Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 14 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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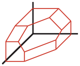
$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

$= f(A_r) + 2f(C) + f(B_r)$ $\quad = f(A_r) + f(C) + f(B_r)$ $\quad = f(A \cap B)$



Cumulative Outstanding Reading

- Good references for today: Birkhoff, "Lattice Theory", 1967.

Announcements, Assignments, and Reminders

- This is a special extra lecture on lattices that is not given during regular lecture time.

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, Combinatorial Geometries
- L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
- L9: From Matroid Polytopes to Polymatroids.
- L10: Polymatroids and Submodularity
- L11: More properties of polymatroids, SFM special cases
- L12: polymatroid properties, extreme points polymatroids,
- L13: sat, dep, supp, exchange capacity, examples
- L14: Lattice theory: partially ordered sets; lattices; distributive, modular, submodular, and boolean lattices; ideals and join irreducibles.
- L15: Supp, Base polytope, polymatroids and entropic Venn diagrams, exchange capacity,
- L16: proof that minimum norm point yields min of submodular function, and the lattice of minimizers of a submodular function, Lovasz extension
- L17: Lovasz extension, Choquet Integration, more properties/examples of Lovasz extension, convex minimization and SFM.
- L18: Lovasz extension examples and structured convex norms, The Min-Norm Point Algorithm detailed.
- L19: symmetric submodular function minimization, maximizing monotone submodular function w. card constraints.
- L20: maximizing monotone submodular function w. other constraints, non-monotone maximization.

Finals Week: June 9th-13th, 2014.

Outline

- We're next going to study lattices and submodular functions.
- In doing so, we'll better be able to understand certain properties of polymatroidal extreme points and ultimately SFM.

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- In a poset, for any $x, y, z \in V$ the following conditions hold (by definition):

For all $x, x \preceq x$.	(Reflexive)	(P1.)
If $x \preceq y$ and $y \preceq x$, then $x = y$	(Antisymmetriy)	(P2.)
If $x \preceq y$ and $y \preceq z$, then $x \preceq z$.	(Transitivity)	(P3.)

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- We can use the above to get other operators as well such as "less than" via $x \preceq y$ and $x \neq y$ implies $x \prec y$. Also, we get $x \succ y$ if not $x \preceq y$. And $x \succeq y$ is read "x contains y". And so on.

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- The order $n(P)$ of a poset P is meant the (cardinal) number of its elements.

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- Given two elements, we need not have either $x \preceq y$ or $y \preceq x$ be true, i.e., these elements might not be comparable. If for all $x, y \in V$ we have $x \preceq y$ or $y \preceq x$ then the poset is **totally ordered**.

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- There may exist only one element x which satisfies $x \preceq y$ for all y. Since if $x \preceq y$ for all y and $z \preceq y$ for all y then $z \preceq x$ and $x \preceq z$ implying $x = z$. If it exists, we can name this element 0 (zero). The dual maximal element is called 1 (one).

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- We define a set of elements $x_1, x_2, \ldots, x_n$ as a chain if $x_1 \preceq x_2 \preceq \cdots \preceq x_n$, which means $x_1 \preceq x_2$ and $x_2 \preceq x_3$ and $\ldots x_{n-1} \preceq x_n$. While we normally thing of the elements of a chain as distinct they need not be. The length of a chain of n elements is $n - 1$.

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Example 14.3.1

Let $V = \mathbb{Z}^+$ be the set of positive integers and let $x \preceq y$ mean that x is less than y in the usual sense. Then we have a poset that is actually totally ordered.

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Let V consist of all real single-valued functions $f(x)$ defined on the closed interval $[-1, 1]$, and let $g \leq f$ mean that $g(x) \leq f(x)$ for all $x \in [-1, 1]$. Again poset, but not total order.

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- Any subset of a chain is a chain.
- Two posets V_1 and V_2 are isomorphic if there is an isomorphism between them (i.e., a 1-1 order preserving (isotone) function that has an order preserving inverse). We write that two posets U and V are isomorphic by $U \simeq V$.

Partially ordered set

- **duality**. The dual poset is formed by exchanging $\preceq$ with $\succeq$. This is called the converse of a partial ordering. The converse of a PO is also a PO. We write the dual of V as V^d. U and V are dually isomorphic if $U = V^d$ or equivalently $V = U^d$. When $U = U^d$ then U is self-dual.

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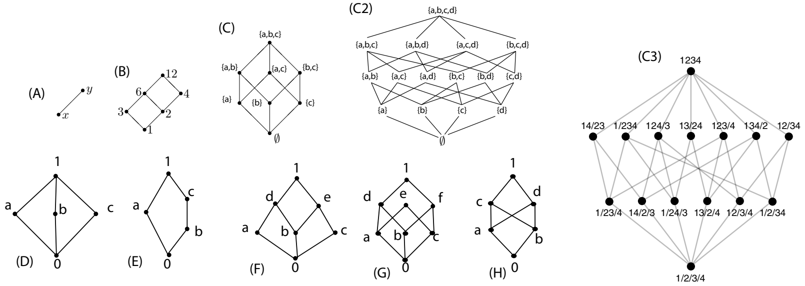
Example 14.3.4

Given an n-dimensional linear (Euclidean) space $\mathbb{R}^n$. A subset of $M \subseteq \mathbb{R}^n$ is an affine set if $(1-\lambda)x + \lambda y \in M$ whenever $x, y \in M$ and $\lambda \in \mathbb{R}$. A *linear subspace* of $\mathbb{R}^n$ is an affine set that contains the origin. Subspaces can be obtained via some A, b such that for every $y \in M$, $y = Ax + b$ for some $x \in \mathbb{R}^n$.
The set of all linear subspaces of $\mathbb{R}^n$ is a poset (ordered by inclusion), and such a set is self-dual.

Partially ordered set



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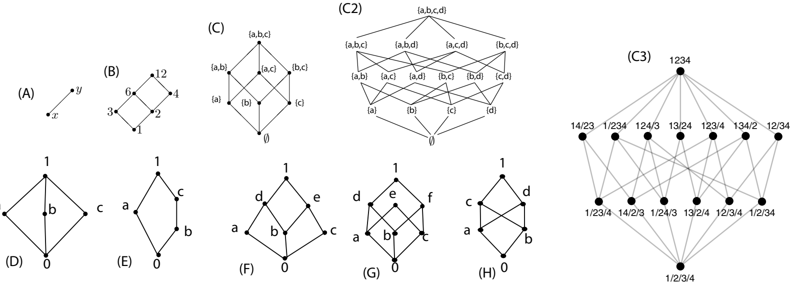

- **cover** y covers x if $x \prec y$ and there exists no z such that $x \prec z \prec y$. Note that the inequalities are strict here. We write $x \sqsubset y$ if y covers x.

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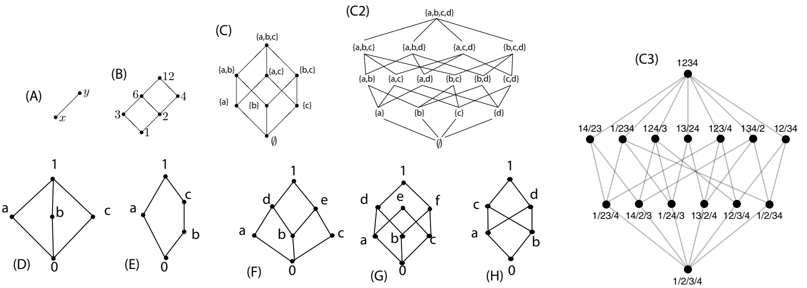

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- A chain is saturated if it is a chain of the form $x_1 \prec x_2 \prec \cdots \prec x_n$ such that $x_1 \sqsubset x_2 \sqsubset \cdots \sqsubset x_n$ (i.e., we have a sequence of coverage relationships where x_{i+1} covers x_i for each $i < n$).

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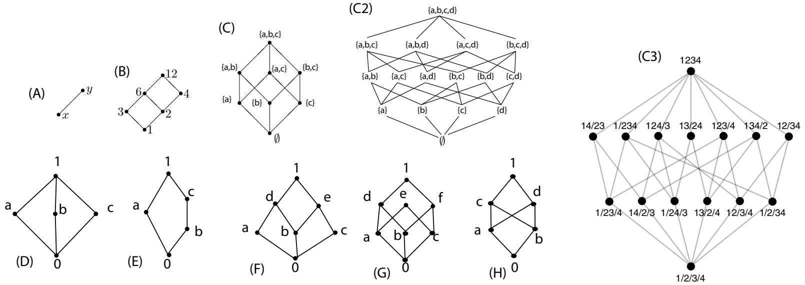

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- Hasse diagram: We can draw a poset using a graph where each $x \in V$ is a node, and if $x \sqsubset y$ we draw y directly above x with a connecting edge, but no other edges.

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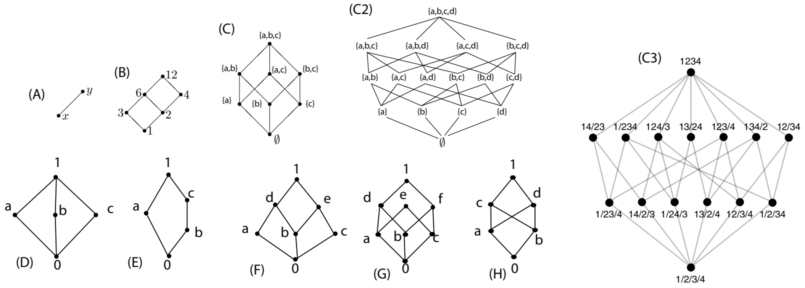

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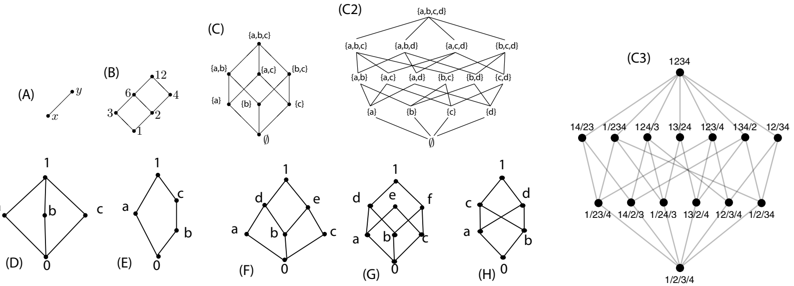

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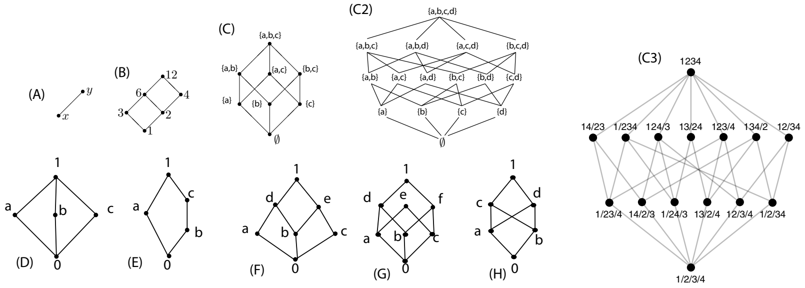

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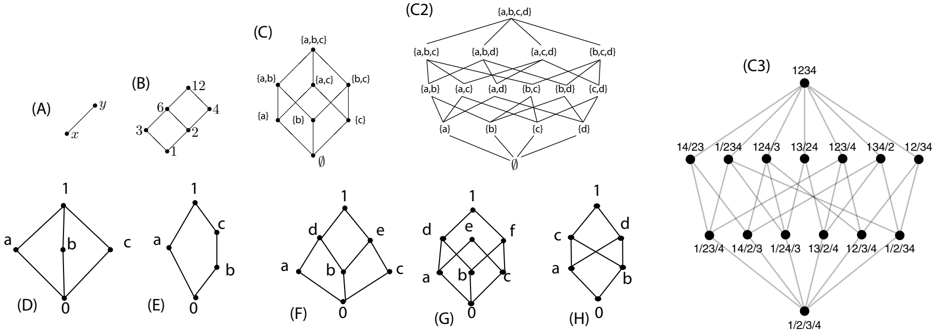
Every non-empty finite subset $X \subseteq V$ has a minimal (and maximal) element.

Partially ordered set/Hasse diagrams examples



For example, in example (A), we see that $x \sqsubset y$. In example (B) we have $3 \preceq 12$ and $3 \sqsubset 6$ but not $3 \sqsubset 12$. Hasse diagram for dual order is obtained by turning Hasse diagram upside down.

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Theorem 14.3.7

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Proof.

Let $X = \{x_1, \dots, x_n\}$. Define $m_1 = x_1$ and

$$m_k = \begin{cases} x_k & \text{if } x_k \prec m_{k-1} \\ m_{k-1} & \text{otherwise} \end{cases} \tag{14.1}$$

Then we have constructed $m_n \preceq m_{n-1} \preceq \cdots \preceq m_1$ meaning there is no m_k for $k < n$ such that $m_k \prec m_n$. By construction, we also have that there is no $x \in X$ with $x \prec m_n$, thus m_n is minimal. Analogously, X has a maximal element. □

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Theorem 14.3.8

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$$m_k = \begin{cases} x_k & \text{if } x_k \prec m_{k-1} \\ m_{k-1} & \text{otherwise} \end{cases} \tag{14.2}$$

Then we have constructed $m_n \preceq m_{n-1} \preceq \cdots \preceq m_1$ meaning there is no m_k for $k < n$ such that $m_k \prec m_n$. Let $M = \{m_1, \ldots, m_n\}$. By construction, we also have that there is no $x \in X$ with $x \prec m_n$, thus m_n is minimal. □

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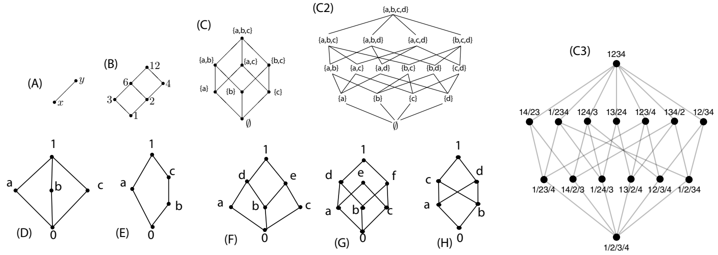
- For example, $\ell(A) = 1$, $\ell(B) = 3$, $\ell(C) = 3$, $\ell(C2) = 4$, $\ell(D) = 2$, $\ell(E) = 3$, $\ell(F) = 3$, and so on.



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- The height or *dimension* of an element $x \in V$, or $l = h(x)$ is the l.u.b. of the lengths of the chains $0 = x_0 \prec x_1 \prec \dots x_l = x$ between 0 and x. Note that $h(1) = \ell(V)$ when they exist.

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- graded posets. Posets might be able to be "graded" by a function $g : V \to \mathbb{Z}$ in the following way

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- A maximal chain is a chain of unique elements between two elements that can not be made any longer

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Proof.

Grading by $h(x)$ makes JDCC true since the length of any chain between $a \succ b$ is $h(a) - h(b)$. Conversely, given JDCC, and given $h(x)$ as defined (length of the maximal length chain from 0 to x), then G1 and G2 follow with $h(\cdot)$ as the grading function. $\square$

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(or JDCC) All maximal length chains between the same endpoints have the same finite length.

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Let V be a poset with $0 \in V$ and where all chains are finite. Then V satisfies JDCC iff it is graded by $h(x)$ (the height function).

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- The height (rank) function in this case is unique. If $x \preceq y$ then $\ell(x, y) = h(y) - h(x)$ is the length between x and y.
- In fact, any graded poset satisfies JDCC, and hence is graded by the height function. Hence, it sufficies for poset to be graded if it is graded by the height function.

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- We say a poset is "graded" if it is graded by the height function.

Lattices

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- Early work on lattices occurred around 1990 by Richard Dedekind, on what were called "modular" lattices (we'll see this below).
- Birkhoff concentrated on the mathematics of lattices in the 20th century.
- Lattices are a useful algebraic structure, but also shed some light on one of the origins of submodular functions (including their name).

Lattice defined

- Given $X \subset V$, $y \in V$ is an upper bound of X if $x \preceq y$ for all $x \in X$. Note that y need not be in X.

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Definition 14.4.1 (lattice)

A *lattice* is a poset V such that any two elements $x, y \in v$ have a g.l.b. or meet denoted by $x \wedge y \in V$, and also have a l.u.b. or join denoted by $x \vee y \in V$. A lattice is complete when all subsets $X \subseteq V$ have both a l.u.b. and a g.l.b. (note that join and meet is defined on pairs, but l.u.b. and g.l.b. can be defined on any subset of V, even of size 1).

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- Note again, that such l.u.b.'s and g.l.b.'s are unique if they exist.

Lattices

- Any finite lattice or lattice of finite length is complete. Note that the reverse need not hold (a complete lattice need not be finite). The reals are not complete but the extended reals are complete. The rationals are not complete (but the rationals extended with a $\pm\infty$ is complete). 2^E for some set E is complete. Note that E can be countably or uncountably infinite.

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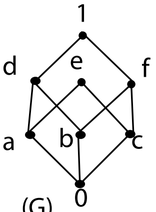
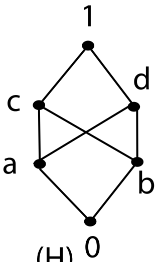


(G)



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- Any non-empty lattice contains a greatest element $1 \in V$ and a least element $0 \in V$.
- The dual of a lattice is a lattice, and the dual of a complete lattice is a complete lattice.
- In a chain, $x \wedge y$ is the smaller of the two, and $x \vee y$ is the larger of the two.

Lattices

Definition 14.4.2 (sublattice)

A sublattice of a lattice is a subset $X \subseteq V$ such that join and meet are closed within X (for all $x, y \in X$, $x \vee y \in X$ and $x \wedge y \in X$). A sublattice is a lattice.

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- Given any $x \preceq y$, then all elements $\{z : x \preceq z \preceq y\}$ form a sublattice. We note that in such case, we say that $[x, y]$ form a (closed) interval in the lattice, and we have that the (closed) interval $[x, y]$ of all elements $z \in L$ such that $x \preceq z \preceq y$ is a sublattice.

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- A convex subset X of a poset V is a subset such that for all $x, y \in V$ with $x \preceq y$, $\{z : x \preceq z \preceq y\} \subseteq X$. A subset X of a *lattice* V is a convex sublattice if $x, y \in X$ imply that $\{z : x \wedge y \preceq z \preceq x \vee y\} \subseteq X$.

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- Obviously, 2^E for some set E is a lattice, with join/meet being union/intersection. See Figure(C).

Lattices

Example 14.4.3

Given lattices U, V, we can form the direct product UV by forming pairs $\{(u,v) : u \in U, v \in V\}$ and ordered so that $(u_1, v_1) \preceq (u_2, v_2)$ iff $u_1 \preceq u_2$ in U and $v_1 \preceq v_2$ in V. The direct product of two lattices is a lattice.

Theorem 14.4.4

In any poset V, the operations of meet and join satisfy the following laws, whenever the associated expressions exist.

$x \wedge x = x, x \vee x = x$	*(Idempotent)*	(L1)
$x \wedge y = y \wedge x, x \vee y = y \vee x$	*(Commutative)*	(L2)
$x \wedge (y \wedge z) = (x \wedge y) \wedge z, x \vee (y \vee z) = (x \vee y) \vee z$	*(Associative)*	(L3)
$x \wedge (x \vee y) = x \vee (x \wedge y) = x$	*(Absorption)*	(L4)
$x \preceq y \iff x \wedge y = x$ *and* $x \vee y = y$	*(Consistency)*	(CON)

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$x \preceq y \iff x \wedge y = x$ *and* $x \vee y = y$	*(Consistency)*	(CON)

Note the above works for posets, not necessary for it to be a lattice.

Lattices

Theorem 14.4.5

Given a poset V with $0 \in V$, then for all $x \in V$,

$$0 \wedge x = 0 \text{ and } 0 \vee x = x \tag{14.3}$$

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Theorem 14.4.6

In any lattice, the operations of join and meet are order-preserving in the following sense:

$$y \preceq z \Rightarrow x \wedge y \preceq x \wedge z \text{ and } x \vee y \preceq x \vee z \tag{14.4}$$

Distributive Inequalities

Theorem 14.4.7

In any lattice, the following distributive inequalities hold for all $x, y, z \in V$*:*

$$x \wedge (y \vee z) \succeq (x \wedge y) \vee (x \wedge z) \tag{14.5a}$$

$$x \vee (y \wedge z) \preceq (x \vee y) \wedge (x \vee z) \tag{14.5b}$$

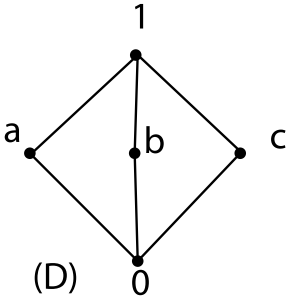
Proof.

We have $x \wedge y \preceq x$ and $x \wedge y \preceq y \preceq y \vee z$. Therefore, $x \wedge y \preceq x \wedge (y \vee z)$. Also, since $x \wedge z \preceq x$ and $x \wedge z \preceq z \preceq y \vee z$, we have $x \wedge z \preceq x \wedge (y \vee z)$. Thus, $x \wedge (y \vee z)$ is an upper bound of both $x \wedge y$ and $x \wedge z$, which means that it is an upper bound of their join.
Eq 14.5b is by duality. □

- Note, this does not mean the lattice is distributive, which we define in a few slides below.

Distributive Inequalities

- Note that these are inequalities and they hold in any lattice. Equality might not hold for all lattices, consider figures (D) and (E).



(D)

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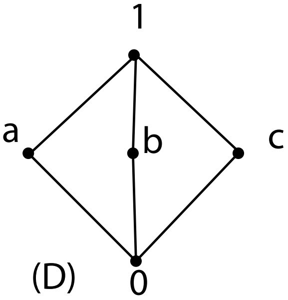

(D)

- For example, in (D), we have that $a \wedge (b \vee c) = a \wedge 1 = a$ but $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$ and obviously $a \succ 0$.

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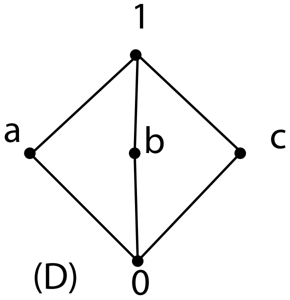

(D)

- For example, in (D), we have that $a \wedge (b \vee c) = a \wedge 1 = a$ but $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$ and obviously $a \succ 0$.
- Also, in (D) we have $a \vee (b \wedge c) = a \vee 0 = a \prec (a \vee b) \wedge (a \vee c) = 1 \wedge 1 = 1$.

Distributive Inequalities

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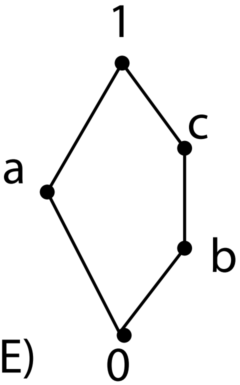

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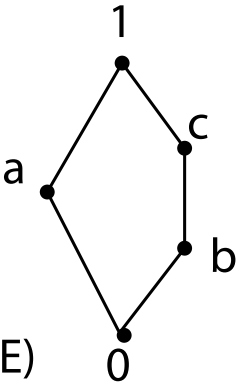

(E)

- In (E), we have that $c \wedge (a \vee b) = c \wedge 1 = c \succ (c \wedge a) \vee (c \wedge b) = 0 \vee b = b$

Distributive Inequalities

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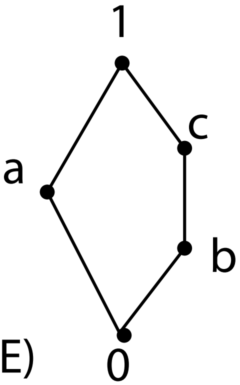

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- Also, in (E), we have $b \vee (a \wedge c) b \vee 0 = b \prec (b \vee a) \wedge (b \vee c) = 1 \wedge c = c$

Modular inequality

Theorem 14.4.8

In any lattice, the following *modular inequalities* *holds for all* $x, y, z \in V$*:*

$$x \preceq z \Rightarrow x \vee (y \wedge z) \preceq (x \vee y) \wedge z \tag{14.6}$$

Modular inequality

Theorem 14.4.8

In any lattice, the following modular inequalities *holds for all* $x, y, z \in V$*:*

$$x \preceq z \Rightarrow x \vee (y \wedge z) \preceq (x \vee y) \wedge z \tag{14.6}$$

Proof.

We have that $x \preceq x \vee y$ and are given $x \preceq z$. Then, $x \preceq (x \vee y) \wedge z$. Also, $y \wedge z \preceq y \preceq x \vee y$ and with $y \wedge z \preceq z$ gives $y \wedge z \preceq (x \vee y) \wedge z$. Then $x \vee (y \wedge z) \preceq (x \vee y) \wedge z$.

□

Modular inequality

Theorem 14.4.8

In any lattice, the following *modular inequalities* *holds for all* $x, y, z \in V$:

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- The term "modular" somehow comes from abstract algebra, where a R-module is an abstract system that generalizes $(\mathbb{R}, \mathbb{R}^n)$ (i.e., a vector field with scalar multiplication). An R-module ends up being a lattice that satisfies this property with equality (see below).

Distributive Lattices

- A lattice is distributive if the aforementioned distributive inequality is an equality. Note that as mentioned above, the distributive inequality holds for all lattices, but not with equality.

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Theorem 14.5.1

In any lattice, the following are equivalent:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad \forall x, y, z \tag{14.7a}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad \forall x, y, z \tag{14.7b}$$

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It is important to note the $\forall x, y, z$ since this is not true only for individual elements. Note moreover that this means that the operators $\vee = +$ and $\wedge = \cdot$ do not form a lattice over $\mathbb{R}$.

Distributive Lattices

Theorem 14.5.2

In any lattice, the following are equivalent:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad \forall x, y, z \tag{14.8a}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad \forall x, y, z \tag{14.8b}$$

Distributive Lattices

Theorem 14.5.2

In any lattice, the following are equivalent:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad \forall x, y, z \tag{14.8a}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad \forall x, y, z \tag{14.8b}$$

Proof.

Take as given the 2nd equation and show the first. Then

$$(x \wedge y) \vee (x \wedge z) = [(x \wedge y) \vee x] \wedge [(x \wedge y) \vee z] \quad \text{by the 2nd eq} \tag{14.9}$$

$$= x \wedge [(x \wedge y) \vee z] \quad x \wedge y \preceq x \tag{14.10}$$

$$= x \wedge [(x \vee z) \wedge (y \vee z)] \quad \text{by the 2nd eq} \tag{14.11}$$

$$= x \wedge (x \vee z) \wedge (y \vee z) \quad \text{associative} \tag{14.12}$$

$$= x \wedge (y \vee z) \quad x \vee z \succeq x \tag{14.13}$$

Distributive Lattices

- Note that any chain is a distributive lattice. The dual of any distributive lattice is distributive.

Distributive Lattices

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Theorem 14.5.4 (identity)

In a distributive lattice, if $z \wedge x = z \wedge y$ and $z \vee x = z \vee y$ then $x = y$.

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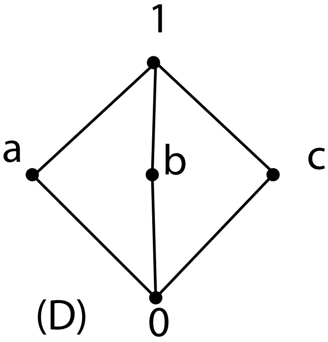
- Easy way to remember. x,y,z and $x \vee \Big(y \wedge z\Big) = \Big(x \vee y\Big) \wedge z$
- The term "modular" comes from abstract algebra, where a R-module is an abstract system that generalizes $(\mathbb{R}, \mathbb{R}^n)$ (i.e., a vector field with scalar multiplication). An R-module ends up being a lattice that satisfies this identity.

Modular Lattices

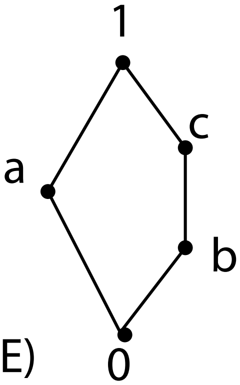
- Not every lattice is modular. Figure (D) is modular but not distributive. We already saw that (D) is not distributive since it is strict for certain assignments. It is modular though.



(D)

Modular Lattices

- Figure (E) is neither modular nor distributive. We saw that it was not distributive since it achieved strictness in the distributive inequalities. It is not modular since: take $b \preceq c$, then $b \vee (a \wedge c) = b \vee 0 = b \prec (b \vee a) \wedge c = 1 \wedge c = c$, so modular equality is violated.



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- Thus, the structure (E) is fundamental to non-modular lattices.

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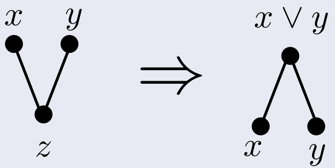

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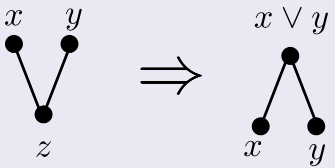

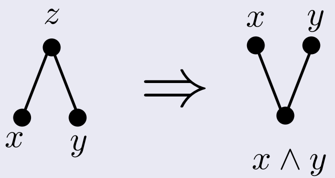

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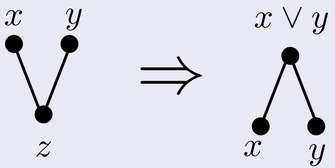

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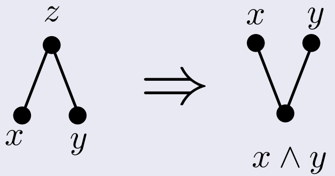

- Thus, upper-semimodularity means that if $z \sqsubset x$ and $z \sqsubset y$, and if $x \neq y$, then $x \sqsubset (x \vee y)$ and $y \sqsubset (x \vee y)$.

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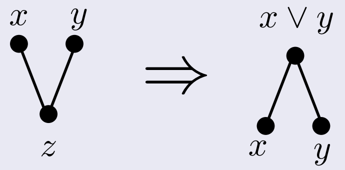

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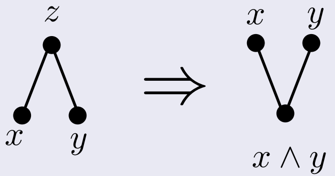

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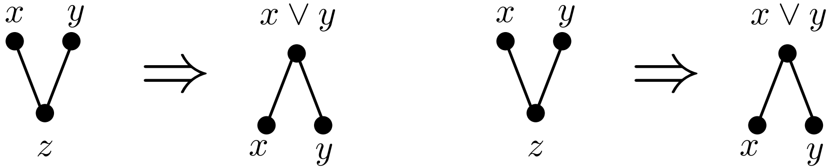
- Thus, lower-semimodularity means that if $x \sqsubset z$ and $y \sqsubset z$, and if $x \neq y$, then $(x \wedge y) \sqsubset x$ and $(x \wedge y) \sqsubset y$.

Upper-Semimodularity vs. Lower-Semimodularity



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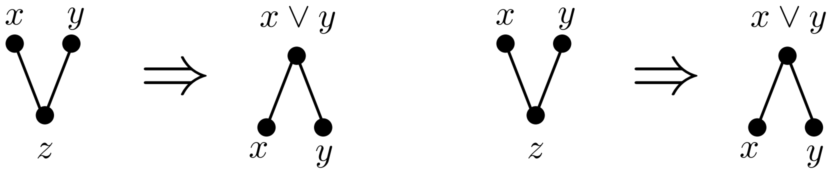

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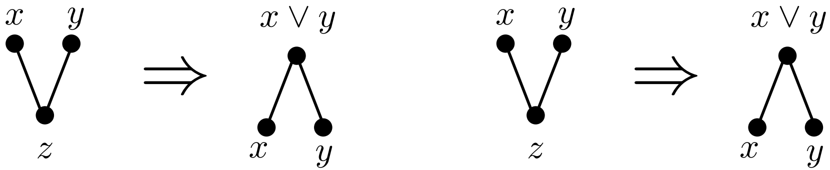

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- Both together imply modularity on the dimension function.

Semi-modular/Submodular Lattices

Theorem 14.7.1

Let L be a finite lattice. The following two conditions are equivalent:

(i) *L is graded, and the height function $h(\cdot)$ of L satisfies the (what we know as the submodular) inequality for all $x, y \in L$.*

$$h(x) + h(y) \geq h(x \vee y) + h(x \wedge y) \tag{14.14}$$

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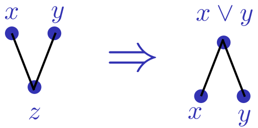
- Condition (ii) is visualized as:



Semi-modular/Submodular Lattices: (i) $\Rightarrow$(ii)

h submodular $\Rightarrow \left\{ (z \sqsubset x, z \sqsubset y) \Rightarrow (x \sqsubset (x \vee y)), (y \sqsubset (x \vee y)) \right\}$.

- Suppose x and y cover z.

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- giving $h(x \vee y) = h(x \wedge y) + 2 = h(x) + 1 = h(y) + 1$, so that $x \vee y$ covers both x and y.

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Semi-modular/Submodular Lattices: (ii) $\Rightarrow$ (i)

$$\left\{(z \sqsubset x, z \sqsubset y) \Rightarrow (x \sqsubset (x \vee y)), (y \sqsubset (x \vee y))\right\} \Rightarrow h \text{ submodular.}$$

- Suppose L is not graded, and let $[u, v]$ be an interval of L of minimal length that is not graded (so all smaller length intervals are graded).

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- Hence L is graded (i.e., every maximal chain has the same length, i.e., JDCC holds).

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$\left\{ (z \sqsubset x, z \sqsubset y) \Rightarrow (x \sqsubset (x \vee y)), (y \sqsubset (x \vee y)) \right\} \Rightarrow h$ submodular.

- Now suppose there is a pair $x, y \in L$ violating the submodularity inequality, i.e., with

$$h(x) + h(y) < h(x \vee y) + h(x \wedge y) \tag{14.17}$$

and choose such a pair first with $\ell(x \wedge y, x \vee y)$ minimal, and then (second) with $h(x) + h(y)$ minimal.

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  and choose such a pair first with $\ell(x \wedge y, x \vee y)$ minimal, and then (second) with $h(x) + h(y)$ minimal.
- By (ii), we cannot have both x and y covering $x \wedge y$ (because if we did, then $h(x) = h(x \wedge y) + 1$, $h(y) = h(x \wedge y) + 1$, and (ii) gives that $h(x \vee y) = h(x) + 1 = h(y) + 1$, and we would have the submodular inequality at equality).

...

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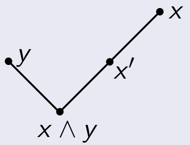

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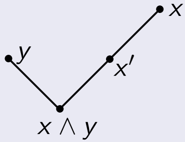

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- By the minimality of $\ell(x \wedge y, x \vee y)$ and $h(x) + h(y)$, we have

$$h(x') + h(y) \geq h(x' \wedge y) + h(x' \vee y). \tag{14.18}$$

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Semi-modular/Submodular Lattices: (ii) $\Rightarrow$ (i)

$\Big\{ (z \sqsubset x, z \sqsubset y) \Rightarrow (x \sqsubset (x \vee y)), (y \sqsubset (x \vee y)) \Big\} \Rightarrow h$ submodular.

- Now $x' \wedge y = x \wedge y$, so Eq. 14.17 and Eq. 14.18 together imply that

$$h(x) + h(x' \vee y) < h(x') + h(x \vee y). \tag{14.19}$$

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Semi-modular/Submodular Lattices: (ii) $\Rightarrow$ (i)

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$$h(x) + h(x' \vee y) < h(x') + h(x \vee y). \tag{14.19}$$

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- Thus, we have found a pair $X, Y \in L$ with $h(X) + h(Y) < h(X \wedge Y) + h(X \vee Y)$ and a strictly shorter length $\ell(X \wedge Y, X \vee Y) < \ell(x \wedge y, x \vee y)$,

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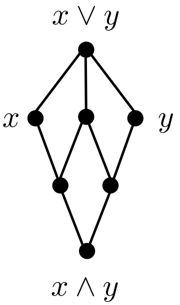
- The proof is complete.

Submodular Lattices

The next figure is an example of an upper-semimodular (or a "submodular") lattice over 7 elements.

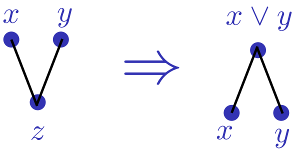


height
3
2
1
0

submodularity
h(x)+h(y)
$>$ h($x \vee y$)
$+$ h($x \wedge y$)

$2 + 2 > 3 + 0$

- Can verify that for all x, y, z that:



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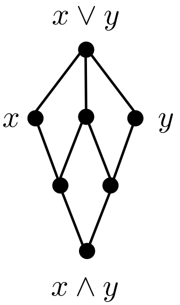

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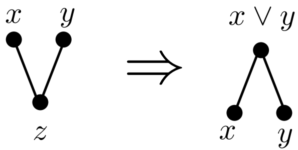

- Hence, the lattice is upper-semimodular (submodular), as the height function is submodular on the lattice.

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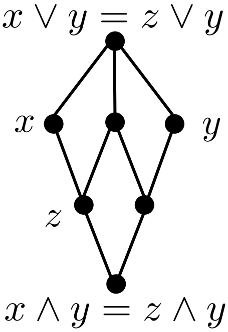

- Note, the lattice is not modular since it violates the modular equality $(\forall x, y, z, \;\; z \preceq x \Rightarrow (z \vee (y \wedge x) = (z \vee y) \wedge x))$.

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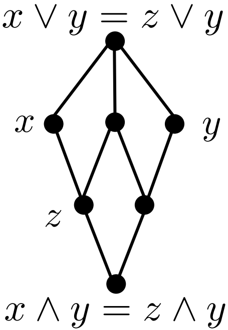

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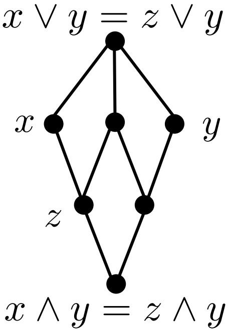

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- I.e., on the left, $z \vee (y \wedge x) = z$ but $(z \vee y) \wedge x = x$
- Flip the Hasse diagram up-side down to get a lower-semimodular (or "supermodular") lattice and corresponding height function.

Ideal in a Lattice

Definition 14.8.1 (ideal)

An ideal is a nonvoid subset J of a lattice L with the properties

$$\forall a \in J, x \in L, \quad x \preceq a \Rightarrow x \in J \tag{14.20}$$

$$\forall a \in J, \quad b \in J \Rightarrow a \vee b \in J. \tag{14.21}$$

The dual concept (in a lattice) is called a dual ideal (or a meet ideal).

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Example 14.8.3

In 2^E, take any $A \subseteq E$, then $L(A) = \{B : B \subseteq A\}$ is an ideal in a set lattice.

Ideal in a Lattice

Definition 14.8.4

Given an element $a \in L$ in a lattice, the set $L(a)$ of all elements $\{x : x \preceq a, x \in L\}$ is an ideal, and is called a principle ideal.

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Theorem 14.8.5

The set of all ideals'of any lattice L, ordered by inclusion, itself forms a lattice. The set of all principal ideals in L forms a sublattice of this lattice, which is isomorphic with L.

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Example 14.8.6

Consider 2^E. Then for any $A \subseteq E$, we see that $L(A) = \{B : B \subseteq A\}$ is an ideal. Also, we can see that the set of sets $\{L(A) : A \subseteq E\}$ is isomorphic to 2^E and also forms a lattice.

Complement and Complemented Lattices

Definition 14.8.7

A lattice with a 0 and 1 is complemented if for all $x \in L$ there exists a $y \in L$ such that $x \vee y = 1$ and $x \wedge y = 0$.

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Proposition 14.8.8

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Proposition 14.8.9

In a complemented modular lattice of finite length, every element is the join of those elements which it contains.

Boolean Lattices

Definition 14.8.10

A Boolean lattice is a complemented distributive lattice.

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Theorem 14.8.11

In any Boolean lattice, each element x has a unique complement x'. Moreover, we have

$$x \wedge x' = 0, \qquad x \vee x' = 1 \tag{L1}$$

$$(x')' = x, \tag{L2}$$

$$(x \wedge y)' = x' \vee y', \qquad (x \vee y)' = x' \wedge y' \tag{L3}$$

Join Irreducible

Definition 14.8.12

An element x of a lattice is called join irreducible if $y \vee z = x$ implies $y = x$ or $z = x$ (ie, if x is the join of two elements, it must be one of those elements).

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Proposition 14.8.14

In any complemented modular lattice, all join irreducible elements are atoms.

Ring of sets

Definition 14.8.15 (ring family)

A ring of sets (or ring family) is a family Φ of subsets of a set E which contains with any two sets S and T also their (set-theoretic) intersection $S \cap T$ and union $S \cup T$. A field of sets is a ring of sets which contains with any S also its set complement $E \setminus S$'

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- Thus, any ring of sets under the natural ordering $S \subset T$ forms a distributive lattice.

Join irreducible, ground elements, Boolean lattices

Theorem 14.8.16

Let L be any distributive lattice of length n. Then the poset X of join-irreducible elements $x \succ 0$ has order n and, moreover, $L \simeq \mathbf{2}^X$

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Theorem 14.8.17

Every Boolean lattice of finite length n is isomorphic with the field of all subsets of a set of $|E| = n$ elements, namely 2^E.

Sources for Today's Lecture

- Birkhoff, "Lattice Theory", 1967.