Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 19 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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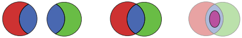
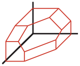
June 4th, 2014

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

$= f(A_r) + 2f(C) + f(B_r)$ $\quad = f(A_r) + f(C) + f(B_r)$ $\quad = f(A \cap B)$





Cumulative Outstanding Reading

- Good references for today: Schrijver-2003, Oxley-1992/2011, Welsh-1973, Goemans-2010, Cunningham-1984, Edmonds-1969, Choquet-1955, Grabisch/Marichal/Mesiar/Pap "Aggregation Functions", Lovász-1983, Bach-2011.
- Read Tom McCormick's overview paper on SFM http://people.commerce.ubc.ca/faculty/mccormick/sfmchap8a.pdf
- Read chapters 1 - 4 from Fujishige book.
- Matroid properties http://www-math.mit.edu/~goemans/18433S09/matroid-notes.pdf
- Read lecture 14 slides on lattice theory at our web page (http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/)
- Wolfe "Finding the Nearest Point in a Polytope", 1976.
- Fujishige & Isotani, "A Submodular Function Minimization Algorithm Based on the Minimum-Norm Base", 2009.

Sources for Today's Lecture

- "Submodular Function Maximization", Krause and Golovin.
- Chekuri, Vondrak, Zenklusen, "Submodular Function Maximization via the Multilinear Relaxation and Contention Resolution Schemes", 2011 (a recent paper (appeared yesterday) that, among other things, has a nice up-to-date summary on all the results on submodular max).
- Minoux, "Accelerated Greedy Algorithms for Maximizing Submodular Set Functions", 1977.
- Feige, Mirrokni, Vondrak, "Maximizing non-monotone submodular functions", 2007.
- Fujishige, "Submodular Functions and Optimization", 2005.
- Fujishige, "Submodular Systems and Related Topics", 1984.
- Fisher, Nemhauser, Wolsey, "An Analysis of Approximations for Maximizing Submodular Set Functions - II", 1978.
- Lin & Bilmes, "A Class Of Submodular Functions for Document Summarization", 2011.

Other readings

- J. Vondrak, "Submodularity and curvature: the optimal algorithm" in RIMS Kokyuroku Bessatsu B23, Kyoto, 2010.
- M. Conforti and G. Cornuéjols. Submodular set functions, matroids and the greedy algorithm: tight worst-case bounds and some generalizations of the Rado-Edmonds theorem. Discrete Applied Math, 7(3):251-274, 1984.

Announcements, Assignments, and Reminders

- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, Combinatorial Geometries
- L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
- L9: From Matroid Polytopes to Polymatroids.
- L10: Polymatroids and Submodularity
- L11: More properties of polymatroids, SFM special cases
- L12: polymatroid properties, extreme points polymatroids,
- L13: sat, dep, supp, exchange capacity, examples
- L14: Lattice theory: partially ordered sets; lattices; distributive, modular, submodular, and boolean lattices; ideals and join irreducibles.
- L15: Supp, Base polytope, polymatroids and entropic Venn diagrams, exchange capacity,
- L16: proof that minimum norm point yields min of submodular function, and the lattice of minimizers of a submodular function, Lovasz extension
- L17: Lovasz extension, Choquet Integration, more properties/examples of Lovasz extension, convex minimization and SFM.
- L18: Lovasz extension examples and structured convex norms, The Min-Norm Point Algorithm detailed.
- L19: symmetric submodular function minimization, maximizing monotone submodular function w. card and other constraints.
- L20:

Finals Week: June 9th-13th, 2014.

Symmetric Submodular Functions

- Given: $\breve{f} : 2^E \to \mathbb{R}$, if $\breve{f}$ is submodular and also has the property that $\breve{f}(A) = \breve{f}(E \setminus A)$ for all A, then $\breve{f}$ is said to be symmetric submodular

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- *Symmetrize* and *normalize* f as $f \to \breve{f}$ via the operation: $\breve{f}(A) = f(A) + f(E \setminus A) - f(E)$, so that $\breve{f}(\emptyset) = 0$ if $f(\emptyset) = 0$.

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- Such an $\breve{f}$ is also non-negative since

$$2\breve{f}(A) = \breve{f}(A) + \breve{f}(E \setminus A) \geq \breve{f}(\emptyset) + \breve{f}(E) = 2\breve{f}(\emptyset) \geq 0 \tag{19.1}$$

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- Combinatorial mutual information function, so $\breve{f}(A) = I_f(A; V \setminus A)$ where $I_f(A; B) = f(A) + f(B) - f(A \cup B) - f(A \cap B)$.

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- Example: $f(A) = H(X_A) =$ entropy, then $\breve{f} = I(X_A; X_{E \setminus A}) =$ symmetric mutual information.

Separators of submodular function via symmetrized version

- Such a symmetrized submodular function measures a form of "dependence" between A and $\bar{A} \triangleq E \setminus A$.

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We are given an f that is normalized & submodular. If $\exists A$ s.t. $\breve{f}(A) \triangleq f(A) + f(\bar{A}) - f(E) = 0$ then f is "decomposable" w.r.t. A — this means $f(B) = f(B \cap A) + f(B \cap \bar{A}), \forall B$.

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Proof.

- By submodularity (subadditivity for non-intersecting sets), we have:

$$f(B) = f\Big((B \cap A) \cup (B \cap \bar{A})\Big) \leq f(B \cap A) + f(B \cap \bar{A}) \tag{19.2}$$

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- Hence, $f(B) \leq f(B \cap A) + f(B \cap \bar{A})$.

...

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. . . proof of Theorem 19.3.1 cont.

□

Separators of submodular function via symmetrized version

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- By submodularity

$f(B) - f(B \cap A) - f(B \cap \bar{A})$

$\square$

Separators of submodular function via symmetrized version

. . . proof of Theorem 19.3.1 cont.

- By submodularity

$$f(B) - f(B \cap A) - f(B \cap \bar{A}) \geq f(A \cup B) - f(A) - f(B \cap \bar{A}) \tag{19.3}$$

$\square$

Separators of submodular function via symmetrized version

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$$f(B) - f(B \cap A) - f(B \cap \bar{A}) \geq f(A \cup B) - f(A) - f(B \cap \bar{A}) \quad (19.3)$$
$$\geq f((A \cup B) \cup \bar{A}) - f(A) - f(\bar{A}) \quad (19.4)$$

$\square$

Separators of submodular function via symmetrized version

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$$= f(E) - f(A) + f(\bar{A}) = 0 \quad (19.5)$$

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- Eqn. (19.3) follows since $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$,

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- Eqn. (19.3) follows since $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$, and Eqn. (19.4) follows since $B \cap \bar{A} = (A \cup B) \cap \bar{A}$ and $f(A \cup B) + f(\bar{A}) \geq f((A \cup B) \cup \bar{A}) + f((A \cup B) \cap \bar{A})$.

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$$f(B) - f(B \cap A) - f(B \cap \bar{A}) \geq f(A \cup B) - f(A) - f(B \cap \bar{A}) \tag{19.3}$$
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- Eqn. (19.3) follows since $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$, and Eqn. (19.4) follows since $B \cap \bar{A} = (A \cup B) \cap \bar{A}$ and $f(A \cup B) + f(\bar{A}) \geq f((A \cup B) \cup \bar{A}) + f((A \cup B) \cap \bar{A})$.
- Hence, both $f(B) \geq f(B \cap A) + f(B \cap \bar{A})$ (from above) and $f(B) \leq f(B \cap A) + f(B \cap \bar{A})$ (previous slide).

□

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- In following slides, $\breve{f}$ is symmetrized & normalized version of f.

Review

Next slide is from Lecture 4.

Many (Equivalent) Definitions of Submodularity

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B), \quad \forall A, B \subseteq V \tag{19.6}$$

$$f(j|S) \geq f(j|T), \; \forall S \subseteq T \subseteq V, \text{ with } j \in V \setminus T \tag{19.7}$$

$$f(C|S) \geq f(C|T), \forall S \subseteq T \subseteq V, \text{ with } C \subseteq V \setminus T \tag{19.8}$$

$$f(j|S) \geq f(j|S \cup \{k\}), \; \forall S \subseteq V \text{ with } j \in V \setminus (S \cup \{k\}) \tag{19.9}$$

$$f(A \cup B|A \cap B) \leq f(A|A \cap B) + f(B|A \cap B), \quad \forall A, B \subseteq V \tag{19.10}$$

$$f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S) - \sum_{j \in S \setminus T} f(j|S \cup T - \{j\}), \; \forall S, T \subseteq V \tag{19.11}$$

$$f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S), \; \forall S \subseteq T \subseteq V \tag{19.12}$$

$$f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|S \setminus \{j\}) + \sum_{j \in T \setminus S} f(j|S \cap T) \; \forall S, T \subseteq V \tag{19.13}$$

$$f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|S \setminus \{j\}), \; \forall T \subseteq S \subseteq V \tag{19.14}$$

Minimization of a Symmetric Submodular Functions

- Minimizing symmetric submodular functions can be done in strongly polynomial time $O(n^3)$. The algorithm by Nagamochi & Ibaracki 1992 for graph cuts shown by Queyranne in 1995 to work for sym. SFM.

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- The algorithm finds (as a subroutine) MA (maximum adjacency) or a maximum back orders (not same as greedy order).

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Choose v_1 arbitrarily ;
W_1 ← (v_1)                  /* The first of an ordered list W_i. */ ;
for i ← 1 ... |V| − 1 do
    Choose v_{i+1} ∈ argmin_{u∈V\W_i} f(W_i|{u}) ;
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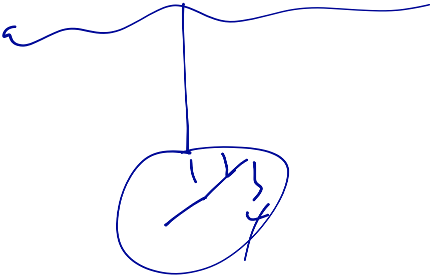
- Note algorithm operates on non-symmetric function f. If f is already symmetric and normalized, then $f = \breve{f}$.
- The final ordered set $W_n = (v_1, v_2, \dots, v_n)$ is special in that the last two nodes (v_{n-1}, v_n) serve as a surrogate minimizer for a special case.

Pendent pair

- A ordered pair of elements (t, u) is called a pendent pair if u is a minimizer amongst all sets that separate u and t.



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- That is,

$$\breve{f}(\{u\}) \leq \breve{f}(A) \ \ \forall A \text{ s.t. } t \notin A \ni u \tag{19.7}$$

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$$\{u\} \in \operatorname{argmin} \left\{ \breve{f}(A) : u \in A, t \notin A \right\} \tag{19.6}$$

- That is,

$$\breve{f}(\{u\}) \leq \breve{f}(A) \;\; \forall A \text{ s.t. } t \notin A \ni u \tag{19.7}$$

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- Interestingly, this algorithm is the same as maximum cardinality search (MCS), when f represents a graph cut function (recall, MCS is used to efficiently test graph chordality).

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- We store the score (min value) in the first case, then, consider a new element "tu" and clustered ground set $V' = V \setminus \{t, u\} \cup \{tu\}$, and new symmetric submodular function $f' : 2^{V'} \to \mathbb{R}$ with

$$\breve{f}'(X) = \begin{cases} \breve{f}(X) & \text{if } tu \notin X \\ \breve{f}(X \cup \{t, u\} \setminus \{tu\}) & \text{if } tu \in X \end{cases} \tag{19.8}$$

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- We do this n times. We take the min over all of the stored values.
- The pendent pair corresponding to the min element, say (t', u') will (most probability) correspond to nested clusters, so we use the original ground elements corresponding to u'.

Minimization of a Symmetric Submodular Functions

Theorem 19.3.3

The final resultant u' when expanded to original ground elements minimizes the symmetric submodular function f in $O(n^3)$ time.

- This has become known as Queyranne's algorithm for symmetric submodular function minimization.
- This was done in 1995 and it is said that this result, at that time, rekindled the efforts to find general combinatorial SFM.
- The actual algorithm was originally developed by Nagamochi and Ibaraki for a simple algorithm for finding graph cut. Queyranne showed it worked for any symmetric submodular function.
- Hence, it seems reasonable that symmetric SFM is faster than general SFM (although this question is still unknown).
- Quoting Fujishige from NIPS 2012, he said that he "hopes general purpose SFM is $O(n^4)$" ☺.

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- Thus, when we do monotone submodular maximization, we either
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  - Do both.
- There is also a sort of dual problem that is often considered together with max, and those are minimum cover problems (to be defined).

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- The set cover problem asks for the smallest subset X of V such that $f(X) = |E|$ (smallest subset of the subsets of E) where E is still covered. I.e.,

$$\text{minimize} |X| \text{ subject to } f(X) \geq |E| \tag{19.9}$$

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- This problem is NP-hard, and Feige in 1998 showed that it cannot be approximated with a ratio better than $(1 - \epsilon) \log n$ unless NP is slightly superpolynomial ($n^{O(\log \log n)}$).

What About Non-monotone

- So even simple case of cardinality constrained submodular function maximization is NP-hard.
- This will be true of most submodular max (and related) problems.
- Hence, the only hope is approximation algorithms. Question is, what is the tradeoff between running time and approximation quality, and is it possible to get tight bounds (i.e., an algorithm that achieves an approximation ratio, and a proof that one can't do better than that unless some extremely unlike event were to be true, such as P=NP).

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- The max k cover problem asks, given a k, what sized k set of sets X can we choose that covers the most? I.e., that maximizes $f(X)$ as in:

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- An important result by Nemhauser et. al. (1978) states that for normalized ($f(\emptyset) = 0$) monotone submodular functions (i.e., polymatroids) can be approximately maximized using a simple greedy algorithm.
- Starting with $S_0 = \emptyset$, we repeat the following greedy step for $i = 0 \dots (k-1)$:

$$S_{i+1} = S_i \cup \left\{ \operatorname*{argmax}_{v \in V \setminus S_i} f(S_i \cup \{v\}) \right\} \tag{19.11}$$

The Greedy Algorithm for Submodular Max

A bit more precisely:

Algorithm 2: The Greedy Algorithm

Set $S_0 \leftarrow \emptyset$;
for $i \leftarrow 0 \dots |E| - 1$ **do**
  Choose v_i as follows:
  $v_i \in \left\{ \operatorname{argmax}_{v \in V \setminus S_i} f(\{v\} | S_i) \right\} = \left\{ \operatorname{argmax}_{v \in V \setminus S_i} f(S_i \cup \{v\}) \right\}$;
  Set $S_{i+1} \leftarrow S_i \cup \{v_i\}$;

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Given a polymatroid function f, the above greedy algorithm returns sets S_i such that for each i we have $f(S_i) \geq (1 - 1/e) \max_{|S| \leq i} f(S)$.

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- Again, since this generalizes max k-cover, Feige (1998) showed that this can't be improved. Unless $P = NP$, no polynomial time algorithm can do better than $(1 - 1/e + \epsilon)$ for any $\epsilon > 0$.

The Greedy Algorithm: $1 - 1/e$ intuition.

- At step $i < k$, greedy chooses v_i +1 to maximize $f(v|S_i)$.

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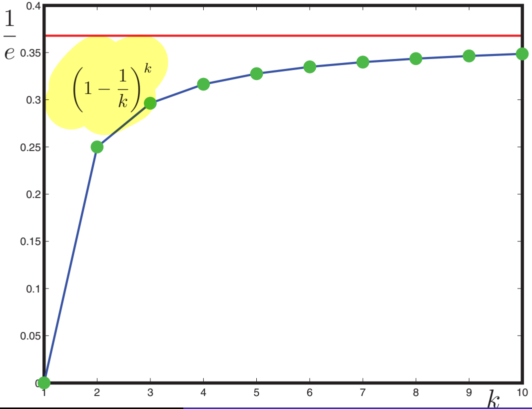

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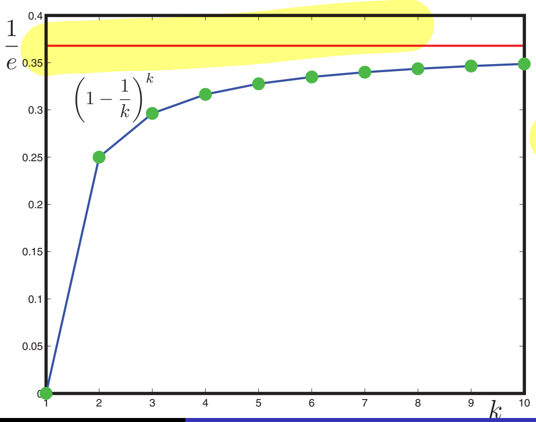

Equation (19.21) will show that: (19.12) $\Rightarrow$

$$\begin{aligned}
&\mathsf{OPT} - f(S_{i+1}) \\
&\quad \leq (1 - 1/k)(\mathsf{OPT} - f(S_i)) \\
\Rightarrow\ &\mathsf{OPT} - f(S_k) \\
&\quad \leq (1 - 1/k)^k \mathsf{OPT} \\
&\quad \leq 1/e\mathsf{OPT} \\
\Rightarrow\ &\mathsf{OPT}(1 - 1/e) \leq f(S_k)
\end{aligned}$$

Cardinality Constrained Polymatroid Max Theorem

Theorem 19.4.2 (Nemhauser et al. 1978)

Given non-negative monotone submodular function $f : 2^V \to \mathbb{R}_+$, define $\{S_i\}_{i \geq 0}$ to be the chain formed by the greedy algorithm (Eqn. (19.11)). Then for all $k, \ell \in \mathbb{Z}_{++}$, we have:

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- Bound is how well does S_ℓ (of size ℓ) do relative to S^*, the optimal set of size k.

Cardinality Constrained Polymatroid Max Theorem

Theorem 19.4.2 (Nemhauser et al. 1978)

Given non-negative monotone submodular function $f : 2^V \to \mathbb{R}_+$, define $\{S_i\}_{i \geq 0}$ to be the chain formed by the greedy algorithm (Eqn. (19.11)). Then for all $k, \ell \in \mathbb{Z}_{++}$, we have:

$$f(S_\ell) \geq (1 - e^{-\ell/k}) \max_{S:|S| \leq k} f(S) \tag{19.13}$$

and in particular, for $\ell = k$, we have $f(S_k) \geq (1 - 1/e) \max_{S:|S| \leq k} f(S)$.

- k is size of optimal set, i.e., OPT $= f(S^*)$ with $|S^*| = k$
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- Intuitively, bound should get worse when $\ell < k$ and get better when $\ell > k$.

Cardinality Constrained Polymatroid Max Theorem

Proof of Theorem 19.4.2.

...

Cardinality Constrained Polymatroid Max Theorem

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- Order $S^* = (v_1^*, v_2^*, \ldots, v_k^*)$ arbitrarily.

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- w.l.o.g. assume $|S^*| = k$.
- Order $S^* = (v_1^*, v_2^*, \ldots, v_k^*)$ arbitrarily.
- Let $(v_1, v_2, \ldots, v_\ell)$ be the greedy order chosen by the algorithm.

$= S_\ell$

...

Cardinality Constrained Polymatroid Max Theorem

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- Fix ℓ (number of items greedy will chose) and k (size of optimal set to compare against).
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- Order $S^* = (v_1^*, v_2^*, \ldots, v_k^*)$ arbitrarily.
- Let $(v_1, v_2, \ldots, v_k)$ be the greedy order chosen by the algorithm.
- Then the following inequalities (on the next slide) follow:

. . .

Cardinality Constrained Polymatroid Max Theorem

... proof of Theorem 19.4.2 cont.

...

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 19.4.2 cont.

- For all $i < \ell$, we have

$$f(S^*)$$

. . .

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 19.4.2 cont.

- For all $i < \ell$, we have

$$f(S^*) \leq f(S^* \cup S_i) \tag{19.14}$$

. . .

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 19.4.2 cont.

- For all $i < \ell$, we have

$$f(S^*) \leq f(S^* \cup S_i) \tag{19.14}$$

$$= f(S_i) + \sum_{j=1}^{k} f(v_j^* | S_i \cup \{v_1^*, v_2^*, \dots, v_{j-1}^*\}) \tag{19.15}$$

. . .

Cardinality Constrained Polymatroid Max Theorem

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$$\leq f(S_i) + \sum_{v \in S^*} f(v | S_i) \tag{19.16}$$

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$$\leq f(S_i) + \sum_{v \in S^*} f(v_{i+1}|S_i)$$

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Cardinality Constrained Polymatroid Max Theorem

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$$\leq f(S_i) + \sum_{v \in S^*} f(v_{i+1}|S_i) = f(S_i) + \sum_{v \in S^*} f(S_{i+1}|S_i) \tag{19.17}$$

...

Cardinality Constrained Polymatroid Max Theorem

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$$= f(S_i) + kf(S_{i+1}|S_i) \tag{19.18}$$

. . .

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$$= f(S_i) + k f(S_{i+1} | S_i) \tag{19.18}$$

- Therefore, we have Equation 19.12, i.e.,:

$$f(S^*) - f(S_i) \leq k f(S_{i+1} | S_i) = k(f(S_{i+1}) - f(S_i)) \tag{19.19}$$

δ_i $\leq k(\delta_i - \delta_{i+1})$

. . .

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 19.4.2 cont.

Cardinality Constrained Polymatroid Max Theorem

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Cardinality Constrained Polymatroid Max Theorem

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- Define $\delta_i \triangleq f(S^*) - f(S_i)$, so $\delta_i - \delta_{i+1} = f(S_{i+1}) - f(S_i)$, giving

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Cardinality Constrained Polymatroid Max Theorem

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- The relationship between δ_0 and δ_ℓ is then

$$\delta_l \leq (1 - \frac{1}{k})^\ell \delta_0 \tag{19.22}$$

Cardinality Constrained Polymatroid Max Theorem

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$$\delta_l \leq (1 - \frac{1}{k})^\ell \delta_0 \tag{19.22}$$
- Now, $\delta_0 = f(S^*) - f(\emptyset) \leq f(S^*)$ since $f \geq 0$.

Cardinality Constrained Polymatroid Max Theorem

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$$\delta_l \leq (1 - \frac{1}{k})^\ell \delta_0 \tag{19.22}$$
$x = 1/k$
- Now, $\delta_0 = f(S^*) - f(\emptyset) \leq f(S^*)$ since $f \geq 0$.
- Also, by variational bound $1 - x \leq e^{-x}$ for $x \in \mathbb{R}$, we have
$$\delta_\ell \leq (1 - \frac{1}{k})^\ell \delta_0 \leq e^{-\ell/k} f(S^*) \tag{19.23}$$

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 19.4.2 cont.

□

Cardinality Constrained Polymatroid Max Theorem

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- When we identify $\delta_l = f(S^*) - f(S_\ell)$, a bit of rearranging then gives:

$$f(S_\ell) \geq (1 - e^{-\ell/k}) f(S^*) \tag{19.24}$$

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$\square$

- With $\ell = k$, when picking k items, greedy gets $(1 - 1/e) \approx 0.6321$ bound. This means that if S_k is greedy solution of size k, and S^* is an optimal solution of size k, $f(S_k) \geq (1 - 1/e) f(S^*) \approx 0.6321 f(S^*)$.

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- What if we want to guarantee a solution no worse than $.95 f(S^*)$ where $|S^*| = k$?

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- What if we want to guarantee a solution no worse than $.95 f(S^*)$ where $|S^*| = k$? Set $0.95 = (1 - e^{-\ell/k})$, which gives $\ell = \lceil -k \ln(1 - 0.95) \rceil = 4k$.

Cardinality Constrained Polymatroid Max Theorem

... proof of Theorem 19.4.2 cont.

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- What if we want to guarantee a solution no worse than $.95 f(S^*)$ where $|S^*| = k$? Set $0.95 = (1 - e^{-\ell/k})$, which gives $\ell = \lceil -k \ln(1 - 0.95) \rceil = 4k$. And $\lceil - \ln(1 - 0.999) \rceil = 7$.
- So solution, in the worst case, quickly gets very good. Typical/practical case is much better.

Greedy running time

- Greedy computes a new maximum $n = |V|$ times, and each maximum computation requires $O(n)$ comparisons, leading to $O(n^2)$ computation for greedy.

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- This is called Minoux's 1977 Accelerated Greedy strategy (and has been rediscovered a few times, e.g., "Lazy greedy"), and runs much faster (typically $n \log n$) while still producing same answer.

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- This is called Minoux's 1977 Accelerated Greedy strategy (and has been rediscovered a few times, e.g., "Lazy greedy"), and runs much faster (typically $n \log n$) while still producing same answer.
- We describe it next:

Minoux's Accelerated Greedy for Submodular Functions

- At stage i in the algorithm, we have a set of gains $f(v|S_i)$ for all $v \notin S_i$. Store these values $\alpha_v \leftarrow f(v|S_i)$ in sorted priority queue.

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- For $v \notin S_{i+1}$ we have $f(v|S_{i+1}) \leq f(v|S_i)$ by submodularity.

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- Priority queue, $O(1)$ to find max, $O(\log n)$ to insert in right place.
- Once we choose a max v, then set $S_{i+1} \leftarrow S_i + v$.
- For $v \notin S_{i+1}$ we have $f(v|S_{i+1}) \leq f(v|S_i)$ by submodularity.
- Therefore, if we find a v' such that $f(v'|S_{i+1}) \geq \alpha_v$ for all $v \neq v'$, then since

$$f(v'|S_{i+1}) \geq \alpha_v = f(v|S_i) \geq f(v|S_{i+1}) \tag{19.25}$$

we have the true max, and we need not re-evaluate gains of other elements again.

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- Therefore, if we find a v' such that $f(v'|S_{i+1}) \geq \alpha_v$ for all $v \neq v'$, then since

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  we have the true max, and we need not re-evaluate gains of other elements again.
- Strategy is: find the $\operatorname{argmax}_{v' \in V \setminus S_{i+1}} \alpha_{v'}$, and then compute the real $f(v'|S_{i+1})$. If it is greater than all other α_v's then that's the next greedy step. Otherwise, replace $\alpha_{v'}$ with its real value, resort, and repeat.

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- Minoux's algorithm is exact, in that it has the same guarantees as does the $O(n^2)$ greedy Algorithm 2 (this means it will return either the same answers, or answers that have the $1 - 1/e$ guarantee).

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- Algorithm has been rediscovered (I think) independently (CELF - cost-effective lazy forward selection, Leskovec et al., 2007)
- Can be used used for "big data" sets (e.g., social networks, selecting blogs of greatest influence, document summarization, etc.).

Priority Queue

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  - Query the value of the max item in the queue
  $$\text{MAX}(Q) \in \mathbb{R} \tag{19.28}$$
- On next slide, we call a popped item "fresh" if the value (v, α) popped has the correct value $\alpha = f(v|S_i)$. Use extra "bit" to store this info

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  - Pop the item (v, α) with maximum value α off the queue.

$$(v, \alpha) \leftarrow \text{POP}(Q) \tag{19.27}$$

  - Query the value of the max item in the queue

$$\text{MAX}(Q) \in \mathbb{R} \tag{19.28}$$

- On next slide, we call a popped item "fresh" if the value (v, α) popped has the correct value $\alpha = f(v|S_i)$. Use extra "bit" to store this info
- If a popped item is fresh, it must be the maximum — this can happen if, at given iteration, v was first popped and neither fresh nor maximum so placed back in the queue, and it then percolates back to the top at which point it is fresh — thereby avoid extra queue check.

Minoux's Accelerated Greedy Algorithm Submodular Max

Algorithm 3: Minoux's Accelerated Greedy Algorithm

1 Set $S_0 \leftarrow \emptyset$; $i \leftarrow 0$; Initialize priority queue Q ;
2 **for** $v \in E$ **do**
3     $\text{INSERT}(Q, f(v))$
4 **repeat**
5     $(v, \alpha) \leftarrow \text{POP}(Q)$;
6   **if** α *not "fresh"* **then**
7   recompute $\alpha \leftarrow f(v|S_i)$
8   **if** *(popped α in line 5 was "fresh") OR ($\alpha \geq \text{MAX}(Q)$)* **then**
9   Set $S_{i+1} \leftarrow S_i \cup \{v\}$;
10         $i \leftarrow i + 1$;
11   **else**
12         $\text{INSERT}(Q, (v, \alpha))$
13 **until** $i = |E|$;

Minimum Submodular Cover

- Given polymatroid f, goal is to find a covering set of minimum cost:

$$S^* \in \operatorname*{argmin}_{S \subseteq V} |S| \text{ such that } f(S) \geq \alpha \tag{19.29}$$

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- For integer valued f, this greedy algorithm an $O(\log(\max_{s \in V} f(\{s\})))$ approximation. Set cover is hard to approximate with a factor better than $(1 - \epsilon) \log \alpha$, where α is the desired cover constraint.

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Generalizations

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- Combinations of the above (e.g., knapsack & multiple matroid constraints).

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- For one matroid, we have a 1/2 approximation.
- Very easy algorithm, Minoux trick still possible, while addresses multiple matroid constraints — but the bound is not that good when there are many matroids.

Review

- Recall bipartite matching via matroid intersection from Lecture 16.

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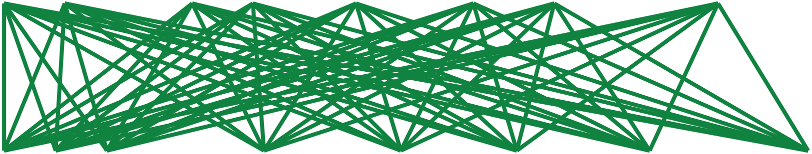
- Consider bipartite graph $G = (E, F, V)$ where E and F are the left/right set of nodes, respectively, and V is the set of edges.
- E corresponds to, say, an English language sentence and F corresponds to a French language sentence — goal is to form a matching (an alignment) between the two.

Greedy over > 1 matroids: Multiple Language Alignment

- Consider English string and French string, set up as a bipartite graph.

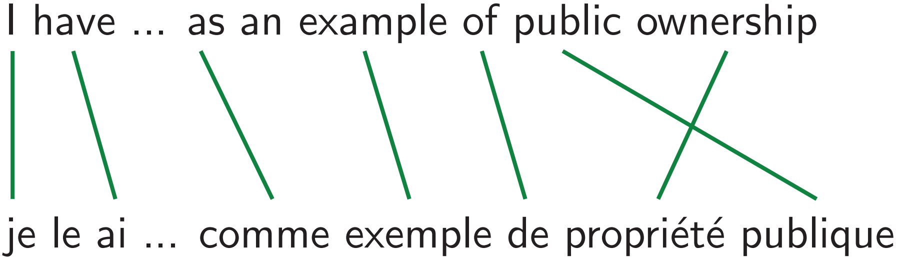
I have ... as an example of public ownership



je le ai ... comme exemple de propriété publique

Greedy over > 1 matroids: Multiple Language Alignment

- One possible alignment, a matching, with score as sum of edge weights.



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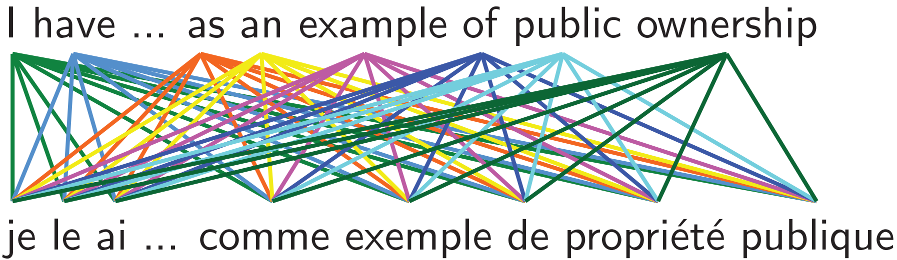
- Edges incident to English words constitute an edge partition



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- For each matroid, a set of edges is independent only if the set intersects each partition block no more than one time.

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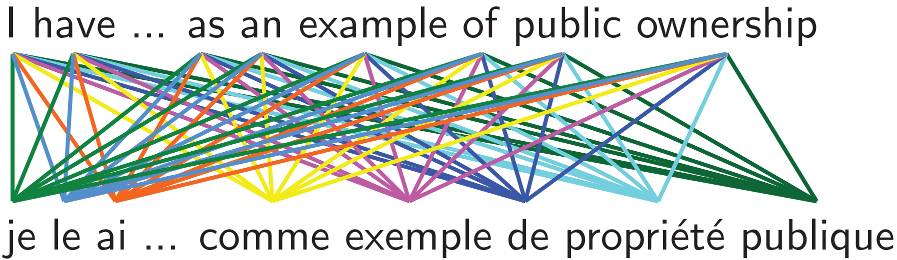

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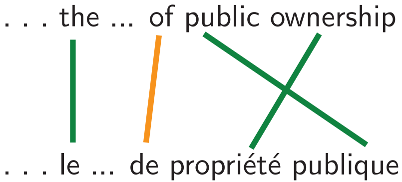

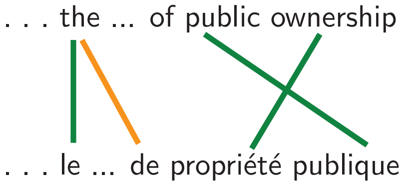

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- We can solve this via submodular maximization subject to multiple matroid independence constraints as we next describe ...

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- For any $e \in E$, the corresponding element in $E^{(i)}$ is called $(e, i) \in E^{(i)}$ (each original element is tagged by integer).
- For $e \in E$, define $E_e = \{(e', i) \in E' : e' = e\}$.
- Hence, $\{E_e\}_{e \in E}$ is a partition of E', each block of the partition for one of the original elements in E.
- Create a 1-partition matroid $\mathcal{M} = (E', \mathcal{I})$ where

$$\mathcal{I} = \left\{ S \subseteq E' : \forall e \in E, |S \cap E_e| \le 1 \right\} \tag{19.34}$$

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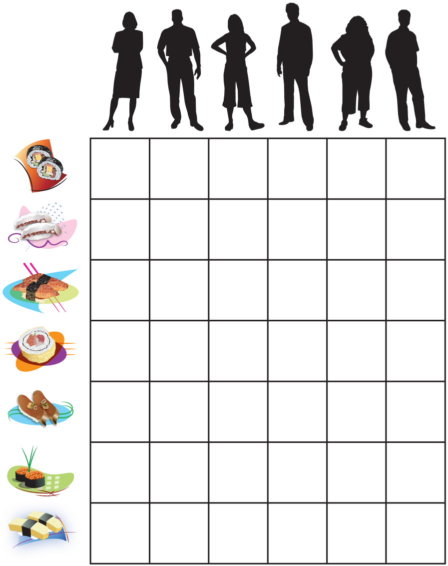
- Submodular welfare maximization becomes matroid constrained submodular max $\max\{f'(S) : S \in \mathcal{I}\}$, so greedy algorithm gives a $1/2$ approximation.

Submodular Social Welfare



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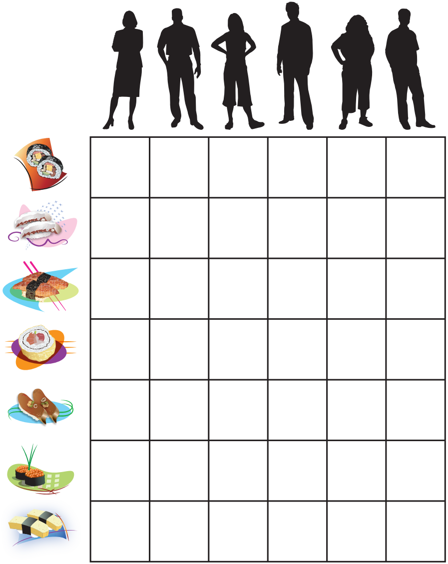

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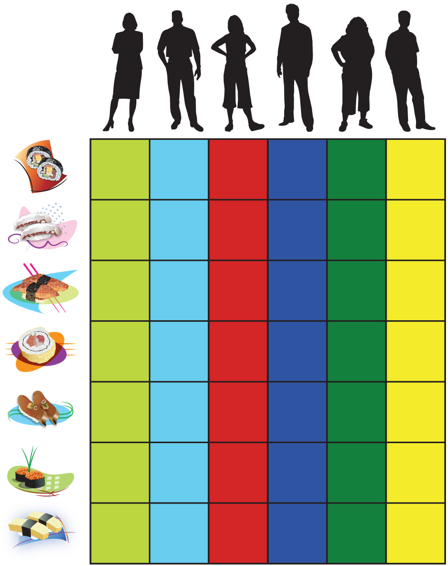

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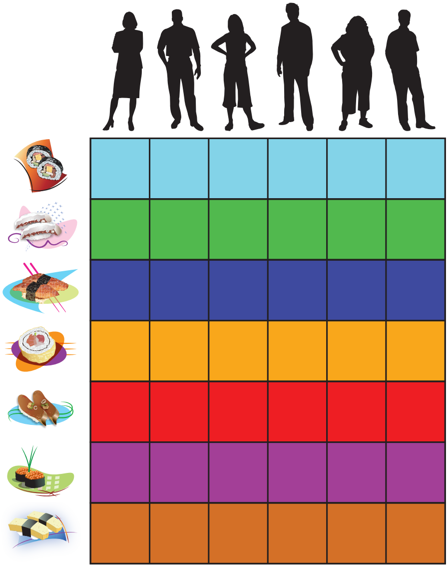

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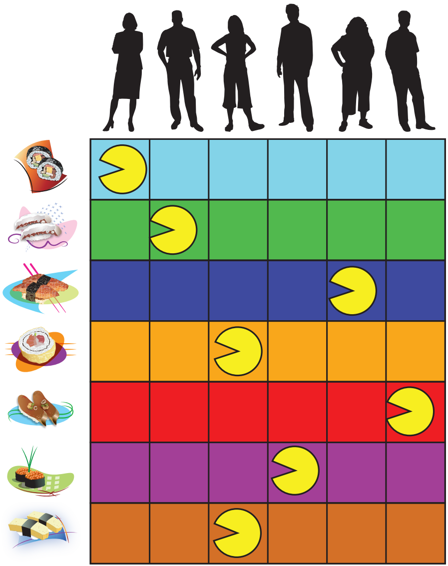

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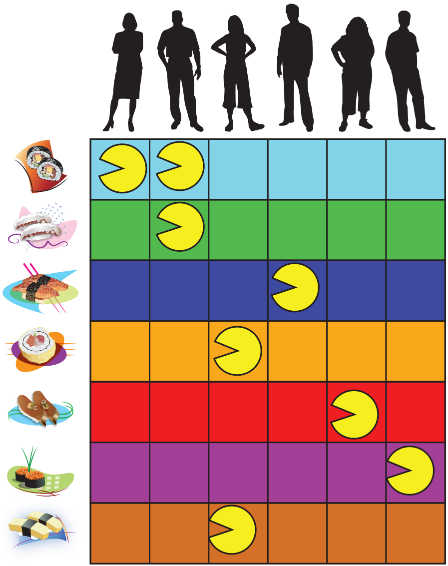

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- Important: A knapsack constraint yields an independence system (down closed) but it is not a matroid!
- $c(e)$ may be seen as the cost of item e and if $c(e) = 1$ for all e, then we recover the cardinality constraint we saw earlier.

Monotone Submodular over Knapsack Constraint

- Greedy can be seen as choosing the best gain: Starting with $S_0 = \emptyset$, we repeat the following greedy step

$$S_{i+1} = S_i \cup \left\{ \underset{v \in V \setminus S_i}{\operatorname{argmax}} \Big(f(S_i \cup \{v\}) - f(S_i) \Big) \right\} \tag{19.35}$$

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- Core idea in knapsack case: Greedy can be extended to choose next whatever looks cost-normalized best, i.e., Starting some initial set S_0, we repeat the following cost-normalized greedy step

$$S_{i+1} = S_i \cup \left\{ \operatorname*{argmax}_{v \in V \setminus S_i} \frac{f(S_i \cup \{v\}) - f(S_i)}{c(v)} \right\} \tag{19.36}$$

which we repeat until $c(S_{i+1}) > b$ and then take S_i as the solution.

A Knapsack Constraint

- There are a number of ways of getting approximation bounds using this strategy.
- If we run the normalized greedy procedure starting with $S_0 = \emptyset$, and compare the solution found with the max of the singletons $\max_{v \in V} f(\{v\})$, choosing the max, then we get a $(1 - e^{-1/2}) \approx 0.39$ approximation, in $O(n^2)$ time (Minoux trick also possible for further speed)
- Partial enumeration: On the other hand, we can get a $(1 - e^{-1}) \approx 0.63$ approximation in $O(n^5)$ time if we run the above procedure starting from all sets of cardinality three (so restart for all S_0 such that $|S_0| = 3$), and compare that with the best singleton and pairwise solution.
- Extending something similar to this to d simultaneous knapsack constraints is possible as well.

Local Search Algorithms

From J. Vondrak

- Local search involves switching up to t elements, as long as it provides a (non-trivial) improvement; can iterate in several phases. Some examples follow:
- $1/3$ approximation to unconstrained non-monotone maximization [Feige, Mirrokni, Vondrak, 2007]
- $1/(k+2+\frac{1}{k}+\delta_t)$ approximation for non-monotone maximization subject to k matroids [Lee, Mirrokni, Nagarajan, Sviridenko, 2009]
- $1/(k+\delta_t)$ approximation for monotone submodular maximization subject to $k \geq 2$ matroids [Lee, Sviridenko, Vondrak, 2010].

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- Thus, any approximation algorithm must be for unipolar submodular functions. E.g., non-negative but otherwise arbitrary submodular functions.
- We may get a $(\frac{1}{3} - \frac{\epsilon}{n})$ approximation for maximizing non-monotone non-negative submodular functions, with most $O(\frac{1}{\epsilon}n^3 \log n)$ function calls using approximate local maxima.

Submodularity and local optima

- Given any submodular function f, a set $S \subseteq V$ is a local maximum of f if $f(S - v) \leq f(S)$ for all $v \in S$ and $f(S + v) \leq f(S)$ for all $v \in V \setminus S$.

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- Similarly, given $v_1, v_2 \notin S$, and $f(S + v_1) \leq f(S)$ and $f(S + v_2) \leq f(S)$. Submodularity requires $f(S + v_1) + f(S + v_2) \geq f(S) + f(S + v_1 + v_2)$ which requires $f(S + v_1 + v_2) \leq f(S)$.

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- This is the approach that yields the $(\frac{1}{3} - \frac{\epsilon}{n})$ approximation algorithm.

Linear time algorithm unconstrained non-monotone max

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Algorithm 7: Randomized Linear-time non-monotone submodular max

```
Set L ← ∅ ; U ← V      /* Lower L, upper U. Invariant: L ⊆ U */ ;
Order elements of V = (v_1, v_2, ..., v_n) arbitrarily ;
for i ← 0 ... |V| do
    a ← [f(v_i|L)]_+; b ← [−f(U|U \ {v_i})]_+ ;
    if a = b = 0 then p ← 1/2 ;
    ;
    else p ← a/(a + b);
    ;
    if Flip of coin with Pr(heads) = p draws heads then
        L ← L ∪ {v_i} ;
    Otherwise /* if the coin drew tails, an event with prob. 1 − p */
        U ← U \ {v}
```

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- It may be possible to choose the random order smartly to get better results in practice.

More general still: multiple constraints different types

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- Often the computational costs of the algorithms are prohibitive (e.g., exponential in k) with large constants, so these algorithms might not scale.
- On the other hand, these algorithms offer deep and interesting intuition into submodular functions, beyond what we have covered here.

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- More general constraints are possible too, as we see on the next table (for references, see Jan Vondrak's publications `http://theory.stanford.edu/~jvondrak/`).

Submodular Max Summary - 2012: From J. Vondrak

Monotone Maximization

Constraint	Approximation	Hardness	Technique
$\lvert S\rvert \leq k$	$1-1/e$	$1-1/e$	greedy
matroid	$1-1/e$	$1-1/e$	multilinear ext.
$O(1)$ knapsacks	$1-1/e$	$1-1/e$	multilinear ext.
k matroids	$k+\epsilon$	$k/\log k$	local search
k matroids and $O(1)$ knapsacks	$O(k)$	$k/\log k$	multilinear ext.

Nonmonotone Maximization

Constraint	Approximation	Hardness	Technique
Unconstrained	$1/2$	$1/2$	combinatorial
matroid	$1/e$	0.48	multilinear ext.
$O(1)$ knapsacks	$1/e$	0.49	multilinear ext.
k matroids	$k+O(1)$	$k/\log k$	local search
k matroids and $O(1)$ knapsacks	$O(k)$	$k/\log k$	multilinear ext.

Curvature of a Submodular function

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- The total curvature of a submodular function is defined as follows:

$$c \triangleq 1 - \min_{S, j \notin S : f(j|\emptyset) \neq 0} \frac{f(j|S)}{f(j|\emptyset)} \tag{19.37}$$

Curvature of a Submodular function

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- Polymatroid functions are, in this sense, more nuanced, in that they allow non-extreme curvature, with $c \in [0, 1]$.
- It will be remembered the notion of "partial dependence" within polymatroid functions.

Curvature and approximation

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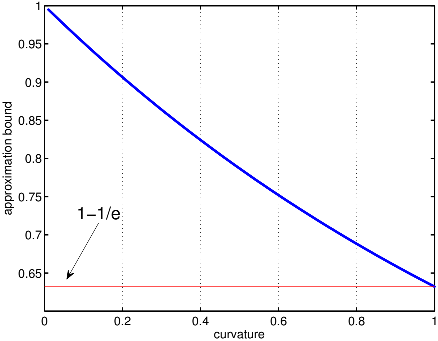
- For k-uniform matroid (i.e., k-cardinality constraints), then approximation factor becomes $\frac{1}{c}(1-e^{-c})$



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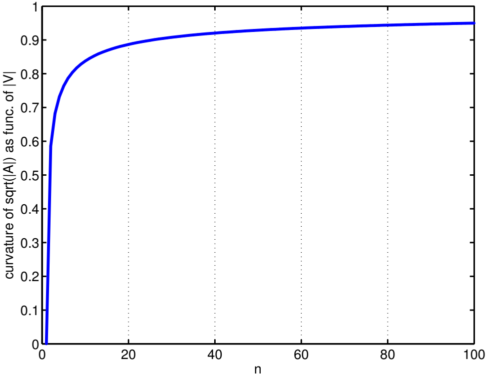

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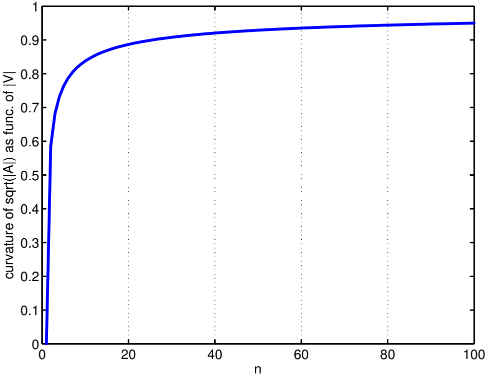

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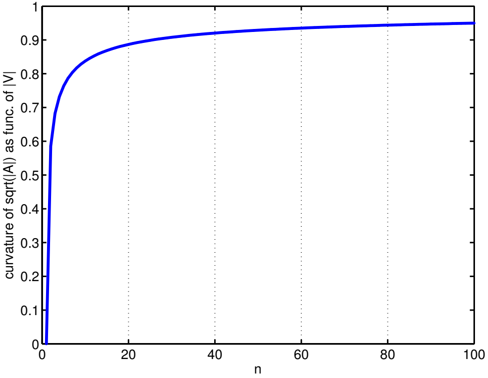

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- Functions of the form $f(S) = \sqrt{m(S)}$ where $m : V \to \mathbb{R}_+$, approximation worse with n if $\min_{i,j} |m(i) - m(j)|$ has a fixed lower bound with increasing n.

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- Most of the approaches for submodular max have not used such an approach, probably due to the difficulty in computing the "concave extension" of a submodular function (the convex extension is easy, namely the Lovász extension).
- A paper by Chekuri, Vondrak, and Zenklusen (2011) make some progress on this front using multilinear extensions.

Multilinear extension

Definition 19.5.3

For a set function $f : 2^V \to \mathbb{R}$, define its multilinear extension $F : [0,1]^V \to \mathbb{R}$ by

$$F(x) = \sum_{S \subseteq V} f(S) \prod_{i \in S} x_i \prod_{j \in V \setminus S} (1 - x_j) \tag{19.39}$$

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- but note, unlike the Lovász extension, this function is neither.

Submodular Max and polyhedral approaches

- Basic idea: Given a set of constraints $\mathcal{I}$, we form a polytope $P_{\mathcal{I}}$ such that $\{\mathbf{1}_I : I \in \mathcal{I}\} \subseteq P_{\mathcal{I}}$
- We find $\max_{x \in P_{\mathcal{I}}} F(x)$ where $F(x)$ is the multi-linear extension of f, to find a fractional solution x^*
- We then round x^* to a point on the hypercube, thus giving us a solution to the discrete problem.

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- In practice, one needs to do Monte-Carlo methods to estimate the multilinear extension (so further approximations would apply).