Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 19 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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University of Washington, Seattle
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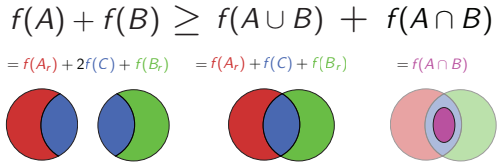
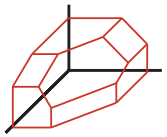
June 4th, 2014





Cumulative Outstanding Reading

- Good references for today: Schrijver-2003, Oxley-1992/2011, Welsh-1973, Goemans-2010, Cunningham-1984, Edmonds-1969, Choquet-1955, Grabisch/Marichal/Mesiar/Pap "Aggregation Functions", Lovász-1983, Bach-2011.
- Read Tom McCormick's overview paper on SFM `http://people.commerce.ubc.ca/faculty/mccormick/sfmchap8a.pdf`
- Read chapters 1 - 4 from Fujishige book.
- Matroid properties `http://www-math.mit.edu/~goemans/18433S09/matroid-notes.pdf`
- Read lecture 14 slides on lattice theory at our web page (`http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/`)
- Wolfe "Finding the Nearest Point in a Polytope", 1976.
- Fujishige & Isotani, "A Submodular Function Minimization Algorithm Based on the Minimum-Norm Base", 2009.

Sources for Today's Lecture

- "Submodular Function Maximization", Krause and Golovin.
- Chekuri, Vondrak, Zenklusen, "Submodular Function Maximization via the Multilinear Relaxation and Contention Resolution Schemes", 2011 (a recent paper (appeared yesterday) that, among other things, has a nice up-to-date summary on all the results on submodular max).
- Minoux, "Accelerated Greedy Algorithms for Maximizing Submodular Set Functions", 1977.
- Feige, Mirrokni, Vondrak, "Maximizing non-monotone submodular functions", 2007.
- Fujishige, "Submodular Functions and Optimization", 2005.
- Fujishige, "Submodular Systems and Related Topics", 1984.
- Fisher, Nemhauser, Wolsey, "An Analysis of Approximations for Maximizing Submodular Set Functions - II", 1978.
- Lin & Bilmes, "A Class Of Submodular Functions for Document Summarization", 2011.

Other readings

- J. Vondrak, "Submodularity and curvature: the optimal algorithm" in RIMS Kokyuroku Bessatsu B23, Kyoto, 2010.
- M. Conforti and G. Cornuéjols. Submodular set functions, matroids and the greedy algorithm: tight worst-case bounds and some generalizations of the Rado-Edmonds theorem. Discrete Applied Math, 7(3):251-274, 1984.

Announcements, Assignments, and Reminders

- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, Combinatorial Geometries
- L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
- L9: From Matroid Polytopes to Polymatroids.
- L10: Polymatroids and Submodularity
- L11: More properties of polymatroids, SFM special cases
- L12: polymatroid properties, extreme points polymatroids,
- L13: sat, dep, supp, exchange capacity, examples
- L14: Lattice theory: partially ordered sets; lattices; distributive, modular, submodular, and boolean lattices; ideals and join irreducibles.
- L15: Supp, Base polytope, polymatroids and entropic Venn diagrams, exchange capacity,
- L16: proof that minimum norm point yields min of submodular function, and the lattice of minimizers of a submodular function, Lovasz extension
- L17: Lovasz extension, Choquet Integration, more properties/examples of Lovasz extension, convex minimization and SFM.
- L18: Lovasz extension examples and structured convex norms, The Min-Norm Point Algorithm detailed.
- L19: symmetric submodular function minimization, maximizing monotone submodular function w. card constraints.
- L20: maximizing monotone submodular function w. other constraints, non-monotone maximization.

Finals Week: June 9th-13th, 2014.

Symmetric Submodular Functions

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- *Symmetrize* and *normalize* f as $f \to \breve{f}$ via the operation: $\breve{f}(A) = f(A) + f(E \setminus A) - f(E)$, so that $\breve{f}(\emptyset) = 0$ if $f(\emptyset) = 0$.

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- Such an $\breve{f}$ is also non-negative since

$$2\breve{f}(A) = \breve{f}(A) + \breve{f}(E \setminus A) \geq \breve{f}(\emptyset) + \breve{f}(E) = 2\breve{f}(\emptyset) \geq 0 \quad (19.1)$$

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- Combinatorial mutual information function, so $\breve{f}(A) = I_f(A; V \setminus A)$ where $I_f(A; B) = f(A) + f(B) - f(A \cup B) - f(A \cap B)$.

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- Example: $f(A) = H(X_A) =$ entropy, then $\breve{f} = I(X_A; X_{E \setminus A}) =$ symmetric mutual information.

Separators of submodular function via symmetrized version

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We are given an f that is normalized & submodular. If $\exists A$ s.t. $\breve{f}(A) \triangleq f(A) + f(\bar{A}) - f(E) = 0$ then f is "decomposable" w.r.t. A — this means $f(B) = f(B \cap A) + f(B \cap \bar{A})$, $\forall B$.

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Proof.

- By submodularity (subadditivity for non-intersecting sets), we have:

$$f(B) = f\Big((B \cap A) \cup (B \cap \bar{A})\Big) \leq f(B \cap A) + f(B \cap \bar{A}) \tag{19.2}$$

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- Hence, $f(B) \leq f(B \cap A) + f(B \cap \bar{A})$.

...

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. . . proof of Theorem 19.3.1 cont.

□

Separators of submodular function via symmetrized version

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- By submodularity

$f(B) - f(B \cap A) - f(B \cap \bar{A})$

□

Separators of submodular function via symmetrized version

. . . proof of Theorem 19.3.1 cont.

- By submodularity

$$f(B) - f(B \cap A) - f(B \cap \bar{A}) \geq f(A \cup B) - f(A) - f(B \cap \bar{A}) \quad (19.3)$$

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$$f(B) - f(B \cap A) - f(B \cap \bar{A}) \geq f(A \cup B) - f(A) - f(B \cap \bar{A}) \quad (19.3)$$

$$\geq f((A \cup B) \cup \bar{A}) - f(A) - f(\bar{A}) \quad (19.4)$$

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$$= f(E) - f(A) + f(\bar{A}) = 0 \quad (19.5)$$

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- Eqn. (19.3) follows since $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$,

□

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- Eqn. (19.3) follows since $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$, and Eqn. (19.4) follows since $B \cap \bar{A} = (A \cup B) \cap \bar{A}$ and $f(A \cup B) + f(\bar{A}) \geq f((A \cup B) \cup \bar{A}) + f((A \cup B) \cap \bar{A})$.

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- By submodularity

$$f(B) - f(B \cap A) - f(B \cap \bar{A}) \geq f(A \cup B) - f(A) - f(B \cap \bar{A}) \quad (19.3)$$
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- Hence, both $f(B) \geq f(B \cap A) + f(B \cap \bar{A})$ (from above) and $f(B) \leq f(B \cap A) + f(B \cap \bar{A})$ (previous slide).

□

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- Set of separators of $\breve{f}$ is closed under intersection, union, and complementation. Hence, the separators partition E.
- In following slides, $\breve{f}$ is symmetrized & normalized version of f.

Review

Next slide is from Lecture 4.

Many (Equivalent) Definitions of Submodularity

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B), \quad \forall A, B \subseteq V \tag{19.6}$$

$$f(j|S) \geq f(j|T), \ \forall S \subseteq T \subseteq V, \text{ with } j \in V \setminus T \tag{19.7}$$

$$f(C|S) \geq f(C|T), \forall S \subseteq T \subseteq V, \text{ with } C \subseteq V \setminus T \tag{19.8}$$

$$f(j|S) \geq f(j|S \cup \{k\}), \ \forall S \subseteq V \text{ with } j \in V \setminus (S \cup \{k\}) \tag{19.9}$$

$$f(A \cup B|A \cap B) \leq f(A|A \cap B) + f(B|A \cap B), \quad \forall A, B \subseteq V \tag{19.10}$$

$$f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S) - \sum_{j \in S \setminus T} f(j|S \cup T - \{j\}), \ \forall S, T \subseteq V \tag{19.11}$$

$$f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S), \ \forall S \subseteq T \subseteq V \tag{19.12}$$

$$f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|S \setminus \{j\}) + \sum_{j \in T \setminus S} f(j|S \cap T) \ \forall S, T \subseteq V \tag{19.13}$$

$$f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|S \setminus \{j\}), \ \forall T \subseteq S \subseteq V \tag{19.14}$$

Minimization of a Symmetric Submodular Functions

- Minimizing symmetric submodular functions can be done in strongly polynomial time $O(n^3)$. The algorithm by Nagamochi & Ibaracki 1992 for graph cuts shown by Queyranne in 1995 to work for sym. SFM.

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- The algorithm finds (as a subroutine) MA (maximum adjacency) or a maximum back orders (not same as greedy order).

Choose v_1 arbitrarily ;
$W_1 \leftarrow (v_1)$ /* The first of an ordered list W_i. */ ;
for $i \leftarrow 1 \dots |V| - 1$ **do**
  Choose $v_{i+1} \in \operatorname{argmin}_{u \in V \setminus W_i} f(W_i | \{u\})$;
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- Note algorithm operates on non-symmetric function f. If f is already symmetric and normalized, then $f = \breve{f}$.
- The final ordered set $W_n = (v_1, v_2, \dots, v_n)$ is special in that the last two nodes (v_{n-1}, v_n) serve as a surrogate minimizer for a special case.

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Theorem 19.3.2

In the ordered set $W = (v_1, \ldots, v_n)$ generated by the MA algorithm, then (v_{n-1}, v_n) is a pendent pair.

Pendent pair

- A ordered pair of elements (t, u) is called a pendent pair if u is a minimizer amongst all sets that separate u and t.
- That is (t, u) is a pendent pair if

$$\{u\} \in \operatorname*{argmin}_{A \subseteq V : u \in A, t \notin A} \breve{f}(A) \tag{19.6}$$

- That is,

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- Interestingly, this algorithm is the same as maximum cardinality search (MCS), when f represents a graph cut function (recall, MCS is used to efficiently test graph chordality).

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- We store the score (min value) in the first case, then, consider a new element "tu" and clustered ground set $V' = V \setminus \{t, u\} \cup \{tu\}$, and new symmetric submodular function $f' : 2^{V'} \to \mathbb{R}$ with

$$\breve{f}'(X) = \begin{cases} \breve{f}(X) & \text{if } tu \notin X \\ \breve{f}(X \cup \{t, u\} \setminus \{tu\}) & \text{if } tu \in X \end{cases} \tag{19.8}$$

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- We do this n times. We take the min over all of the stored values.
- The pendent pair corresponding to the min element, say (t', u') will (most probability) correspond to nested clusters, so we use the original ground elements corresponding to u'.

Minimization of a Symmetric Submodular Functions

Theorem 19.3.3

The final resultant u' when expanded to original ground elements minimizes the symmetric submodular function f in $O(n^3)$ time.

- This has become known as Queyranne's algorithm for symmetric submodular function minimization.
- This was done in 1995 and it is said that this result, at that time, rekindled the efforts to find general combinatorial SFM.
- The actual algorithm was originally developed by Nagamochi and Ibaraki for a simple algorithm for finding graph cut. Queyranne showed it worked for any symmetric submodular function.
- Hence, it seems reasonable that symmetric SFM is faster than general SFM (although this question is still unknown).
- Quoting Fujishige from NIPS 2012, he said that he "hopes general purpose SFM is $O(n^4)$" ☺.

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  - Do both.
- There is also a sort of dual problem that is often considered together with max, and those are minimum cover problems (to be defined).

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- The set cover problem asks for the smallest subset X of V such that $f(X) = |E|$ (smallest subset of the subsets of E) where E is still covered. I.e.,

$$\text{minimize} |X| \text{ subject to } f(X) \geq |E| \tag{19.9}$$

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- This problem is NP-hard, and Feige in 1998 showed that it cannot be approximated with a ratio better than $(1 - \epsilon) \log n$ unless NP is slightly superpolynomial ($n^{O(\log \log n)}$).

What About Non-monotone

- So even simple case of cardinality constrained submodular function maximization is NP-hard.
- This will be true of most submodular max (and related) problems.
- Hence, the only hope is approximation algorithms. Question is, what is the tradeoff between running time and approximation quality, and is it possible to get tight bounds (i.e., an algorithm that achieves an approximation ratio, and a proof that one can't do better than that unless some extremely unlike event were to be true, such as P=NP).

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- The max k cover problem asks, given a k, what sized k set of sets X can we choose that covers the most? I.e., that maximizes $f(X)$ as in:

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- An important result by Nemhauser et. al. (1978) states that for normalized ($f(\emptyset) = 0$) monotone submodular functions (i.e., polymatroids) can be approximately maximized using a simple greedy algorithm.
- Starting with $S_0 = \emptyset$, we repeat the following greedy step for $i = 0 \ldots (k-1)$:

$$S_{i+1} = S_i \cup \left\{ \operatorname*{argmax}_{v \in V \setminus S_i} f(S_i \cup \{v\}) \right\} \tag{19.11}$$

The Greedy Algorithm for Submodular Max

A bit more precisely:

Algorithm 2: The Greedy Algorithm

Set $S_0 \leftarrow \emptyset$;
for $i \leftarrow 0 \dots |E| - 1$ **do**
  Choose v_i as follows:
  $v_i \in \left\{ \text{argmax}_{v \in V \setminus S_i} f(\{v\}|S_i) \right\} = \left\{ \text{argmax}_{v \in V \setminus S_i} f(S_i \cup \{v\}) \right\}$;
  Set $S_{i+1} \leftarrow S_i \cup \{v_i\}$;

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Given a polymatroid function f , the above greedy algorithm returns sets S_i such that for each i we have $f(S_i) \geq (1 - 1/e) \max_{|S| \leq i} f(S)$.

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- Again, since this generalizes max k-cover, Feige (1998) showed that this can't be improved. Unless $P = NP$, no polynomial time algorithm can do better than $(1 - 1/e + \epsilon)$ for any $\epsilon > 0$.