Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 13 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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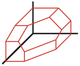
$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

$= f(A_r) + 2f(C) + f(B_r)$ $\quad = f(A_r) + f(C) + f(B_r)$ $\quad = f(A \cap B)$



Cumulative Outstanding Reading

- Good references for today: Schrijver-2003, Oxley-1992/2011, Welsh-1973, Goemans-2010, Cunningham-1984, Edmonds-1969.
- Read Tom McCormick's overview paper on SFM http://people.commerce.ubc.ca/faculty/mccormick/sfmchap8a.pdf
- Read chapters 1 - 4 from Fujishige book.
- Matroid properties http://www-math.mit.edu/~goemans/18433S09/matroid-notes.pdf

Announcements, Assignments, and Reminders

- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, Combinatorial Geometries
- L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
- L9: From Matroid Polytopes to Polymatroids.
- L10: Polymatroids and Submodularity
- L11: More properties of polymatroids, SFM special cases
- L12: polymatroid properties, extreme points polymatroids,
- L13: sat, dep, supp, exchange capacity, examples
- L14:
- L15:
- L16:
- L17:
- L18:
- L19:
- L20:

Finals Week: June 9th-13th, 2014.

Tight sets $\mathcal{D}(y)$ are closed, and max tight set $\text{sat}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{A : A \subseteq E,\ y(A) = f(A)\} \qquad (13.18)$$

Theorem 13.2.1

For any $y \in P_f^+$, with f a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

Proof.

We have already proven this as part of Theorem **??** □

Also recall the definition of $\text{sat}(y)$, the maximal set of tight elements relative to $y \in \mathbb{R}_+^E$.

$$\text{sat}(y) \stackrel{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\} \qquad (13.19)$$

Fundamental circuits in matroids

Lemma 13.2.3

Let $I \in \mathcal{I}(M)$*, and* $e \in E$*, then* $I \cup \{e\}$ *contains at most one circuit in* M*.*

Proof.

- Suppose, to the contrary, that there are two distinct circuits C_1, C_2 such that $C_1 \cup C_2 \subseteq I \cup \{e\}$.
- Then $e \in C_1 \cap C_2$, and by (C2), there is a circuit C_3 of M s.t. $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\} \subseteq I$
- This contradicts the independence of I.

□

In general, let $C(I, e)$ be the unique circuit associated with $I \cup \{e\}$ (commonly called the fundamental circuit in M w.r.t. I and e).

Matroid Partition Problem

Theorem 13.2.1

Let M_i be a collection of k matroids as described. Then, a set $S \subseteq E$ can be partitioned into k subsets $I_i, i = 1 \dots k$ where $I_i \in \mathcal{I}_i$ is independent in matroid i, if and only if, for all $A \subseteq S$

$$|A| \leq \sum_{i=1}^{k} r_i(A) \tag{13.1}$$

where r_i is the rank function of M_i.

- Now, if all matroids are the same $M_i = M$ for all i, we get condition

$$|A| \leq kr(A) \;\; \forall A \subseteq E \tag{13.2}$$

- But considering vector of all ones $\mathbf{1} \in \mathbb{R}_+^E$, this is the same as

$$\frac{1}{k}|A| = \frac{1}{k}\mathbf{1}(A) \leq r(A) \;\; \forall A \subseteq E \tag{13.3}$$

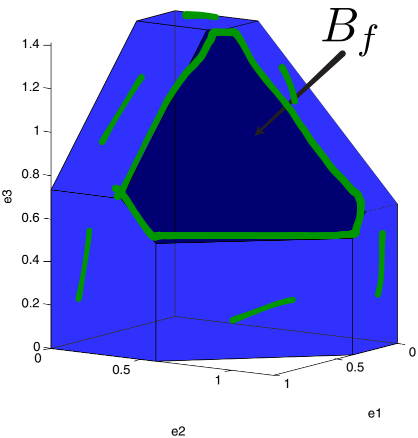
Polymatroidal polyhedron and greedy

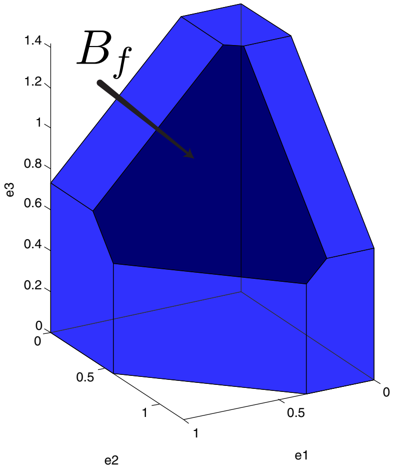
- Thus, restating the above results into a single complete theorem, we have a result very similar to what we saw for matroids (i.e., Theorem **??**)

Theorem 13.2.1

If $f : 2^E \to \mathbb{R}_+$ is given, and P is a polytope in $\mathbb{R}_+^E$ of the form $P = \{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\}$, then the greedy solution to the problem $\max(wx : x \in P)$ is $\forall w$ optimum iff f is monotone non-decreasing submodular (i.e., iff P is a polymatroid).

Base Polytope in 3D





$$P_f = \{x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E\} \tag{13.5}$$

$$B_f = P_f \cap \{x \in \mathbb{R}^E : x(E) = f(E)\} \tag{13.6}$$

Polymatroid extreme points

Theorem 13.2.1

For a given ordering $E = (e_1, \dots, e_m)$ of E and a given $E_i = (e_1, \dots, e_i)$ and x generated by E_i using the greedy procedure ($x(e_i) = f(e_i|E_{i-1})$), then x is an extreme point of P_f

Proof.

- We already saw that $x \in P_f$ (Theorem **??**).
- To show that x is an extreme point of P_f, note that it is the unique solution of the following system of equations

$$x(E_j) = f(E_j) \text{ for } 1 \leq j \leq i \leq m \tag{13.9}$$

$$x(e) = 0 \text{ for } e \in E \setminus E_i \tag{13.10}$$

There are $i \leq m$ equations and $i \leq m$ unknowns, and simple Gaussian elimination gives us back the x constructed via the Greedy algorithm!!

Polymatroid extreme points

- Moreover, we have (and will ultimately prove)

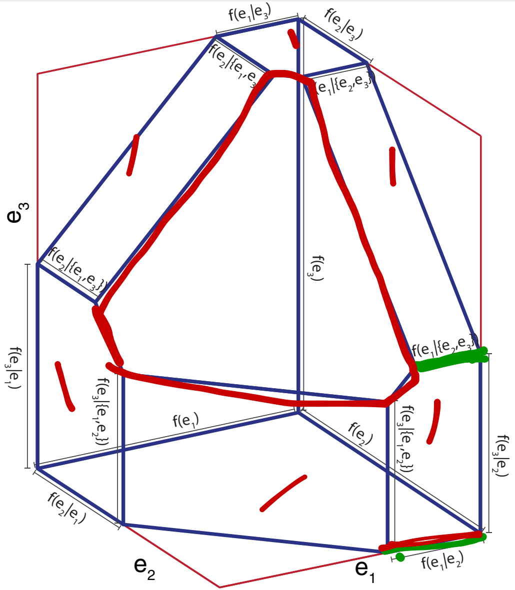
Corollary 13.2.2

If x is an extreme point of P_f and $B \subseteq E$ is given such that $\text{supp}(x) = \{e \in E : x(e) \neq 0\} \subseteq B \subseteq \cup(A : x(A) = f(A)) = \text{sat}(x)$, then x is generated using greedy by some ordering of B.

- Note, $\text{sat}(x) = \text{cl}(x) = \cup(A : x(A) = f(A))$ is also called the closure of x (recall that sets A such that $x(A) = f(A)$ are called tight, and such sets are closed under union and intersection, as seen in Lecture 8, Theorem **??**)
- Thus, $\text{cl}(x)$ is a tight set.
- Also, $\text{supp}(x) = \{e \in E : x(e) \neq 0\}$ is called the support of x.
- For arbitrary x, $\text{supp}(x)$ is not necessarily tight, but for an extreme point, $\text{supp}(x)$ is.

Polymatroid with labeled edge lengths

- Recall $f(e|A) = f(A+e) - f(A)$
- Notice how submodularity, $f(e|B) \leq f(e|A)$ for $A \subseteq B$, defines the shape of the polytope.
- In fact, we have strictness here $f(e|B) < f(e|A)$ for $A \subset B$.
- Also, consider how the greedy algorithm proceeds along the edges of the polytope.



Minimizers of a Submodular Function form a lattice

Theorem 13.2.2

For arbitrary submodular f, the minimizers are closed under union and intersection. That is, let $\mathcal{M} = \operatorname{argmin}_{X \subseteq E} f(X)$ be the set of minimizers of f. Let $A, B \in \mathcal{M}$. Then $A \cup B \in \mathcal{M}$ and $A \cap B \in \mathcal{M}$.

Proof.

Since A and B are minimizers, we have $f(A) = f(B) \leq f(A \cap B)$ and $f(A) = f(B) \leq f(A \cup B)$.
By submodularity, we have

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \tag{13.8}$$

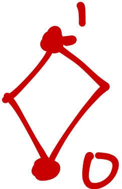
Hence, we must have $f(A) = f(B) = f(A \cup B) = f(A \cap B)$. □

Thus, the minimizers of a submodular function form a lattice, and there is a maximal and a minimal minimizer of every submodular function.

The sat function = Polymatroid Closure

- Matroid closure is generalized by the unique maximal element in $\mathcal{D}(x)$, also called the polymatroid closure or sat (saturation function).
- For some $x \in P_f$, we have defined:

$f(A) - x(A) = 0$



$$\mathrm{cl}(x) \stackrel{\text{def}}{=} \mathrm{sat}(x) \stackrel{\text{def}}{=} \bigcup \{A : A \in \mathcal{D}(x)\} \tag{13.8}$$

$$= \bigcup \{A : A \subseteq E, x(A) = f(A)\} \tag{13.9}$$

$$= \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\} \tag{13.10}$$

- Hence, $\mathrm{sat}(x)$ is the maximal (zero-valued) minimizer of the submodular function $f_x(A) \triangleq f(A) - x(A)$.
- Eq. (**??**) says that sat consists of any point x that is P_f saturated (any additional positive movement, in that dimension, leaves P_f). We'll revisit this in a few slides.
- First, we see how sat generalizes matroid closure.

The sat function = Polymatroid Closure

- Consider matroid $(E, \mathcal{I}) = (E, r)$, some $I \in \mathcal{I}$. Then $\mathbf{1}_I \in P_r$ and

$$\mathcal{D}(\mathbf{1}_I) = \{A : \mathbf{1}_I(A) = r(A)\} \tag{13.1}$$

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and

$$\mathrm{sat}(\mathbf{1}_I)$$

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$$= \bigcup \{A : A \subseteq E, |I \cap A| = r(A)\} \tag{13.4}$$

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- Notice that $\mathbf{1}_I(A) = |I \cap A| \leq |I|$.

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$I \cap A \subset A$

- Notice that $\mathbf{1}_I(A) = |I \cap A| \leq |I|$.
- Intuitively, consider an $A \supset I \in \mathcal{I}$ that doesn't increase rank, meaning $r(A) = r(I)$. If $r(A) = |I \cap A| = r(I \cap A)$, as in Eqn. (13.4), then A is in I's span, so should get $\text{sat}(\mathbf{1}_I) = \text{span}(I)$.

$A \cap I$'s span,

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- We formalize this next.

The sat function = Polymatroid Closure

Lemma 13.3.1 (Matroid $\mathrm{sat} : \mathbb{R}_+^E \to 2^E$ is the same as closure.)

For $I \in \mathcal{I}$, we have $$\mathrm{sat}(\mathbf{1}_I) = \mathrm{span}(I) \tag{13.5}$$

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Proof.

- For $\mathbf{1}_I(I) = |I| = r(I)$, so $I \in \mathcal{D}(\mathbf{1}_I)$ and $I \subseteq \text{sat}(\mathbf{1}_I)$. Also, $I \subseteq \text{span}(I)$.

I is tight for 1_I

since $1_I(I) = r(I)$

...

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- Consider some $b \in \text{span}(I) \setminus I$.

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- Consider some $b \in \text{span}(I) \setminus I$.
- Then $I \cup \{b\} \in \mathcal{D}(\mathbf{1}_I)$ since $\mathbf{1}_I(I \cup \{b\}) = |I| = r(I \cup \{b\}) = r(I)$.

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- Thus, $b \in \text{sat}(\mathbf{1}_I)$.
- Therefore, $\text{sat}(\mathbf{1}_I) \supseteq \text{span}(I)$.

...

The sat function = Polymatroid Closure

. . . proof continued.

- Now, consider $b \in \text{sat}(\mathbf{1}_I) \setminus I$.

$\square$

The sat function = Polymatroid Closure

. . . proof continued.

- Now, consider $b \in \text{sat}(\mathbf{1}_I) \setminus I$.
- Choose any $A \in \mathcal{D}(\mathbf{1}_I)$ with $b \in A$, thus $b \in A \setminus I$.

□

The sat function = Polymatroid Closure

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- Now, consider $b \in \text{sat}(\mathbf{1}_I) \setminus I$.
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- Then $\mathbf{1}(A) = |A \cap I| = r(A)$.
- Now $r(A) = |A \cap I| \leq |I| = r(I)$.

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- Then $\mathbf{1}(A) = |A \cap I| = r(A)$.
- Now $r(A) = |A \cap I| \leq |I| = r(I)$.
- Also, $r(A \cap I) = |A \cap I|$ since $A \cap I \in \mathcal{I}$.

$\square$

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- Also, $r(A \cap I) = |A \cap I|$ since $A \cap I \in \mathcal{I}$.
- Hence, $r(A \cap I) = r(A) = r((A \cap I) \cup (A \setminus I))$ meaning $(A \setminus I) \subseteq \text{span}(A \cap I) \subseteq \text{span}(I)$.

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- Since $b \in A \setminus I$, we get $b \in \mathrm{span}(I)$.

□

The sat function = Polymatroid Closure

... proof continued.

- Now, consider $b \in \text{sat}(\mathbf{1}_I) \setminus I$.
- Choose any $A \in \mathcal{D}(\mathbf{1}_I)$ with $b \in A$, thus $b \in A \setminus I$.
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- Since $b \in A \setminus I$, we get $b \in \text{span}(I)$.
- Thus, $\text{sat}(\mathbf{1}_I) \subseteq \text{span}(I)$.

$\square$

The sat function = Polymatroid Closure

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- Since $b \in A \setminus I$, we get $b \in \text{span}(I)$.
- Thus, $\boxed{\text{sat}(\mathbf{1}_I) \subseteq \text{span}(I)}$.
- Hence $\boxed{\text{sat}(\mathbf{1}_I) = \text{span}(I)}$

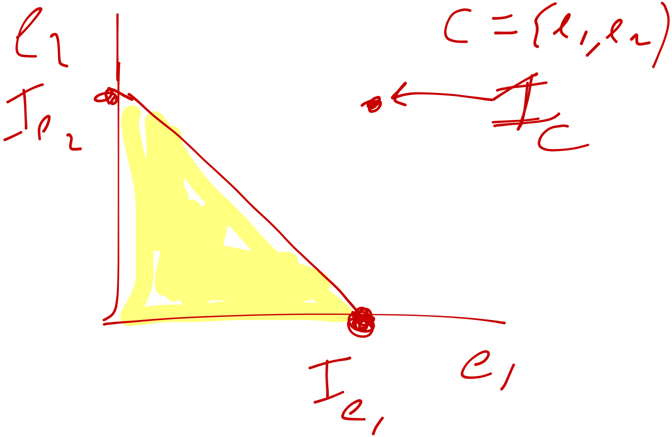
$\square$

The sat function = Polymatroid Closure

- Now, consider a matroid (E, r) and some $C \subseteq E$ with $C \notin \mathcal{I}$, and consider $\mathbf{1}_C$.

The sat function = Polymatroid Closure

- Now, consider a matroid (E, r) and some $C \subseteq E$ with $C \notin \mathcal{I}$, and consider $\mathbf{1}_C$. Is $\mathbf{1}_C \in P_r$?



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- Then we have $\mathbf{1}_B \leq \mathbf{1}_C \leq \mathbf{1}_{\text{span}(C)}$, and that $\mathbf{1}_B \in P_r$. We can then make the definition:

$$\text{sat}(\mathbf{1}_C) \triangleq \text{sat}(\mathbf{1}_B) \text{ for } B \in \mathcal{B}(C) \tag{13.6}$$

In which case, we also get $\text{sat}(\mathbf{1}_C) = \text{span}(C)$ (in general, could define $\text{sat}(y) = \text{sat}(\text{P-basis}(y))$).

Uniform points are their own P-basis.

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Exercise: is $\text{span}(C) = \text{sat}(\mathbf{1}_C)$? Prove or disprove it.

The sat function, span, and submodular function minimization

- Thus, for a matroid, $\mathrm{sat}(\mathbf{1}_I)$ is exactly the closure (or span) of I in the matroid. I.e., for matroid (E, r), we have $\mathrm{span}(I) = \mathrm{sat}(\mathbf{1}_B)$.

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- Recall, for $x \in P_f$ and polymatroidal f, $\mathrm{sat}(x)$ is the maximal (by inclusion) minimizer of $f(A) - x(A)$, and thus in a matroid, span(I) is the maximal minimizer of the submodular function formed by $r(A) - \mathbf{1}_I(A)$.

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- Submodular function minimization can solve "span" queries in a matroid or "sat" queries in a polymatroid.

sat, as tight polymatroidal elements

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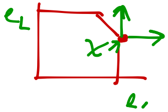

$\text{sat}(x) = \{e_1, e_2\}$

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- We next show more formally that these are the same.

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$\alpha \cdot \mathbf{1}_e(A) = \alpha$

$e \in A \Rightarrow$

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- So now, if A is any set such that $x(A) = f(A)$, then we clearly have

$$\forall e \in A, e \in \text{sat}(x), \text{ and therefore that } \text{sat}(x) \supseteq A \tag{13.16}$$

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- . . . and therefore, with sat as defined in Eq. (**??**),

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- On the other hand, for any $e \in \operatorname{sat}(x)$ defined as in Eq. (13.15), since e is itself a member of a tight set, there is a set $A \ni e$ such that $x(A) = f(A)$, giving

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- Therefore, the two definitions of sat are identical.

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or

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Saturation Capacity

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- Also, for $e \in \mathrm{sat}(x)$, we have that $\hat{c}(x;e) = 0$.

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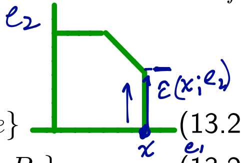
- Thus we have for $x \in P_f$,

$$\hat{c}(x;e) \stackrel{\text{def}}{=} \min \{ f(A) - x(A), \forall A \ni e \} \tag{13.24}$$

$$= \max \{ \alpha : \alpha \in \mathbb{R}, x + \alpha \mathbf{1}_e \in P_f \} \tag{13.25}$$



- We immediately see that for $e \in E \setminus \text{sat}(x)$, we have that $\hat{c}(x;e) > 0$.
- Also, for $e \in \text{sat}(x)$, we have that $\hat{c}(x;e) = 0$.
- Note that any α with $0 \leq \alpha \leq \hat{c}(x;e)$ we have $x + \alpha \mathbf{1}_e \in P_f$.

Saturation Capacity

- The max is achieved when

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- We also see that computing $\hat{c}(x; e)$ is a form of submodular function minimization.

Dependence Function

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- Given $x \in P_f$, and $e \in \text{sat}(x)$, define

$$\mathcal{D}(x,e) = \{A : e \in A \subseteq E, x(A) = f(A)\} \tag{13.26}$$
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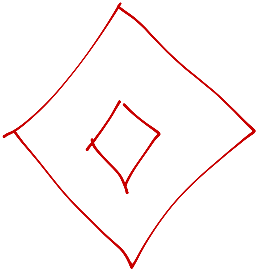

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$$\text{dep}(x,e) = \begin{cases} \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases} \tag{13.28}$$

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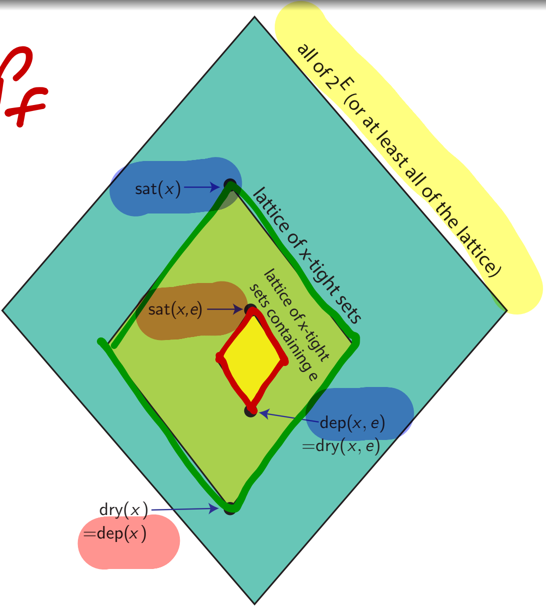
- I.e., $\text{dep}(x, e)$ is the minimal element in $\mathcal{D}(x)$ that contains e (the minimal x-tight set containing e).

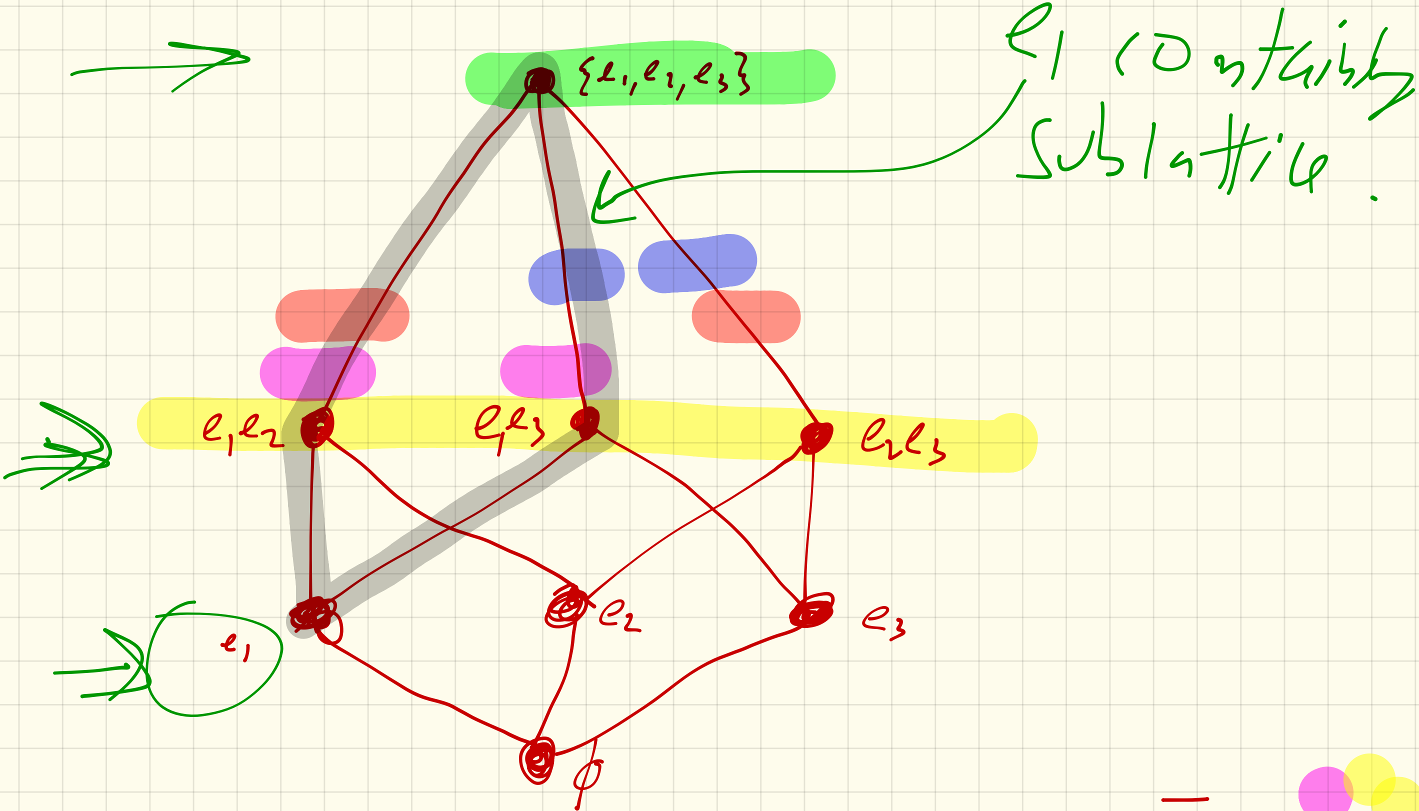
dep and sat in a lattice

for some $X \in P_f$

- The picture on the right summarizes the relationships between the lattices and sublattices.
- Note, $\bigcap_e \operatorname{dep}(x, e) = \operatorname{dep}(x)$.



$E = \{e_1, e_2, e_3\}$



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- Note that dry need not be the empty set. Exercise: give example.

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- Now, given $x \in P_f$, and $e \in \text{sat}(x)$, recall distributive sub-lattice of e-containing tight sets $\mathcal{D}(x, e) = \{A : e \in A, x(A) = f(A)\}$

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- But actually, $\text{dry}(x, e) = \text{dep}(x, e)$, so we have derived another expression for $\text{dep}(x, e)$ in Eq. (13.30).

Dependence Function and Fundamental Matroid Circuit

- Now, let $(E, \mathcal{I}) = (E, r)$ be a matroid, and let $I \in \mathcal{I}$ giving $\mathbf{1}_I \in P_r$. We have $\text{sat}(\mathbf{1}_I) = \text{span}(I) = \text{closure}(I)$.

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- By SFM lattice, $\exists$ a unique minimal $A \ni e$ with $|I \cap A| = r(A)$.
- Thus, $\text{dep}(\mathbf{1}_I, e)$ must be a circuit since if it included more than a circuit, it would not be minimal in this sense.

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- Therefore, when $e \in \text{sat}(\mathbf{1}_I) \setminus I$, then $\text{dep}(\mathbf{1}_I, e) = C(I, e)$ where $C(I, e)$ is the unique circuit contained in $I + e$ in a matroid (the fundamental circuit of e and I that we encountered before).

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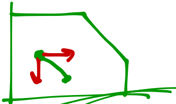
- We are thus free to take subsets of I as A, all of which must contain e, but all of which have rank equal to size.
- Also note: in general for $x \in P_f$ and $e \in \text{sat}(x)$, we have $\text{dep}(x, e)$ is tight by definition.

Summary of sat, and dep

- For $x \in P_f$, $\text{sat}(x)$ (span, closure) is the maximal saturated (x-tight) set w.r.t. x. I.e., $\text{sat}(x) = \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\}$. That is,



$$\text{cl}(x) \stackrel{\text{def}}{=} \text{sat}(x) \triangleq \bigcup \{A : A \in \mathcal{D}(x)\} \tag{13.34}$$

$$= \bigcup \{A : A \subseteq E, x(A) = f(A)\} \tag{13.35}$$

$$= \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\} \tag{13.36}$$

- For $e \in \text{sat}(x)$, we have $\text{dep}(x, e)$ (fundamental circuit) is the minimal (common) saturated (x-tight) set w.r.t. x containing e. That is,

$$\text{dep}(x, e) = \begin{cases} \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases}$$

$$= \{e' : \exists \alpha > 0, \text{ s.t. } x + \alpha(\mathbf{1}_e - \mathbf{1}_{e'}) \in P_f\} \tag{13.37}$$

Dependence Function and exchange

- For $e \in \text{span}(I) \setminus I$, we have that $I + e \notin \mathcal{I}$. This is a set addition restriction property.

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- But, analogous to the circuit case, is there an exchange property for $\text{dep}(x,e)$ in the form of vector movement restriction?
- We might expect the vector $\text{dep}(x,e)$ property to take the form: a positive move in the e-direction stays within P_f^+ only if we simultaneously take a negative move in one of the $\text{dep}(x,e)$ directions.

Dependence Function and exchange in 2D

- $\text{dep}(x, e)$ is set of neg. directions we must move if we want to move in pos. e direction, starting at x and staying within P_f.

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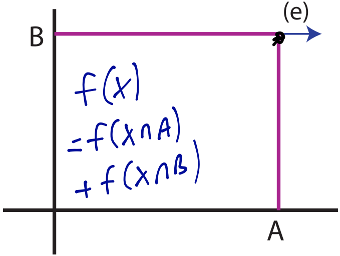
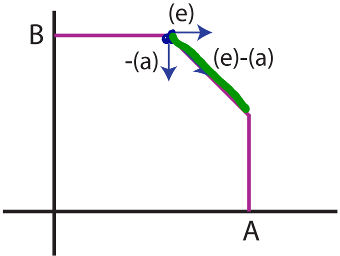
- Viewable in 2D, we have for $A, B \subseteq E$, $A \cap B = \emptyset$:





Left: $A \cap \text{dep}(x,e) = \emptyset$, and we can't move further in (e) direction, and moving in any negative $a \in A$ direction doesn't change that. Notice no dependence between (e) and any element in A.

Right: $A \subseteq \text{dep}(x,e)$, and we can't move further in the (e) direction, but we can move further in (e) direction by moving in some $a \in A$ negative direction. Notice dependence between (e) and elements in A.

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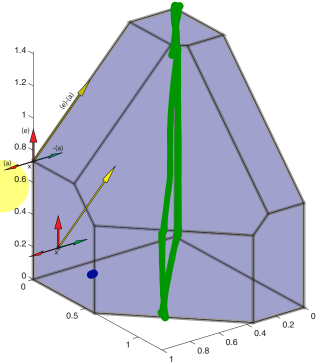
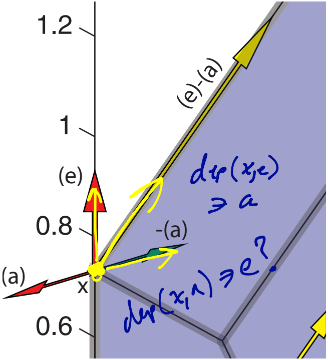
- In 3D, we have:

$V = V_1 \cup V_2$ partition

$\bar{f} : 2^{\{1,2\}} \to \mathbb{R}$

$f(A) = f\left(\bigcup_{i \in A} V_i\right)$





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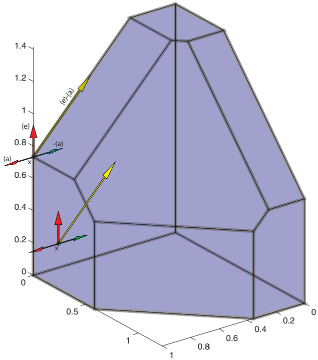

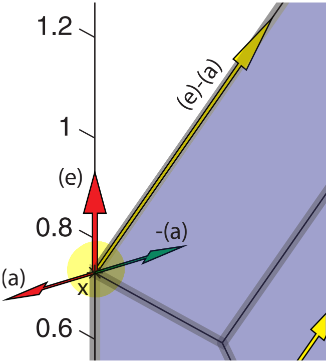

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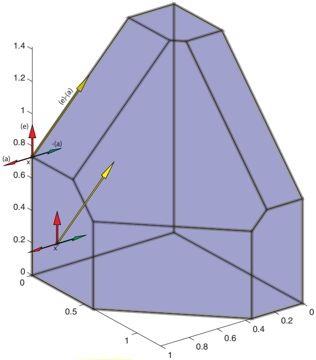

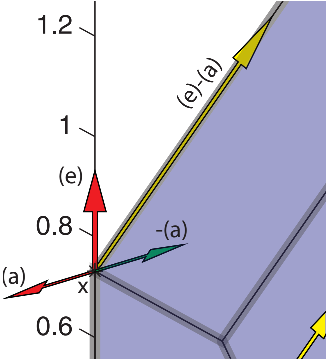

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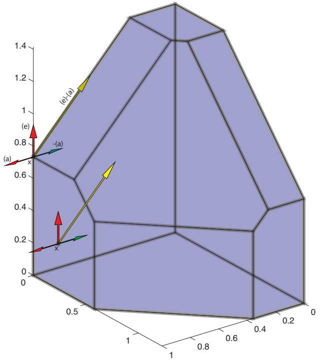

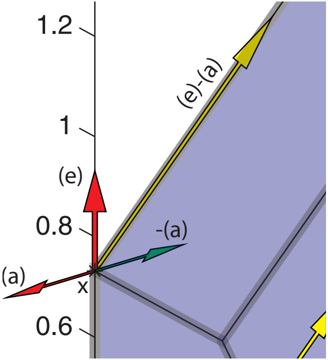

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- We next show this formally . . .

dep and exchange derived

- The derivation for $\mathrm{dep}(x, e)$ involves turning a strict inequality into a non-strict one with a strict explicit slack variable α:

$$\mathrm{dep}(x, e) = \mathsf{ntight}(x, e) = \tag{13.40}$$

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$$= \{e' : \exists \alpha > 0, \text{ s.t. } x(A) + \alpha(\mathbf{1}_e(A) - \mathbf{1}_{e'}(A)) \leq f(A), \forall A \not\ni e', e \in A\} \tag{13.45}$$

- Now, $1_e(A) - \mathbf{1}_{e'}(A) = 0$ if either $\{e, e'\} \subseteq A$, or $\{e, e'\} \cap A = \emptyset$.
- Also, if $e' \in A$ but $e \notin A$, then $x(A) + \alpha(\mathbf{1}_e(A) - \mathbf{1}_{e'}(A)) = x(A) - \alpha \leq f(A)$ since $x \in P_f$.

dep and exchange derived

- thus, we get the same in the above if we remove the constraint $A \not\ni e', e \in A$, that is we get

$$\text{dep}(x, e) = \{e' : \exists \alpha > 0, \text{ s.t. } x(A) + \alpha(\mathbf{1}_e(A) - \mathbf{1}_{e'}(A)) \leq f(A), \forall A\} \tag{13.46}$$

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- Compare with original, the minimal element of $\mathcal{D}(x, e)$, with $e \in \text{sat}(x)$:

$$\text{dep}(x, e) = \begin{cases} \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases} \tag{13.48}$$

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- Recall: $\text{sat}(x) = \{e : \hat{c}(x; e) = 0\}$ and $E \setminus \text{sat}(x) = \{e : \hat{c}(x; e) > 0\}$.

Summary important definitions so far: tight, dep, & sat

- x-tight sets: For $x \in P_f$, $\mathcal{D}(x) = \{A \subseteq E : x(A) = f(A)\}$.
- Polymatroid closure/maximal x-tight set: For $x \in P_f$, $\text{sat}(x) = \cup\{A : A \in \mathcal{D}(x)\} = \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\}$.
- Saturation capacity: for $x \in P_f$, $0 \leq \hat{c}(x; e) = \min\{f(A) - x(A) | \forall A \ni e\} = \max\{\alpha : \alpha \in \mathbb{R}, x + \alpha \mathbf{1}_e \in P_f\}$
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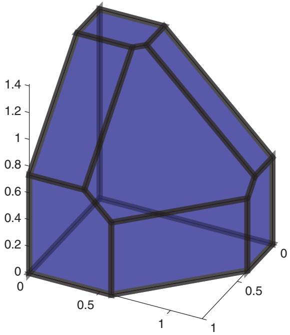

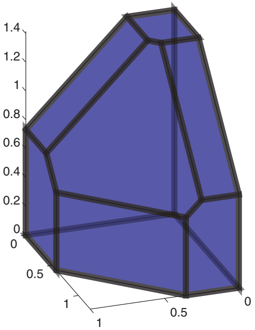

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- Since $\text{supp}(x)$ is tight, we immediately have that $\text{sat}(x) \supseteq \text{supp}(x)$.

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- For modular functions, they are always equal (e.g., think of "hyperrectangular" polymatroids).

Summary of supp, sat, and dep

- For $x \in P_f$, $\text{supp}(x) = \{e : x(e) \neq 0\} \subseteq \text{sat}(x)$
- For $x \in P_f$, $\text{sat}(x)$ (span, closure) is the maximal saturated (x-tight) set w.r.t. x. I.e., $\text{sat}(x) = \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\}$. That is,

$$\text{cl}(x) \stackrel{\text{def}}{=} \text{sat}(x) \triangleq \bigcup \{A : A \in \mathcal{D}(x)\} \tag{13.34}$$

$$= \bigcup \{A : A \subseteq E, x(A) = f(A)\} \tag{13.35}$$

$$= \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\} \tag{13.36}$$

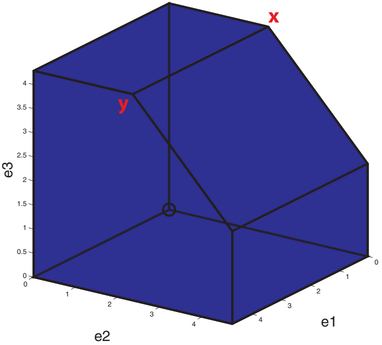
- For $e \in \text{sat}(x)$, we have $\text{dep}(x, e)$ (fundamental circuit) is the minimal (common) saturated (x-tight) set w.r.t. x containing e. That is,

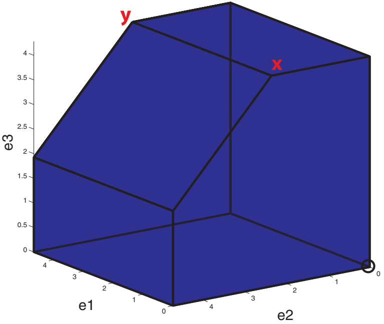
$$\text{dep}(x, e) = \begin{cases} \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases}$$

$$= \left\{e' : \exists \alpha > 0, \text{ s.t. } x + \alpha(\mathbf{1}_e - \mathbf{1}_{e'}) \in P_f\right\} \tag{13.37}$$

supp, sat, dep, example with perfect independence

- Example polymatroid where there is perfect independence between $A = \{e_2, e_3\}$ and $B = \{e_1\}$, i.e., $e_1 \perp\!\!\!\perp \{e_2, e_3\}$.





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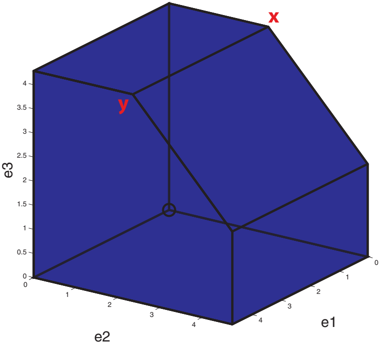

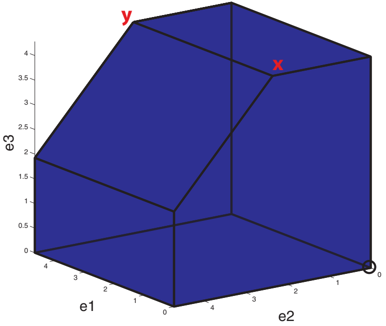

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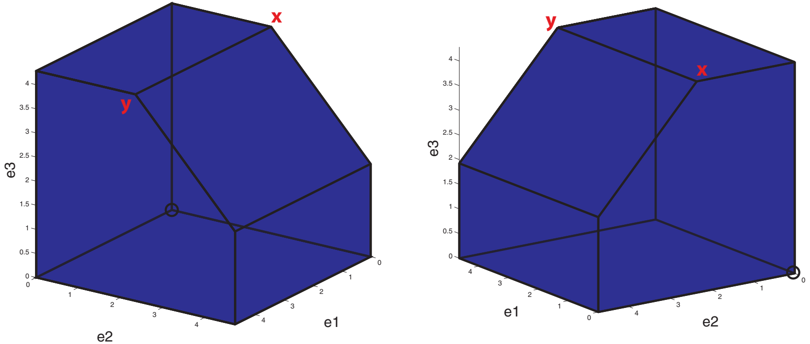

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- But $x(\{e_1, e_2, e_3\}) = x(\{e_2, e_3\}) < f(e_1, e_2, e_3) = f(e_1) + f(e_2, e_3)$. Thus, $\mathrm{supp}(x) = \mathrm{sat}(x) = \{e_2, e_3\}$.

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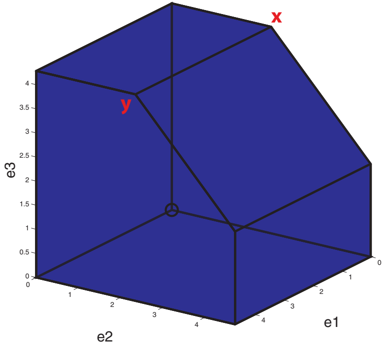

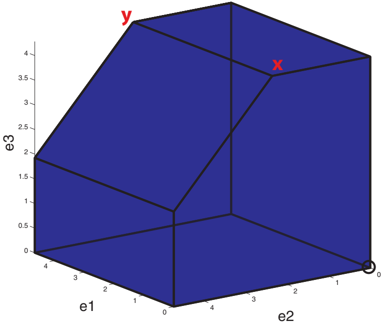

- Note that considering a submodular function on clustered ground set $E = \{e_1, e_{23}\}$ where $f'(e_1) = f(e_1)$, $f'(e_{23}) = f(e_2, e_3)$ leads to a rectangle (no dependence between $\{e1\}$ and $\{e2, e3\}$).

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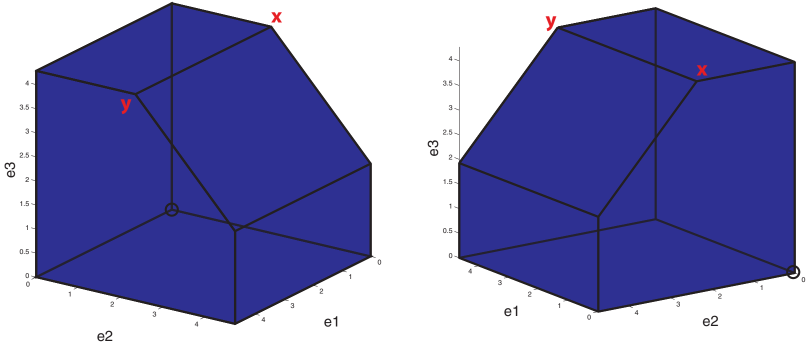
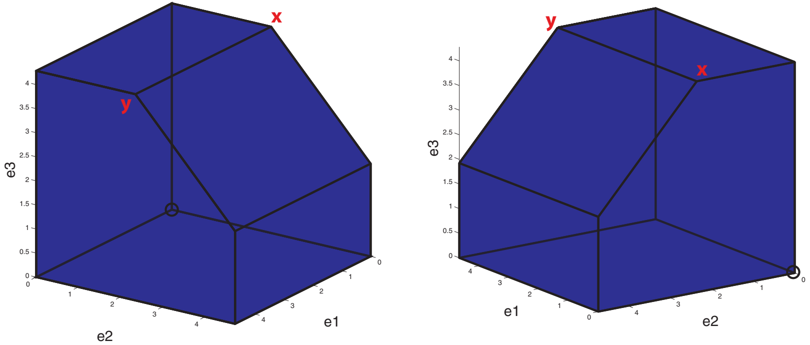


- We also have $\mathrm{sat}(x) = \{e_3, e_2\}$. So $\mathrm{dep}(x, e_1)$ is not defined, $\mathrm{dep}(x, e_2) = \{e_3\}$, and $\mathrm{dep}(x, e_3) = \emptyset$.
- $\mathrm{sat}(y) = \{e_1, e_2, e_3\}$. So $\mathrm{dep}(y, e_1) = \emptyset$, $\mathrm{dep}(y, e_2) = e_3$, and $\mathrm{dep}(y, e_3) = \emptyset$.

supp, sat, dep, example with perfect independence

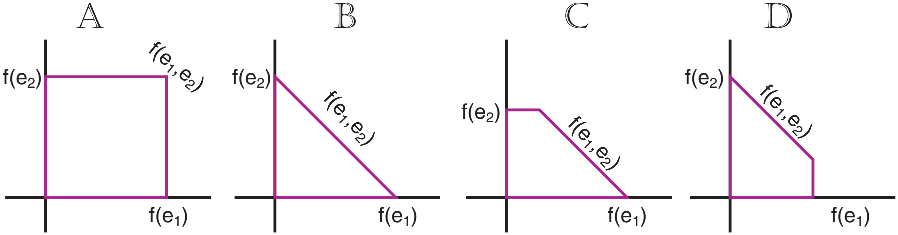
- Example polymatroid where there is perfect independence between $A = \{e_2, e_3\}$ and $B = \{e_1\}$, i.e., $e_1 \perp\!\!\!\perp \{e_2, e_3\}$.



- We also have $\text{sat}(x) = \{e_3, e_2\}$. So $\text{dep}(x, e_1)$ is not defined, $\text{dep}(x, e_2) = \{e_3\}$, and $\text{dep}(x, e_3) = \emptyset$.
- $\text{sat}(y) = \{e_1, e_2, e_3\}$. So $\text{dep}(y, e_1) = \emptyset$, $\text{dep}(y, e_2) = e_3$, and $\text{dep}(y, e_3) = \emptyset$.

supp, sat, and polymatroid dependence in 2D

- Ex: various amounts of "dependence" between e_1 and e_2.



supp, sat, and polymatroid dependence in 2D

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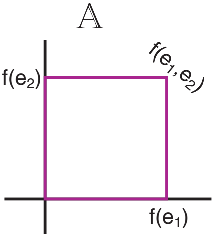

- Case A: perfect independence.

supp, sat, and polymatroid dependence in 2D

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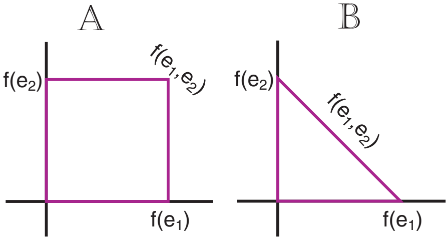

- Case A: perfect independence.
- Case B: perfect dependence. Since slope is -45°, we must have $f(e_1) = f(e_2) = f(e_1, e_2)$. Entropy case: deterministic bijection between random variables e_1 and e_2.

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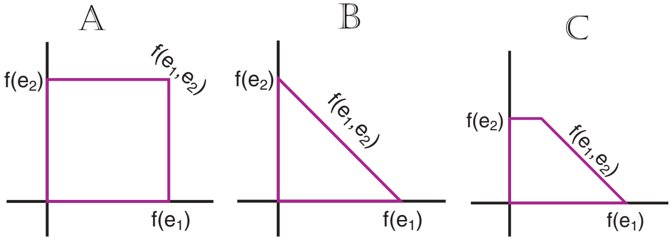

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- Case C: $f(e_2) < f(e_1) = f(e_1, e_2)$. Entropy case: random variable e_2 a deterministic function of e_1 which has higher entropy.

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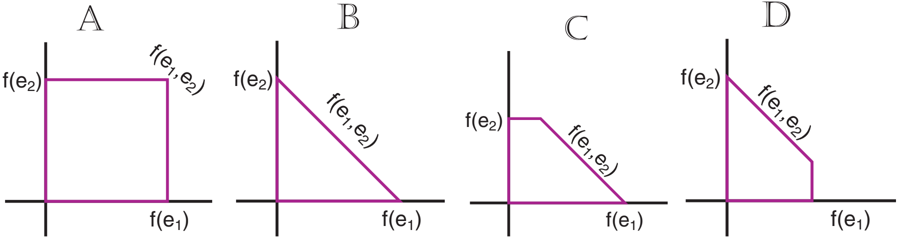

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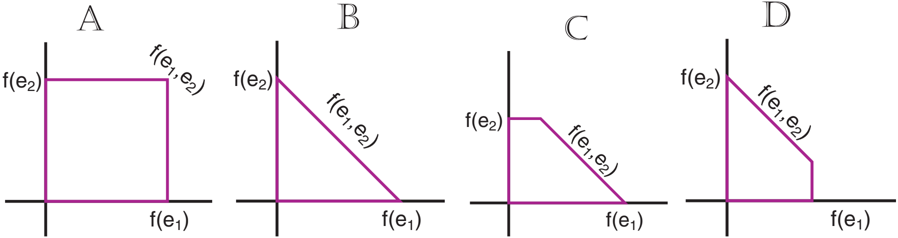
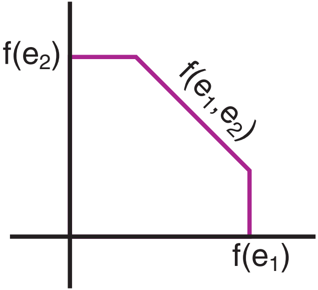


- In each case, we see points x where $\text{supp}(x) \subseteq \text{sat}(x)$.
- Example: Case B or C, let $x = (f(e_1), 0)$ so $\text{supp}(x) = \{e_1\}$ but since $x(\{e_1, e_2\}) = x(\{e_1\}) = f(e_1) = f(e_1, e_2)$ we have $\text{sat}(x) = \{e_1, e_2\}$.
- Similar for case D with $x = (0, f(e_2))$.

supp, sat, and dependence in 2D

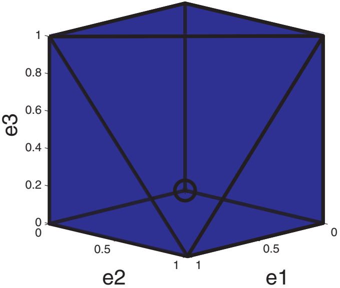
- General case, $f(e_1, e_2) < f(e_1) + f(e_2)$, $f(e_1) < f(e_1, e_2)$, and $f(e_2) < f(e_1, e_2)$.

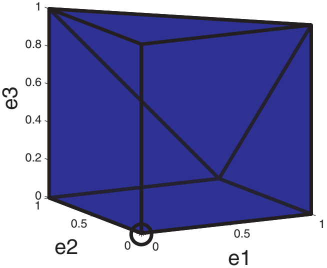


- Entropy case: We have a random variable Z and two separate deterministic functions $e_1 = h_1(Z)$ and $e_2 = h_2(Z)$ such that the entropy $H(e_1, e_2) = H(Z)$, but each deterministic function gives a different "view" of Z, each contains more than half the information, and the two are redundant w.r.t. each other.

supp, sat, and perfect dependence in 3D

- Ex: polymatroid with perfect independence between e_2 and e_3, so $f(e_2, e_3) = f(e_2) + f(e_3)$, but perfect dependence between $A = \{e_2, e_3\}$ and $B = \{e_1\}$, so $f(e_1, e_2, e_3) = f(e_2, e_3)$





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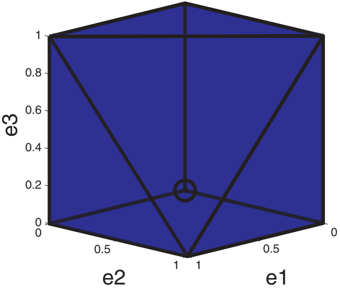

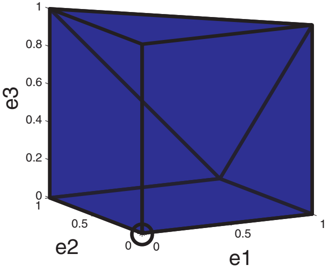

- Entropy case: xor V-structure Bayesian network $e_1 = h(e_2, e_3)$ where h is the xor function ($e_2 \rightarrow e_1 \leftarrow e_3$), and e_2, e_3 are both independent binary with unity entropy.

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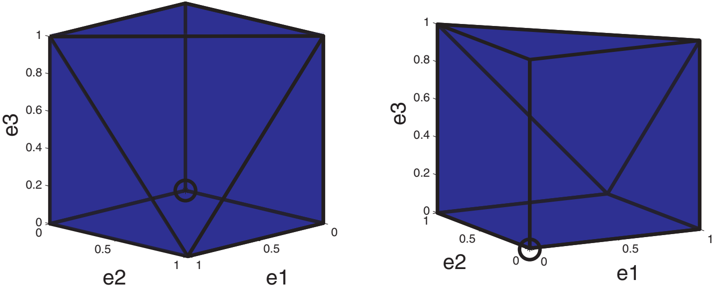

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- Q: Why does the polytope have a symmetry? Notice independence (square) for any pair.

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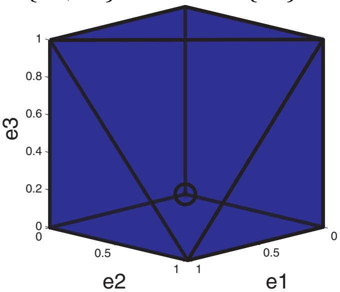
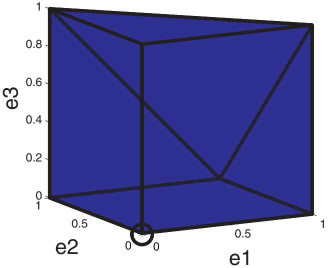
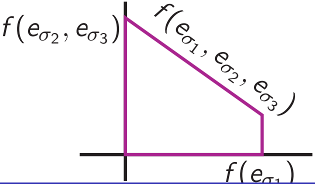




- For any permutation σ of $\{1, 2, 3\}$, considering $\{e_{\sigma_1}, e_{\sigma_2}\}$ vs. $\{e_{\sigma_3}\}$:

e_{σ_3} is a deterministic function of $\{e_{\sigma_1}, e_{\sigma_2}\}$



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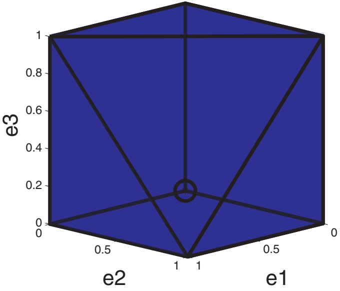

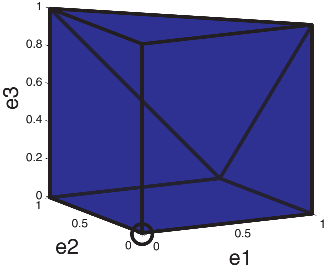

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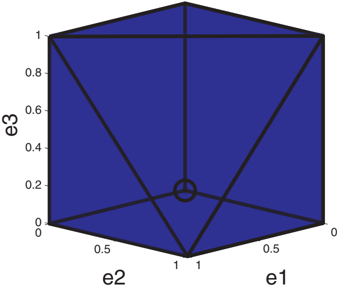

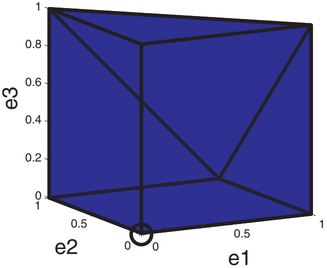

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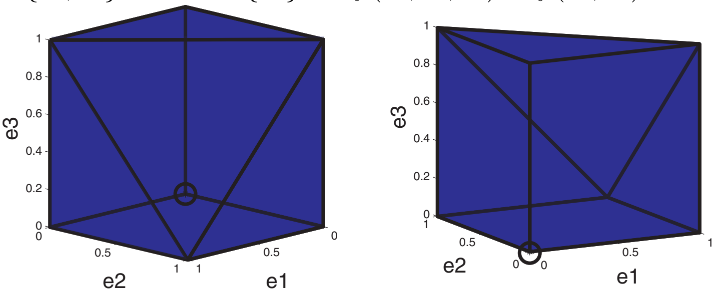

- Note also, that for some of the extreme points, multiple orders generate them.
- Consider extreme point $x = (x_1, x_2, x_3) = (1, 1, 0)$. Then we get this either with orders (e_1, e_2, e_3), or (e_2, e_1, e_3). This is true since $f(e_{\sigma_e}|\{e_{\sigma_1}, e_{\sigma_2}\}) = 0$ for all permutations σ of $\{1, 2, 3\}$.

perfect dependence in 3D, entropy, and Bayesian networks

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- Consequently, $X_i \perp\!\!\!\perp X_j$ for any $i \neq j$.
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- The entropy function $f(A) = H(X_A)$ is a submodular function that will have the symmetric 3D polytope of the previous example.

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- Thus, for any extremal x, with $\text{sat}(x) \supset \text{supp}(x)$, we see that for $e \in \text{sat}(x) \setminus \text{supp}(x)$, we have $\text{supp}(x) + e$ is also tight.
- Note also, for any $A \subseteq \text{sat}(x) \setminus \text{supp}(x)$, we have $f(A|\,\text{supp}(x)) = 0$.

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- Note that all of these results hold when f is monotone non-decreasing submodular (e.g., for a polymatroid function).

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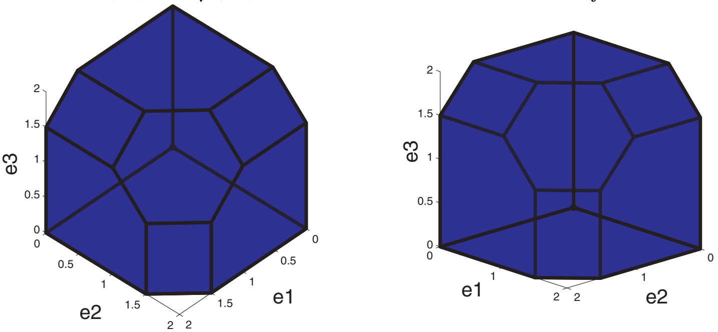
- For the most part, we are interested in these quantities when x is extremal as we will see.

supp and sat, example under limited curvature

- Strict monotone f polymatroids, where $f(e|E \setminus e) > 0, \forall e$.
- Example: $f(A) = \sqrt{|A|}$, where all $m!$ vertices of B_f are unique.



- In such cases, taking any extremal point $x \in P_f$ based on prefix order $E = (e_1, \dots)$, where $\text{supp}(x) \subset E$, we have that $\text{sat}(x) = \text{supp}(x)$ since the largest tight set corresponds to $x(E_i) = f(E_i)$ for some i, and while any $e \in E \setminus E_i$ is such that $x(E_i + e) = x(E_i)$, there is **no** such e with $f(E_i + e) = f(E_i)$.

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- Thus, any point $x \in B_f$ is a convex combination of at most $|E| - k + 1$ vertices of B_f.
- And if f does not have such independence, dimension of B_f is $|E| - 1$ and any point $x \in B_f$ is a convex combination of at most $|E|$ vertices of B_f.

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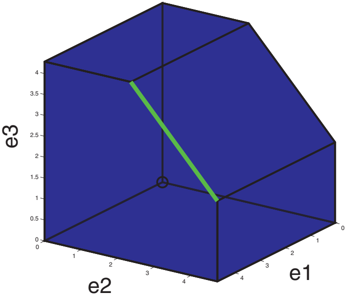
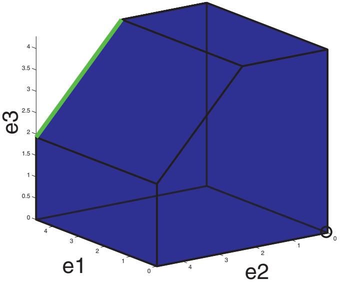
- Example f with independence between $A = \{e_2, e_3\}$ and $B = \{e_1\}$, i.e., $e_1 \perp\!\!\!\perp \{e_2, e_3\}$, with B_f marked in green.





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- Given polymatroid function f, the base polytope $B_f = \left\{x \in \mathbb{R}_+^E : x(A) \leq f(A)\ \forall A \subseteq E, \text{ and } x(E) = f(E)\right\}$ always exists.

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- Moreover, in this case, x is a vertex of B_f since it is extremal.

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- In words, B_f intersects all "multi-axis congruent" hyperplanes within R^E of the form $\left\{ x \in \mathbb{R}^E : x(A) = f(A) \right\}$ <u>for all</u> $A \subseteq E$.

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- In fact, every $x \in P_f$ is dominated by $x \leq y \in B_f$.

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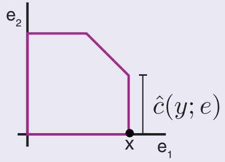
- Consider following algorithm:

```
T' ← T ;
for e ∈ E \ T do
    y ← y + c(y; e)1_e ; T' ← T' ∪ {e};
```



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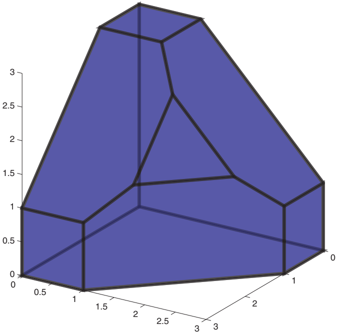
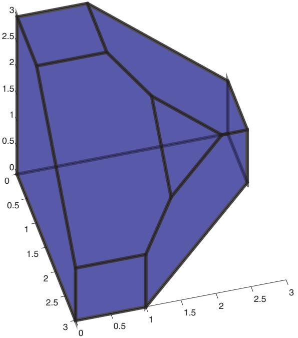
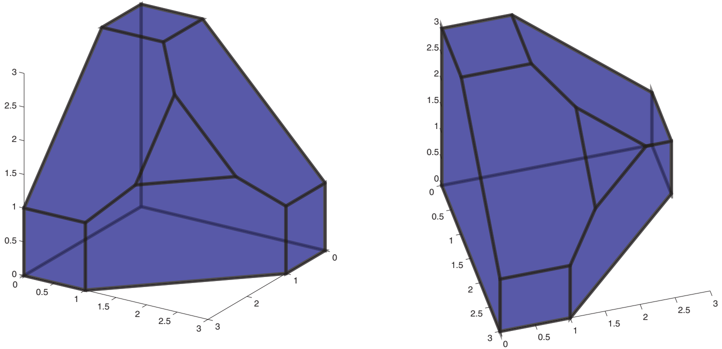




Polytope example 1

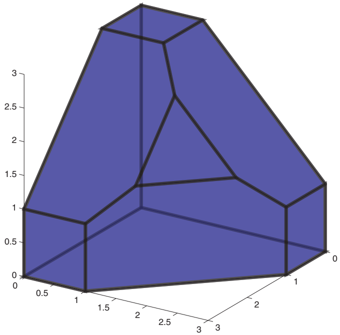
- Observe: P_f (at two views):

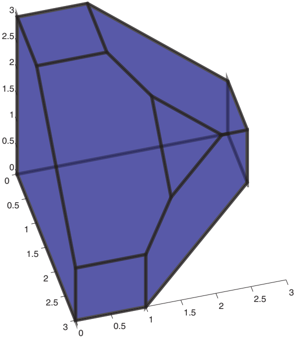


- Is this a polymatroidal polytope?

Polytope example 1

- Observe: P_f (at two views):

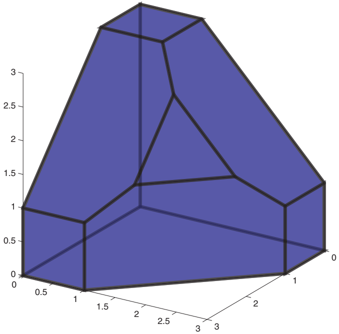


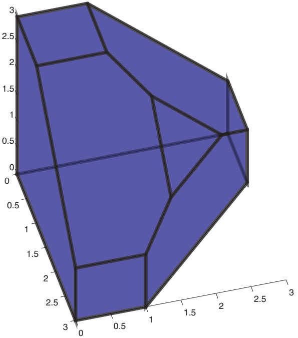


- Is this a polymatroidal polytope?
- No, "B_f" doesn't intersect sets of the form $\{x : x(e) = f(e)\}$ for $e \in E$.

Polytope example 1

- Observe: P_f (at two views):

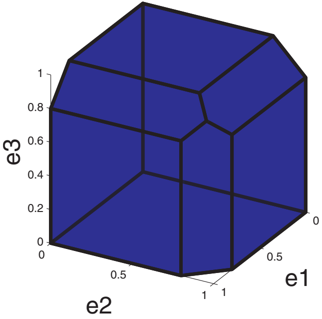


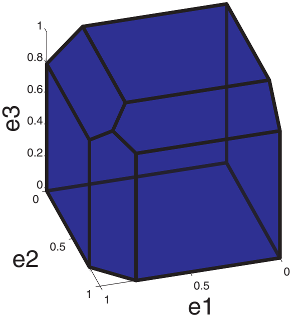


- Is this a polymatroidal polytope?
- No, "B_f" doesn't intersect sets of the form $\{x : x(e) = f(e)\}$ for $e \in E$.
- This was generated using function $g(0) = 0$, $g(1) = 3$, $g(2) = 4$, and $g(3) = 5.5$. Then $f(S) = g(|S|)$ is not submodular since (e.g.) $f(\{e_1, e_3\}) + f(\{e_1, e_2\}) = 4 + 4 = 8$ but

Polytope example 2

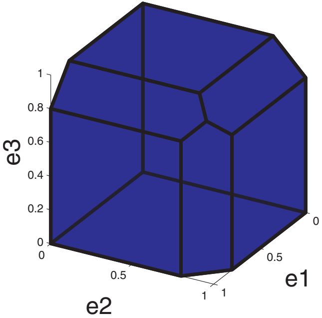
- Observe: P_f (at two views):

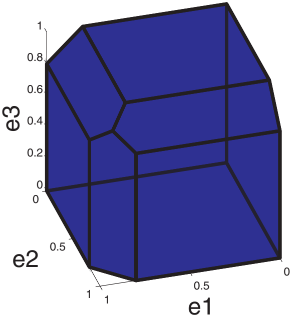




Polytope example 2

- Observe: P_f (at two views):

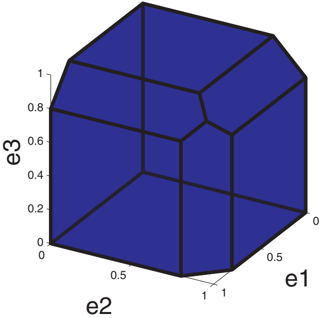


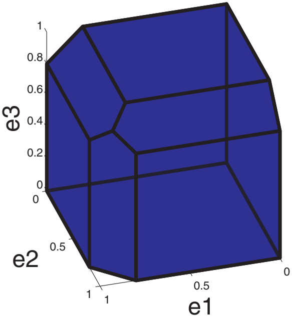


- Is this a polymatroidal polytope?

Polytope example 2

- Observe: P_f (at two views):

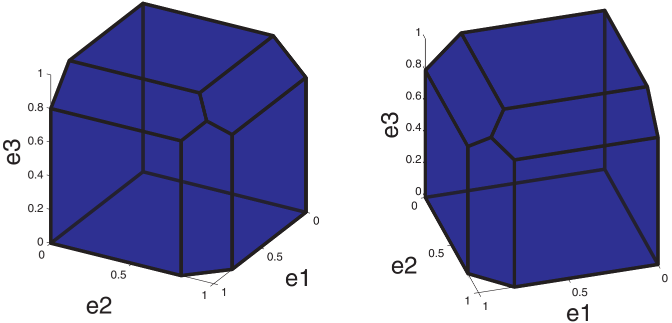




- Is this a polymatroidal polytope?
- No, "B_f" (which would be a single point in this case) doesn't intersect sets of the form $\{x : x(e) = f(e)\}$ for $e \in E$.

Polytope example 2

- Observe: P_f (at two views):



- Is this a polymatroidal polytope?
- No, "B_f" (which would be a single point in this case) doesn't intersect sets of the form $\{x : x(e) = f(e)\}$ for $e \in E$.
- This was generated using function $g(0) = 0$, $g(1) = 1$, $g(2) = 1.8$, and $g(3) = 3$. Then $f(S) = g(|S|)$ is not submodular since (e.g.) $f(\{e_1, e_3\}) + f(\{e_1, e_2\}) = 1.8 + 1.8 = 3.6$ but $f(\{e_1, e_2, e_3\}) + f(\{e_1\}) = 3 + 1 = 4$.