Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 17 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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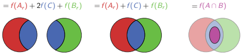
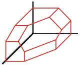
$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$





Cumulative Outstanding Reading

- Good references for today: Schrijver-2003, Oxley-1992/2011, Welsh-1973, Goemans-2010, Cunningham-1984, Edmonds-1969, Choquet-1955, Grabisch/Marichal/Mesiar/Pap "Aggregation Functions", Lovász-1983, Bach-2011.
- Read Tom McCormick's overview paper on SFM `http://people.commerce.ubc.ca/faculty/mccormick/sfmchap8a.pdf`
- Read chapters 1 - 4 from Fujishige book.
- Matroid properties `http://www-math.mit.edu/~goemans/18433S09/matroid-notes.pdf`
- Read lecture 14 slides on lattice theory at our web page (`http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/`)
- Wolfe "Finding the Nearest Point in a Polytope", 1976.
- Fujishige & Isotani, "A Submodular Function Minimization Algorithm Based on the Minimum-Norm Base", 2009.

Announcements, Assignments, and Reminders

- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, Combinatorial Geometries
- L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
- L9: From Matroid Polytopes to Polymatroids.
- L10: Polymatroids and Submodularity
- L11: More properties of polymatroids, SFM special cases
- L12: polymatroid properties, extreme points polymatroids,
- L13: sat, dep, supp, exchange capacity, examples
- L14: Lattice theory: partially ordered sets; lattices; distributive, modular, submodular, and boolean lattices; ideals and join irreducibles.
- L15: Supp, Base polytope, polymatroids and entropic Venn diagrams, exchange capacity,
- L16: proof that minimum norm point yields min of submodular function, and the lattice of minimizers of a submodular function, Lovasz extension
- L17: Lovasz extension, Choquet Integration, more properties/examples of Lovasz extension, convex minimization and SFM.
- L18: Lovasz extension examples and structured convex norms, The Min-Norm Point Algorithm detailed.
- L19: symmetric submodular function minimization, maximizing monotone submodular function w. card constraints.
- L20: maximizing monotone submodular function w. other constraints, non-monotone maximization.

Finals Week: June 9th-13th, 2014.

Min-Norm Point and SFM

Theorem 17.2.1

*Let y^*, A_-, and A_0 be as given. Then y^* is a maximizer of the l.h.s. of Eqn. (**??**). Moreover, A_- is the unique minimal minimizer of f and A_0 is the unique maximal minimizer of f.*

Proof.

- First note, since $x^* \in B_f$, we have $x^*(E) = f(E)$, meaning $\text{sat}(x^*) = E$. Thus, we can consider any $e \in E$ within $\text{dep}(x^*, e)$.
- Consider any pair (e, e') with $e' \in \text{dep}(x^*, e)$ and $e \in A_-$. Then $x^*(e) < 0$, and $\exists \alpha > 0$ s.t. $x^* + \alpha \mathbf{1}_e - \alpha \mathbf{1}_{e'} \in P_f$.
- We have $x^*(E) = f(E)$ and x^* is minimum in l2 sense. We have $(x^* + \alpha \mathbf{1}_e - \alpha \mathbf{1}_{e'}) \in P_f$, and in fact

$$(x^* + \alpha \mathbf{1}_e - \alpha \mathbf{1}_{e'})(E) = x^*(E) + \alpha - \alpha = f(E) \tag{17.1}$$

so $x^* + \alpha \mathbf{1}_e - \alpha \mathbf{1}_{e'} \in B_f$ also.

...

Min-norm point and other minimizers of f

- Recall, that the set of minimizers of f forms a lattice.
- In fact, with x^* the min-norm point, and A_- and A_0 as defined above, we have the following theorem:

Theorem 17.2.1

Let $A \subseteq E$ be any minimizer of submodular f, and let x^* be the minimum-norm point. Then A has the form:

$$A = A_- \cup \bigcup_{a \in A_m} \mathrm{dep}(x^*, a) \tag{17.7}$$

for some set $A_m \subseteq A_0 \setminus A_-$.

A continuous extension of submodular f

- That is, given a submodular function f, a $w \in \mathbb{R}^E$, and defining $E_i = \{e_1, e_2, \ldots, e_i\}$ and where we choose the element order $(e_1, e_2, \ldots, e_m)$ based on decreasing w,so that $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$, we have

$$\tilde{f}(w) = \max(wx : x \in P_f) \tag{17.11}$$

$$= \sum_{i=1}^{m} w(e_i) f(e_i | E_{i-1}) \tag{17.12}$$

$$= \sum_{i=1}^{m} w(e_i)(f(E_i) - f(E_{i-1})) \tag{17.13}$$

$$= w(e_m) f(E_m) + \sum_{i=1}^{m-1} (w(e_i) - w(e_{i+1})) f(E_i) \tag{17.14}$$

- We say that $\emptyset \triangleq E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m = E$ forms a chain based on w.

A continuous extension of submodular f

- Definition of the continuous extension, once again, for reference:

$$\tilde{f}(w) = \max(wx : x \in P_f) \tag{17.11}$$

- Therefore, if f is a submodular function, we can write

$$\tilde{f}(w) = w(e_m)f(E_m) + \sum_{i=1}^{m-1}(w(e_i) - w(e_{i+1}))f(E_i) \tag{17.12}$$

$$= \sum_{i=1}^{m} \lambda_i f(E_i) \tag{17.13}$$

  where $\lambda_m = w(e_m)$ and otherwise $\lambda_i = w(e_i) - w(e_{i+1})$, where the elements are sorted descending according to w as before.
- From convex analysis, we know $\tilde{f}(w) = \max(wx : x \in P)$ is always convex in w for any set $P \subseteq R^E$, since it is the maximum of a set of linear functions (true even when f is not submodular or P is not a convex set).

An extension of an arbitrary $f : 2^V \to \mathbb{R}$

- Thus, for any $f : 2^E \to \mathbb{R}$, even non-submodular f, we can define an extension, having $\tilde{f}(\mathbf{1}_A) = f(A)$, $\forall A$, in this way where

$$\tilde{f}(w) = \sum_{i=1}^{m} \lambda_i f(E_i) \tag{17.20}$$

with the $E_i = \{e_1, \ldots, e_i\}$'s defined based on sorted descending order of w as in $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$, and where

$$\text{for } i \in \{1, \ldots, m\}, \quad \lambda_i = \begin{cases} w(e_i) - w(e_{i+1}) & \text{if } i < m \\ w(e_m) & \text{if } i = m \end{cases} \tag{17.21}$$

so that $w = \sum_{i=1}^{m} \lambda_i \mathbf{1}_{E_i}$.

- $w = \sum_{i=1}^{m} \lambda_i \mathbf{1}_{E_i}$ is an interpolation of certain hypercube vertices.
- $\tilde{f}(w) = \sum_{i=1}^{m} \lambda_i f(E_i)$ is the associated interpolation of the values of f at sets corresponding to each hypercube vertex.

Summary: comparison of the two extension forms

- So if f is submodular, then we can write $\tilde{f}(w) = \max(wx : x \in P_f)$ (which is clearly convex) in the form:

$$\tilde{f}(w) = \max(wx : x \in P_f) = \sum_{i=1}^{m} \lambda_i f(E_i) \tag{17.24}$$

where $w = \sum_{i=1}^{m} \lambda_i \mathbf{1}_{E_i}$ and $E_i = \{e_1, \dots, e_i\}$ defined based on sorted descending order $w(e_1) \geq w(e_2) \geq \dots \geq w(e_m)$.

- On the other hand, for any f (even non-submodular), we can produce an extension $\tilde{f}$ having the form

$$\tilde{f}(w) = \sum_{i=1}^{m} \lambda_i f(E_i) \tag{17.25}$$

where $w = \sum_{i=1}^{m} \lambda_i \mathbf{1}_{E_i}$ and $E_i = \{e_1, \dots, e_i\}$ defined based on sorted descending order $w(e_1) \geq w(e_2) \geq \dots \geq w(e_m)$.

- In both Eq. (**??**) and Eq. (**??**), we have $\tilde{f}(\mathbf{1}_A) = f(A)$, $\forall A$, but Eq. (**??**), might not be convex.
- Submodularity is sufficient for convexity of but is it necessary?

Lovász Extension, Submodularity and Convexity

Theorem 17.2.1

A function $f : 2^E \to \mathbb{R}$ is submodular iff its Lovász extension $\tilde{f}$ of f is convex.

Proof.

- We've already seen that if f is submodular, its extension can be written via Eqn.(**??**) due to the greedy algorithm, and therefore is also equivalent to $\tilde{f}(w) = \max\{wx : x \in P_f\}$, and thus is convex.
- Conversely, suppose the Lovász extension $\tilde{f}(w) = \sum_i \lambda_i f(E_i)$ of some function $f : 2^E \to \mathbb{R}$ is a convex function.
- We note that, based on the extension definition, in particular the definition of the $\{\lambda_i\}_i$, we have that $\tilde{f}(\alpha w) = \alpha \tilde{f}(w)$ for any $\alpha \in \mathbb{R}_+$. I.e., f is a positively homogeneous convex function.

. . .

Integration and Aggregation

- Integration is just summation (e.g., the $\int$ symbol has as its origins a sum).

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- Lebesgue integration allows integration w.r.t. an underlying measure μ of sets. E.g., given measurable function f, we can define

$$\int_X f du = \sup I_X(s) \tag{17.1}$$

where $I_X(s) = \sum_{i=1}^n c_i \mu(X \cap X_i)$, and where we take the sup over all measurable functions s such that $0 \leq s \leq f$ and $s(x) = \sum_{i=1}^n c_i I_{X_i}(x)$ and where $I_{X_i}(x)$ is indicator of membership of set X_i, with $c_i > 0$.

Integration, Aggregation, and Weighted Averages

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- I.e., given a weight vector $w \in [0,1]^E$ for some finite ground set E, then for any $x \in \mathbb{R}^E$ we have the weighted average of x as:

$$\mathrm{WAVG}(x) = \sum_{e \in E} x(e)w(e) \tag{17.2}$$

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so seen as a function on the hypercube vertices, the entire WAVG function is given based on values on a size $m = |E|$ subset of the vertices of this hypercube, i.e., $\{\mathbf{1}_e : e \in E\}$.

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so seen as a function on the hypercube vertices, the entire WAVG function is given based on values on a size $m = |E|$ subset of the vertices of this hypercube, i.e., $\{\mathbf{1}_e : e \in E\}$. Moreover, we are interpolating as in

$$\text{WAVG}(x) = \sum_{e \in E} x(e)w(e) = \sum_{e \in E} x(e)\text{WAVG}(\mathbf{1}_e) \tag{17.4}$$

Integration, Aggregation, and Weighted Averages

$$\mathsf{WAVG}(x) = \sum_{e \in E} x(e)w(e) \tag{17.5}$$

- Clearly, WAVG function is linear in weights w, in the argument x, and is homogeneous. That is, for all $w, w_1, w_2, x, x_1, x_2 \in \mathbb{R}^E$ and $\alpha \in \mathbb{R}$,

$$\mathsf{WAVG}_{w_1+w_2}(x) = \mathsf{WAVG}_{w_1}(x) + \mathsf{WAVG}_{w_2}(x), \tag{17.6}$$

$$\mathsf{WAVG}_{w}(x_1+x_2) = \mathsf{WAVG}_{w}(x_1) + \mathsf{WAVG}_{w}(x_2), \tag{17.7}$$

and,

$$\mathsf{WAVG}(\alpha x) = \alpha \mathsf{WAVG}(x). \tag{17.8}$$

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- We will see: The Lovász extension is still be linear in "weights" (i.e., the submodular function f), but will not be linear in x and will only be positively homogeneous (for $\alpha \geq 0$).

Integration, Aggregation, and Weighted Averages

- More complex "nonlinear" aggregation functions can be constructed by defining the aggregation function on all vertices of the hypercube. I.e., for each $\mathbf{1}_A : A \subseteq E$ we might have (for all $A \subseteq E$):

$$\text{AG}(\mathbf{1}_A) = w_A \tag{17.9}$$

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$$\text{AG}(x) = \sum_{A \subseteq E} x(A) w_A = \sum_{A \subseteq E} x(A) \text{AG}(\mathbf{1}_A) \tag{17.10}$$

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- Note, we can define $w(e) = w'(e)$ and $w(A) = 0, \forall A : |A| > 1$ and get back previous (normal) weighted average, in that

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- Set function $f : 2^E \to \mathbb{R}$ is a game if f is normalized $f(\emptyset) = 0$.

Integration, Aggregation, and Weighted Averages

- Set function $f : 2^E \to \mathbb{R}$ is called a capacity if it is monotone non-decreasing, i.e., $f(A) \leq f(B)$ whenever $A \subseteq B$.

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- A Boolean function f is any function $f : \{0,1\}^m \to \{0,1\}$ and is a pseudo-Boolean function if $f : \{0,1\}^m \to \mathbb{R}$.

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- Any set function corresponds to a pseudo-Boolean function. I.e., given $f : 2^E \to \mathbb{R}$, form $f_b : \{0,1\}^m \to \mathbb{R}$ as $f_b(x) = f(A_x)$ where the A, x bijection is $A = \{e \in E : x_e = 1\}$ and $x = \mathbf{1}_A$.

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- Also, if we have an expression for f_b we can construct a set function f as $f(A) = f_b(\mathbf{1}_A)$. We can also often relax f_b to any $x \in [0,1]^m$.

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- We saw this for Lovász extension.

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- Also, if we have an expression for f_b we can construct a set function f as $f(A) = f_b(\mathbf{1}_A)$. We can also often relax f_b to any $x \in [0,1]^m$.
- We saw this for Lovász extension.
- It turns out that a concept essentially identical to the Lovász extension was derived much earlier, in 1954, and using this derivation (via integration) leads to deeper intuition.

Choquet integral

Definition 17.3.1

Let f be any capacity on E and $w \in \mathbb{R}_+^E$. The Choquet integral (1954) of w w.r.t. f is defined by

$$C_f(w) = \sum_{i=1}^{m} (w_{e_i} - w_{e_{i+1}}) f(E_i) \tag{17.12}$$

where in the sum, we have sorted and renamed the elements of E so that $w_{e_1} \geq w_{e_2} \geq \cdots \geq w_{e_m} \geq w_{e_{m+1}} \triangleq 0$, and where $E_i = \{e_1, e_2, \ldots, e_i\}$.

- We immediately see that an equivalent formula is as follows:

$$C_f(w) = \sum_{i=1}^{m} w(e_i)(f(E_i) - f(E_{i-1})) \tag{17.13}$$

where $E_0 \stackrel{\text{def}}{=} \emptyset$.

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- BTW: this again essentially Abel's partial summation formula: Given two arbitrary sequences $\{a_n\}$ and $\{b_n\}$ with $A_n = \sum_{k=1}^{n} a_k$, we have

$$\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n} A_k(b_k - b_{k+1}) + A_n b_{n+1} - A_{m-1} b_m \tag{17.14}$$

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- For any $w_{e_i} > \alpha \geq w_{e_{i+1}}$ we also have $E_i = \{e_1, e_2, \ldots, e_i\} = \{e \in E : w_e > \alpha\}$.

The "integral" in the Choquet integral

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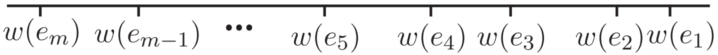
- First note, assuming E is ordered according to descending w, so that $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_{m-1}) \geq w(e_m)$, then $E_i = \{e_1, e_2, \ldots, e_i\} = \{e \in E : w_e \geq w_{e_i}\}$.
- For any $w_{e_i} > \alpha \geq w_{e_{i+1}}$ we also have $E_i = \{e_1, e_2, \ldots, e_i\} = \{e \in E : w_e > \alpha\}$.
- Consider segmenting the real-axis at boundary points w_{e_i}, right most is w_{e_1}.



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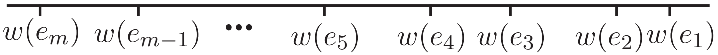

- A function can be defined on a segment of $\mathbb{R}$, namely $w_{e_i} > \alpha \geq w_{e_{i+1}}$. This function $F_i : [w_{e_{i+1}}, w_{e_i}) \to \mathbb{R}$ is defined as

$$F_i(\alpha) = f(\{e \in E : w_e > \alpha\}) = f(E_i) \tag{17.15}$$

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- We can generalize this to multiple segments of $\mathbb{R}$ (for now, take $w \in \mathbb{R}_+^E$). The piecewise-constant function is defined as:

$$F(\alpha) = \begin{cases} f(E) & \text{if } 0 \leq \alpha < w_m \\ f(\{e \in E : w_e > \alpha\}) & \text{if } w_{e_{i+1}} \leq \alpha < w_{e_i},\ i \in \{1, \dots, m-1\} \\ 0 \ (= f(\emptyset)) & \text{if } w_1 < \alpha \end{cases}$$

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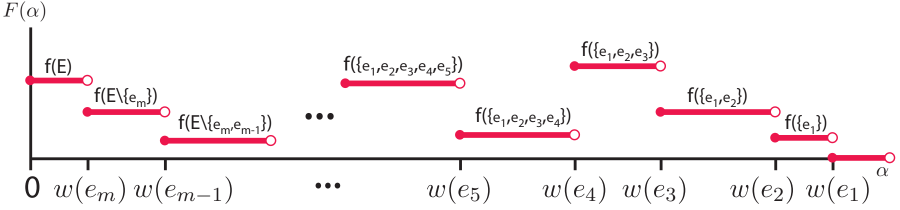
- Visualizing a piecewise constant function, where the constant values are given by f evaluated on E_i for each i



Note, what is depicted may be a game but not a capacity. Why?

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- Now consider the integral, with $w \in \mathbb{R}_+^E$, and normalized f so that $f(\emptyset) = 0$. Recall $w_{m+1} \stackrel{\text{def}}{=} 0$.

$$\tilde{f}(w) \stackrel{\text{def}}{=} \int_0^\infty F(\alpha) d\alpha \tag{17.16}$$

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Given $w \in \mathbb{R}_+^E$, the Lovász extension (equivalently Choquet integral) may be defined as follows:

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- The above integral will be further generalized a bit later.

Choquet integral and aggregation

- Recall, we want to produce some notion of generalized aggregation function having the flavor of:

$$\mathsf{AG}(x) = \sum_{A \subseteq E} x(A) w_A = \sum_{A \subseteq E} x(A) \mathsf{AG}(\mathbf{1}_A) \tag{17.22}$$

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- This forms a "triangulation" of the hypercube.
- For any $x \in [0,1]^m$ there is a (not necessarily unique) $\mathcal{V}(x) = \mathcal{V}_j$ for some j such that $x \in \operatorname{conv}(\mathcal{V}(x))$.

Choquet integral and aggregation

- Most generally, for $x \in [0,1]^m$, let us define the (unique) coefficients $\alpha_0^x(A)$ and $\alpha_i^x(A)$ that define the affine transformation of the coefficients of x to be used with the particular hypercube vertex $\mathbf{1}_A \in \text{conv}(\mathcal{V}(x))$. The affine transformation is as follows:

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- From this, we can define an aggregation function of the form

$$\mathsf{AG}(x) \stackrel{\text{def}}{=} \sum_{A : \mathbf{1}_A \in \mathcal{V}(x)} \Big(\alpha_0^x(A) + \sum_{j=1}^{m} \alpha_j^x(A) x_j \Big) \mathsf{AG}(\mathbf{1}_A) \tag{17.24}$$

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- We can define a canonical triangulation of the hypercube in terms of permutations of the coordinates. I.e., given some permutation σ, define

$$\mathrm{conv}(\mathcal{V}_\sigma) = \left\{ x \in [0,1]^n | x_{\sigma(1)} \geq x_{\sigma(2)} \geq \cdots \geq x_{\sigma(m)} \right\} \tag{17.25}$$

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The above linear interpolation in Eqn. (17.24) using the canonical partition yields the Lovász extension with $\alpha_0^x(A) + \sum_{j=1}^m \alpha_j^x(A) x_j$ $= x_{\sigma_i} - x_{\sigma_{i-1}}$ *for* $A = E_i = \{e_{\sigma_1}, \dots, e_{\sigma_i}\}$ *for appropriate order* σ.

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- Hence, Lovász extension is a generalized aggregation function.

Lovász extension as max over orders

- We can also write the Lovász extension as follows:

$$\tilde{f}(w) = \max_{\sigma \in \Pi_{[m]}} w^\intercal c^\sigma \tag{17.26}$$

where $\Pi_{[m]}$ is the set of $m!$ permutations of $[m] = E$, $\sigma \in \Pi_{[m]}$ is a particular permutation, and c^σ is a vector associated with permutation σ defined as:

$$c_i^\sigma = f(E_{\sigma_i}) - f(E_{\sigma_{i-1}}) \tag{17.27}$$

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- Note this immediately follows from the definition of the Lovász extension in the form:

$$\tilde{f}(w) = \max_{x \in P_f} w^\intercal x = \max_{x \in B_f} w^\intercal x \tag{17.28}$$

  since we know that the maximum is achieved by an extreme point of the base B_f and all extreme points are obtained by a permutation-of-E-parameterized greedy instance.

Lovász extension, defined in multiple ways

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- We have already seen how we can define the Lovász extension for any (not necessarily submodular) function f in the following equivalent ways:

$$\tilde{f}(w) = \sum_{i=1}^{m} w(e_i) f(e_i | E_{i-1}) \tag{17.29}$$

$$= \sum_{i=1}^{m-1} f(E_i)(w(e_i) - w(e_{i+1})) + f(E)w(e_m)a \tag{17.30}$$

$$= \sum_{i=1}^{m-1} \lambda_i f(E_i) \tag{17.31}$$

Lovász extension, as integral

- Additional ways we can define the Lovász extension for any (not necessarily submodular) but normalized function f include:

$$\tilde{f}(w) = \sum_{i=1}^{m} w(e_i) f(e_i|E_{i-1}) = \sum_{i=1}^{m} \lambda_i f(E_i) \tag{17.32}$$

$$= \sum_{i=1}^{m-1} f(E_i)(w(e_i) - w(e_{i+1})) + f(E)w(e_m) \tag{17.33}$$

$$= \int_{\min\{w_1,\ldots,w_m\}}^{+\infty} f(\{w \geq \alpha\})d\alpha + f(E)\min\{w_1,\ldots,w_m\} \tag{17.34}$$

$$= \int_{0}^{+\infty} f(\{w \geq \alpha\})d\alpha + \int_{-\infty}^{0} [f(\{w \geq \alpha\}) - f(E)]d\alpha \tag{17.35}$$

general Lovász extension, as simple integral

- In fact, we have that, given function f, and any $w \in \mathbb{R}^E$:

$$\tilde{f}(w) = \int_{-\infty}^{+\infty} \hat{f}(\alpha) d\alpha \tag{17.36}$$

where

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- So we can write it as a simple integral over the right function.
- These make it easier to see certain properties of the Lovász extension. But first, we show the above.

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- To show Eqn. (17.34), first note that the r.h.s. terms are the same since $w(e_m) = \min\{w_1, \ldots, w_m\}$.

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$$f(\{w \geq \alpha\}) = \begin{cases} 0 & \text{if } \alpha > w(e_1) \\ f(E_k) & \text{if } \alpha \in (w(e_{k+1}), w(e_k)), k \in \{1, \dots, m-1\} \\ f(E) & \text{if } \alpha < w(e_m) \end{cases} \tag{17.38}$$

we may use open intervals since sets of zero measure don't change integration.

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- Inside the integral, then, this recovers Eqn. (17.33).

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- To show Eqn. (17.35), start with Eqn. (17.34), note $w_m = \min\{w_1, \dots, w_m\}$, take any $\beta \leq \min\{0, w_1, \dots, w_m\}$, and form:

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- Useful for showing results for randomized rounding schemes in solving submodular opt. problems subject to constraints via relaxations to convex optimization problems subject to linear constraints.

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Given $c \in \mathbb{R}^V$, find a vector $x \in C$ that maximizes $c^\intercal x$ on C. I.e., solve

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Definition 17.5.2 ((strong) separation problem)

Given a vector $y \in \mathbb{R}^V$, decide if $y \in C$, and if not, find a hyperplane that separates y from C. I.e., find vector $c \in \mathbb{R}^V$ such that:

$$c^\intercal y > \max_{x \in C} c^\intercal x \tag{17.44}$$

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- See also, the book: Grötschel, Lovász, and Schrijver, "Geometric Algorithms and Combinatorial Optimization"

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- Also, since we can recover f from $\tilde{f}$ via $f(A) = \tilde{f}(\mathbf{1}_A)$, and (as we will see) get discrete solutions from continuous convex minimization solution.
- Is this the only convex extension of a submodular function? Are there others that have more attractive properties?

Continuous Extensions of Discrete Set Functions

- Any function $f : 2^V \to \mathbb{R}$ (equivalently $f : \{0,1\}^V \to \mathbb{R}$) can be extended to a continuous function $\tilde{f} : [0,1]^V \to \mathbb{R}$.

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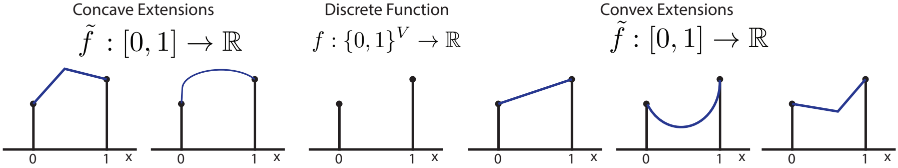

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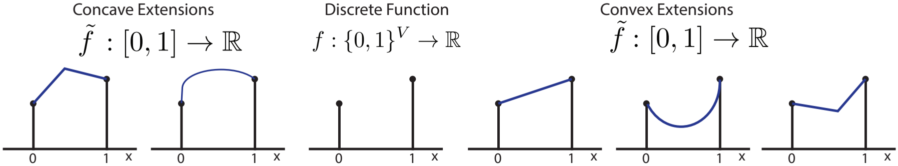

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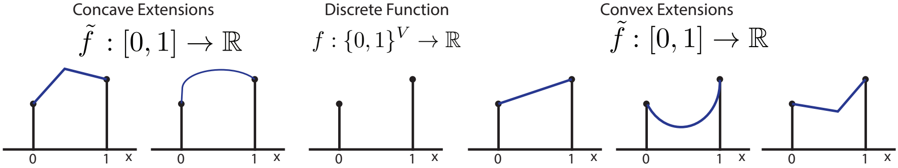

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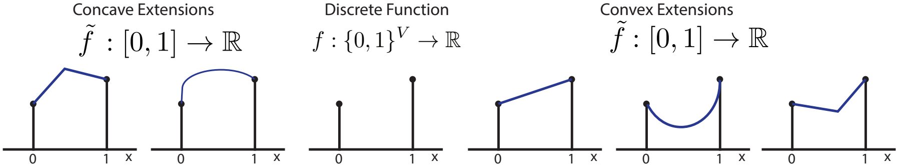

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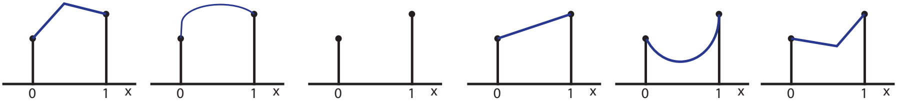

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Minimizing $\tilde{f}$ vs. minimizing f

In fact, we have:

Theorem 17.5.4

Let f be submodular and $\tilde{f}$ be its Lovász extension. Then $\min\{f(A)|A \subseteq E\} = \min_{w \in \{0,1\}^E} \tilde{f}(w) = \min_{w \in [0,1]^E} \tilde{f}(w)$.

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- Also, $\sum_i \lambda_i = w(e_1) \leq 1$.

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Minimizing $\tilde{f}$ vs. minimizing f

. . . cont. proof of Thm. 17.5.4.

- Note that since $f(\emptyset) = 0$, $\min \{f(A) | A \subseteq E\} \leq 0$.

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- Note that since $f(\emptyset) = 0$, $\min\{f(A) | A \subseteq E\} \leq 0$.
- Then we have

$$\tilde{f}(w) = \int_0^1 f(\{w \geq \alpha\}) d\alpha = \sum_{i=1}^m \lambda_i f(E_i) \tag{17.45}$$

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Other minimizers based on min of $\tilde{f}$

- Let $w^* \in \operatorname{argmin}\left\{\tilde{f}(w) | w \in [0,1]^E\right\}$ and let $A^* \in \operatorname{argmin}\{f(A) | A \subseteq V\}$.

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- Case 1: $f(A^*) = 0$. Then for any i with $\lambda_i > 0$ we must have $f(E_i) = 0$ as well for all i since $f(A^*) \leq f(E_i)$.
- Case 2 is where $f(A^*) < 0$. In this second case, we have

$$0 > f(A^*) = \sum_i \lambda_i f(E_i) \geq \sum_i \lambda_i f(A^*) \tag{17.51}$$

$$\overset{(a)}{\geq} \sum_i \lambda_i f(A^*) + (1 - \bar{\lambda}) f(A^*) = f(A^*) \tag{17.52}$$

where $\bar{\lambda} = \sum_i \lambda_i$ and $(1 - \bar{\lambda}) \geq 0$ and where (a) follows since $f(A^*) < 0$.

Alternate way to see Equation 17.49

- We know $f(A^*) \leq 0$. Consider two cases in Equation 17.49.
- Case 1: $f(A^*) = 0$. Then for any i with $\lambda_i > 0$ we must have $f(E_i) = 0$ as well for all i since $f(A^*) \leq f(E_i)$.
- Case 2 is where $f(A^*) < 0$. In this second case, we have

$$0 > f(A^*) = \sum_i \lambda_i f(E_i) \geq \sum_i \lambda_i f(A^*) \tag{17.51}$$

$$\stackrel{(a)}{\geq} \sum_i \lambda_i f(A^*) + (1 - \bar{\lambda}) f(A^*) = f(A^*) \tag{17.52}$$

where $\bar{\lambda} = \sum_i \lambda_i$ and $(1 - \bar{\lambda}) \geq 0$ and where (a) follows since $f(A^*) < 0$.

- Hence, all inequalities must be equalities, which means that we must have that $\bar{\lambda} = 1$.

θ-rounding the L.E. minimum

We can also view the above as a form of rounding a continuous convex relaxation to the problem.

Definition 17.5.5 (θ-rounding)

Given vector $x \in [0,1]^E$, choose $\theta \in (0,1)$ and define a set corresponding to elements above θ, i.e.,

$$\hat{X}_\theta = \{i : \hat{x}(i) \geq \theta\} \triangleq \{\hat{x} \geq \theta\} \tag{17.53}$$

Lemma 17.5.6 (Fujishige-2005)

Given a continuous minimizer $x^* \in \operatorname{argmin}_{x \in [0,1]^n} \tilde{f}(x)$, the discrete minimizers are exactly the maximal chain of sets $\emptyset \subset X_{\theta_1} \subset \ldots X_{\theta_k}$ obtained by θ-rounding x^*, for $\theta_j \in (0,1)$.