Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 15 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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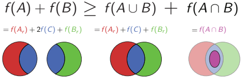
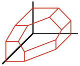
May 19th, 2014





Cumulative Outstanding Reading

- Good references for today: Schrijver-2003, Oxley-1992/2011, Welsh-1973, Goemans-2010, Cunningham-1984, Edmonds-1969.
- Read Tom McCormick's overview paper on SFM http://people.commerce.ubc.ca/faculty/mccormick/sfmchap8a.pdf
- Read chapters 1 - 4 from Fujishige book.
- Matroid properties http://www-math.mit.edu/~goemans/18433S09/matroid-notes.pdf
- Read lecture 14 slides on lattice theory at our web page (http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/)

Announcements, Assignments, and Reminders

- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, Combinatorial Geometries
- L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
- L9: From Matroid Polytopes to Polymatroids.
- L10: Polymatroids and Submodularity
- L11: More properties of polymatroids, SFM special cases
- L12: polymatroid properties, extreme points polymatroids,
- L13: sat, dep, supp, exchange capacity, examples
- L14: Lattice theory: partially ordered sets; lattices; distributive, modular, submodular, and boolean lattices; ideals and join irreducibles.
- L15: Supp, Base polytope, polymatroids and entropic Venn diagrams, exchange capacity,
- L16: proof that minimum norm point yields min of submodular function, and the lattice of minimizers of a submodular function, Lovasz extension
- L17: Lovasz extension, Choquet Integration, more properties/examples of Lovasz extension, convex minimization and SFM.
- L18: Lovasz extension examples and structured convex norms, The Min-Norm Point Algorithm detailed.
- L19: symmetric submodular function minimization, maximizing monotone submodular function w. card constraints.
- L20: maximizing monotone submodular function w. other constraints, non-monotone maximization.

Finals Week: June 9th-13th, 2014.

Summary of Concepts

- Most violated inequality $\max\{x(A) - f(A) : A \subseteq E\}$
- Matroid by circuits, and the fundamental circuit $C(I, e) \subseteq I + e$.
- Minimizers of submodular functions form a lattice.
- Minimal and maximal element of a lattice.
- x-tight sets, maximal and minimal tight set.
- sat function & Closure
- Saturation Capacity
- e-containing tight sets
- dep function & fundamental circuit of a matroid

Summary important definitions so far: tight, dep, & sat

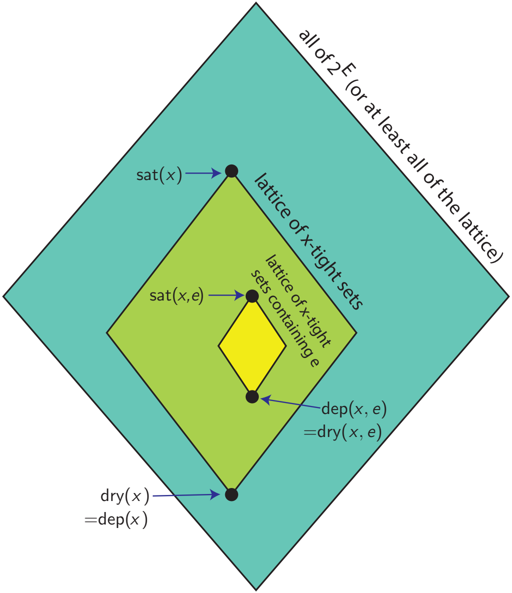
- x-tight sets: For $x \in P_f$, $\mathcal{D}(x) = \{A \subseteq E : x(A) = f(A)\}$.
- Polymatroid closure/maximal x-tight set: For $x \in P_f$, $\text{sat}(x) = \cup\{A : A \in \mathcal{D}(x)\} = \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\}$.
- Saturation capacity: for $x \in P_f$, $0 \leq \hat{c}(x; e) = \min \{f(A) - x(A) | \forall A \ni e\} = \max \{\alpha : \alpha \in \mathbb{R}, x + \alpha \mathbf{1}_e \in P_f\}$
- Recall: $\text{sat}(x) = \{e : \hat{c}(x; e) = 0\}$ and $E \setminus \text{sat}(x) = \{e : \hat{c}(x; e) > 0\}$.
- e-containing x-tight sets: For $x \in P_f$, $\mathcal{D}(x, e) = \{A : e \in A \subseteq E, x(A) = f(A)\} \subseteq \mathcal{D}(x)$.
- Minimal e-containing x-tight set/polymatroidal fundamental circuit/: For $x \in P_f$,

$$\text{dep}(x, e) = \begin{cases} \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases}$$

$$= \left\{e' : \exists \alpha > 0, \text{ s.t. } x + \alpha(\mathbf{1}_e - \mathbf{1}_{e'}) \in P_f\right\}$$

dep and sat in a lattice

- Given some $x \in P_f$,
- The picture on the right summarizes the relationships between the lattices and sublattices.
- Note, $\bigcap_e \mathrm{dep}(x, e) = \mathrm{dep}(x)$.



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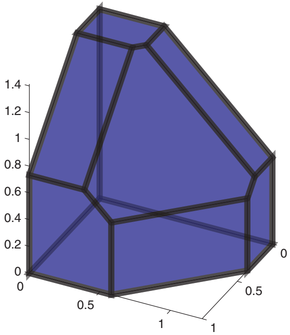
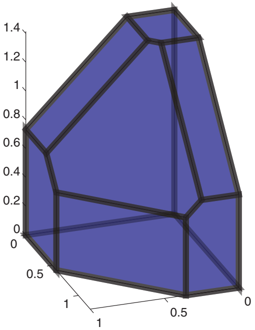
- Example





Tightness of supp at polymatroidal extreme point

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- Since $\mathrm{supp}(x)$ is tight, we immediately have that $\mathrm{sat}(x) \supseteq \mathrm{supp}(x)$.

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- For modular functions, they are always equal at extreme points (e.g., think of "hyperrectangular" polymatroids).

Summary of supp, sat, and dep

- For $x \in P_f$, $\text{supp}(x) = \{e : x(e) \neq 0\} \subseteq \text{sat}(x)$
- For $x \in P_f$, $\text{sat}(x)$ (span, closure) is the maximal saturated (x-tight) set w.r.t. x. I.e., $\text{sat}(x) = \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\}$. That is,

$$\text{cl}(x) \stackrel{\text{def}}{=} \text{sat}(x) \triangleq \bigcup \{A : A \in \mathcal{D}(x)\} \tag{15.29}$$

$$= \bigcup \{A : A \subseteq E, x(A) = f(A)\} \tag{15.30}$$

$$= \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\} \tag{15.31}$$

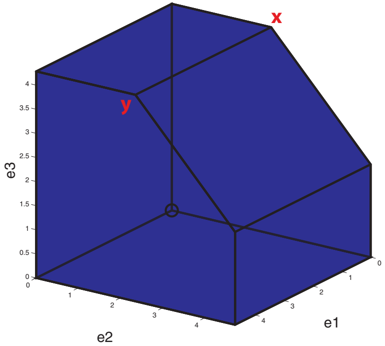
- For $e \in \text{sat}(x)$, we have $\text{dep}(x, e) \subseteq \text{sat}(x)$ (fundamental circuit) is the minimal (common) saturated (x-tight) set w.r.t. x containing e. I.e.,

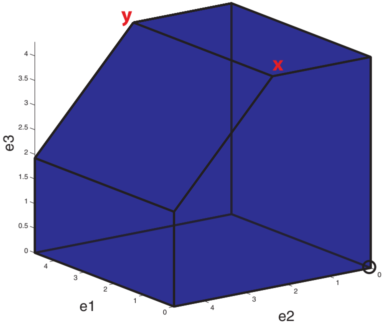
$$\text{dep}(x, e) = \begin{cases} \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases}$$

$$= \{e' : \exists \alpha > 0, \text{ s.t. } x + \alpha(\mathbf{1}_e - \mathbf{1}_{e'}) \in P_f\} \tag{15.32}$$

supp, sat, dep, example with perfect independence

- Example polymatroid where there is perfect independence between $A = \{e_2, e_3\}$ and $B = \{e_1\}$, i.e., $e_1 \perp\!\!\!\perp \{e_2, e_3\}$.





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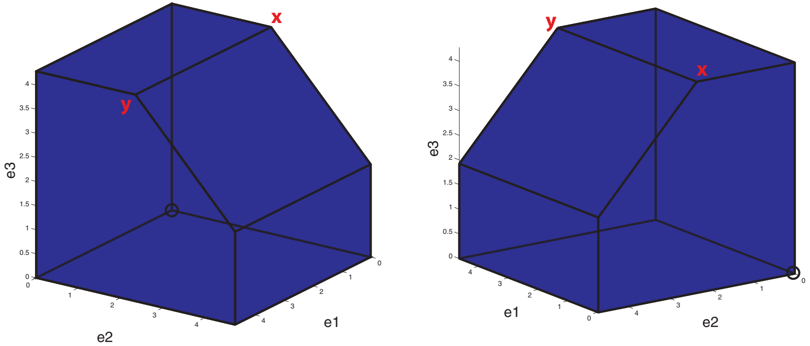

- Point x is extreme and $x(\{e_2, e_3\}) = f(e_2, e_3)$ (why?).

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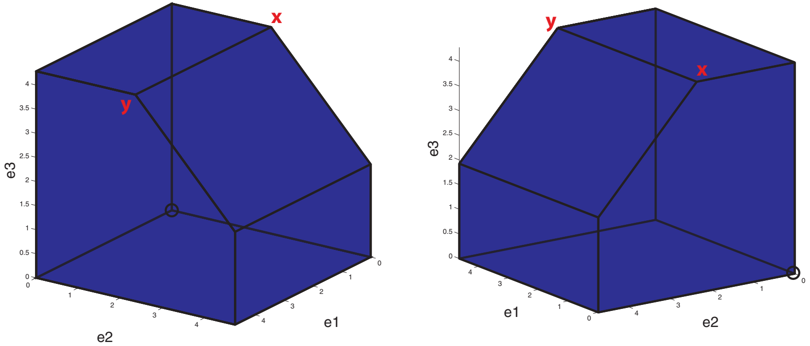

- Point x is extreme and $x(\{e_2, e_3\}) = f(e_2, e_3)$ (why?).
- But $x(\{e_1, e_2, e_3\}) = x(\{e_2, e_3\}) < f(e_1, e_2, e_3) = f(e_1) + f(e_2, e_3)$. Thus, $\mathrm{supp}(x) = \mathrm{sat}(x) = \{e_2, e_3\}$.

supp, sat, dep, example with perfect independence

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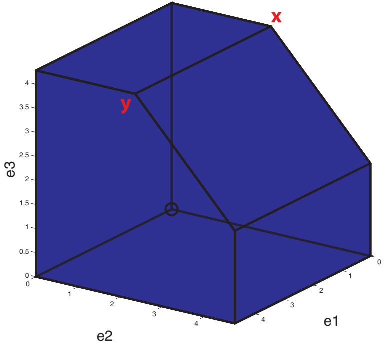

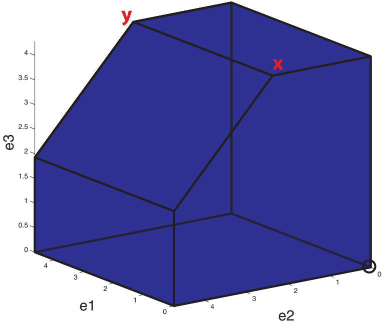

- Note that considering a submodular function on clustered ground set $E = \{e_1, e_{23}\}$ where $f'(e_1) = f(e_1)$, $f'(e_{23}) = f(e_2, e_3)$ leads to a rectangle (no dependence between $\{e1\}$ and $\{e2, e3\}$).

supp, sat, dep, example with perfect independence

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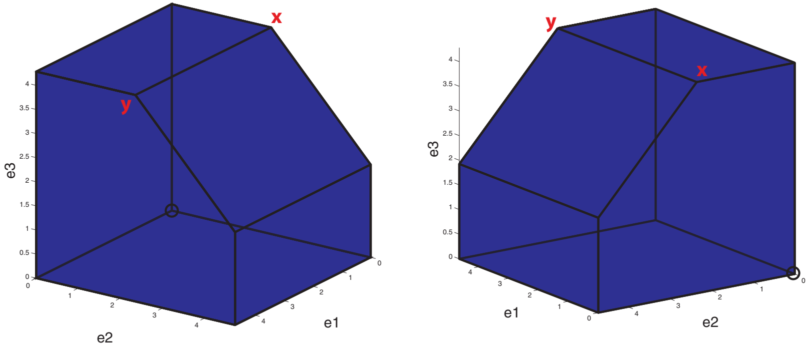

- We also have $\mathrm{sat}(x) = \{e_3, e_2\}$. So $\mathrm{dep}(x, e_1)$ is not defined, $\mathrm{dep}(x, e_2) = \{e_3\}$, and $\mathrm{dep}(x, e_3) = \emptyset$.
- $\mathrm{sat}(y) = \{e_1, e_2, e_3\}$. So $\mathrm{dep}(y, e_1) = \emptyset$, $\mathrm{dep}(y, e_2) = e_3$, and $\mathrm{dep}(y, e_3) = \emptyset$.

supp, sat, dep, example with perfect independence

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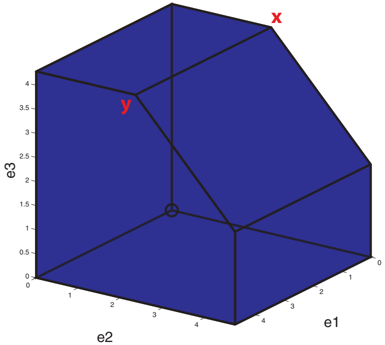

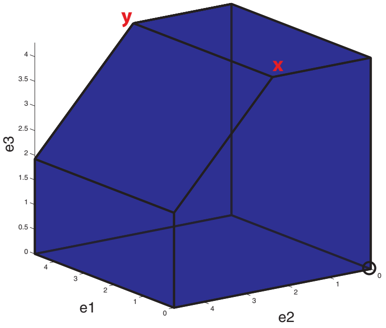

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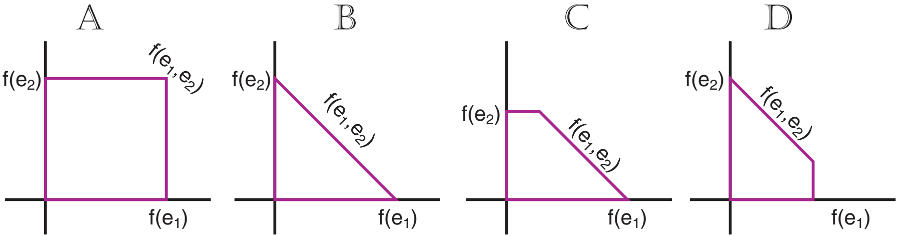
supp, sat, and polymatroid dependence in 2D

- Ex: various amounts of "dependence" between e_1 and e_2.



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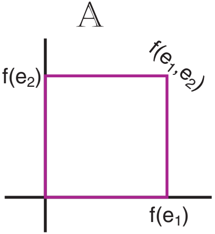

- Case A: perfect independence/irredunancy.

supp, sat, and polymatroid dependence in 2D

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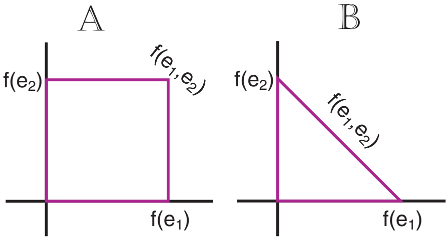

- Case A: perfect independence/irredunancy.
- Case B: perfect dependence/redundancy. Since slope is -45°, we must have $f(e_1) = f(e_2) = f(e_1, e_2)$. Entropy case: deterministic bijection between random variables e_1 and e_2.

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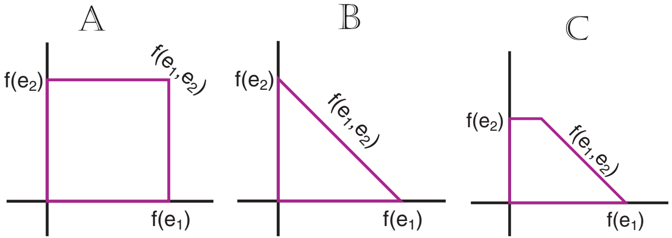

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- Case C: $f(e_2) < f(e_1) = f(e_1, e_2)$. Entropy case: random variable e_2 a deterministic function of e_1 which has higher entropy.

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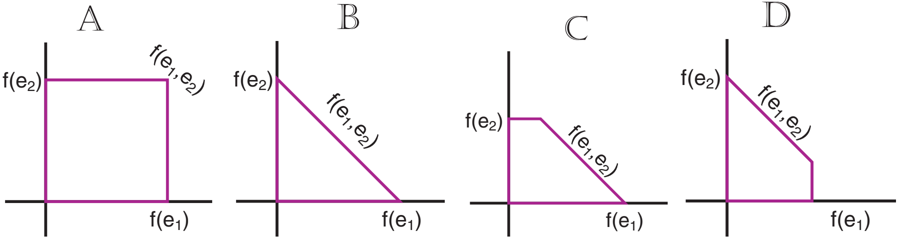

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- Case D: $f(e_1) < f(e_2) = f(e_1, e_2)$. Entropy case: random variable e_1 a deterministic function of e_2 which has higher entropy.

supp, sat, and polymatroid dependence in 2D

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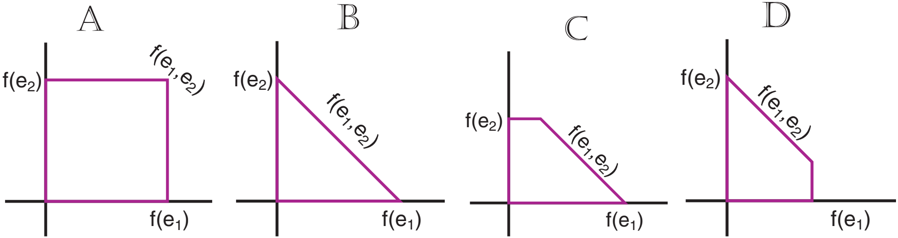
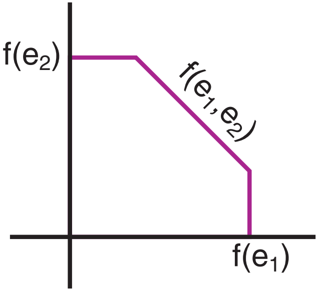


- In each case, we see points x where $\mathrm{supp}(x) \subseteq \mathrm{sat}(x)$.
- Example: Case B or C, let $x = (f(e_1), 0)$ so $\mathrm{supp}(x) = \{e_1\}$ but since $x(\{e_1, e_2\}) = x(\{e_1\}) = f(e_1) = f(e_1, e_2)$ we have $\mathrm{sat}(x) = \{e_1, e_2\}$.
- Similar for case D with $x = (0, f(e_2))$.

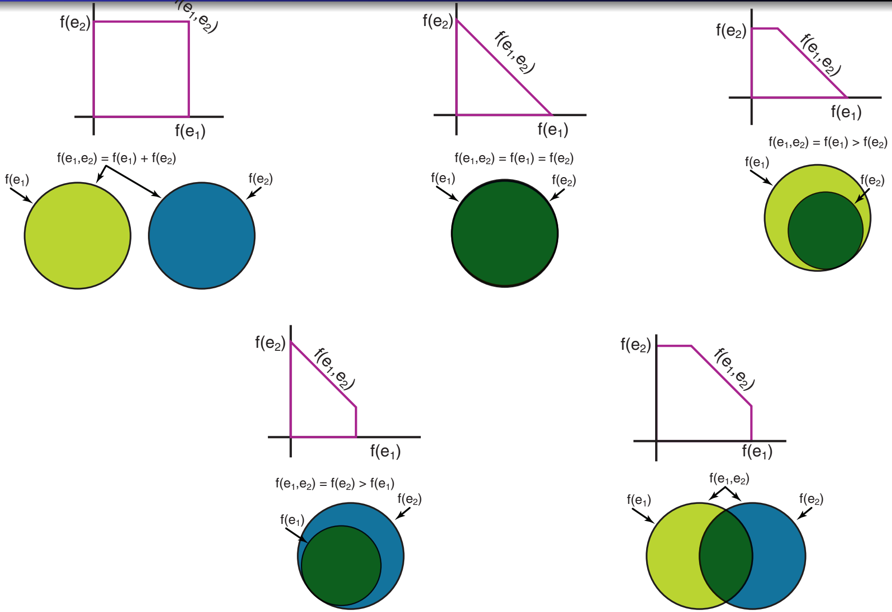
supp, sat, and dependence in 2D

- General case, $f(e_1, e_2) < f(e_1) + f(e_2)$, $f(e_1) < f(e_1, e_2)$, and $f(e_2) < f(e_1, e_2)$.



- Entropy case: We have a random variable Z and two separate deterministic functions $e_1 = h_1(Z)$ and $e_2 = h_2(Z)$ such that the entropy $H(e_1, e_2) = H(Z)$, but each deterministic function gives a different "view" of Z, each contains more than half the information, and the two are redundant w.r.t. each other $(H(e_1) + H(e_2) > H(Z))$.

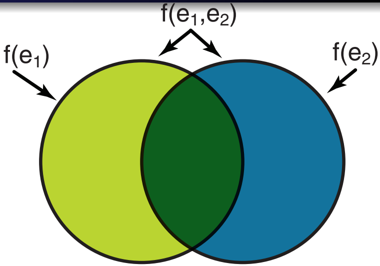
2D polymatroids and Information Venn Diagrams

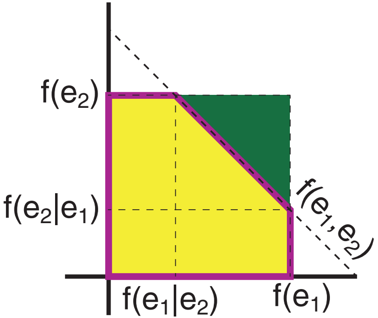


2D polymatroids and Information Venn Diagrams

- Consider symmeterized combinatorial mutual information function:

$$I_f(e_1, e_2) = f(e_1) + f(e_2) - f(e_1, e_2)$$





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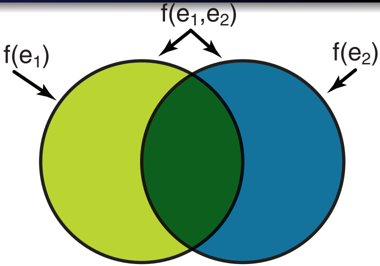
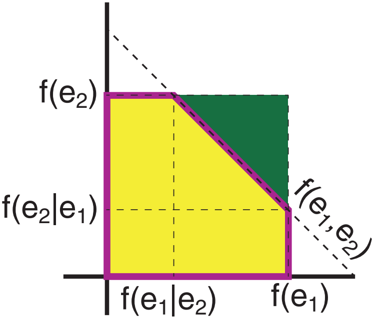
- Consider symmeterized combinatorial mutual information function:

$$I_f(e_1, e_2) = f(e_1) + f(e_2) - f(e_1, e_2)$$

- Consider area of green triangle:

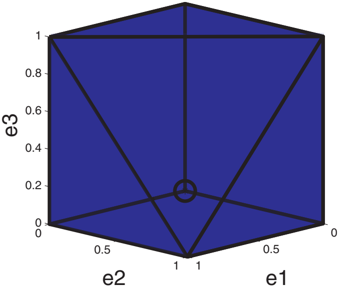
$$\frac{1}{2}\Big(f(e_2) - f(e_2|e_1)\Big)\Big(f(e_1) - f(e_1|e_2)\Big)$$
$$= \frac{1}{2}(f(e_1) + f(e_2) - f(e_1, e_2))^2$$
$$= \frac{1}{2}\Big(I_f(e_1, e_2)\Big)^2$$

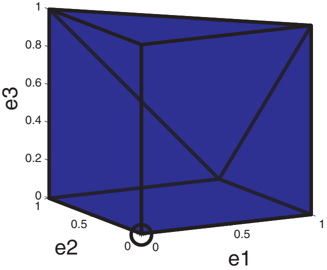




supp, sat, and perfect dependence in 3D

- Ex: polymatroid with perfect independence between e_2 and e_3, so $f(e_2, e_3) = f(e_2) + f(e_3)$, but perfect dependence between $A = \{e_2, e_3\}$ and $B = \{e_1\}$, so $f(e_1, e_2, e_3) = f(e_2, e_3)$





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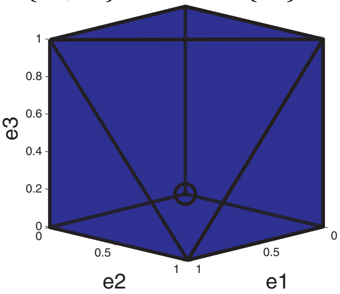

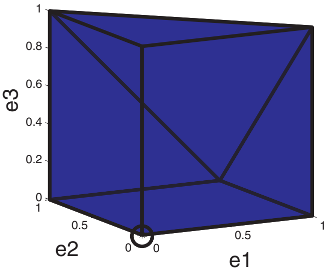

- Entropy case: xor V-structure Bayesian network $e_1 = h(e_2, e_3)$ where h is the xor function ($e_2 \to e_1 \leftarrow e_3$), and e_2, e_3 are both independent binary with unity entropy.

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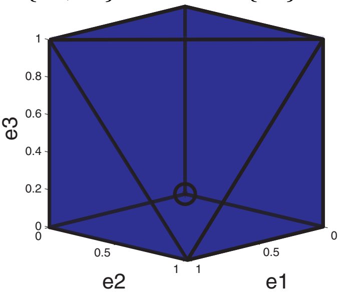

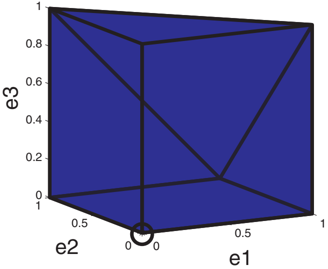

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- Q: Why does the polytope have a symmetry? Notice independence (square) for any pair.

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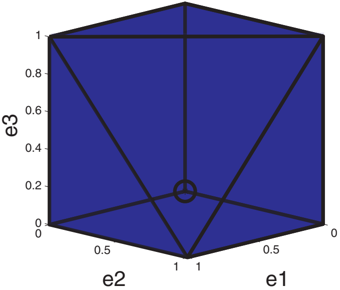
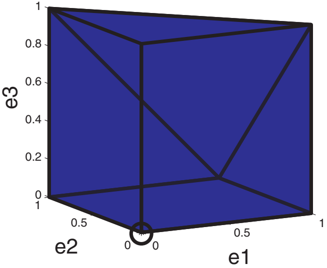
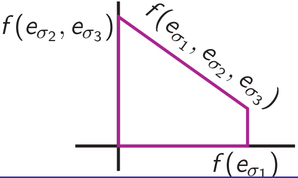




- For any permutation σ of $\{1, 2, 3\}$, considering $\{e_{\sigma_1}, e_{\sigma_2}\}$ vs. $\{e_{\sigma_3}\}$:

e_{σ_3} is a deterministic function of $\{e_{\sigma_1}, e_{\sigma_2}\}$



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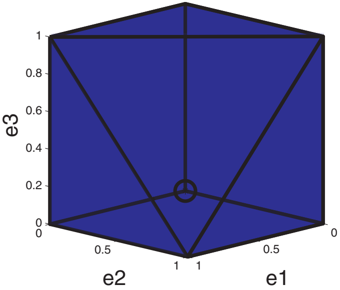

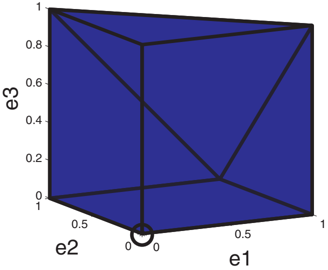

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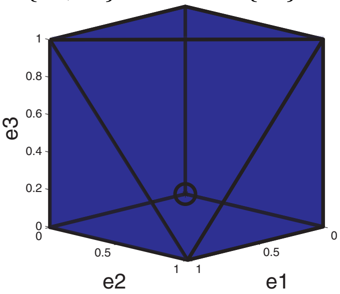

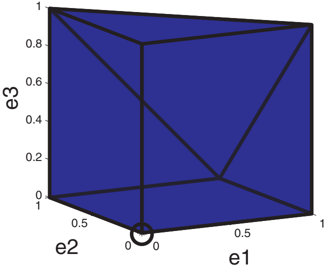

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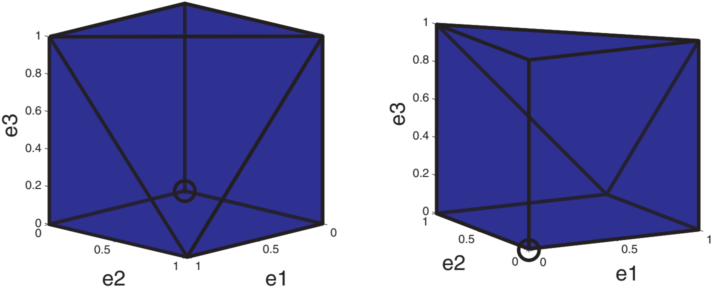

- Note also, that for some of the extreme points, multiple orders generate them.
- Consider extreme point $x = (x_1, x_2, x_3) = (1, 1, 0)$. Then we get this either with orders (e_1, e_2, e_3), or (e_2, e_1, e_3). This is true since $f(e_{\sigma_e} | \{e_{\sigma_1}, e_{\sigma_2}\}) = 0$ for all permutations σ of $\{1, 2, 3\}$.

perfect dependence in 3D, entropy, and Bayesian networks

- The example in the previous slides can be realized with entropy of random variables and a Bayesian network.

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- Consider three binary random variables $X_1, X_2, X_3 \in \{0, 1\}$ that factor w.r.t., the V-structure $X_1 \to X_3 \leftarrow X_2$, where $X_3 = X_1 \oplus X_2$, where $\oplus$ is the X-OR operator, and where $X_1 \perp\!\!\!\perp X_2$.

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- Consequently, $X_i \perp\!\!\!\perp X_j$ for any $i \neq j$.

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- Moreover, for any permutation σ of $\{1, 2, 3\}$, we have the relationship $X_{\sigma_1} = X_{\sigma_2} \oplus X_{\sigma_3}$.

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- Moreover, for any permutation σ of $\{1, 2, 3\}$, we have the relationship $X_{\sigma_1} = X_{\sigma_2} \oplus X_{\sigma_3}$.
- The entropy function $f(A) = H(X_A)$ is a submodular function that will have the symmetric 3D polytope of the previous example.

supp, sat, extremal x, perfect dependence

- In general, for extremal x, $\mathrm{sat}(x) \supseteq \mathrm{supp}(x)$.

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- Note also, for any $A \subseteq \text{sat}(x) \setminus \text{supp}(x)$, we have $f(A | \text{supp}(x)) = 0$.

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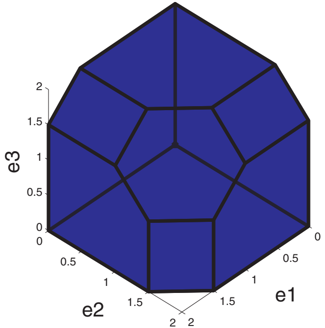
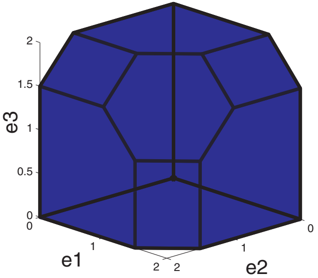
- For general $x \in P_f$ (not nec. extremal), $sat(x)$ and $\mathrm{supp}(x)$ might have an arbitrary relationship (but we want to strengthen this relationship further, and we will do so below).
- For the most part, we are interested in these quantities when x is extremal as we will see.

supp and sat, example under limited curvature

- Strict monotone f polymatroids, where $f(e|E \setminus e) > 0, \forall e$.
- Example: $f(A) = \sqrt{|A|}$, where all $m!$ vertices of B_f are unique.





- In such cases, taking any extremal point $x \in P_f$ based on prefix order $E = (e_1, \dots)$, where $\text{supp}(x) \subset E$, we have that $\text{sat}(x) = \text{supp}(x)$ since the largest tight set corresponds to $x(E_i) = f(E_i)$ for some i, and while any $e \in E \setminus E_i$ is such that $x(E_i + e) = x(E_i)$, there is **no** such e with $f(E_i + e) = f(E_i)$.

Another revealing theorem

Theorem 15.5.1

Let f be a polymatroid function and suppose that E can be partitioned into $(E_1, E_2, \ldots, E_k)$ such that $f(A) = \sum_{i=1}^{k} f(A \cap E_i)$ for all $A \subseteq E$, and k is maximum. Then the base polytope $B_f = \{x \in P_f : x(E) = f(E)\}$ (the E-tight subset of P_f) has dimension $|E| - k$.

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- Thus, any point $x \in B_f$ is a convex combination of at most $|E| - k + 1$ vertices of B_f.
- And if f does not have such independence, dimension of B_f is $|E| - 1$ and any point $x \in B_f$ is a convex combination of at most $|E|$ vertices of B_f.

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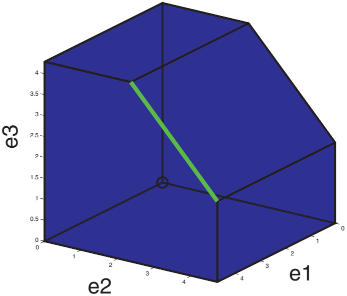
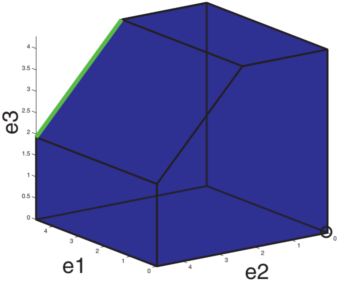
- Example f with independence between $A = \{e_2, e_3\}$ and $B = \{e_1\}$, i.e., $e_1 \perp\!\!\!\perp \{e_2, e_3\}$, with B_f marked in green.





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- Given polymatroid function f, the base polytope $B_f = \left\{x \in \mathbb{R}_+^E : x(A) \leq f(A) \; \forall A \subseteq E, \text{ and } x(E) = f(E)\right\}$ always exists.

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- Moreover, in this case, x is a vertex of B_f since it is extremal.

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- In words, B_f intersects all "multi-axis congruent" hyperplanes within R^E of the form $\left\{x \in \mathbb{R}^E : x(A) = f(A)\right\}$ <u>for all</u> $A \subseteq E$.

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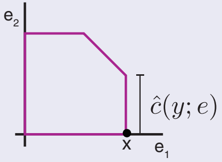
- Consider following algorithm:

$T' \leftarrow T$;
for $e \in E \setminus T$ **do**
  $y \leftarrow y + c(y;e)\mathbf{1}_e$; $T' \leftarrow T' \cup \{e\}$;



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- We set $y(e) \leftarrow y(e) + \hat{c}(y; e) \leq y(e) + f(T' + e) - y(T' + e)$.

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- We set $y(e) \leftarrow y(e) + \hat{c}(y;e) \leq y(e) + f(T'+e) - y(T'+e)$. Hence, after each step, $y \in P_f$ and $\hat{c}(y;e) \geq 0$. (also, consider r.h. version of $\hat{c}(y;e)$).

B_f dominates P_f

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- We set $y(e) \leftarrow y(e) + \hat{c}(y; e) \leq y(e) + f(T' + e) - y(T' + e)$. Hence, after each step, $y \in P_f$ and $\hat{c}(y; e) \geq 0$. (also, consider r.h. version of $\hat{c}(y; e)$).
- Also, only $y(e)$ for $e \notin T$ changed, final y has $y(e) = x(e)$ for $e \in T$.

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- Also, only $y(e)$ for $e \notin T$ changed, final y has $y(e) = x(e)$ for $e \in T$.
- Let $S_e \ni e$ be a set that achieves $c(y;e) = f(S_e) - y(S_e)$.

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- Let $S_e \ni e$ be a set that achieves $c(y; e) = f(S_e) - y(S_e)$.
- At iteration e, let $y'(e)$ (resp. $y(e)$) be new (resp. old) entry for e, then

$$y'(S_e) = y(S_e \setminus \{e\}) + y'(e) \tag{15.3}$$
$$= y(S_e \setminus \{e\}) + [y(e) + f(S_e) - y(S_e)] = f(S_e)$$

B_f dominates P_f

. . . proof of Thm. 15.5.2 cont.

- Each step maintains feasibility: consider one step adding e to T' — for $e \notin T'$, feasibility requires $y(T' + e) = y(T') + y(e) \leq f(T' + e)$, or $y(e) \leq f(T' + e) - y(T') = y(e) + f(T' + e) - y(T' + e)$.
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So, S_e is tight for y'. It remains tight in further iterations since y doesn't decrease and it stays within P_f.

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...proof of Thm. 15.5.2 cont.

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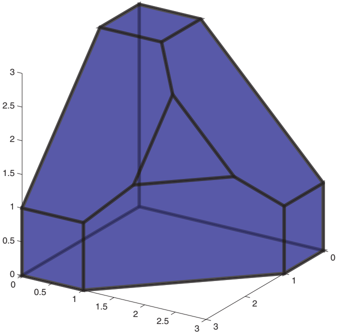
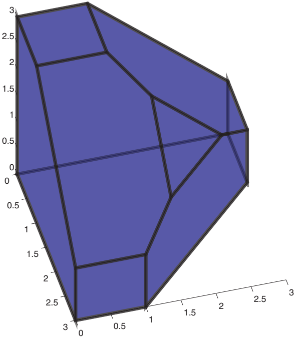
So, S_e is tight for y'. It remains tight in further iterations since y doesn't decrease and it stays within P_f.
- Also, $E = T \cup \bigcup_{e \notin T} S_e$ is also tight, meaning the final y has $y \in B_f$.

Polytope example 1

- Observe: P_f (at two views):





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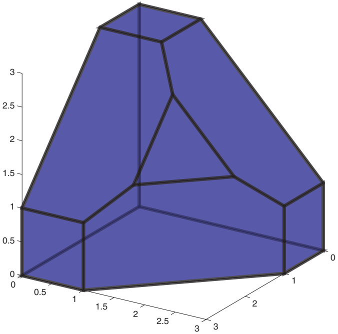

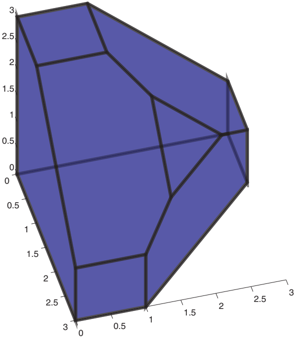

- Is this a polymatroidal polytope?

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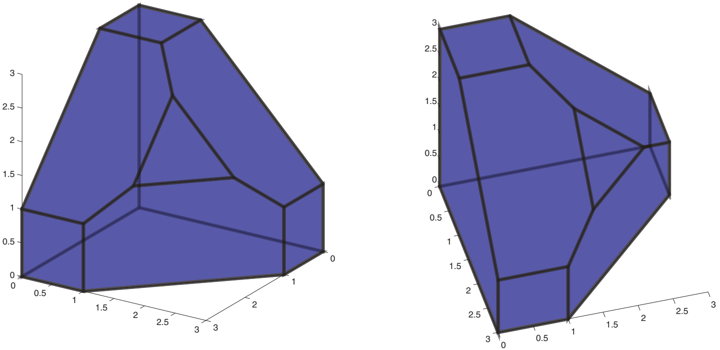

- Is this a polymatroidal polytope?
- No, "B_f" doesn't intersect sets of the form $\{x : x(e) = f(e)\}$ for $e \in E$.

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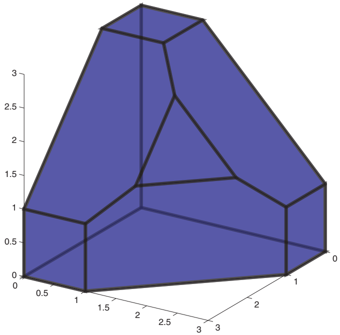
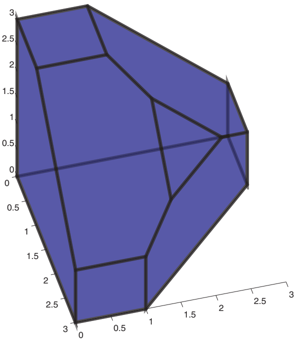
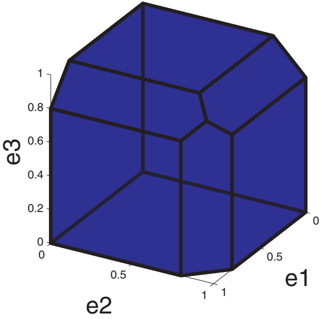


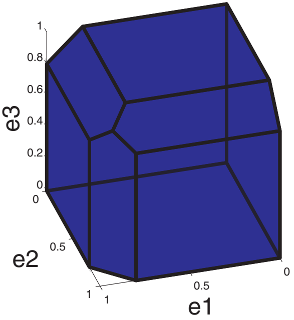


- Is this a polymatroidal polytope?
- No, "B_f" doesn't intersect sets of the form $\{x : x(e) = f(e)\}$ for $e \in E$.
- This was generated using function $g(0) = 0$, $g(1) = 3$, $g(2) = 4$, and $g(3) = 5.5$. Then $f(S) = g(|S|)$ is not submodular since (e.g.) $f(\{e_1, e_3\}) + f(\{e_1, e_2\}) = 4 + 4 = 8$ but

Polytope example 2

- Observe: P_f (at two views):





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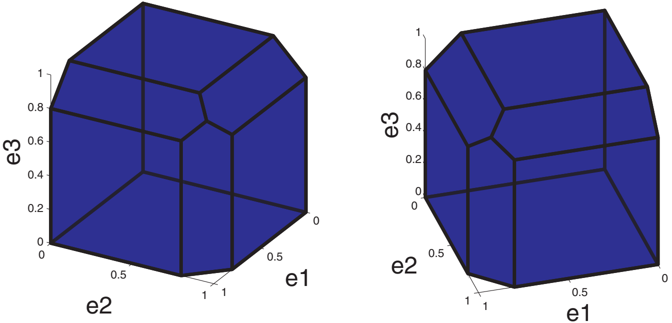

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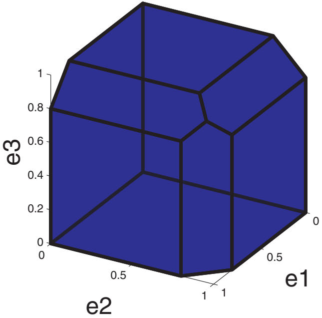

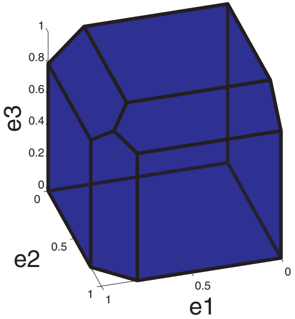

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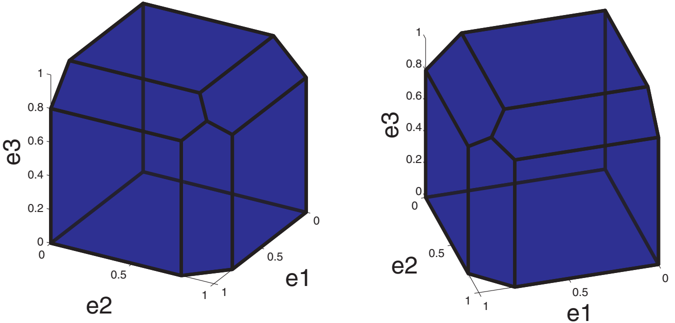

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- No, "B_f" (which would be a single point in this case) doesn't intersect sets of the form $\{x : x(e) = f(e)\}$ for $e \in E$.
- This was generated using function $g(0) = 0$, $g(1) = 1$, $g(2) = 1.8$, and $g(3) = 3$. Then $f(S) = g(|S|)$ is not submodular since (e.g.) $f(\{e_1, e_3\}) + f(\{e_1, e_2\}) = 1.8 + 1.8 = 3.6$ but $f(\{e_1, e_2, e_3\}) + f(\{e_1\}) = 3 + 1 = 4$.

Review

The next slide is review from lecture 13.

Saturation Capacity

- The max is achieved when

$$\alpha = \hat{c}(x;e) \stackrel{\text{def}}{=} \min \{ f(A) - x(A), \forall A \supseteq \{e\} \} \tag{15.22}$$

- $\hat{c}(x;e)$ is known as the saturation capacity associated with $x \in P_f$ and e.

Matroids and Exchange

- Recall, matroids have a number of "exchange" properties.

Matroids and Exchange

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- Also, recall that given a matroid $\mathcal{M} = (E, \mathcal{I})$, if $I \in \mathcal{I}$ is independent, and $e \in \text{span}(I)$, and $e' \in C(I, e)$ where $C(I, e)$ is the fundamental circuit created when adding e to I, then we have:

$$I + e - e' \in \mathcal{I} \tag{15.4}$$

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- Since $\text{dep}(x, e)$ generalizes the fundamental circuit of a matroid to polymatroids, we saw (last lecture) that this a property exists for polymatroids as well.

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- As there is saturation capacity for elements that are not saturated, is there is a corresponding concept for degree of polymatroidal exchange?
- Yes, and it is called the "exchange capacity"

Exchange Capacity

- Consider $x \in P_f$, $e \in \mathrm{sat}(x)$ and $e' \in \mathrm{dep}(x, e) \setminus \{e\}$

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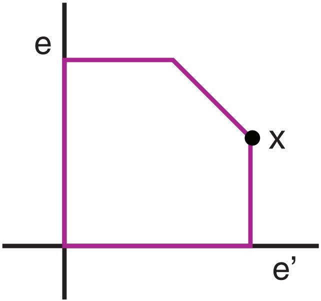
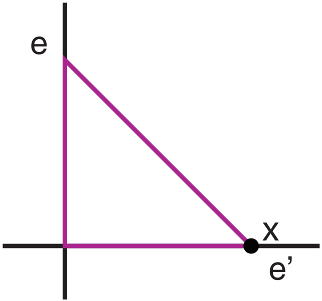
- Examples:





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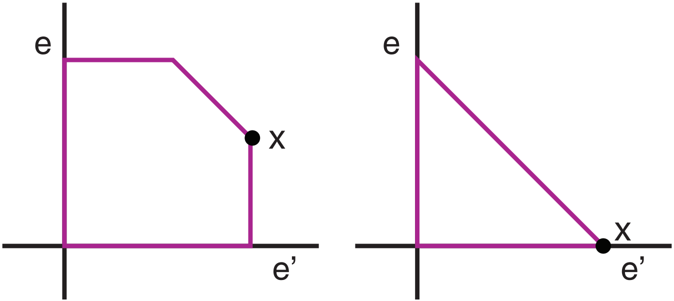

- How much can we move in positive e direction if we simultaneously move in negative e' direction?

Exchange Capacity

- $x \in P_f$, $e \in \text{sat}(x)$ and $e' \in \text{dep}(x, e) \setminus \{e\}$, consider

$$\max \{\alpha : \alpha \in \mathbb{R}, x + \alpha(\mathbf{1}_e - \mathbf{1}_{e'}) \in P_f\} \tag{15.5}$$

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$$\max \{\alpha : \alpha \in \mathbb{R}, (x + \alpha(\mathbf{1}_e - \mathbf{1}_{e'}))(A) \leq f(A), \forall A\} \tag{15.6}$$

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- Which is identical to:

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- In such case, we get $\mathbf{1}_{e'}(A) = 0$, thus above identical to

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- For any α with $0 \le \alpha \le \hat{c}(x; e, e')$, we have that $x + \alpha(\mathbf{1}_e - \mathbf{1}_{e'}) \in P_f$.

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$$\max\left\{\alpha : \alpha \in \mathbb{R}, \alpha \le f(A) - x(A), \forall A \supseteq \{e\}, e' \notin A\right\} \quad (15.10)$$

- This max is achieved when

$$\alpha = \hat{c}(x; e, e') \stackrel{\text{def}}{=} \min\left\{f(A) - x(A), \forall A \supseteq \{e\}, e' \notin A\right\} \quad (15.11)$$

- $\hat{c}(x; e, e')$ is known as the exchange capacity associated with $x \in P_f$ and e.
- For any α with $0 \le \alpha \le \hat{c}(x; e, e')$, we have that $x + \alpha(\mathbf{1}_e - \mathbf{1}_{e'}) \in P_f$.
- As we will see, if e and e' are successive in an order that generates extreme point x, then we get a "neighbor" extreme point via $x' = x + \hat{c}(x; e, e')(\mathbf{1}_e - \mathbf{1}_{e'})$.

Exchange Capacity

- In such case, we get $\mathbf{1}_{e'}(A) = 0$, thus above identical to

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- Note that Eqn. (15.11) is a form of SFM.

A polymatroid function's polyhedron is a polymatroid.

Theorem 15.7.1

Let f be a submodular function defined on subsets of E. For any $x \in \mathbb{R}^E$, we have:

$$\text{rank}(x) = \max\left(y(E) : y \leq x, y \in P_f\right) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{15.5}$$

Essentially the same theorem as Theorem **??**. Taking $x = 0$ we get:

Corollary 15.7.2

Let f be a submodular function defined on subsets of E. $x \in \mathbb{R}^E$, we have:

$$\text{rank}(0) = \max\left(y(E) : y \leq 0, y \in P_f\right) = \min\left(f(A) : A \subseteq E\right) \tag{15.6}$$

Min-Norm Point: Definition

- Restating what we saw before, we have:

$$\max \{y(E) | y \in P_f, y \leq 0\} = \min \{f(X) | X \subseteq V\} \tag{15.12}$$

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- Consider the optimization:

$$\begin{aligned} &\text{minimize} && \|x\|_2^2 && (15.13a)\\ &\text{subject to} && x \in B_f && (15.13b)\end{aligned}$$

where B_f is the base polytope of submodular f, and $\|x\|_2^2 = \sum_{e \in E} x(e)^2$ is the squared 2-norm. Let x^* be the optimal solution.

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- Note, x^* is the unique optimal solution since we have a strictly convex objective over a set of convex constraints.

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- Note, x^* is the unique optimal solution since we have a strictly convex objective over a set of convex constraints.
- x^* is called the minimum norm point of the base polytope.

Min-Norm Point: Examples