Submodular Functions, Optimization, and Applications to Machine Learning
— Spring Quarter, Lecture 12 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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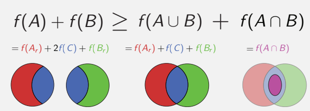

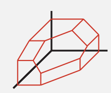

Cumulative Outstanding Reading

- Read chapters 1 and 2, and sections 3.1-3.2 from Fujishige's book.
- Good references for today: Schrijver-2003, Oxley-1992/2011, Welsh-1973, Goemans-2010, Cunningham-1984, Edmonds-1969.
- Read Tom McCormick's overview paper on SFM http://people.commerce.ubc.ca/faculty/mccormick/sfmchap8a.pdf
- Read chapters 1 - 3 from Fujishige book.
- Matroid properties http://www-math.mit.edu/~goemans/18433S09/matroid-notes.pdf

Announcements, Assignments, and Reminders

- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, Combinatorial Geometries
- L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
- L9: From Matroid Polytopes to Polymatroids.
- L10: Polymatroids and Submodularity
- L11: More properties of polymatroids, SFM special cases
- L12: polymatroid properties, extreme points polymatroids,
- L13: sat, dep, supp, exchange capacity, examples
- L14: Lattice theory: partially ordered sets; lattices; distributive, modular, submodular, and boolean lattices; ideals and join irreducibles.
- L15: Supp, Base polytope, polymatroids and entropic Venn diagrams, exchange capacity,
- L16: minimum norm point algorithm and the lattice of minimizers of a submodular function, Lovasz extension
- L17: Lovasz extension, Choquet Integration, more properties/examples of Lovasz extension, convex minimization and SFM.
- L18:
- L19:
- L20:

Finals Week: June 9th-13th, 2014.

A polymatroid function's polyhedron is a polymatroid.

Theorem 12.2.4

Let f be a polymatroid function defined on subsets of E. For any $x \in \mathbb{R}_+^E$, and any P_f^+-basis $y^x \in \mathbb{R}_+^E$ of x, the component sum of y^x is

$$y^x(E) = \text{rank}(x) = \max\left(y(E) : y \le x, y \in P_f^+\right) \\ = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \quad (12.34)$$

As a consequence, P_f^+ is a polymatroid, since r.h.s. is constant w.r.t. y^x.

By taking $B = \text{supp}(x)$ (so elements $E \setminus B$ are zero in x), and for $b \in B$, $x(b)$ is big enough, the r.h.s. min has solution $A^* = E \setminus B$. We recover submodular function from the polymatroid polyhedron via the following:

$$f(B) = \max\left\{y(B) : y \in P_f^+\right\} \quad (12.35)$$

In fact, we will ultimately see a number of important consequences of this theorem (other than just that P_f^+ is a polymatroid)

Tight sets $\mathcal{D}(y)$ are closed, and max tight set $\text{sat}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{A : A \subseteq E,\ y(A) = f(A)\} \quad (12.18)$$

Theorem 12.2.1

For any $y \in P_f^+$, with f a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

Proof.

We have already proven this as part of Theorem 9.4.5 □

Also recall the definition of $\text{sat}(y)$, the maximal set of tight elements relative to $y \in \mathbb{R}_+^E$.

$$\text{sat}(y) \stackrel{\text{def}}{=} \bigcup\{T : T \in \mathcal{D}(y)\} \quad (12.19)$$

Bipartite Matching

- Given a matching $A \subseteq E$ (which might be empty), we can increase the matching if we can find an augmenting path S.
- The updated matching becomes $A' = A \setminus S \cup S \setminus A = A \ominus S$, where $\ominus$ is the symmetric difference operator.
- The algorithm becomes:

Algorithm 8.1: Alternating Path Bipartite Matching

Let A be an arbitrary (including empty) matching in $G = (V, F, E)$;
while *There exists an augmenting path S in G* **do**
  $A \leftarrow A \ominus S$;

- This can easily be made to run in $O(m^2 n)$, where $|V| = m$, $|F| = n$, $m \leq n$, but it can be made to run much faster as well (see Schrijver-2003).

Matroid Intersection

- Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$.
- While $(V, \mathcal{I}_1 \cap \mathcal{I}_2)$ is typically not a matroid (Exercise: show graphical example.), we might be interested in finding the maximum size common independent set. That is, find $\max |X|$ such that both $X \in \mathcal{I}_1$ and $X \in \mathcal{I}_2$.

Theorem 12.2.5

Let M_1 and M_2 be given as above, with rank functions r_1 and r_2. Then the size of the maximum size set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by

$$(r_1 * r_2)(V) \triangleq \min_{X \subseteq V} \Big(r_1(X) + r_2(V \setminus X) \Big) \tag{12.7}$$

This is an instance of the convolution of two submodular functions, f_1 and f_2 that, evaluated at $Y \subseteq V$, is written as:

$$(f_1 * f_2)(Y) = \min_{X \subseteq Y} \Big(f_1(X) + f_2(Y \setminus X) \Big) \tag{12.8}$$

Fundamental circuits in matroids

Lemma 12.2.3

Let $I \in \mathcal{I}(M)$, and $e \in E$, then $I \cup \{e\}$ contains at most one circuit in M.

Proof.

- Suppose, to the contrary, that there are two distinct circuits C_1, C_2 such that $C_1 \cup C_2 \subseteq I \cup \{e\}$.
- Then $e \in C_1 \cap C_2$, and by (C2), there is a circuit C_3 of M s.t. $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\} \subseteq I$
- This contradicts the independence of I.

□

In general, let $C(I, e)$ be the unique circuit associated with $I \cup \{e\}$ (commonly called the fundamental circuit in M w.r.t. I and e).

Matroid Intersection Algorithm Idea

- Consider two matroids $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ and start with any $I \in \mathcal{I}_1 \cap \mathcal{I}_2$.
- Consider some $v_1 \notin \text{span}_1(I)$, so that $I + v_1 \in \mathcal{I}_1$.
- If $I + v_1 \in \mathcal{I}_2$, then v_1 is "augmenting", and we can augment I to $I + v_1$ and still be independent in both M_1 and M_2.
- If $I + v_1 \notin \mathcal{I}_2$, $\exists C_2(I, v_1)$ a circuit in M_2, and choosing $v_2 \in C_2(I, v_1)$ s.t. $v_2 \neq v_1$ leads to $I + v_1 - v_2$ which (because $\text{span}_2(I) = \text{span}_2(I + v_1 - v_2)$) is again independent in M_2. $I + v_1 - v_2$ is also independent in M_1. Note, $v_2 \in I$.
- Next choose a $v_3 \in \text{span}_1(I) - \text{span}_1(I - v_2)$ to recover what was lost in $I \cup \{v_1\}$ when we removed v_2 from it. Note, $v_3 \notin I$.
- Then $\text{span}_1(I) = \text{span}_1(I - v_2 + v_3)$.
- Moreover, since $I + v_1 \in \mathcal{I}_1$, $v_1 \notin \text{span}_1(I)$, so $\text{span}_1(I + v_1) = \text{span}_1(I + v_1 - v_2 + v_3)$.
- But $I + v_1 - v_2 + v_3$ might not be independent in M_2 again, so need to find an $v_4 \in C_2(I + v_1 - v_2, v_3)$, $v_4 \in I$ to remove, and so on.

- Hopefully (eventually) we'll find an odd length sequence $S = (v_1, v_2, \dots, v_s)$ such that we will be independent in both M_1

Identifying Augmenting Sequences

Theorem 12.2.6

Let I_p and I_{p+1} be intersections of M_1 and M_2 with p and $p+1$ elements respectively. Then there exists an augmenting sequence $S \subseteq I_p \ominus I_{p+1}$ w.r.t. I_p.

Theorem 12.2.7

An intersection is of maximum cardinality iff it admits no augmenting sequence.

Theorem 12.2.8

For any intersection I, there exists a maximum cardinality intersection I^* such that $\mathrm{span}_1(I) \subseteq \mathrm{span}_1(I^*)$ and $\mathrm{span}_2(I) \subseteq \mathrm{span}_2(I^*)$.

All this can be made to run in poly time.

Matroid Partition Problem

- Suppose $M_i = (E, \mathcal{I}_i)$ is a matroid and that we have k of them on the same ground set E.
- We wish to, if possible, partition E into k blocks, $I_i, i \in \{1, 2, \ldots, k\}$ where $I_i \in \mathcal{I}_i$.
- Moreover, we want partition to be lexicographically maximum, that is $|I_1|$ is maximum, $|I_2|$ is maximum given $|I_1|$, and so on.

Matroid Partition Problem

Theorem 12.3.1

Let M_i be a collection of k matroids as described. Then, a set $S \subseteq E$ can be partitioned into k subsets $I_i, i = 1 \dots k$ where $I_i \in \mathcal{I}_i$ is independent in matroid i, if and only if, for all $A \subseteq S$

$$|A| \leq \sum_{i=1}^{k} r_i(A) \tag{12.1}$$

where r_i is the rank function of M_i.

- Now, if all matroids are the same $M_i = M$ for all i, we get condition
$$|A| \leq kr(A) \;\; \forall A \subseteq E \tag{12.2}$$
- But considering vector of all ones $\mathbf{1} \in \mathbb{R}_+^E$, this is the same as
$$\frac{1}{k}|A| = \frac{1}{k}\mathbf{1}(A) \leq r(A) \;\; \forall A \subseteq E \tag{12.3}$$

Matroid Partition Problem

- Recall definition of matroid polytope
$$P_r^+ = \left\{ y \in \mathbb{R}_+^E : y(A) \leq r(A) \text{ for all } A \subseteq E \right\} \tag{12.4}$$
- Then we see that this special case of the matroid partition problem is just testing if $\frac{1}{k}\mathbf{1} \in P_r^+$, a problem of testing the membership in matroid polyhedra.
- This is therefore a special case of submodular function minimization.

Review

- The next two slides from respectively from Lecture 9 and Lecture 8.

Polymatroidal polyhedron (or a "polymatroid")

Definition 12.4.4 (polymatroid)

A polymatroid is a compact set $P \subseteq \mathbb{R}_+^E$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For every $x \in \mathbb{R}_+^E$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any P-basis of x), has the same component sum $y(E)$

Maximum weight independent set via greedy weighted rank

Theorem 12.4.6

Let $M = (V, \mathcal{I})$ be a matroid, with rank function r, then for any weight function $w \in \mathbb{R}_+^V$, there exists a chain of sets $U_1 \subset U_2 \subset \cdots \subset U_n \subseteq V$ such that

$$\max\{w(I) | I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i) \tag{12.19}$$

where $\lambda_i \geq 0$ satisfy

$$w = \sum_{i=1}^{n} \lambda_i \mathbf{1}_{U_i} \tag{12.20}$$

Polymatroidal polyhedron and greedy

- Let $(E, \mathcal{I})$ be a set system and $w \in \mathbb{R}_+^E$ be a weight vector.
- Recall greedy algorithm: Set $A = \emptyset$, and repeatedly choose $y \in E \setminus A$ such that $A \cup \{y\} \in \mathcal{I}$ with $w(y)$ as large as possible, stopping when no such y exists.
- For a matroid, we saw that set system $(E, \mathcal{I})$ is a matroid iff for each weight function $w \in \mathbb{R}_+^E$, the greedy algorithm leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.
- Stated succinctly, considering $\max\{w(I) : I \in \mathcal{I}\}$, then $(E, \mathcal{I})$ is a matroid iff greedy works for this maximization.
- Can we also characterize a polymatroid in this way?
- That is, if we consider $\max\left\{wx : x \in P_f^+\right\}$, where P_f^+ represents the "independent vectors", is it the case that P_f^+ is a polymatroid iff greedy works for this maximization?
- Can we even relax things so that $w \in \mathbb{R}^E$?

Polymatroidal polyhedron and greedy

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- What is the greedy solution in this setting, when $w \in \mathbb{R}^E$?
- Sort elements of E w.r.t. w so that, w.l.o.g. $E = (e_1, e_2, \ldots, e_m)$ with $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$.
- Let $k+1$ be the first point (if any) at which we are non-positive, i.e., $w(e_k) > 0$ and $0 \geq w(e_{k+1})$.
  That is, we have
  $$w(e_1) \geq w(e_2) \geq \cdots \geq w(e_k) > 0 \geq w(e_{k+1}) \geq \cdots \geq w(e_m) \quad (12.5)$$
- Next define partial accumulated sets E_i, for $i = 0 \ldots m$, we have w.r.t. the above sorted order:
  $$E_i \stackrel{\text{def}}{=} \{e_1, e_2, \ldots e_i\} \quad (12.7)$$
  (note $E_0 = \emptyset$, $f(E_0) = 0$, and E and E_i is always sorted w.r.t w).
- The greedy solution is the vector $x \in \mathbb{R}_+^E$ with elements defined as:

$$x(e_1) \stackrel{\text{def}}{=} f(E_1) = f(e_1) = f(e_1|E_0) = f(e_1|\emptyset) \quad (12.8)$$

$$x(e_i) \stackrel{\text{def}}{=} f(E_i) - f(E_{i-1}) = f(e_i|E_{i-1}) \text{ for } i = 2 \ldots k \quad (12.9)$$

$$x(e_i) \stackrel{\text{def}}{=} 0 \text{ for } i = k+1 \ldots m = |E| \quad (12.10)$$

Some Intuition: greedy and gain

- Note $x(e_i) = f(e_i|E_{i-1}) \leq f(e_i|E')$ for any $E' \subseteq E_{i-1}$
- So $x(e_1) = f(e_1)$ and this corresponds to $w(e_1) \geq w(e_i)$ for all $i \neq 1$.
- Hence, for the largest value of w (namely $w(e_1)$), we use for $x(e_1)$ the largest possible gain value of e_1 (namely $f(e_1|\emptyset) \geq f(e_1|A)$ for any $A \subseteq E \setminus \{e_1\}$).
- For the next largest value of w (namely $w(e_2)$), we use for $x(e_2)$ the next largest gain value of e_2 (namely $f(e_2|e_1)$), while still ensuring (as we will soon see in Theorem 12.4.1) that the resulting $x \in P_f$.
- This process continues, using the next largest possible gain of e_i for $x(e_i)$ while ensuring we do not leave the polytope, given the values we've already chosen for $x(e_{i'})$ for $i' < i$.

Polymatroidal polyhedron and greedy

Theorem 12.4.1

The vector $x \in \mathbb{R}_+^E$ as previously defined using the greedy algorithm maximizes wx over P_f^+, with $w \in \mathbb{R}_+^E$, if f is submodular.

Proof.

- Consider the LP strong duality equation:

$$\max(wx : x \in P_f^+) = \min\Big(\sum_{A \subseteq E} y_A f(A) : y \in \mathbb{R}_+^{2^E}, \sum_{A \subseteq E} y_A \mathbf{1}_A \geq w\Big) \tag{12.11}$$

- Define the following vector $y \in \mathbb{R}_+^{2^E}$ as

$$y_{E_i} \leftarrow w(e_i) - w(e_{i+1}) \text{ for } i = 1 \ldots (m-1), \tag{12.12}$$

$$y_E \leftarrow w(e_m), \text{ and} \tag{12.13}$$

$$y_A \leftarrow 0 \text{ otherwise} \tag{12.14}$$

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Polymatroidal polyhedron and greedy

Proof.

- We first will see that greedy $x \in P_f^+$ (that is $x(A) \leq f(A), \forall A$).
- Order $A = (a_1, a_2, \ldots, a_k)$ based on order $(e_1, e_2, \ldots, e_m)$.

		a_1		a_2	a_3			a_4		a_5	$\ldots$	
e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	$\ldots$	e_m

- Define $e^{-1} : E \to \{1, \ldots, m\}$ so that $e^{-1}(e_i) = i$.
- Then, we have $x \in P_f^+$ since for all A:

$$f(A) = \sum_{i=1}^{k} f(a_i | a_{1:i-1}) \tag{12.15}$$

$$\geq \sum_{i=1}^{k} f(a_i | e_{1:e^{-1}(a_i)-1}) \tag{12.16}$$

$$= \sum_{a \in A} f(a | e_{1:e^{-1}(a)-1}) = x(A) \tag{12.17}$$

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Polymatroidal polyhedron and greedy

Proof.

- Next, y is also feasible for the dual constraints in Eq. 12.11 since:
- Next, we check that y is dual feasible. Clearly, $y \geq 0$,
- and also, considering y component wise, for any i, we have that

$$\sum_{A: e_i \in A} y_A = \sum_{j \geq i} y_{E_j} = \sum_{j=i}^{m-1} (w(e_j) - w(e_{j+1})) + w(e_m) = w(e_i).$$

- Now optimality for x and y follows from strong duality, i.e.:

$$wx = \sum_{e \in E} w(e)x(e) = \sum_{i=1}^{m} w(e_i) f(e_i | E_{i-1}) = \sum_{i=1}^{m} w(e_i)\Big(f(E_i) - f(E_{i-1})\Big)$$
$$= \sum_{i=1}^{n-1} f(E_i)\Big(w(e_i) - w(e_{i+1})\Big) + f(E)w(e_m) = \sum_{A \subseteq E} y_A f(A)$$

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Mtrd. Partitioning | Polymatroids and Greedy | Possible Polytopes | Extreme Points | Most Violated ≤ | Matroids cont. | Closure/Sat

Polymatroidal polyhedron and greedy

Proof.

- The equality in prev. Eq. follows via Abel summation:

$$wx = \sum_{i=1}^{m} w_i x_i \tag{12.18}$$
$$= \sum_{i=1}^{m} w_i \Big(f(E_i) - f(E_{i-1})\Big) \tag{12.19}$$
$$= \sum_{i=1}^{m} w_i f(E_i) - \sum_{i=1}^{m-1} w_{i+1} f(E_i) \tag{12.20}$$
$$= w_m f(E_m) + \sum_{i=1}^{m-1} (w_i - w_{i+1}) f(E_i) \tag{12.21}$$

□

What about $w \in \mathbb{R}^E$

- When w contains negative elements, we have $x(e_i) = 0$ for $i = k+1, \dots, m$, where k is the last positive element of w when it is sorted in decreasing order.
- Exercise: show a modification of the previous proof that works for arbitrary $w \in \mathbb{R}^E$

Polymatroidal polyhedron and greedy

Theorem 12.4.1

Conversely, suppose P_f^+ is a polytope of form $P_f^+ = \{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\}$, then the greedy solution to $\max(wx : x \in P)$ is optimum only if f is submodular.

Proof.

- Order elements of E arbitrarily as $(e_1, e_2, \dots, e_m)$ and define $E_i = (e_1, e_2, \dots, e_i)$. Also, choose A and B arbitrarily.
- For $1 \leq p \leq q \leq m$, define $A = \{e_1, e_2, \dots, e_k, e_{k+1}, \dots, e_p\} = E_p$ and $B = \{e_1, e_2, \dots, e_k, e_{p+1}, \dots, e_q\} = E_k \cup (E_q \setminus E_p)$
- Note, then we have $A \cap B = \{e_1, \dots, e_k\} = E_k$, and $A \cup B = E_q$.
- Define $w \in \{0,1\}^m$ as:

$$w \stackrel{\text{def}}{=} \sum_{i=1}^{q} \mathbf{1}_{e_i} = \mathbf{1}_{A \cup B} \tag{12.22}$$

- Suppose optimum solution x is given by the greedy procedure.

Polymatroidal polyhedron and greedy

Proof.

- Then

$$\sum_{i=1}^{k} x_i = f(E_1) + \sum_{i=2}^{k} (f(E_i) - f(E_{i-1})) = f(E_k) = f(A \cap B) \tag{12.23}$$

- and

$$\sum_{i=1}^{p} x_i = f(E_1) + \sum_{i=2}^{p} (f(E_i) - f(E_{i-1})) = f(E_p) = f(A) \tag{12.24}$$

- and

$$\sum_{i=1}^{q} x_i = f(E_1) + \sum_{i=2}^{q} (f(E_i) - f(E_{i-1})) = f(E_q) = f(A \cup B) \tag{12.25}$$

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Mtrd. Partitioning | Polymatroids and Greedy | Possible Polytopes | Extreme Points | Most Violated $\leq$ | Matroids cont. | Closure/Sat

Polymatroidal polyhedron and greedy

Proof.

- Thus, we have

$$x(B) = \sum_{i \in 1, \dots, k, p+1, \dots, q} x_i = \sum_{i : e_i \in B} x_i = f(A \cup B) + f(A \cap B) - f(A) \tag{12.26}$$

- But given that the greedy algorithm gives the optimal solution to $\max(wx : x \in P_f^+)$, we have that $x \in P_f^+$ and thus $x(B) \leq f(B)$.
- Thus,

$$x(B) = f(A \cup B) + f(A \cap B) - f(A) = \sum_{i : e_i \in B} x_i \leq f(B) \tag{12.27}$$

ensuring the submodularity of f, since A and B are arbitrary.

□

Polymatroidal polyhedron and greedy

- Thus, restating the above results into a single complete theorem, we have a result very similar to what we saw for matroids (i.e., Theorem 8.6.1)

Theorem 12.4.1

If $f : 2^E \to \mathbb{R}_+$ is given, and P is a polytope in $\mathbb{R}_+^E$ of the form $P = \{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\}$, then the greedy solution to the problem $\max(wx : x \in P)$ is $\forall w$ optimum iff f is monotone non-decreasing submodular (i.e., iff P is a polymatroid).

Multiple Polytopes associated with arbitrary f

- Given an arbitrary submodular function $f : 2^V \to R$ (not necessarily a polymatroid function, so it need not be positive, monotone, etc.).
- If $f(\emptyset) \neq 0$, we can set $f'(A) = f(A) - f(\emptyset)$ without destroying submodularity. This also does not change any minima, so we assume all functions are normalized $f(\emptyset) = 0$.
  Note that due to constraint $x(\emptyset) \leq f(\emptyset)$, we must have $f(\emptyset) \geq 0$ since if not (i.e., if $f(\emptyset) < 0$), then P_f^+ doesn't exist.
  Another form of normalization can do is:

$$f'(A) = \begin{cases} f(A) & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset \end{cases} \tag{12.28}$$

  This preserves submodularity due to $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$, and if $A \cap B = \emptyset$ then r.h.s. only gets smaller when $f(\emptyset) \geq 0$.
- We can define several polytopes:

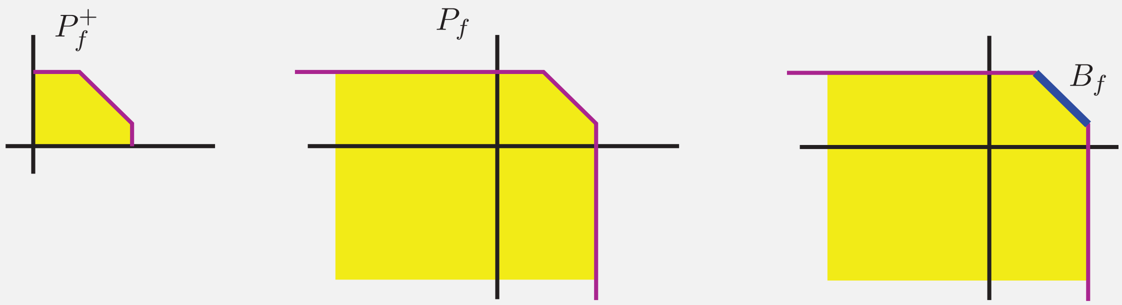
$$P_f = \{x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E\} \tag{12.29}$$

$$P_f^+ = P_f \cap \{x \in \mathbb{R}^E : x \geq 0\} \tag{12.30}$$

$$B_f = P_f \cap \{x \in \mathbb{R}^E : x(E) = f(E)\} \tag{12.31}$$

- P_f is what is sometimes called the extended polytope (sometimes notated as EP_f.

Multiple Polytopes associated with f

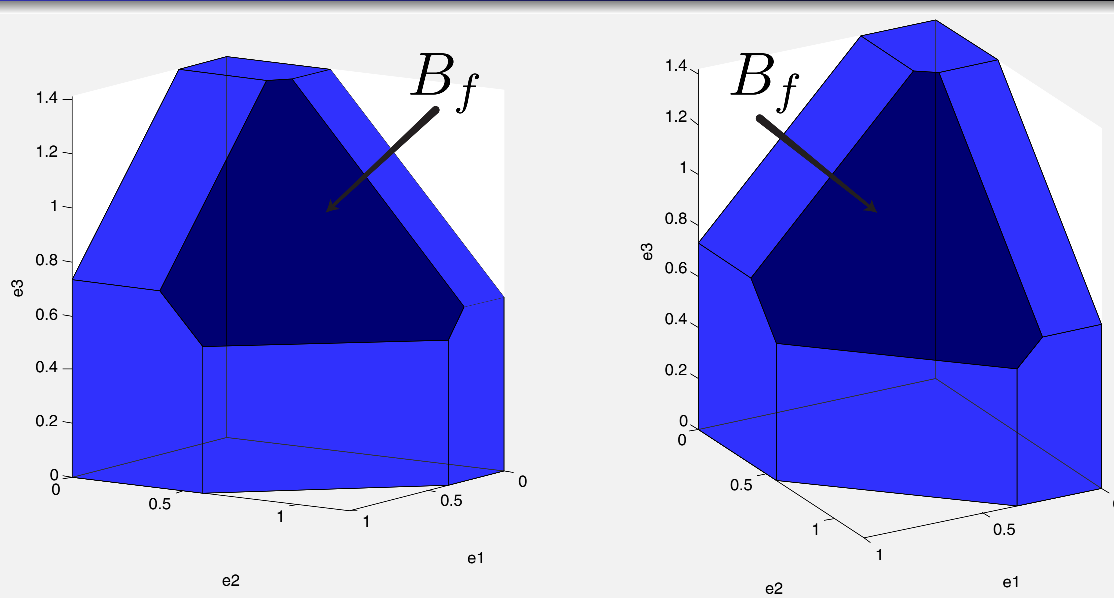


$$P_f^+ = P_f \cap \left\{x \in \mathbb{R}^E : x \geq 0\right\} \tag{12.32}$$

$$P_f = \left\{x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E\right\} \tag{12.33}$$

$$B_f = P_f \cap \left\{x \in \mathbb{R}^E : x(E) = f(E)\right\} \tag{12.34}$$

Base Polytope in 3D



$$P_f = \left\{x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E\right\} \tag{12.35}$$

$$B_f = P_f \cap \left\{x \in \mathbb{R}^E : x(E) = f(E)\right\} \tag{12.36}$$

A polymatroid function's polyhedron is a polymatroid.

Theorem 12.5.1

Let f be a submodular function defined on subsets of E. For any $x \in \mathbb{R}^E$, we have:

$$\text{rank}(x) = \max\left(y(E) : y \leq x, y \in P_f\right) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{12.37}$$

If we take x to be zero, we get:

Corollary 12.5.2

Let f be a submodular function defined on subsets of E. $x \in \mathbb{R}^E$, we have:

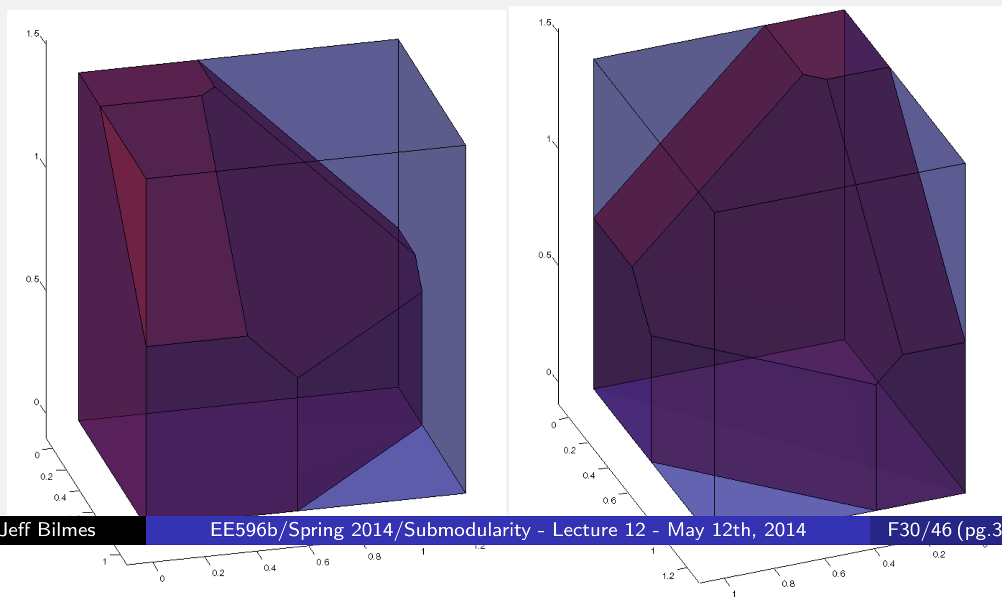
$$\text{rank}(0) = \max\left(y(E) : y \leq 0, y \in P_f\right) = \min\left(f(A) : A \subseteq E\right) \tag{12.38}$$

Greedy and P_f

- In Theorem 12.4.1, we can relax P_f^+ to P_f.
- If $\exists e$ such that $w(e) < 0$ then $\max(wx : x \in P_f) = \infty$ since we can let $x_e \to \infty$, unless we ignore the negative elements or assume $w \geq 0$.
- The proof, moreover, showed also that $x \in P_f$, not just P_f^+.
- Moreover, in polymatroidal case, since the greedy constructed x has $x(E) = f(E)$, we have that the greedy $x \in B_f$.
- In fact, we next will see that the greedy x is a vertex of B_f.

Polymatroid extreme points

- The greedy algorithm does more than solve $\max(wx : x \in P_f^+)$. We can use it to generate vertices of polymatroidal polytopes.
- Consider P_f^+ and also $C_f^+ \stackrel{\text{def}}{=} \{x : x \in \mathbb{R}_+^E, x(e) \leq f(e), \forall e \in E\}$
- Then ordering $A = (a_1, \dots, a_{|A|})$ arbitrarily with $A_i = \{a_1, \dots, a_i\}$, $f(A) = \sum_i f(a_i | A_{i-1}) \leq \sum_i f(a_i)$, and hence $P_f^+ \subseteq C_f^+$.



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Polymatroid extreme points

- Since $w \in \mathbb{R}_+^E$ is arbitrary, it may be that any $e \in E$ is max (i.e., is such that $w(e) > w(e')$ for $e' \in E \setminus \{e\}$).
- Thus, intuitively, any first vertex of the polytope away from the origin might be obtained by advancing along the corresponding axis.
- Recall, base polytope defined as the extreme face of P_f. I.e.,

$$B_f = P_f \cap \left\{x \in \mathbb{R}_+^E : x(E) = f(E)\right\} \tag{12.39}$$

- Also, intuitively, we can continue advancing along the skeletal edges of the polytope to reach any other vertex, given the appropriate ordering. If we advance in all dimensions, we'll reach a vertex in B_f, and if we advance only in some dimensions, we'll reach a vertex in $P_f \setminus B_f$.
- We formalize this next:

Polymatroid extreme points

- Given any arbitrary order of $E = (e_1, e_2, \dots, e_m)$, define $E_i = (e_1, e_2, \dots, e_i)$.
- As before, a vector x is generated by E_i using the greedy procedure as follows

$$x(e_1) = f(E_1) = f(e_1) \tag{12.40}$$
$$x(e_j) = f(E_j) - f(E_{j-1}) = f(e_j|E_{j-1}) \text{ for } 2 \leq j \leq i \tag{12.41}$$
$$x(e) = 0 \text{ for } e \in E \setminus E_i \tag{12.42}$$

- An extreme point of P_f is a point that is not a convex combination of two other distinct points in P_f. Equivalently, an extreme point corresponds to setting certain inequalities in the specification of P_f to be equalities, so that there is a unique single point solution.

Polymatroid extreme points

Theorem 12.6.1

For a given ordering $E = (e_1, \dots, e_m)$ of E and a given $E_i = (e_1, \dots, e_i)$ and x generated by E_i using the greedy procedure ($x(e_i) = f(e_i|E_{i-1})$), then x is an extreme point of P_f

Proof.

- We already saw that $x \in P_f$ (Theorem 12.4.1).
- To show that x is an extreme point of P_f, note that it is the unique solution of the following system of equations

$$x(E_j) = f(E_j) \text{ for } 1 \leq j \leq i \leq m \tag{12.43}$$
$$x(e) = 0 \text{ for } e \in E \setminus E_i \tag{12.44}$$

There are $i \leq m$ equations and $i \leq m$ unknowns, and simple Gaussian elimination gives us back the x constructed via the Greedy algorithm!!

□

Polymatroid extreme points

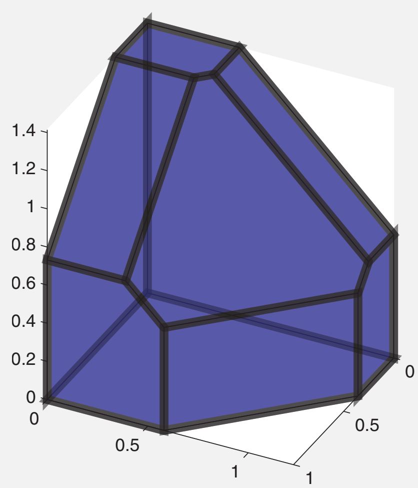
- As an example, we have $x(E_1) = x(e_1) = f(e_1)$
- $x(E_2) = x(e_1) + x(e_2) = f(e_1, e_2)$ so $x(e_2) = f(e_1, e_2) - x(e_1) = f(e_1, e_2) - f(e_1) = f(e_2|e_1)$.
- $x(E_3) = x(e_1) + x(e_2) + x(e_3) = f(e_1, e_2, e_3)$ so $x(e_3) = f(e_1, e_2, e_3) - x(e_2) - x(e_1) = f(e_1, e_2, e_3) - f(e_1, e_2) = f(e_3|e_1, e_2)$
- And so on . . . , but we see that this is just Gaussian elimination.
- Also, since $x \in P_f$, for each i, we see that,

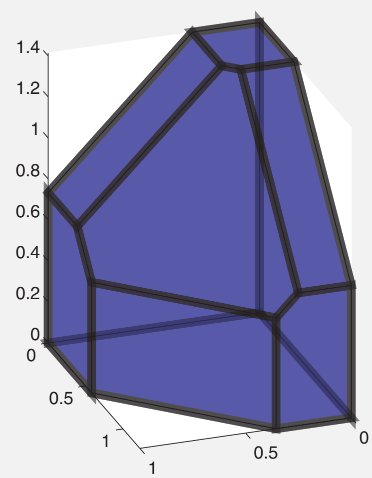
$$x(E_j) = f(E_j) \quad \text{for } 1 \leq j \leq i \tag{12.45}$$
$$x(A) \leq f(A), \forall A \subseteq E \tag{12.46}$$

- Thus, the greedy procedure provides a modular function lower bound on f that is tight on all points E_i in the order. This can be useful in its own right.

Polymatroid extreme points

some examples





Polymatroid extreme points

- Moreover, we have (and will ultimately prove)

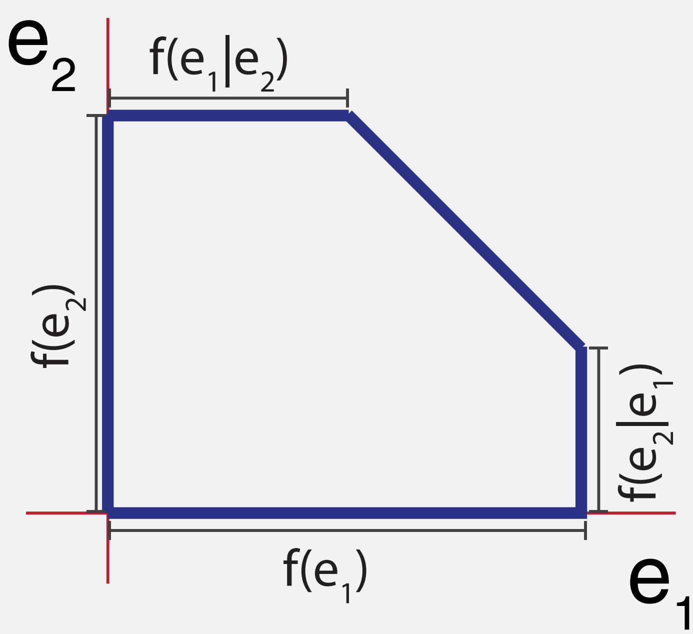
Corollary 12.6.2

If x is an extreme point of P_f and $B \subseteq E$ is given such that $\text{supp}(x) = \{e \in E : x(e) \neq 0\} \subseteq B \subseteq \cup(A : x(A) = f(A)) = \text{sat}(x)$, then x is generated using greedy by some ordering of B.

- Note, $\text{sat}(x) = \text{cl}(x) = \cup(A : x(A) = f(A))$ is also called the closure of x (recall that sets A such that $x(A) = f(A)$ are called tight, and such sets are closed under union and intersection, as seen in Lecture 8, Theorem **??**)
- Thus, $\text{cl}(x)$ is a tight set.
- Also, $\text{supp}(x) = \{e \in E : x(e) \neq 0\}$ is called the support of x.
- For arbitrary x, $\text{supp}(x)$ is not necessarily tight, but for an extreme point, $\text{supp}(x)$ is.

Polymatroid with labeled edge lengths

- Recall $f(e|A) = f(A + e) - f(A)$
- Notice how submodularity, $f(e|B) \leq f(e|A)$ for $A \subseteq B$, defines the shape of the polytope.
- In fact, we have strictness here $f(e|B) < f(e|A)$ for $A \subset B$.
- Also, consider how the greedy algorithm proceeds along the edges of the polytope.



Polymatroid with labeled edge lengths

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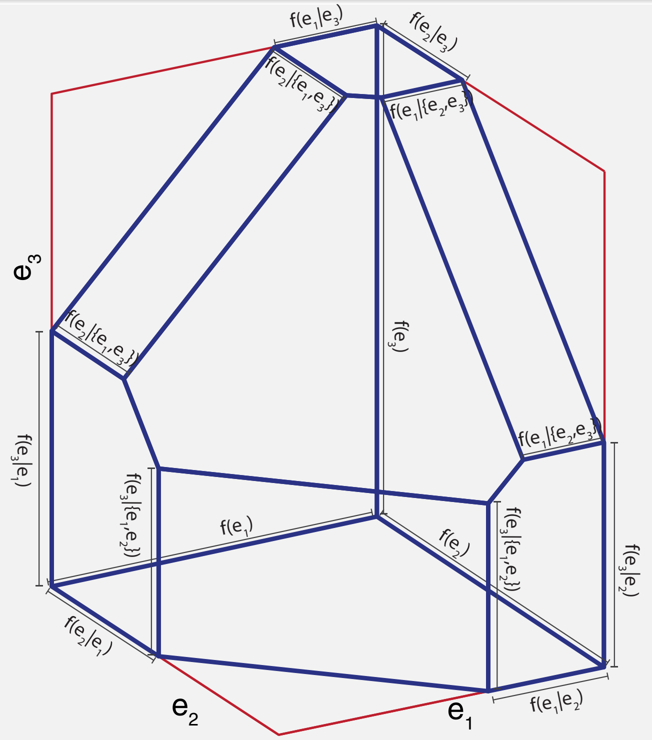
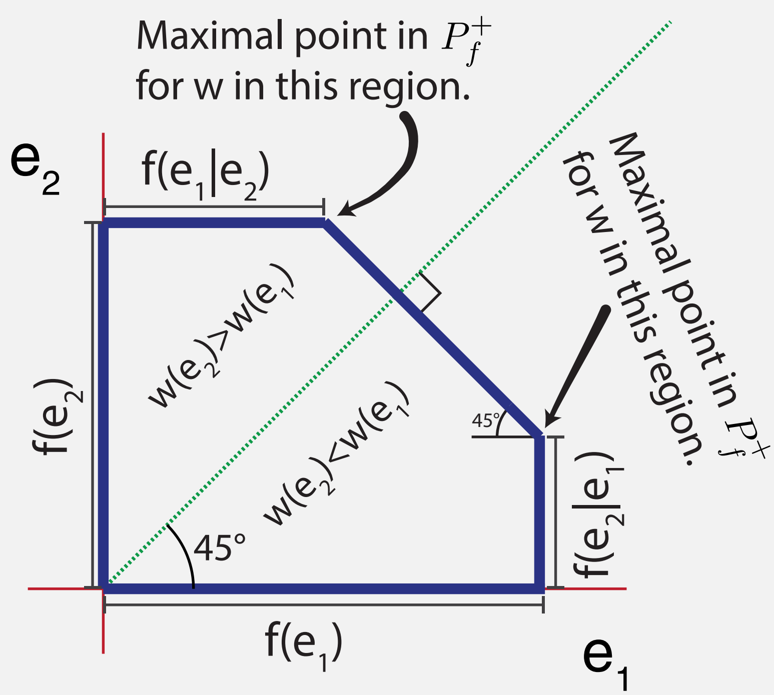


Intuition: why greedy works with polymatroids

- Given w, the goal is to find $x = (x(e_1), x(e_2))$ that maximizes $x^\intercal w = x(e_1)w(e_1) + x(e_2)w(e_2)$.
- If $w(e_2) > w(e_1)$ the upper extreme point indicated maximizes $x^\intercal w$ over $x \in P_f^+$.
- If $w(e_2) < w(e_1)$ the lower extreme point indicated maximizes $x^\intercal w$ over $x \in P_f^+$.



A polymatroid function's polyhedron is a polymatroid.

Theorem 12.7.4

Let f be a polymatroid function defined on subsets of E. For any $x \in \mathbb{R}_+^E$, and any P_f^+-basis $y^x \in \mathbb{R}_+^E$ of x, the component sum of y^x is

$$y^x(E) = \textit{rank}(x) = \max\Big(y(E) : y \leq x, y \in P_f^+\Big) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{12.34}$$

As a consequence, P_f^+ is a polymatroid, since r.h.s. is constant w.r.t. y^x.

By taking $B = \text{supp}(x)$ (so elements $E \setminus B$ are zero in x), and for $b \in B$, $x(b)$ is big enough, the r.h.s. min has solution $A^* = E \setminus B$. We recover submodular function from the polymatroid polyhedron via the following:

$$f(B) = \max\left\{y(B) : y \in P_f^+\right\} \tag{12.35}$$

In fact, we will ultimately see a number of important consequences of this theorem (other than just that P_f^+ is a polymatroid)

Matroid instance of Theorem 9.4.5

- Considering Theorem 9.4.5, the matroid case is now a special case, where we have that:

Corollary 12.7.2

We have that:

$$\max\left\{y(E) : y \in P_{\textit{ind. set}}(M), y \leq x\right\} = \min\left\{r_M(A) + x(E \setminus A) : A \subseteq E\right\} \tag{12.2}$$

where r_M is the matroid rank function of some matroid.

Most violated inequality problem in matroid polytope case

- Consider

$$P_r^+ = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r_M(A), \forall A \subseteq E\right\} \tag{12.47}$$

- Suppose we have any $x \in \mathbb{R}_+^E$ such that $x \notin P_r^+$.
- Hence, there must be a set of $\mathcal{W} \subseteq 2^V$, each member of which corresponds to a violated inequality, i.e., equations of the form $x(A) > r_M(A)$ for $A \in \mathcal{W}$.
- The most violated inequality when x is considered w.r.t. P_r^+ corresponds to the set A that maximizes $x(A) - r_M(A)$, i.e., the most violated inequality is valuated as:

$$\max\left\{x(A) - r_M(A) : A \in \mathcal{W}\right\} = \max\left\{x(A) - r_M(A) : A \subseteq E\right\} \tag{12.48}$$

- Since x is modular and $x(E \setminus A) = x(E) - x(A)$, we can express this via a min as in;:

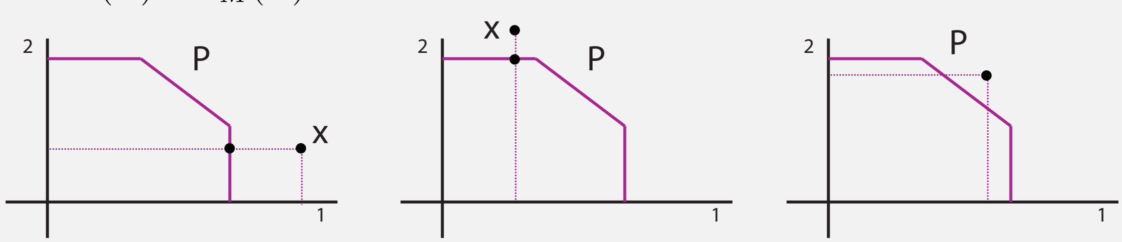
$$\min\left\{r_M(A) + x(E \setminus A) : A \subseteq E\right\} \tag{12.49}$$

Most violated inequality/polymatroid membership/SFM

- Consider

$$P_f^+ = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\right\} \tag{12.50}$$

- Suppose we have any $x \in \mathbb{R}_+^E$ such that $x \notin P_f^+$.
- Hence, there must be a set of $\mathcal{W} \subseteq 2^V$, each member of which corresponds to a violated inequality, i.e., equations of the form $x(A) > r_M(A)$ for $A \in \mathcal{W}$.



Left: $\mathcal{W} = \{\{1\}\}$ Center: $\mathcal{W} = \{\{2\}\}$ Right: $\mathcal{W} = \{\{1, 2\}\}$

Most violated inequality/polymatroid membership/SFM

- The most violated inequality when x is considered w.r.t. P_f^+ corresponds to the set A that maximizes $x(A) - f(A)$, i.e., the most violated inequality is valuated as:

$$\max\{x(A) - f(A) : A \in \mathcal{W}\} = \max\{x(A) - f(A) : A \subseteq E\} \quad (12.51)$$

- Since x is modular and $x(E \setminus A) = x(E) - x(A)$, we can express this via a min as in;:

$$\min\{f(A) + x(E \setminus A) : A \subseteq E\} \quad (12.52)$$

- More importantly, $\min\{f(A) + x(E \setminus A) : A \subseteq E\}$ is a form of submodular function minimization, namely $\min\{f(A) - x(A) : A \subseteq E\}$ for a submodular f and $x \in \mathbb{R}_+^E$, consisting of a difference of polymatroid and modular function (so $f - x$ is no longer necessarily monotone, nor positive).
- We will ultimatley answer how general this form of SFM is.

Matroids, other definitions using matroid rank $r : 2^V \to \mathbb{Z}_+$

Definition 12.8.1 (closed/flat/subspace)

A subset $A \subseteq E$ is closed (equivalently, a flat or a subspace) of matroid M if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

A hyperplane is a flat of rank $r(M) - 1$.

Definition 12.8.2 (closure)

Given $A \subseteq E$, the closure (or span) of A, is defined by $\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}$.

Therefore, a closed set A has $\text{span}(A) = A$.

Definition 12.8.3 (circuit)

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise-minimal dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

Matroids by circuits

Several circuit definitions for matroids.

Theorem 12.8.1 (Matroid by circuits)

Let E be a set and $\mathcal{C}$ be a collection of nonempty subsets of E, such that no two sets in $\mathcal{C}$ are contained in each other. Then the following are equivalent.

1. *$\mathcal{C}$ is the collection of circuits of a matroid;*
2. *if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$;*
3. *if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, and $y \in C \setminus C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$ containing y;*

Again, think about this for a moment in terms of linear spaces and matrices, and spanning trees.

Fundamental circuits in matroids

Lemma 12.8.3

Let $I \in \mathcal{I}(M)$, and $e \in E$, then $I \cup \{e\}$ contains at most one circuit in M.

Proof.

- Suppose, to the contrary, that there are two distinct circuits C_1, C_2 such that $C_1 \cup C_2 \subseteq I \cup \{e\}$.
- Then $e \in C_1 \cap C_2$, and by (C2), there is a circuit C_3 of M s.t. $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\} \subseteq I$
- This contradicts the independence of I.

□

In general, let $C(I, e)$ be the unique circuit associated with $I \cup \{e\}$ (commonly called the fundamental circuit in M w.r.t. I and e).

Matroids: The Fundamental Circuit

- Define $C(I, e)$ be the unique circuit associated with $I \cup \{e\}$ (the fundamental circuit in M w.r.t. I and e, if it exists).
- If $e \in \text{span}(I) \setminus I$, then $C(I, e)$ is well defined ($I + e$ creates one circuit).
- If $e \in I$, then $I + e = I$ doesn't create a circuit. In such cases, $C(I, e)$ is not really defined.
- In such cases, we define $C(I, e) = \{e\}$, and we will soon see why. why we do this.
- If $e \notin \text{span}(I)$, then $C(I, e) = \emptyset$, since no circuit is created in this case.

Union of matroid bases of a set

Lemma 12.8.1

Let $\mathcal{B}(C)$ be the set of bases of C. Then, given matroid $\mathcal{M} = (E, \mathcal{I})$, and any loop-free set $C \subseteq E$, we have that:

$$\bigcup_{B \in \mathcal{B}(C)} B = C. \tag{12.53}$$

Proof.

- Define $C' \triangleq \bigcup_{B \in \mathcal{B}(C)}$, and suppose $\exists c \in C$ such that $c \notin C'$.
- Hence, $\forall B \in \mathcal{B}(C)$ we have $c \notin B$, and $B + c$ contains a single circuit for any B, namely $C(B, c)$.
- Then choose $c' \in C(B, c)$ with $c' \neq c$.
- Then $B + c - c'$ is independent size $|B|$ subset of C and hence spans C, and thus is a c-containing member of $\mathcal{B}(C)$, contradicting $c \notin C'$.

□

The sat function = Polymatroid Closure

- Thus, in a matroid, closure (span) of a set A are all items that A spans (eq. that depend on A).
- We wish to generalize closure to polymatroids.
- Consider $x \in P_f$ for polymatroid function f.
- Again, recall, tight sets are closed under union and intersection, and therefore form a distributive lattice.
- That is, we saw in Lecture 7 that for any $A, B \in \mathcal{D}(x)$, we have that $A \cup B \in \mathcal{D}(x)$ and $A \cap B \in \mathcal{D}(x)$, which can constitute a join and meet.
- Recall, for a given $x \in P_f$, we have defined this tight family as

$$\mathcal{D}(x) = \{A : A \subseteq E, x(A) = f(A)\} \tag{12.54}$$

Text

The sat function = Polymatroid Closure

- Now given $x \in P_f^+$:

$$\mathcal{D}(x) = \{A : A \subseteq E, x(A) = f(A)\} \tag{12.55}$$
$$= \{A : f(A) - x(A) = 0\} \tag{12.56}$$

- Since $x \in P_f^+$ and f is presumed to be polymatroid function, we see $f'(A) = f(A) - x(A)$ is a non-negative submodular function, and $\mathcal{D}(x)$ are the zero-valued minimizers (if any) of $f'(A)$.
- The zero-valued minimizers of f' are thus closed under union and intersection.
- In fact, this is true for all minimizers of a submodular function as stated in the next theorem.

Minimizers of a Submodular Function form a lattice

Theorem 12.9.1

For arbitrary submodular f, the minimizers are closed under union and intersection. That is, let $\mathcal{M} = \operatorname{argmin}_{X \subseteq E} f(X)$ be the set of minimizers of f. Let $A, B \in \mathcal{M}$. Then $A \cup B \in \mathcal{M}$ and $A \cap B \in \mathcal{M}$.

Proof.

Since A and B are minimizers, we have $f(A) = f(B) \leq f(A \cap B)$ and $f(A) = f(B) \leq f(A \cup B)$.
By submodularity, we have

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \tag{12.57}$$

Hence, we must have $f(A) = f(B) = f(A \cup B) = f(A \cap B)$. □

Thus, the minimizers of a submodular function form a lattice, and there is a maximal and a minimal minimizer of every submodular function.

The sat function = Polymatroid Closure

- Matroid closure is generalized by the unique maximal element in $\mathcal{D}(x)$, also called the polymatroid closure or sat (saturation function).
- For some $x \in P_f$, we have defined:

$$\text{cl}(x) \stackrel{\text{def}}{=} \text{sat}(x) \stackrel{\text{def}}{=} \bigcup \{A : A \in \mathcal{D}(x)\} \tag{12.58}$$

$$= \bigcup \{A : A \subseteq E, x(A) = f(A)\} \tag{12.59}$$

$$= \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\} \tag{12.60}$$

- Hence, $\text{sat}(x)$ is the maximal (zero-valued) minimizer of the submodular function $f_x(A) \triangleq f(A) - x(A)$.
- Eq. (12.60) says that sat consists of any point x that is P_f saturated (any additional positive movement, in that dimension, leaves P_f). We'll revisit this in a few slides.
- First, we see how sat generalizes matroid closure.