Submodular Functions, Optimization, and Applications to Machine Learning
— Spring Quarter, Lecture 13 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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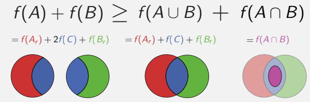
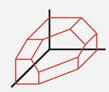
$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$





Cumulative Outstanding Reading

- Good references for today: Schrijver-2003, Oxley-1992/2011, Welsh-1973, Goemans-2010, Cunningham-1984, Edmonds-1969.
- Read Tom McCormick's overview paper on SFM http://people.commerce.ubc.ca/faculty/mccormick/sfmchap8a.pdf
- Read chapters 1 - 4 from Fujishige book.
- Matroid properties http://www-math.mit.edu/~goemans/18433S09/matroid-notes.pdf

Announcements, Assignments, and Reminders

- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, Combinatorial Geometries
- L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
- L9: From Matroid Polytopes to Polymatroids.
- L10: Polymatroids and Submodularity
- L11: More properties of polymatroids, SFM special cases
- L12: polymatroid properties, extreme points polymatroids,
- L13: sat, dep, supp, exchange capacity, examples
- L14: Lattice theory: partially ordered sets; lattices; distributive, modular, submodular, and boolean lattices; ideals and join irreducibles.
- L15: Supp, Base polytope, exchange capacity, minimum norm point algorithm and the lattice of minimizers of a submodular function, Lovasz extension
- L16:
- L17:
- L18:
- L19:
- L20:

Finals Week: June 9th-13th, 2014.

Tight sets $\mathcal{D}(y)$ are closed, and max tight set $\mathrm{sat}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{A : A \subseteq E,\ y(A) = f(A)\} \tag{13.18}$$

Theorem 13.2.1

For any $y \in P_f^+$, with f a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

Proof.

We have already proven this as part of Theorem **??** □

Also recall the definition of $\mathrm{sat}(y)$, the maximal set of tight elements relative to $y \in \mathbb{R}_+^E$.

$$\mathrm{sat}(y) \stackrel{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\} \tag{13.19}$$

Fundamental circuits in matroids

Lemma 13.2.3

Let $I \in \mathcal{I}(M)$, and $e \in E$, then $I \cup \{e\}$ contains at most one circuit in M.

Proof.

- Suppose, to the contrary, that there are two distinct circuits C_1, C_2 such that $C_1 \cup C_2 \subseteq I \cup \{e\}$.
- Then $e \in C_1 \cap C_2$, and by (C2), there is a circuit C_3 of M s.t. $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\} \subseteq I$
- This contradicts the independence of I.

□

In general, let $C(I, e)$ be the unique circuit associated with $I \cup \{e\}$ (commonly called the fundamental circuit in M w.r.t. I and e).

Matroid Partition Problem

Theorem 13.2.1

Let M_i be a collection of k matroids as described. Then, a set $S \subseteq E$ can be partitioned into k subsets $I_i, i = 1 \dots k$ where $I_i \in \mathcal{I}_i$ is independent in matroid i, if and only if, for all $A \subseteq S$

$$|A| \leq \sum_{i=1}^{k} r_i(A) \tag{13.1}$$

where r_i is the rank function of M_i.

- Now, if all matroids are the same $M_i = M$ for all i, we get condition

$$|A| \leq kr(A) \;\; \forall A \subseteq E \tag{13.2}$$

- But considering vector of all ones $\mathbf{1} \in \mathbb{R}_+^E$, this is the same as

$$\frac{1}{k}|A| = \frac{1}{k}\mathbf{1}(A) \leq r(A) \;\; \forall A \subseteq E \tag{13.3}$$

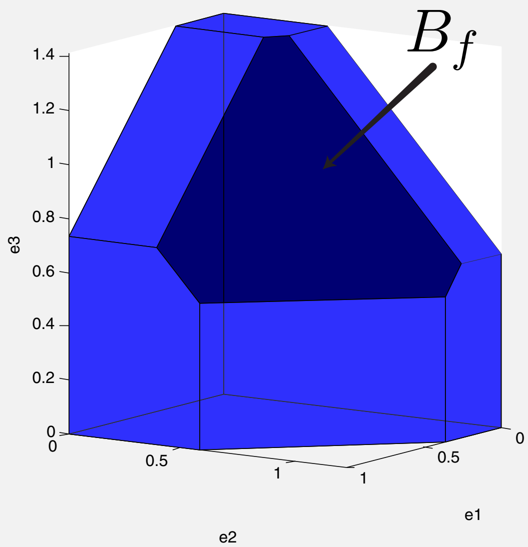
Polymatroidal polyhedron and greedy

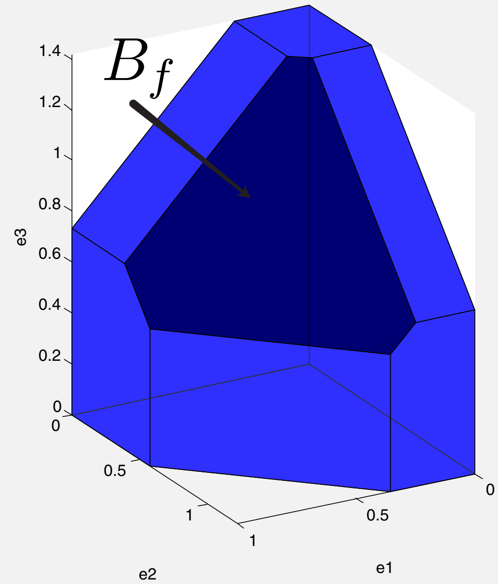
- Thus, restating the above results into a single complete theorem, we have a result very similar to what we saw for matroids (i.e., Theorem **??**)

Theorem 13.2.1

If $f : 2^E \to \mathbb{R}_+$ is given, and P is a polytope in $\mathbb{R}_+^E$ of the form $P = \{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\}$, then the greedy solution to the problem $\max(wx : x \in P)$ is $\forall w$ optimum iff f is monotone non-decreasing submodular (i.e., iff P is a polymatroid).

Base Polytope in 3D





$$P_f = \left\{x \in \mathbb{R}^E : x(S) \le f(S), \forall S \subseteq E\right\} \tag{13.6}$$

$$B_f = P_f \cap \left\{x \in \mathbb{R}^E : x(E) = f(E)\right\} \tag{13.7}$$

Polymatroid extreme points

Theorem 13.2.1

For a given ordering $E = (e_1, \dots, e_m)$ of E and a given $E_i = (e_1, \dots, e_i)$ and x generated by E_i using the greedy procedure ($x(e_i) = f(e_i|E_{i-1})$), then x is an extreme point of P_f

Proof.

- We already saw that $x \in P_f$ (Theorem **??**).
- To show that x is an extreme point of P_f, note that it is the unique solution of the following system of equations

$$x(E_j) = f(E_j) \text{ for } 1 \le j \le i \le m \tag{13.10}$$

$$x(e) = 0 \text{ for } e \in E \setminus E_i \tag{13.11}$$

There are $i \le m$ equations and $i \le m$ unknowns, and simple Gaussian elimination gives us back the x constructed via the Greedy algorithm!!

□

Polymatroid extreme points

- Moreover, we have (and will ultimately prove)

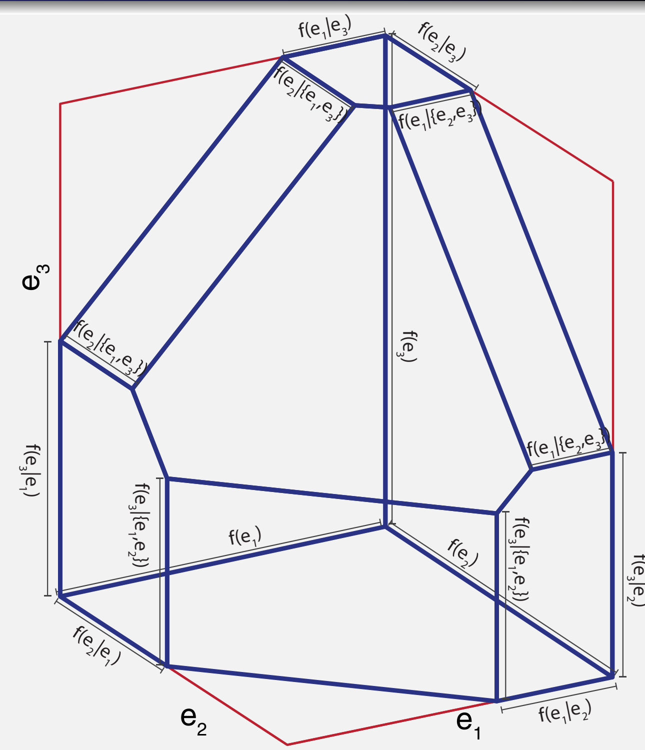
Corollary 13.2.2

If x is an extreme point of P_f and $B \subseteq E$ is given such that $\text{supp}(x) = \{e \in E : x(e) \neq 0\} \subseteq B \subseteq \cup(A : x(A) = f(A)) = \text{sat}(x)$, *then x is generated using greedy by some ordering of B.*

- Note, $\text{sat}(x) = \text{cl}(x) = \cup(A : x(A) = f(A))$ is also called the closure of x (recall that sets A such that $x(A) = f(A)$ are called tight, and such sets are closed under union and intersection, as seen in Lecture 8, Theorem **??**)
- Thus, $\text{cl}(x)$ is a tight set.
- Also, $\text{supp}(x) = \{e \in E : x(e) \neq 0\}$ is called the support of x.
- For arbitrary x, $\text{supp}(x)$ is not necessarily tight, but for an extreme point, $\text{supp}(x)$ is.

Polymatroid with labeled edge lengths

- Recall $f(e|A) = f(A+e) - f(A)$
- Notice how submodularity, $f(e|B) \leq f(e|A)$ for $A \subseteq B$, defines the shape of the polytope.
- In fact, we have strictness here $f(e|B) < f(e|A)$ for $A \subset B$.
- Also, consider how the greedy algorithm proceeds along the edges of the polytope.



Minimizers of a Submodular Function form a lattice

Theorem 13.2.2

For arbitrary submodular f, the minimizers are closed under union and intersection. That is, let $\mathcal{M} = \operatorname{argmin}_{X \subseteq E} f(X)$ be the set of minimizers of f. Let $A, B \in \mathcal{M}$. Then $A \cup B \in \mathcal{M}$ and $A \cap B \in \mathcal{M}$.

Proof.

Since A and B are minimizers, we have $f(A) = f(B) \leq f(A \cap B)$ and $f(A) = f(B) \leq f(A \cup B)$.
By submodularity, we have

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \tag{13.9}$$

Hence, we must have $f(A) = f(B) = f(A \cup B) = f(A \cap B)$. □

Thus, the minimizers of a submodular function form a lattice, and there is a maximal and a minimal minimizer of every submodular function.

The sat function = Polymatroid Closure

- Matroid closure is generalized by the unique maximal element in $\mathcal{D}(x)$, also called the polymatroid closure or sat (saturation function).
- For some $x \in P_f$, we have defined:

$$\text{cl}(x) \stackrel{\text{def}}{=} \mathrm{sat}(x) \stackrel{\text{def}}{=} \bigcup \{A : A \in \mathcal{D}(x)\} \tag{13.9}$$

$$= \bigcup \{A : A \subseteq E, x(A) = f(A)\} \tag{13.10}$$

$$= \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\} \tag{13.11}$$

- Hence, $\mathrm{sat}(x)$ is the maximal (zero-valued) minimizer of the submodular function $f_x(A) \triangleq f(A) - x(A)$.
- Eq. (13.11) says that sat consists of any point x that is P_f saturated (any additional positive movement, in that dimension, leaves P_f). We'll revisit this in a few slides.
- First, we see how sat generalizes matroid closure.

The sat function = Polymatroid Closure

- Consider matroid $(E, \mathcal{I}) = (E, r)$, some $I \in \mathcal{I}$. Then $\mathbf{1}_I \in P_r$ and

$$\mathcal{D}(\mathbf{1}_I) = \{A : \mathbf{1}_I(A) = r(A)\} \tag{13.1}$$

and

$$\begin{aligned} \text{sat}(\mathbf{1}_I) &= \bigcup \{A : A \subseteq E, A \in \mathcal{D}(\mathbf{1}_I)\} && (13.2)\\ &= \bigcup \{A : A \subseteq E, \mathbf{1}_I(A) = r(A)\} && (13.3)\\ &= \bigcup \{A : A \subseteq E, |I \cap A| = r(A)\} && (13.4) \end{aligned}$$

- Notice that $\mathbf{1}_I(A) = |I \cap A| \leq |I|$.
- Intuitively, consider an $A \supset I \in \mathcal{I}$ that doesn't increase rank, meaning $r(A) = r(I)$. If $r(A) = |I \cap A| = r(I \cap A)$, as in Eqn. (13.4), then A is in I's span, so should get $\text{sat}(\mathbf{1}_I) = \text{span}(I)$.
- We formalize this next.

The sat function = Polymatroid Closure

Lemma 13.3.1 (Matroid $\text{sat} : \mathbb{R}_+^E \to 2^E$ is the same as closure.)

For $I \in \mathcal{I}$, we have $\text{sat}(\mathbf{1}_I) = \text{span}(I)$ (13.5)

Proof.

- For $\mathbf{1}_I(I) = |I| = r(I)$, so $I \in \mathcal{D}(\mathbf{1}_I)$ and $I \subseteq \text{sat}(\mathbf{1}_I)$. Also, $I \subseteq \text{span}(I)$.
- Consider some $b \in \text{span}(I) \setminus I$.
- Then $I \cup \{b\} \in \mathcal{D}(\mathbf{1}_I)$ since $\mathbf{1}_I(I \cup \{b\}) = |I| = r(I \cup \{b\}) = r(I)$.
- Thus, $b \in \text{sat}(\mathbf{1}_I)$.
- Therefore, $\text{sat}(\mathbf{1}_I) \supseteq \text{span}(I)$.

...

The sat function = Polymatroid Closure

. . . proof continued.

- Now, consider $b \in \text{sat}(\mathbf{1}_I) \setminus I$.
- Choose any $A \in \mathcal{D}(\mathbf{1}_I)$ with $b \in A$, thus $b \in A \setminus I$.
- Then $\mathbf{1}(A) = |A \cap I| = r(A)$.
- Now $r(A) = |A \cap I| \leq |I| = r(I)$.
- Also, $r(A \cap I) = |A \cap I|$ since $A \cap I \in \mathcal{I}$.
- Hence, $r(A \cap I) = r(A) = r((A \cap I) \cup (A \setminus I))$ meaning $(A \setminus I) \subseteq \text{span}(A \cap I) \subseteq \text{span}(I)$.
- Since $b \in A \setminus I$, we get $b \in \text{span}(I)$.
- Thus, $\text{sat}(\mathbf{1}_I) \subseteq \text{span}(I)$.
- Hence $\text{sat}(\mathbf{1}_I) = \text{span}(I)$

□

The sat function = Polymatroid Closure

- Now, consider a matroid (E, r) and some $C \subseteq E$ with $C \notin \mathcal{I}$, and consider $\mathbf{1}_C$. Is $\mathbf{1}_C \in P_r$? No, it might not be a vertex, or even a member, of P_r.
- $\text{span}(\cdot)$ operates on more than just independent sets, so $\text{span}(C)$ is perfectly sensible.
- Note $\text{span}(C) = \text{span}(B)$ where $\mathcal{I} \ni B \in \mathcal{B}(C)$ is a base of C.
- Then we have $\mathbf{1}_B \leq \mathbf{1}_C \leq \mathbf{1}_{\text{span}(C)}$, and that $\mathbf{1}_B \in P_r$. We can then make the definition:

$$\text{sat}(\mathbf{1}_C) \triangleq \text{sat}(\mathbf{1}_B) \text{ for } B \in \mathcal{B}(C) \tag{13.6}$$

  In which case, we also get $\text{sat}(\mathbf{1}_C) = \text{span}(C)$ (in general, could define $\text{sat}(y) = \text{sat}(\text{P-basis}(y))$).
- However, consider the following form

$$\text{sat}(\mathbf{1}_C) = \bigcup \{A : A \subseteq E, |A \cap C| = r(A)\} \tag{13.7}$$

  Exercise: is $\text{span}(C) = \text{sat}(\mathbf{1}_C)$? Prove or disprove it.

The sat function, span, and submodular function minimization

- Thus, for a matroid, $\mathrm{sat}(\mathbf{1}_I)$ is exactly the closure (or span) of I in the matroid. I.e., for matroid (E, r), we have $\mathrm{span}(I) = \mathrm{sat}(\mathbf{1}_B)$.
- Recall, for $x \in P_f$ and polymatroidal f, $\mathrm{sat}(x)$ is the maximal (by inclusion) minimizer of $f(A) - x(A)$, and thus in a matroid, span(I) is the maximal minimizer of the submodular function formed by $r(A) - \mathbf{1}_I(A)$.
- Submodular function minimization can solve "span" queries in a matroid or "sat" queries in a polymatroid.

sat, as tight polymatroidal elements

- We are given an $x \in P_f^+$ for submodular function f.
- Recall that for such an x, $\mathrm{sat}(x)$ is defined as

$$\mathrm{sat}(x) = \bigcup \{A : x(A) = f(A)\} \tag{13.8}$$

- We also have stated that $\mathrm{sat}(x)$ can be defined as:

$$\mathrm{sat}(x) = \left\{e : \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f^+\right\} \tag{13.9}$$

- We next show more formally that these are the same.

sat, as tight polymatroidal elements

- Lets start with one definition and derive the other.

$$\begin{aligned}\mathrm{sat}(x) &\stackrel{\text{def}}{=} \left\{e : \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f^+\right\} && (13.10)\\ &= \{e : \forall \alpha > 0, \exists A \text{ s.t. } (x + \alpha \mathbf{1}_e)(A) > f(A)\} && (13.11)\\ &= \{e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } (x + \alpha \mathbf{1}_e)(A) > f(A)\} && (13.12)\end{aligned}$$

- this last bit follows since $\mathbf{1}_e(A) = 1 \iff e \in A$. Continuing, we get

$$\mathrm{sat}(x) = \{e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } x(A) + \alpha > f(A)\} \quad (13.13)$$

- given that $x \in P_f^+$, meaning $x(A) \le f(A)$ for all A, we must have

$$\begin{aligned}\mathrm{sat}(x) &= \{e : \forall \alpha > 0, \exists A \ni e \text{ s.t. } x(A) = f(A)\} && (13.14)\\ &= \{e : \exists A \ni e \text{ s.t. } x(A) = f(A)\} && (13.15)\end{aligned}$$

- So now, if A is any set such that $x(A) = f(A)$, then we clearly have

$$\forall e \in A, e \in \mathrm{sat}(x), \text{ and therefore that } \mathrm{sat}(x) \supseteq A \quad (13.16)$$

sat, as tight polymatroidal elements

- ...and therefore, with sat as defined in Eq. (**??**),

$$\mathrm{sat}(x) \supseteq \bigcup \{A : x(A) = f(A)\} \quad (13.17)$$

- On the other hand, for any $e \in \mathrm{sat}(x)$ defined as in Eq. (13.15), since e is itself a member of a tight set, there is a set $A \ni e$ such that $x(A) = f(A)$, giving

$$\mathrm{sat}(x) \subseteq \bigcup \{A : x(A) = f(A)\} \quad (13.18)$$

- Therefore, the two definitions of sat are identical.

Saturation Capacity

- Another useful concept is saturation capacity which we develop next.
- For $x \in P_f$, and $e \in E$, consider finding

$$\max\{\alpha : \alpha \in \mathbb{R}, x + \alpha \mathbf{1}_e \in P_f\} \tag{13.19}$$

- This is identical to:

$$\max\{\alpha : (x + \alpha \mathbf{1}_e)(A) \leq f(A), \forall A \supseteq \{e\}\} \tag{13.20}$$

since any $B \subseteq E$ such that $e \notin B$ does not change in a $\mathbf{1}_e$ adjustment, meaning $(x + \alpha \mathbf{1}_e)(B) = x(B)$.
- Again, this is identical to:

$$\max\{\alpha : x(A) + \alpha \leq f(A), \forall A \supseteq \{e\}\} \tag{13.21}$$

or

$$\max\{\alpha : \alpha \leq f(A) - x(A), \forall A \supseteq \{e\}\} \tag{13.22}$$

Saturation Capacity

- The max is achieved when

$$\alpha = \hat{c}(x; e) \stackrel{\text{def}}{=} \min\{f(A) - x(A), \forall A \supseteq \{e\}\} \tag{13.23}$$

- $\hat{c}(x; e)$ is known as the saturation capacity associated with $x \in P_f$ and e.
- Thus we have for $x \in P_f$,

$$\hat{c}(x; e) \stackrel{\text{def}}{=} \min\{f(A) - x(A), \forall A \ni e\} \tag{13.24}$$

$$= \max\{\alpha : \alpha \in \mathbb{R}, x + \alpha \mathbf{1}_e \in P_f\} \tag{13.25}$$

- We immediately see that for $e \in E \setminus \text{sat}(x)$, we have that $\hat{c}(x; e) > 0$.
- Also, for $e \in \text{sat}(x)$, we have that $\hat{c}(x; e) = 0$.
- Note that any α with $0 \leq \alpha \leq \hat{c}(x; e)$ we have $x + \alpha \mathbf{1}_e \in P_f$.
- We also see that computing $\hat{c}(x; e)$ is a form of submodular function minimization.

Dependence Function

- Tight sets can be restricted to contain a particular element.
- Given $x \in P_f$, and $e \in \text{sat}(x)$, define

$$\mathcal{D}(x,e) = \{A : e \in A \subseteq E, x(A) = f(A)\} \tag{13.26}$$
$$= \mathcal{D}(x) \cap \{A : A \subseteq E, e \in A\} \tag{13.27}$$

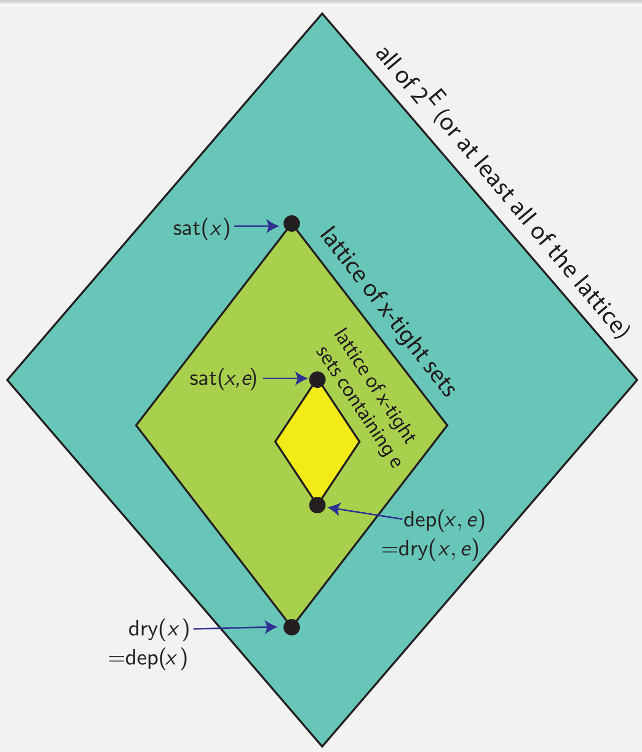
- Thus, $\mathcal{D}(x,e) \subseteq \mathcal{D}(x)$, and $\mathcal{D}(x,e)$ is a sublattice of $\mathcal{D}(x)$.
- Therefore, we can define a unique minimal element of $\mathcal{D}(x,e)$ denoted as follows:

$$\text{dep}(x,e) = \begin{cases} \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases} \tag{13.28}$$

- I.e., $\text{dep}(x,e)$ is the minimal element in $\mathcal{D}(x)$ that contains e (the minimal x-tight set containing e).

dep and sat in a lattice

- Given some $x \in P_f$,
- The picture on the right summarizes the relationships between the lattices and sublattices.
- Note, $\bigcap_e \text{dep}(x,e) = \text{dep}(x)$.



dep and sat in a lattice

- Given $x \in P_f$, recall distributive lattice of tight sets $\mathcal{D}(x) = \{A : x(A) = f(A)\}$
- We had that $\text{sat}(x) = \bigcup \{A : A \in \mathcal{D}(x)\}$ is the "1" element of this lattice.
- Consider the "0" element of $\mathcal{D}(x)$, i.e., $\text{dry}(x) \stackrel{\text{def}}{=} \bigcap \{A : A \in \mathcal{D}(x)\}$
- We can see $\text{dry}(x)$ as the elements that are necessary for tightness.
- That is, we can equivalently define $\text{dry}(x)$ as

$$\text{dry}(x) = \left\{e' : x(A) < f(A), \forall A \not\ni e'\right\} \tag{13.29}$$

- This can be read as, for any $e' \in \text{dry}(x)$, any set that does not contain e' is not tight for x (any set A that is missing any element of $\text{dry}(x)$ is not tight).
- Perhaps, then, a better name for dry is ntight(x), for the necessary for tightness (but we'll actually use neither name).
- Note that dry need not be the empty set. Exercise: give example.

An alternate expression for $\text{dep} = \text{dry}$

- Now, given $x \in P_f$, and $e \in \text{sat}(x)$, recall distributive sub-lattice of e-containing tight sets $\mathcal{D}(x, e) = \{A : e \in A, x(A) = f(A)\}$
- We can define the "1" element of this sub-lattice as $\text{sat}(x, e) \stackrel{\text{def}}{=} \bigcup \{A : A \in \mathcal{D}(x, e)\}$.
- Analogously, we can define the "0" element of this sub-lattice as $\text{dry}(x, e) \stackrel{\text{def}}{=} \bigcap \{A : A \in \mathcal{D}(x, e)\}$.
- We can see $\text{dry}(x, e)$ as the elements that are necessary for e-containing tightness, with $e \in \text{sat}(x)$.
- That is, we can view $\text{dry}(x, e)$ as

$$\text{dry}(x, e) = \left\{e' : x(A) < f(A), \forall A \not\ni e', e \in A\right\} \tag{13.30}$$

- This can be read as, for any $e' \in \text{dry}(x, e)$, any e-containing set that does not contain e' is not tight for x.
- But actually, $\text{dry}(x, e) = \text{dep}(x, e)$, so we have derived another expression for $\text{dep}(x, e)$ in Eq. (13.30).

Dependence Function and Fundamental Matroid Circuit

- Now, let $(E,\mathcal{I}) = (E,r)$ be a matroid, and let $I \in \mathcal{I}$ giving $\mathbf{1}_I \in P_r$. We have $\text{sat}(\mathbf{1}_I) = \text{span}(I) = \text{closure}(I)$.
- Given $e \in \text{sat}(\mathbf{1}_I) \setminus I$ and then consider an $A \ni e$ with $|I \cap A| = r(A)$.
- Then $I \cap A$ serves as a base for A (i.e., $I \cap A$ spans A) and any such A contains a circuit (i.e., we can add $e \in A \setminus I$ to $I \cap A$ w/o increasing rank).
- Given $e \in \text{sat}(\mathbf{1}_I) \setminus I$, and consider $\text{dep}(\mathbf{1}_I, e)$, with

$$\text{dep}(\mathbf{1}_I, e) = \bigcap \{A : e \in A \subseteq E, \mathbf{1}_I(A) = r(A)\} \tag{13.31}$$
$$= \bigcap \{A : e \in A \subseteq E, |I \cap A| = r(A)\} \tag{13.32}$$
$$= \bigcap \{A : e \in A \subseteq E, r(A) - |I \cap A| = 0\} \tag{13.33}$$

- By SFM lattice, $\exists$ a unique minimal $A \ni e$ with $|I \cap A| = r(A)$.
- Thus, $\text{dep}(\mathbf{1}_I, e)$ must be a circuit since if it included more than a circuit, it would not be minimal in this sense.

Dependence Function and Fundamental Matroid Circuit

- Therefore, when $e \in \text{sat}(\mathbf{1}_I) \setminus I$, then $\text{dep}(\mathbf{1}_I, e) = C(I, e)$ where $C(I, e)$ is the unique circuit contained in $I + e$ in a matroid (the fundamental circuit of e and I that we encountered before).
- Now, if $e \in \text{sat}(\mathbf{1}_I) \cap I$ with $I \in \mathcal{I}$, we said that $C(I, e)$ was undefined (since no circuit is created in this case) and so we defined it as $C(I, e) = \{e\}$
- In this case, for such an e, we have $\text{dep}(\mathbf{1}_I, e) = \{e\}$ since all such sets $A \ni e$ with $|I \cap A| = r(A)$ contain e, but in this case no cycle is created, i.e., $|I \cap A| \geq |I \cap \{e\}| = r(e) = 1$.
- We are thus free to take subsets of I as A, all of which must contain e, but all of which have rank equal to size.
- Also note: in general for $x \in P_f$ and $e \in \text{sat}(x)$, we have $\text{dep}(x, e)$ is tight by definition.

Summary of sat, and dep

- For $x \in P_f$, $\text{sat}(x)$ (span, closure) is the maximal saturated (x-tight) set w.r.t. x. I.e., $\text{sat}(x) = \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\}$. That is,

$$\text{cl}(x) \stackrel{\text{def}}{=} \text{sat}(x) \triangleq \bigcup \{A : A \in \mathcal{D}(x)\} \quad (13.34)$$

$$= \bigcup \{A : A \subseteq E, x(A) = f(A)\} \quad (13.35)$$

$$= \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\} \quad (13.36)$$

- For $e \in \text{sat}(x)$, we have $\text{dep}(x, e)$ (fundamental circuit) is the minimal (common) saturated (x-tight) set w.r.t. x containing e. That is,

$$\text{dep}(x, e) = \begin{cases} \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases}$$

$$= \{e' : \exists \alpha > 0, \text{ s.t. } x + \alpha(\mathbf{1}_e - \mathbf{1}_{e'}) \in P_f\} \quad (13.37)$$

Dependence Function and exchange

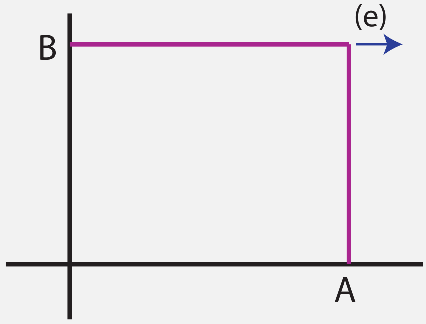
- For $e \in \text{span}(I) \setminus I$, we have that $I + e \notin \mathcal{I}$. This is a set addition restriction property.
- Analogously, for $e \in \text{sat}(x)$, any $x + \alpha \mathbf{1}_e \notin P_f$ for $\alpha > 0$. This is a vector increase restriction property.
- Recall, we have $C(I, e) \setminus e' \in \mathcal{I}$ for $e' \in C(I, e)$. I.e., $C(I, e)$ consists of elements that when removed recover independence.
- In other words, for $e \in \text{span}(I) \setminus I$, we have that

$$C(I, e) = \{a \in E : I + e - a \in \mathcal{I}\} \quad (13.38)$$

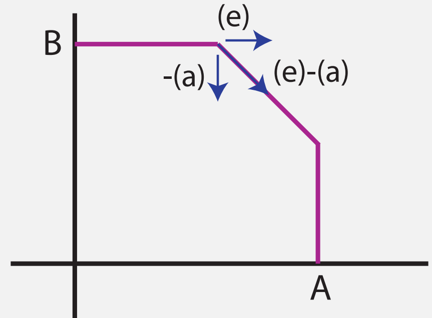
- I.e., <u>an addition of e to I stays within $\mathcal{I}$ only if we simultaneously remove one of the elements of $C(I, e)$</u>.
- But, analogous to the circuit case, is there an exchange property for $\text{dep}(x, e)$ in the form of vector movement restriction?
- We might expect the vector $\text{dep}(x, e)$ property to take the form: <u>a positive move in the e-direction stays within P_f^+ only if we simultaneously take a negative move in one of the $\text{dep}(x, e)$ directions.</u>

Dependence Function and exchange in 2D

- $\text{dep}(x, e)$ is set of neg. directions we must move if we want to move in pos. e direction, starting at x and staying within P_f.
- Viewable in 2D, we have for $A, B \subseteq E$, $A \cap B = \emptyset$:



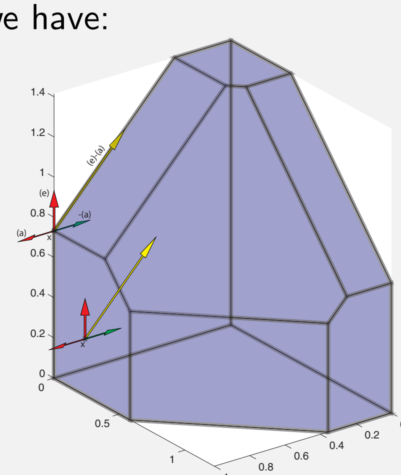
Left: $A \cap \text{dep}(x, e) = \emptyset$, and we can't move further in (e) direction, and moving in any negative $a \in A$ direction doesn't change that. Notice no dependence between (e) and any element in A.

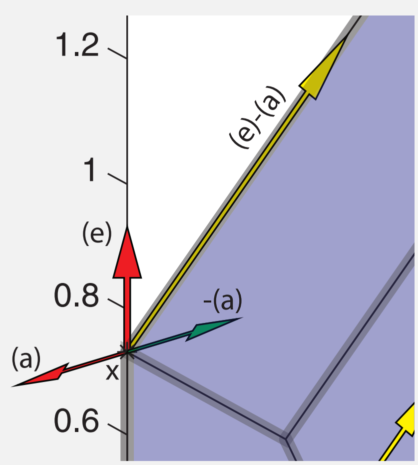


Right: $A \subseteq \text{dep}(x, e)$, and we can't move further in the (e) direction, but we can move further in (e) direction by moving in some $a \in A$ negative direction. Notice dependence between (e) and elements in A.

Dependence Function and exchange in 3D

- We can move neither in the (e) nor the (a) direction, but we can move in the (e) direction if we simultaneously move in the -(a) direction.
- In 3D, we have:





- I.e., for $e \in \text{sat}(x)$, $a \in \text{sat}(x)$, $a \in \text{dep}(x, e)$, $e \notin \text{dep}(x, a)$, and

$$\text{dep}(x, e) = \{a : a \in E, \exists \alpha > 0 : x + \alpha(\mathbf{1}_e - \mathbf{1}_a) \in P_f\} \quad (13.39)$$

- We next show this formally ...

dep and exchange derived

- The derivation for $\mathrm{dep}(x, e)$ involves turning a strict inequality into a non-strict one with a strict explicit slack variable α:

$$\mathrm{dep}(x, e) = \mathsf{ntight}(x, e) = \tag{13.40}$$
$$= \{e' : x(A) < f(A), \forall A \not\ni e', e \in A\} \tag{13.41}$$
$$= \{e' : \exists \alpha > 0, \text{ s.t. } \alpha \leq f(A) - x(A), \forall A \not\ni e', e \in A\} \tag{13.42}$$
$$= \{e' : \exists \alpha > 0, \text{ s.t. } \alpha \mathbf{1}_e(A) \leq f(A) - x(A), \forall A \not\ni e', e \in A\} \tag{13.43}$$
$$= \{e' : \exists \alpha > 0, \text{ s.t. } \alpha(\mathbf{1}_e(A) - \mathbf{1}_{e'}(A)) \leq f(A) - x(A), \forall A \not\ni e', e \in A\} \tag{13.44}$$
$$= \{e' : \exists \alpha > 0, \text{ s.t. } x(A) + \alpha(\mathbf{1}_e(A) - \mathbf{1}_{e'}(A)) \leq f(A), \forall A \not\ni e', e \in A\} \tag{13.45}$$

- Now, $1_e(A) - \mathbf{1}_{e'}(A) = 0$ if either $\{e, e'\} \subseteq A$, or $\{e, e'\} \cap A = \emptyset$.
- Also, if $e' \in A$ but $e \notin A$, then $x(A) + \alpha(\mathbf{1}_e(A) - \mathbf{1}_{e'}(A)) = x(A) - \alpha \leq f(A)$ since $x \in P_f$.

dep and exchange derived

- thus, we get the same in the above if we remove the constraint $A \not\ni e', e \in A$, that is we get

$$\mathrm{dep}(x, e) = \{e' : \exists \alpha > 0, \text{ s.t. } x(A) + \alpha(\mathbf{1}_e(A) - \mathbf{1}_{e'}(A)) \leq f(A), \forall A\} \tag{13.46}$$

- This is then identical to

$$\mathrm{dep}(x, e) = \{e' : \exists \alpha > 0, \text{ s.t. } x + \alpha(\mathbf{1}_e - \mathbf{1}_{e'}) \in P_f\} \tag{13.47}$$

- Compare with original, the minimal element of $\mathcal{D}(x, e)$, with $e \in \mathrm{sat}(x)$:

$$\mathrm{dep}(x, e) = \begin{cases} \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \mathrm{sat}(x) \\ \emptyset & \text{else} \end{cases} \tag{13.48}$$

Summary of Concepts

- Most violated inequality $\max\{x(A) - f(A) : A \subseteq E\}$
- Matroid by circuits, and the fundamental circuit $C(I, e) \subseteq I + e$.
- Minimizers of submodular functions form a lattice.
- Minimal and maximal element of a lattice.
- x-tight sets, maximal and minimal tight set.
- sat function & Closure
- Saturation Capacity
- e-containing tight sets
- dep function & fundamental circuit of a matroid

Summary important definitions so far: tight, dep, & sat

- x-tight sets: For $x \in P_f$, $\mathcal{D}(x) = \{A \subseteq E : x(A) = f(A)\}$.
- Polymatroid closure/maximal x-tight set: For $x \in P_f$, $\mathrm{sat}(x) = \cup\{A : A \in \mathcal{D}(x)\} = \{e : e \in E, \forall\alpha > 0, x + \alpha\mathbf{1}_e \notin P_f\}$.
- Saturation capacity: for $x \in P_f$, $0 \leq \hat{c}(x; e) = \min\{f(A) - x(A) | \forall A \ni e\} = \max\{\alpha : \alpha \in \mathbb{R}, x + \alpha\mathbf{1}_e \in P_f\}$
- Recall: $\mathrm{sat}(x) = \{e : \hat{c}(x; e) = 0\}$ and $E \setminus \mathrm{sat}(x) = \{e : \hat{c}(x; e) > 0\}$.
- e-containing x-tight sets: For $x \in P_f$, $\mathcal{D}(x, e) = \{A : e \in A \subseteq E, x(A) = f(A)\} \subseteq \mathcal{D}(x)$.
- Minimal e-containing x-tight set/polymatroidal fundamental circuit/: For $x \in P_f$,

$$\mathrm{dep}(x, e) = \begin{cases} \bigcap\{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \mathrm{sat}(x) \\ \emptyset & \text{else} \end{cases}$$

$$= \{e' : \exists\alpha > 0,\ \text{s.t. } x + \alpha(\mathbf{1}_e - \mathbf{1}_{e'}) \in P_f\}$$