$

- Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{6, 7\}$, $A_r = \{1\}$, $B_r = \{5\}$.
- Then $r(A) = 3$, $r(B) = 3$, $r(C) = 2$.
- $r(A \cup C) = 3$, $r(B \cup C) = 3$.
- $r(A \cup A_r) = 3$, $r(B \cup B_r) = 3$, $r(A \cup B_r) = 4$, $r(B \cup A_r) = 4$.
- $r(A \cup B) = 4$, $r(A \cap B) = 1 < r(C) = 2$.

Example: Rank function of a matrix

Consider the following 4×8 matrix, so $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 0 & 2 & 2 & 3 & 0 & 1 & 3 & 1 \\ 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} | & | & | & | & | & | & | & | \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ | & | & | & | & | & | & | & | \end{pmatrix} \end{array}$$

- Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{6, 7\}$, $A_r = \{1\}$, $B_r = \{5\}$.
- Then $r(A) = 3$, $r(B) = 3$, $r(C) = 2$.
- $r(A \cup C) = 3$, $r(B \cup C) = 3$.
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$$\begin{array}{c c} & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 2 & 2 & 3 & 0 & 1 & 3 & 1 \\ 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} | & | & | & | & | & | & | & | \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ | & | & | & | & | & | & | & | \end{pmatrix} \end{array}$$

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Consider the following 4×8 matrix, so $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

$$\begin{array}{c c}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 2 & 2 & 3 & 0 & 1 & 3 & 1 \\ 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}
\end{array} = \begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{pmatrix} | & | & | & | & | & | & | & | \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ | & | & | & | & | & | & | & | \end{pmatrix}
\end{array}$$

- Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{6, 7\}$, $A_r = \{1\}$, $B_r = \{5\}$.
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Example: Rank function of a matrix

Consider the following 4×8 matrix, so $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

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- $r(A \cup B) = 4$, $r(A \cap B) = 1 < r(C) = 2$.
- $6 = r(A) + r(B) > r(A \cup B) + r(A \cap B) = 5$

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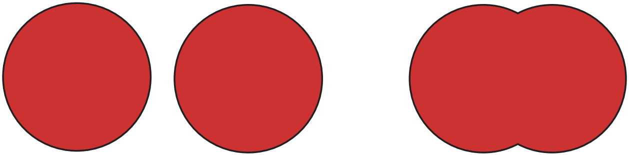

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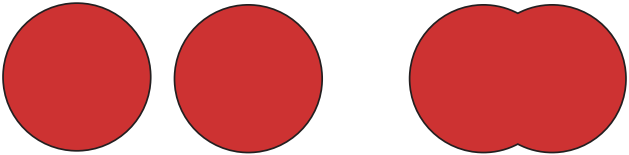

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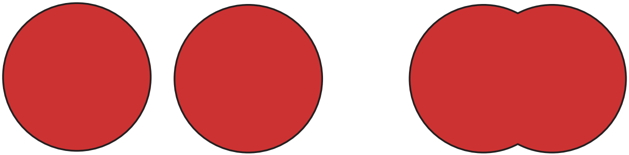

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- Any function where the above inequality is true for all $A, B \subseteq V$ is called subadditive.

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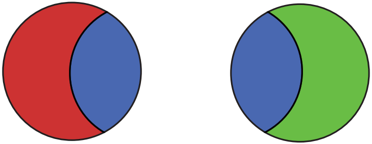

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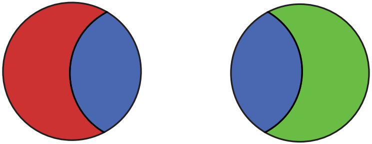

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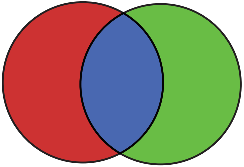

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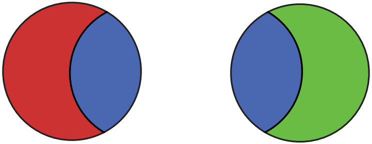

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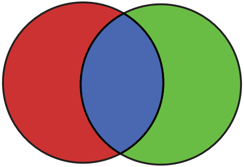

- Thus, we have subadditivity: $r(A) + r(B) \geq r(A \cup B)$. Can we add more to the r.h.s. and still have an inequality? Yes.

Rank function of a matrix

- Note, $r(A \cap B) \leq r(C)$. Why? Vectors indexed by $A \cap B$ (i.e., the common index set) span no more than the dimensions commonly spanned by A and B (namely, those spanned by the professed C).

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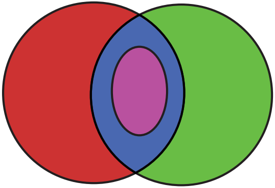

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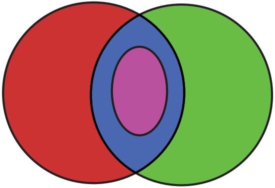

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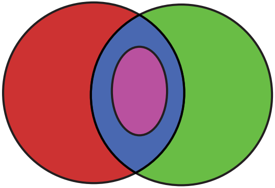

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- More generally, common information (blue) is "more" (no less) than information within common index (magenta).

The Venn and Art of Submodularity

$$\underbrace{r(A) + r(B)}_{= r(A_r) + 2r(C) + r(B_r)} \geq \underbrace{r(A \cup B)}_{= r(A_r) + r(C) + r(B_r)} + \underbrace{r(A \cap B)}_{= r(A \cap B)}$$