Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 12 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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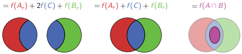
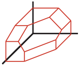
May 12th, 2014

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$





Cumulative Outstanding Reading

- ~~Read chapters 1 and 2, and sections 3.1-3.2 from Fujishige's book.~~
- Good references for today: Schrijver-2003, Oxley-1992/2011, Welsh-1973, Goemans-2010, Cunningham-1984, Edmonds-1969.
- Read Tom McCormick's overview paper on SFM `http://people.commerce.ubc.ca/faculty/mccormick/sfmchap8a.pdf`
- Read chapters 1 - 3 from Fujishige book.
- Matroid properties `http://www-math.mit.edu/~goemans/18433S09/matroid-notes.pdf`

Announcements, Assignments, and Reminders

- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, Combinatorial Geometries
- L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
- L9: From Matroid Polytopes to Polymatroids.
- L10: Polymatroids and Submodularity
- L11: More properties of polymatroids, SFM special cases
- L12: polymatroid properties, extreme points polymatroids, sat, dep.
- L13:
- L14:
- L15:
- L16:
- L17:
- L18:
- L19:
- L20:

Finals Week: June 9th-13th, 2014.

A polymatroid function's polyhedron is a polymatroid.

Theorem 12.2.4

Let f be a polymatroid function defined on subsets of E. For any $x \in \mathbb{R}_+^E$, and any P_f^+-basis $y^x \in \mathbb{R}_+^E$ of x, the component sum of y^x is

$$y^x(E) = \text{rank}(x) = \max\left(y(E) : y \leq x, y \in P_f^+\right)$$
$$= \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \quad (12.34)$$

As a consequence, P_f^+ is a polymatroid, since r.h.s. is constant w.r.t. y^x.

By taking $B = \text{supp}(x)$ (so elements $E \setminus B$ are zero in x), and for $b \in B$, $x(b)$ is big enough, the r.h.s. min has solution $A^* = E \setminus B$. We recover submodular function from the polymatroid polyhedron via the following:

$$f(B) = \max\left\{y(B) : y \in P_f^+\right\} \quad (12.35)$$

In fact, we will ultimately see a number of important consequences of this theorem (other than just that P_f^+ is a polymatroid)

Tight sets $\mathcal{D}(y)$ are closed, and max tight set $\text{sat}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{A : A \subseteq E,\ y(A) = f(A)\} \tag{12.18}$$

Theorem 12.2.1

For any $y \in P_f^+$, with f a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

Proof.

We have already proven this as part of Theorem 9.4.5 □

Also recall the definition of $\text{sat}(y)$, the maximal set of tight elements relative to $y \in \mathbb{R}_+^E$.

$$\text{sat}(y) \stackrel{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\} \tag{12.19}$$

Bipartite Matching

- Given a matching $A \subseteq E$ (which might be empty), we can increase the matching if we can find an augmenting path S.
- The updated matching becomes $A' = A \setminus S \cup S \setminus A = A \ominus S$, where $\ominus$ is the symmetric difference operator.
- The algorithm becomes:

Algorithm 8.1: Alternating Path Bipartite Matching

Let A be an arbitrary (including empty) matching in $G = (V, F, E)$;
while *There exists an augmenting path S in G* **do**
  $A \leftarrow A \ominus S$;

- This can easily be made to run in $O(m^2 n)$, where $|V| = m$, $|F| = n$, $m \leq n$, but it can be made to run much faster as well (see Schrijver-2003).

Matroid Intersection

- Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$.
- While $(V, \mathcal{I}_1 \cap \mathcal{I}_2)$ is typically not a matroid (Exercise: show graphical example.), we might be interested in finding the maximum size common independent set. That is, find $\max |X|$ such that both $X \in \mathcal{I}_1$ and $X \in \mathcal{I}_2$.

Theorem 12.2.5

Let M_1 and M_2 be given as above, with rank functions r_1 and r_2. Then the size of the maximum size set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by

$$(r_1 * r_2)(V) \triangleq \min_{X \subseteq V} \Big(r_1(X) + r_2(V \setminus X) \Big) \tag{12.7}$$

This is an instance of the convolution of two submodular functions, f_1 and f_2 that, evaluated at $Y \subseteq V$, is written as:

$$(f_1 * f_2)(Y) = \min_{X \subseteq Y} \Big(f_1(X) + f_2(Y \setminus X) \Big) \tag{12.8}$$

Fundamental circuits in matroids

Lemma 12.2.3

Let $I \in \mathcal{I}(M)$, and $e \in E$, then $I \cup \{e\}$ contains at most one circuit in M.

Proof.

- Suppose, to the contrary, that there are two distinct circuits C_1, C_2 such that $C_1 \cup C_2 \subseteq I \cup \{e\}$.
- Then $e \in C_1 \cap C_2$, and by (C2), there is a circuit C_3 of M s.t. $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\} \subseteq I$
- This contradicts the independence of I.

$\square$

In general, let $C(I, e)$ be the unique circuit associated with $I \cup \{e\}$ (commonly called the fundamental circuit in M w.r.t. I and e).

Matroid Intersection Algorithm Idea

- Hopefully (eventually) we'll find an odd length sequence $S = (v_1, v_2, \ldots, v_s)$ such that we will be independent in both M_1 and M_2 and thus be one greater in size than I.
- We will have $v_i \notin I$ for i odd (will be shown in blue), and will have $v_i \in I$ for i even (will be shown in green), while $v \in I \setminus S$ will be shown in red.
- We then replace I with $I \ominus S$ (quite analogous to the bipartite matching case), and start again.

Identifying Augmenting Sequences

Theorem 12.2.6

Let I_p and I_{p+1} be intersections of M_1 and M_2 with p and $p+1$ elements respectively. Then there exists an augmenting sequence $S \subseteq I_p \ominus I_{p+1}$ w.r.t. I_p.

Theorem 12.2.7

An intersection is of maximum cardinality iff it admits no augmenting sequence.

Theorem 12.2.8

For any intersection I, there exists a maximum cardinality intersection I^* such that $\mathrm{span}_1(I) \subseteq \mathrm{span}_1(I^*)$ and $\mathrm{span}_2(I) \subseteq \mathrm{span}_2(I^*)$.

All this can be made to run in poly time.

Matroid Partition Problem

- Suppose $M_i = (E, \mathcal{I}_i)$ is a matroid and that we have k of them on the same ground set E.

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- We wish to, if possible, partition E into k blocks, $I_i, i \in \{1, 2, \ldots, k\}$ where $I_i \in \mathcal{I}_i$.

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- We wish to, if possible, partition E into k blocks, $I_i, i \in \{1, 2, \dots, k\}$ where $I_i \in \mathcal{I}_i$.
- Moreover, we want partition to be lexicographically maximum, that is $|I_1|$ is maximum, $|I_2|$ is maximum given $|I_1|$, and so on.

Matroid Partition Problem

Theorem 12.3.1

Let M_i be a collection of k matroids as described. Then, a set $I \subseteq E$ can be partitioned into k subsets $I_i, i = 1 \dots k$ where $I_i \in \mathcal{I}_i$ is independent in matroid i, if and only if, for all $A \subseteq I$

$$|A| \leq \sum_{i=1}^{k} r_i(A) \tag{12.1}$$

where r_i is the rank function of M_i.

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- But considering vector of all ones $\mathbf{1} \in \mathbb{R}_+^E$, this is the same as

$$\frac{1}{k}|A| = \frac{1}{k}\mathbf{1}(A) \leq r(A) \;\; \forall A \subseteq E \tag{12.3}$$

Matroid Partition Problem

- Recall definition of matroid polytope

$$P_r^+ = \{y \in \mathbb{R}_+^E : y(A) \leq r(A) \text{ for all } A \subseteq E\} \tag{12.4}$$

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- This is therefore a special case of submodular function minimization.

Review

- The next two slides from respectively from Lecture 9 and Lecture 8.

Polymatroidal polyhedron (or a "polymatroid")

Definition 12.4.4 (polymatroid)

A polymatroid is a compact set $P \subseteq \mathbb{R}_+^E$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For every $x \in \mathbb{R}_+^E$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any P-basis of x), has the same component sum $y(E)$

- Vectors within P (i.e., any $y \in P$) are called independent, and any vector outside of P is called dependent.
- Since all P-bases of x have the same component sum, if $\mathcal{B}_x$ is the set of P-bases of x, than $\text{rank}(x) = y(E)$ for any $y \in \mathcal{B}_x$.

Maximum weight independent set via greedy weighted rank

Theorem 12.4.6

Let $M = (V, \mathcal{I})$ be a matroid, with rank function r, then for any weight function $w \in \mathbb{R}_+^V$, there exists a chain of sets $U_1 \subset U_2 \subset \cdots \subset U_n \subseteq V$ such that

$$\max \{w(I) | I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i) \tag{12.19}$$

where $\lambda_i \geq 0$ satisfy

$$w = \sum_{i=1}^{n} \lambda_i \mathbf{1}_{U_i} \tag{12.20}$$

Polymatroidal polyhedron and greedy

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- For a matroid, we saw that set system $(E, \mathcal{I})$ is a matroid iff for each weight function $w \in \mathbb{R}_+^E$, the greedy algorithm leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.

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- Stated succinctly, considering $\max\{w(I) : I \in \mathcal{I}\}$, then $(E, \mathcal{I})$ is a matroid iff greedy works for this maximization.

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- That is, if we consider $\max \left\{ wx : x \in P_f^+ \right\}$, where P_f^+ represents the "independent vectors", is it the case that P_f^+ is a polymatroid iff greedy works for this maximization?

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- That is, if we consider $\max \left\{wx : x \in P_f^+\right\}$, where P_f^+ represents the "independent vectors", is it the case that P_f^+ is a polymatroid iff greedy works for this maximization?
- Can we even relax things so that $w \in \mathbb{R}^E$?

Polymatroidal polyhedron and greedy

- What is the greedy solution in this setting, when $w \in \mathbb{R}^E$?

Polymatroidal polyhedron and greedy

- What is the greedy solution in this setting, when $w \in \mathbb{R}^E$?
- Sort elements of E w.r.t. w so that, w.l.o.g. $E = (e_1, e_2, \dots, e_m)$ with $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$.

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- Let $k+1$ be the first point (if any) at which we are non-positive, i.e., $w(e_k) > 0$ and $0 \geq w(e_{k+1})$.

That is, we have

$$w(e_1) \geq w(e_2) \geq \cdots \geq w(e_k) > 0 \geq w(e_{k+1}) \geq \cdots \geq w(e_m) \tag{12.5}$$

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- Let $k+1$ be the first point (if any) at which we are non-positive, i.e., $w(e_k) > 0$ and $0 \geq w(e_{k+1})$.
- Next define partial accumulated sets E_i, for $i = 0 \dots m$, we have w.r.t. the above sorted order:

$$E_i \stackrel{\text{def}}{=} \{e_1, e_2, \dots e_i\} \tag{12.6}$$

(note $E_0 = \emptyset$, $f(E_0) = 0$, and $\underline{E \text{ and } E_i \text{ is always sorted w.r.t } w}$).

Polymatroidal polyhedron and greedy

- What is the greedy solution in this setting, when $w \in \mathbb{R}^E$?
- Sort elements of E w.r.t. w so that, w.l.o.g. $E = (e_1, e_2, \ldots, e_m)$ with $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$.
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- The greedy solution is the vector $x \in \mathbb{R}_+^E$ with elements defined as:

$$x(e_1) \stackrel{\text{def}}{=} f(E_1) = f(e_1) = f(e_1|E_0) = f(e_1|\emptyset) \tag{12.7}$$

$$x(e_i) \stackrel{\text{def}}{=} f(E_i) - f(E_{i-1}) = f(e_i|E_{i-1}) \text{ for } i = 2 \ldots k \tag{12.8}$$

$$x(e_i) \stackrel{\text{def}}{=} 0 \text{ for } i = k + 1 \ldots m = |E| \tag{12.9}$$

Some Intuition: greedy and gain

$x \cdot w$

- Note $x(e_i) = f(e_i|E_{i-1}) \leq f(e_i|E')$ for any $E' \subseteq E_{i-1}$

Some Intuition: greedy and gain

$x \cdot w = x(e_1) \cdot w(e_1) + x(e_2) \cdot w(e_2) + \cdots$

- Note $x(e_i) = f(e_i|E_{i-1}) \leq f(e_i|E')$ for any $E' \subseteq E_{i-1}$
- So $x(e_1) = f(e_1)$ and this corresponds to $w(e_1) \geq w(e_i)$ for all $i \neq 1$.

$f(e_1) \geq f(e_1 | A) \quad \forall A$

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- Hence, for the largest value of w (namely $w(e_1)$), we use for $x(e_1)$ the largest possible gain value of e_1 (namely $f(e_1|\emptyset) \geq f(e_1|A)$ for any $A \subseteq E \setminus \{e_1\}$).

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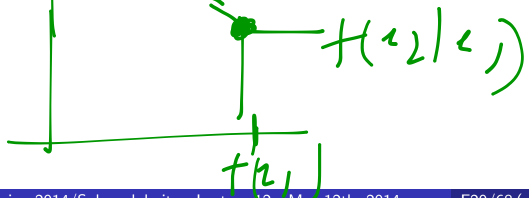
- For the next largest value of w (namely $w(e_2)$), we use for $x(e_2)$ the next largest gain value of e_2 (namely $f(e_2|e_1)$), while still ensuring (as we will soon see in Theorem 12.4.1) that the resulting $x \in P_f$.



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- This process continues, using the next largest possible gain of e_i for $x(e_i)$ while ensuring we do not leave the polytope, given the values we've already chosen for $x(e_{i'})$ for $i' < i$.

Polymatroidal polyhedron and greedy

Theorem 12.4.1

The vector $x \in \mathbb{R}_+^E$ as previously defined using the greedy algorithm maximizes wx over P_f^+, with $w \in \mathbb{R}_+^E$, if f is submodular.

Proof.

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Proof.

- Consider the LP strong duality equation:

$$\max(wx : x \in P_f^+) = \min\left(\sum_{A \subseteq E} y_A f(A) : y \in \mathbb{R}_+^{2^E}, \sum_{A \subseteq E} y_A \mathbf{1}_A \geq w\right) \tag{12.10}$$

$$w(e_i) = \sum_{A \ni e_i} y_A$$

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Proof.

- Consider the LP strong duality equation:

$$\max(wx : x \in P_f^+) = \min\Big(\sum_{A \subseteq E} y_A f(A) : y \in \mathbb{R}_+^{2^E}, \sum_{A \subseteq E} y_A \mathbf{1}_A \geq w\Big) \tag{12.10}$$

- Define the following vector $y \in \mathbb{R}_+^{2^E}$ as

$$y_{E_i} \leftarrow w(e_i) - w(e_{i+1}) \text{ for } i = 1 \ldots (m-1), \tag{12.11}$$

$$y_E \leftarrow w(e_m), \text{ and} \tag{12.12}$$

$$y_A \leftarrow 0 \text{ otherwise} \tag{12.13}$$

Polymatroidal polyhedron and greedy

Proof.

- We first will see that greedy $x \in P_f^+$ (that is $x(A) \leq f(A), \forall A$).

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		a_1		a_2	a_3			a_4		a_5	$\ldots$	
e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	$\ldots$	e_m

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e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	$\ldots$	e_m

- Define $e^{-1} : E \to \{1, \ldots, m\}$ so that $e^{-1}(e_i) = i$.

This means that with $A = \{a_1, a_2, \ldots, a_k\}$, and $\forall j \leq k$

$$\{a_1, a_2, \ldots, a_j\} \subseteq \left\{e_1, e_2, \ldots, e_{e^{-1}(a_j)}\right\} \tag{12.14}$$

and

$$\{a_1, a_2, \ldots, a_{j-1}\} \subseteq \left\{e_1, e_2, \ldots, e_{e^{-1}(a_j)-1}\right\} \tag{12.15}$$

Also recall matlab notation: $a_{1:j} \equiv \{a_1, a_2, \ldots, a_j\}$.
E.g., with $j = 4$ we get $e^{-1}(a_4) = 9$, and

$$\{a_1, a_2, a_3, a_4\} \subseteq \{e_1, e_2, \ldots, e_9\} \tag{12.16}$$

Polymatroidal polyhedron and greedy

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- Then, we have $x \in P_f^+$ since for all A:

$$f(A) = \sum_{i=1}^{k} f(a_i | a_{1:i-1}) \tag{12.14}$$

$$\geq \sum_{i=1}^{k} f(a_i | e_{1:e^{-1}(a_i)-1}) \tag{12.15}$$

$$= \sum_{a \in A} f(a | e_{1:e^{-1}(a)-1}) = x(A) \tag{12.16}$$

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- Next, we check that y is dual feasible. Clearly, $y \geq 0$,
- and also, considering y component wise, for any i, we have that

$$\sum_{A:e_i \in A} y_A = \sum_{j \geq i} y_{E_j} = \underbrace{\sum_{j=i}^{m-1} (w(e_j) - w(e_{j+1})) + w(e_m)}_{} = w(e_i).$$

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Polymatroidal polyhedron and greedy

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- Now optimality for x and y follows from strong duality, i.e.:

$$wx = \sum_{e \in E} w(e)x(e) = \sum_{e \in E} w(e) f(e_i | E_{i-1}) = \sum_{i=1}^{m} w(e_i)\Big(f(E_i) - f(E_{i-1})\Big)$$

$$= \sum_{i=1}^{n-1} f(E_i)\Big(w(e_i) - w(e_{i+1})\Big) + f(E)w(e_m) = \sum_{A \subseteq E} y_A f(A)$$

...

Polymatroidal polyhedron and greedy

Proof.

- The equality in prev. Eq. follows via Abel summation:

$$wx = \sum_{i=1}^{m} w_i x_i \tag{12.17}$$

$$= \sum_{i=1}^{m} w_i \Big(f(E_i) - f(E_{i-1}) \Big) \tag{12.18}$$

$$= \sum_{i=1}^{m} w_i f(E_i) - \sum_{i=1}^{m-1} w_{i+1} f(E_i) \tag{12.19}$$

$$= w_m f(E_m) + \sum_{i=1}^{m-1} \big(w_i - w_{i+1} \big) f(E_i) \tag{12.20}$$

$\square$

What about $w \in \mathbb{R}^E$

- When w contains negative elements, we have $x(e_i) = 0$ for $i = k+1, \dots, m$, where k is the last positive element of w when it is sorted in decreasing order.

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- When w contains negative elements, we have $x(e_i) = 0$ for $i = k+1, \dots, m$, where k is the last positive element of w when it is sorted in decreasing order.
- Exercise: show a modification of the previous proof that works for arbitrary $w \in\in \mathbb{R}^E$

Polymatroidal polyhedron and greedy

Theorem 12.4.1

Conversely, suppose P_f^+ is a polytope of form $P_f^+ = \{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\}$, then the greedy solution to $\max(wx : x \in P)$ is optimum only if f is submodular.

Proof.

- Order elements of E arbitrarily as $(e_1, e_2, \ldots, e_m)$ and define $E_i = (e_1, e_2, \ldots, e_i)$. Also, choose A and B arbitrarily.

Polymatroidal polyhedron and greedy

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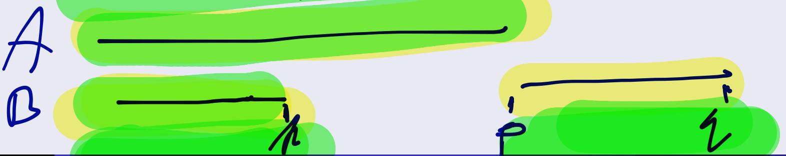
- For $1 \leq p \leq q \leq m$, define $A = \{\overbrace{e_1, e_2, \ldots, e_k}^{A \cap B}, \overbrace{e_{k+1}, \ldots, e_p}^{A \setminus B}\} = E_p$ and $B = \{\underbrace{e_1, e_2, \ldots, e_k}_{A \cap B}, \underbrace{e_{p+1}, \ldots, e_q}_{B \setminus A}\} = E_k \cup (E_q \setminus E_p)$



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- Note, then we have $A \cap B = \{e_1, \ldots, e_k\} = E_k$, and $A \cup B = E_q$.

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- Define $w \in \{0, 1\}^m$ as:

$$w \stackrel{\text{def}}{=} \sum_{i=1}^{q} \mathbf{1}_{e_i} = \mathbf{1}_{A \cup B} \tag{12.21}$$

Polymatroidal polyhedron and greedy

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- Suppose optimum solution x is given by the greedy procedure.

Polymatroidal polyhedron and greedy

Proof.

- Then

$$\sum_{i=1}^{k} x_i = f(E_1) + \sum_{i=2}^{k}(f(E_i) - f(E_{i-1})) = f(E_k) = f(A \cap B) \tag{12.22}$$

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Polymatroidal polyhedron and greedy

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...

Polymatroidal polyhedron and greedy

Proof.

- Thus, we have

$$x(B) = \sum_{i \in 1,\dots,k,p+1,\dots,q} x_i = \sum_{i:e_i \in B} x_i = f(A \cup B) + f(A \cap B) - f(A) \tag{12.25}$$

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Polymatroidal polyhedron and greedy

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- But given that the greedy algorithm gives the optimal solution to $\max(wx : x \in P_f^+)$, we have that $x \in P_f^+$ and thus $x(B) \leq f(B)$.

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- But given that the greedy algorithm gives the optimal solution to $\max(wx : x \in P_f^+)$, we have that $x \in P_f^+$ and thus $x(B) \leq f(B)$.
- Thus,

$$x(B) = f(A \cup B) + f(A \cap B) - f(A) = \sum_{i:e_i \in B} x_i \leq f(B) \tag{12.26}$$

ensuring the submodularity of f, since A and B are arbitrary.

□

Polymatroidal polyhedron and greedy

- Thus, restating the above results into a single complete theorem, we have a result very similar to what we saw for matroids (i.e., Theorem 8.6.1)

Theorem 12.4.1

If $f : 2^E \to \mathbb{R}_+$ is given, and P is a polytope in $\mathbb{R}_+^E$ of the form $P = \{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\}$, then the greedy solution to the problem $\max(wx : x \in P)$ is $\forall w$ optimum iff f is monotone non-decreasing submodular (i.e., iff P is a polymatroid).

Multiple Polytopes associated with arbitrary f

- Given an arbitrary submodular function $f : 2^V \to R$ (not necessarily a polymatroid function, so it need not be positive, monotone, etc.).

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  Note that due to constraint $x(\emptyset) \leq f(\emptyset)$, we must have $f(\emptyset) \geq 0$ since if not (i.e., if $f(\emptyset) < 0$), then P_f^+ doesn't exist.
  Another form of normalization can do is:

$$f'(A) = \begin{cases} f(A) & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset \end{cases} \tag{12.27}$$

  This preserves submodularity due to $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$, and if $A \cap B = \emptyset$ then r.h.s. only gets smaller when $f(\emptyset) \geq 0$.

$0 = x(\emptyset) \leq f(\emptyset)$

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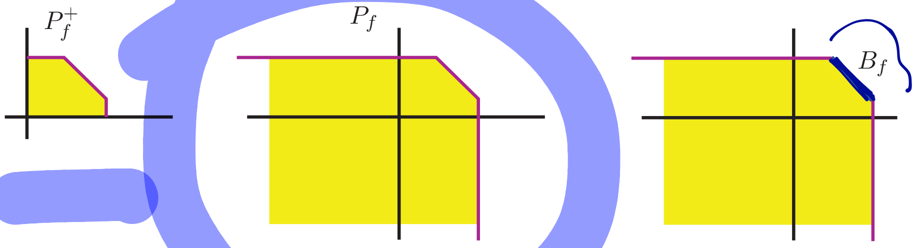
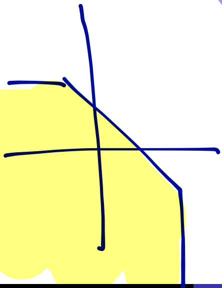
- B_f is called the base polytope

Multiple Polytopes associated with f





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$$P_f = \left\{x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E\right\} \tag{12.31}$$

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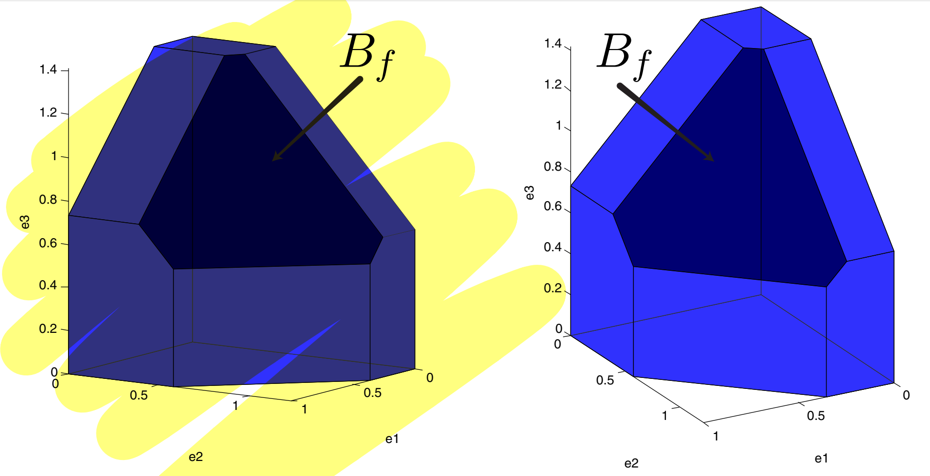
Base Polytope in 3D



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A polymatroid function's polyhedron is a polymatroid.

Theorem 12.5.1

Let f be a submodular function defined on subsets of E. For any $x \in \mathbb{R}^E$, we have:

$$\max\left(y(E) : y \leq x, y \in P_f\right) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \quad (12.35)$$

If we take x to be zero, we get:

Corollary 12.5.2

Let f be a submodular function defined on subsets of E. $x \in \mathbb{R}^E$, we have:

$$\max\left(y(E) : y \leq 0, y \in P_f\right) = \min\left(f(A) : A \subseteq E\right) \quad (12.36)$$

Greedy and P_f

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- Moreover, in polymatroidal case, since the greedy constructed x has $x(E) = f(E)$, we have that the greedy $x \in B_f$.
- In fact, we next will see that the greedy x is a vertex of B_f.

Polymatroid extreme points

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$x \in P_f^+$

$x(A) \leq f(A) \leq \sum_{a \in A} f(a)$

$\Rightarrow \quad x \in C_f^+$

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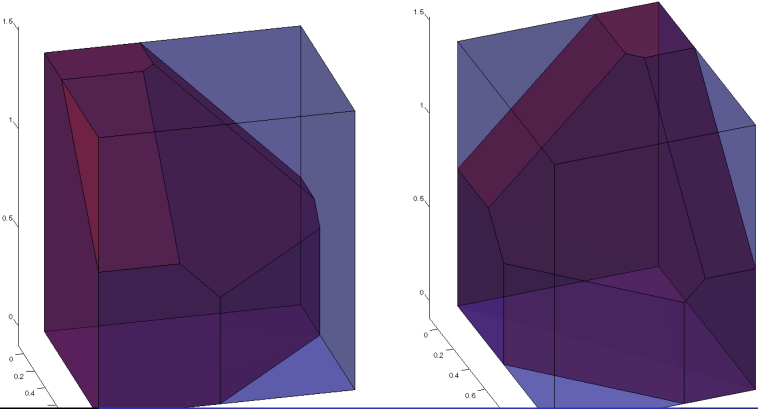

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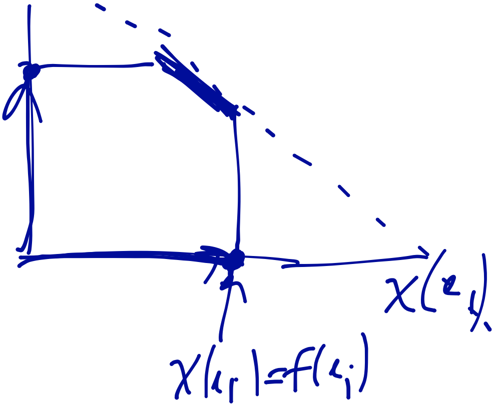

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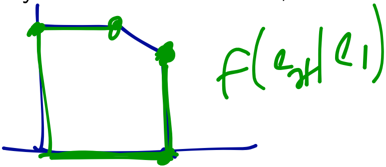

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- We formalize this next:

Polymatroid extreme points

- Given any arbitrary order of $E = (e_1, e_2, \ldots, e_m)$, define $E_i = (e_1, e_2, \ldots, e_i)$.

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$$x(e_j) = f(E_j) - f(E_{j-1}) = f(e_j|E_{j-1}) \text{ for } 2 \leq j \leq i \tag{12.39}$$

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- An extreme point of P_f is a point that is not a convex combination of two other distinct points in P_f. Equivalently, an extreme point corresponds to setting certain inequalities in the specification of P_f to be equalities, so that there is a unique single point solution.

Polymatroid extreme points

Theorem 12.6.1

For a given ordering $E = (e_1, \ldots, e_m)$ of E and a given E_i and x generated by E_i using the greedy procedure, then x is an extreme point of P_f

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Proof.

- We already saw that $x \in P_f$ (Theorem 12.4.1).

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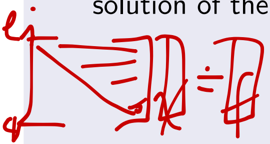
Proof.

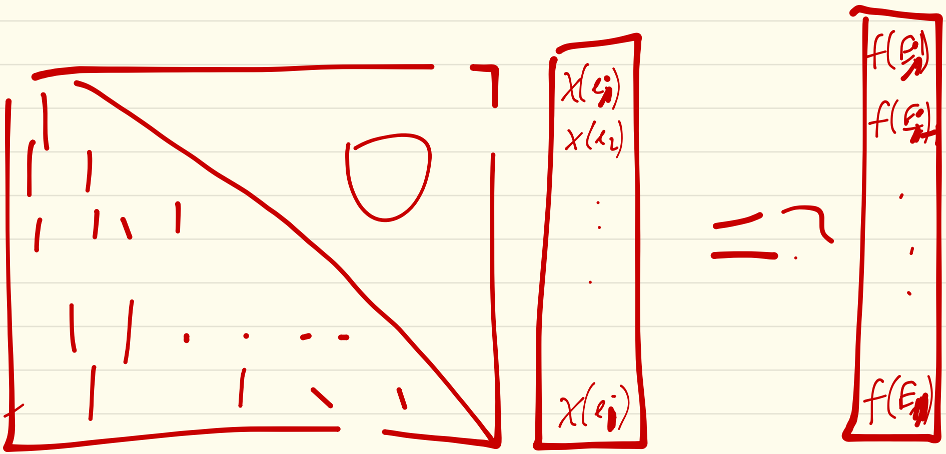
- We already saw that $x \in P_f$ (Theorem 12.4.1).
- To show that x is an extreme point of P_f, note that it is the unique solution of the following system of equations



$$x(E_j) = f(E_j) \text{ for } 1 \leq j \leq i \leq m \tag{12.41}$$

$$x(e) = 0 \text{ for } e \in E \setminus E_i \tag{12.42}$$

There are $i \leq m$ equations and $i \leq m$ unknowns, and simple Gaussian elimination gives us back the x constructed via the Greedy algorithm!!



$$x(e_1) = f(E_1)$$

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$$= f(e_2 \mid e_1)$$

Polymatroid extreme points

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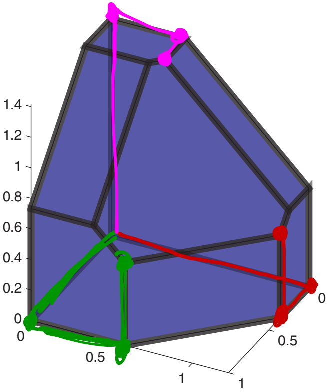
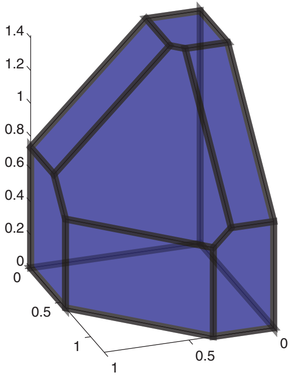
- Thus, the greedy procedure provides a modular function lower bound on f that is tight on all points E_i in the order. This can be useful in its own right.

Polymatroid extreme points

some examples





Polymatroid extreme points

- Moreover, we have (and will ultimately prove)

Corollary 12.6.2

If x is an extreme point of P_f and $B \subseteq E$ is given such that $\text{supp}(x) = \{e \in E : x(e) \neq 0\} \subseteq B \subseteq \cup(A : x(A) = f(A)) = \text{sat}(x)$, *then x is generated using greedy by some ordering of B.*

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- Thus, $\mathrm{cl}(x)$ is a tight set.
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- Thus, $\text{cl}(x)$ is a tight set.
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- For arbitrary x, $\text{supp}(x)$ is not necessarily tight, but for an extreme point, $\text{supp}(x)$ is.

$\text{supp}(x) \in \mathcal{D}(x))$

Polymatroid with labeled edge lengths

- Recall $f(e|A) = f(A+e) - f(A)$
- Notice how submodularity, $f(e|B) \leq f(e|A)$ for $A \subseteq B$, defines the shape of the polytope.
- In fact, we have strictness here $f(e|B) < f(e|A)$ for $A \subset B$.
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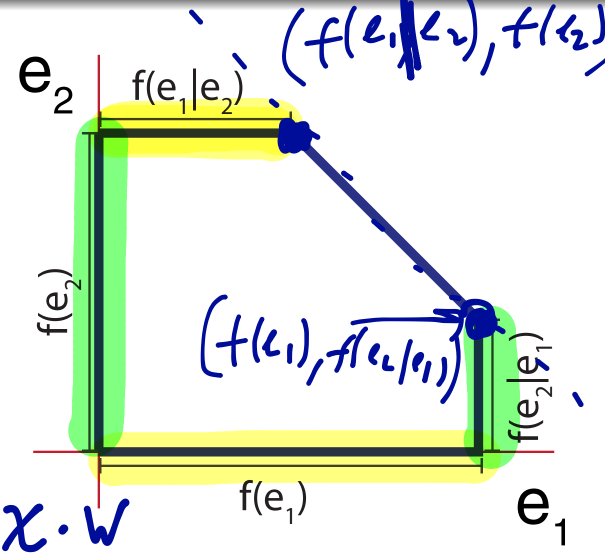

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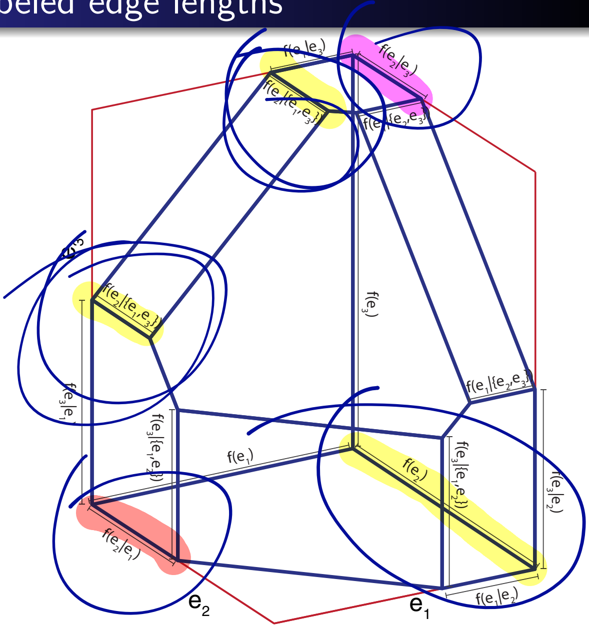
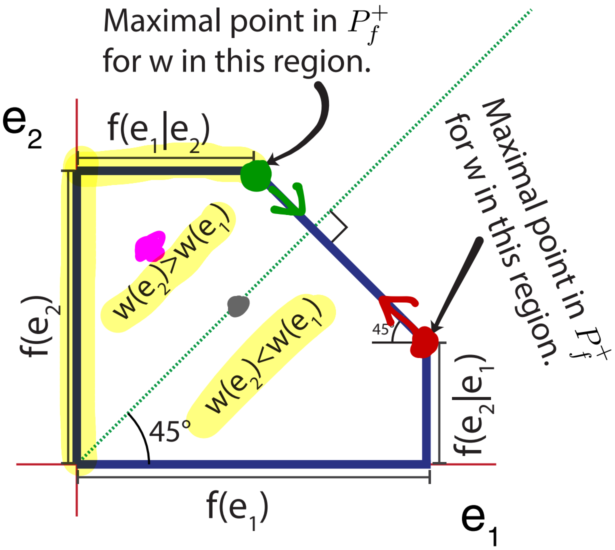


Intuition: why greedy works with polymatroids

- Given w, the goal is to find $x = (x(e_1), x(e_2))$ that maximizes $x^\intercal w = x(e_1)w(e_1) + x(e_2)w(e_2)$.
- If $w(e_2) > w(e_1)$ the upper extreme point indicated maximizes $x^\intercal w$ over $x \in P_f^+$.
- If $w(e_2) < w(e_1)$ the lower extreme point indicated maximizes $x^\intercal w$ over $x \in P_f^+$.



A polymatroid function's polyhedron is a polymatroid.

Theorem 12.7.4

Let f be a polymatroid function defined on subsets of E. For any $x \in \mathbb{R}_+^E$, and any P_f^+-basis $y^x \in \mathbb{R}_+^E$ of x, the component sum of y^x is

$$y^x(E) = \mathit{rank}(x) = \max\left(y(E) : y \leq x, y \in P_f^+\right) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{12.34}$$

As a consequence, P_f^+ is a polymatroid, since r.h.s. is constant w.r.t. y^x.

By taking $B = \text{supp}(x)$ (so elements $E \setminus B$ are zero in x), and for $b \in B$, $x(b)$ is big enough, the r.h.s. min has solution $A^* = E \setminus B$. We recover submodular function from the polymatroid polyhedron via the following:

$$f(B) = \max\left\{y(B) : y \in P_f^+\right\} \tag{12.35}$$

In fact, we will ultimately see a number of important consequences of this theorem (other than just that P_f^+ is a polymatroid)

Matroid instance of Theorem 9.4.5

- Considering Theorem 9.4.5, the matroid case is now a special case, where we have that:

Corollary 12.7.2

We have that:

$$\max\{y(E) : y \in P_{\textit{ind. set}}(M), y \leq x\} = \min\{r_M(A) + x(E \setminus A) : A \subseteq E\} \tag{12.2}$$

where r_M is the matroid rank function of some matroid.

Most violated inequality problem in matroid polytope case

- Consider

$$P_r^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r_M(A), \forall A \subseteq E\} \tag{12.45}$$

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- Hence, there must be a set of $\mathcal{W} \subseteq 2^V$, each member of which corresponds to a violated inequality, i.e., equations of the form $x(A) > r_M(A)$ for $A \in \mathcal{W}$.

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- The most violated inequality when x is considered w.r.t. P_r^+ corresponds to the set A that maximizes $x(A) - r_M(A)$, i.e., the most violated inequality is valuated as:

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$$\max \left\{ x(A) - r_M(A) : A \in \mathcal{W} \right\} = \max \left\{ x(A) - r_M(A) : A \subseteq E \right\} \tag{12.46}$$

- Since x is modular and $x(E \setminus A) = x(E) - x(A)$, we can express this via a min as in;:

$$\min \left\{ r_M(A) + x(E \setminus A) : A \subseteq E \right\} \tag{12.47}$$

Most violated inequality/polymatroid membership/SFM

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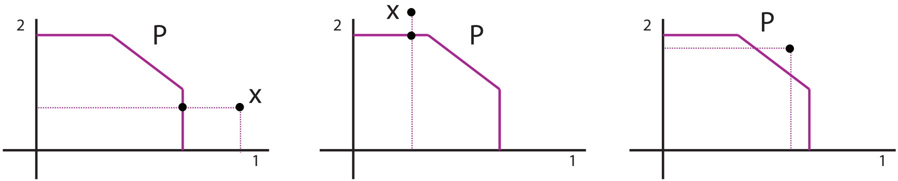
Most violated inequality/polymatroid membership/SFM

- Consider

$$P_f^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\} \tag{12.48}$$

- Suppose we have any $x \in \mathbb{R}_+^E$ such that $x \notin P_f^+$.
- Hence, there must be a set of $\mathcal{W} \subseteq 2^V$, each member of which corresponds to a violated inequality, i.e., equations of the form $x(A) > r_M(A)$ for $A \in \mathcal{W}$.



Left: $\mathcal{W} = \{\{1\}\}$ Center: $\mathcal{W} = \{\{2\}\}$ Right: $\mathcal{W} = \{\{1, 2\}\}$

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- The most violated inequality when x is considered w.r.t. P_f^+ corresponds to the set A that maximizes $x(A) - f(A)$, i.e., the most violated inequality is valuated as:

$$\max \{x(A) - f(A) : A \in \mathcal{W}\} = \max \{x(A) - f(A) : A \subseteq E\} \quad (12.49)$$

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- We will ultimatley answer how general this form of SFM is.

Matroids, other definitions using matroid rank $r : 2^V \to \mathbb{Z}_+$

Definition 12.8.1 (closed/flat/subspace)

A subset $A \subseteq E$ is closed (equivalently, a flat or a subspace) of matroid M if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

A hyperplane is a flat of rank $r(M) - 1$.

Definition 12.8.2 (closure)

Given $A \subseteq E$, the closure (or span) of A, is defined by $\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}$.

$I + b$

Therefore, a closed set A has $\text{span}(A) = A$.

$b \in \text{Span}(I) \setminus I$

Definition 12.8.3 (circuit)

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise-minimal dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

Matroids by circuits

Several circuit definitions for matroids.

Theorem 12.8.1 (Matroid by circuits)

Let E be a set and $\mathcal{C}$ be a collection of nonempty subsets of E, such that no two sets in $\mathcal{C}$ are contained in each other. Then the following are equivalent.

1. *$\mathcal{C}$ is the collection of circuits of a matroid;*
2. *if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$;*
3. *if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, and $y \in C \setminus C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$ containing y;*

Again, think about this for a moment in terms of linear spaces and matrices, and spanning trees.

Fundamental circuits in matroids

Lemma 12.8.3

Let $I \in \mathcal{I}(M)$, and $e \in E$, then $I \cup \{e\}$ contains at most one circuit in M.

Proof.

- Suppose, to the contrary, that there are two distinct circuits C_1, C_2 such that $C_1 \cup C_2 \subseteq I \cup \{e\}$.
- Then $e \in C_1 \cap C_2$, and by (C2), there is a circuit C_3 of M s.t. $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\} \subseteq I$
- This contradicts the independence of I.

□

In general, let $C(I, e)$ be the unique circuit associated with $I \cup \{e\}$ (commonly called the fundamental circuit in M w.r.t. I and e).

Matroids: The Fundamental Circuit

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- In such cases, we define $C(I, e) = \{e\}$, and we will soon see why. why we do this.
- If $e \notin \text{span}(I)$, then $C(I, e) = \emptyset$, since no circuit is created in this case.

Union of matroid bases of a set

Lemma 12.8.1

Let $\mathcal{B}(C)$ be the set of bases of C. Then, given matroid $\mathcal{M} = (E, \mathcal{I})$, and any loop-free set $C \subseteq E$, we have that:

$$\bigcup_{B \in \mathcal{B}(C)} B = C. \tag{12.51}$$

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- Then choose $c' \in C(B, c)$ with $c' \neq c$.
- Then $B + c - c'$ is independent size $|B|$ subset of C and hence spans C, and thus is a c-containing member of $\mathcal{B}(C)$, contradicting $c \notin C'$.

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- Thus, in a matroid, closure (span) of a set A are all items that A spans (eq. that depend on A).

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- That is, we saw in Lecture 7 that for any $A, B \in \mathcal{D}(x)$, we have that $A \cup B \in \mathcal{D}(x)$ and $A \cap B \in \mathcal{D}(x)$, which can constitute a join and meet.

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- Recall, for a given $x \in P_f$, we have defined this tight family as

$$\mathcal{D}(x) = \{A : A \subseteq E, x(A) = f(A)\} \tag{12.52}$$

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- Now given $x \in P_f^+$:

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- Since $x \in P_f^+$ and f is presumed to be polymatroid function, we see $f'(A) = f(A) - x(A)$ is a non-negative submodular function, and $\mathcal{D}(x)$ are the zero-valued minimizers (if any) of $f'(A)$.

$x(A) \leq f(A)$

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- The zero-valued minimizers of f' are thus closed under union and intersection.
- In fact, this is true for all minimizers of a submodular function as stated in the next theorem.

Minimizers of a Submodular Function form a lattice

Theorem 12.9.1

For arbitrary submodular f, the minimizers are closed under union and intersection. That is, let $\mathcal{M} = \text{argmin}_{X \subseteq E} f(X)$ be the set of minimizers of f. Let $A, B \in \mathcal{M}$. Then $A \cup B \in \mathcal{M}$ and $A \cap B \in \mathcal{M}$.

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Proof.

Since A and B are minimizers, we have $f(A) = f(B) \leq f(A \cap B)$ and $f(A) = f(B) \leq f(A \cup B)$.

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By submodularity, we have

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \tag{12.55}$$

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Thus, the minimizers of a submodular function form a lattice, and there is a maximal and a minimal minimizer of every submodular function.

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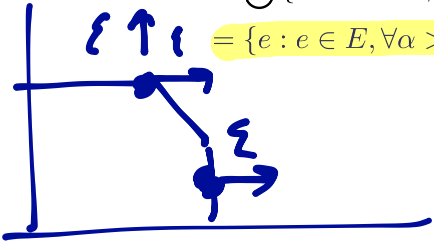

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- First, we see how sat generalizes matroid closure.

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- Consider matroid $(E, \mathcal{I}) = (E, r)$, some $I \in \mathcal{I}$. Then $\mathbf{1}_I \in P_r$ and

$$\mathcal{D}(\mathbf{1}_I) = \{A : \mathbf{1}_I(A) = r(A)\} \tag{12.59}$$

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- We formalize this next.

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Lemma 12.9.2 (Matroid $\text{sat} : \mathbb{R}_+^E \to 2^E$ is the same as closure.)

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In which case, we also get $\text{sat}(\mathbf{1}_C) = \text{span}(C)$ (in general, could define $\text{sat}(y) = \text{sat}(\text{P-basis}(y))$).

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Exercise: is $\text{span}(C) = \text{sat}(\mathbf{1}_C)$? Prove or disprove it.

The sat function, span, and submodular function minimization

- Thus, for a matroid, $\text{sat}(\mathbf{1}_I)$ is exactly the closure (or span) of I in the matroid. I.e., for matroid (E, r), we have $\text{span}(I) = \text{sat}(\mathbf{1}_B)$.

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- Recall, for $x \in P_f$ and polymatroidal f, $\text{sat}(x)$ is the maximal (by inclusion) minimizer of $f(A) - x(A)$, and thus in a matroid, span(I) is the maximal minimizer of the submodular function formed by $r(A) - \mathbf{1}_I(A)$.

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- Find largest minimizer of $g : 2^{V \setminus A} \to \mathbb{R}$ with $g(B) = f(B|A)$. Exercise: give example of greedy failing here.

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- We next show more formally that these are the same.

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- given that $x \in P_f^+$, meaning $x(A) \leq f(A)$ for all A, we must have

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- . . . and therefore, with sat as defined in Eq. (12.58),

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- On the other hand, for any $e \in \operatorname{sat}(x)$ defined as in Eq. (12.73), since e is itself a member of a tight set, there is a set $A \ni e$ such that $x(A) = f(A)$, giving

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- Therefore, the two definitions of sat are identical.

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- We also see that computing $\hat{c}(x;e)$ is a form of submodular function minimization.

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- Given $x \in P_f$, and $e \in \operatorname{sat}(x)$, define

$$\mathcal{D}(x, e) = \{A : e \in A \subseteq E, x(A) = f(A)\} \tag{12.84}$$

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- Therefore, we can define a unique minimal element of $\mathcal{D}(x, e)$ denoted as follows:

$$\text{dep}(x, e) = \begin{cases} \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases} \tag{12.86}$$

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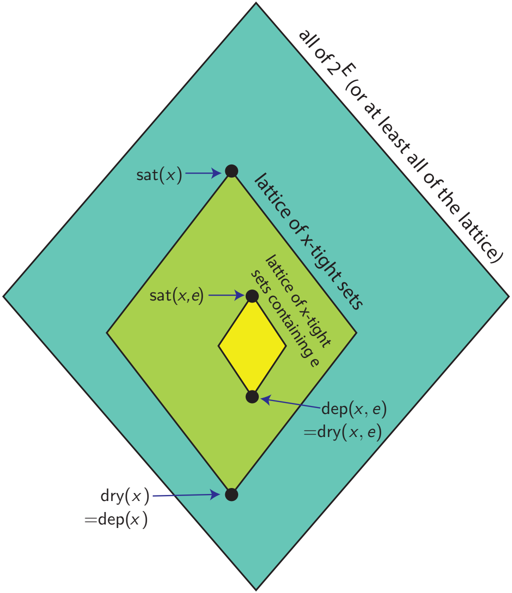
- I.e., $\text{dep}(x, e)$ is the minimal element in $\mathcal{D}(x)$ that contains e (the minimal x-tight set containing e).

dep and sat in a lattice

- The picture on the right summarizes the relationships between the lattices and sublattices.
- Note, $\bigcap_e \text{dep}(x, e) = \text{dep}(x)$.



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- Note that dry need not be empty. Exercise: give example.

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- Now, given $x \in P_f$, and $e \in \text{sat}(x)$, recall distributive sub-lattice of e-containing tight sets $\mathcal{D}(x, e) = \{A : e \in A, x(A) = f(A)\}$

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- We can define the "1" element of this sub-lattice as $\text{sat}(x, e) \stackrel{\text{def}}{=} \bigcup \{A : A \in \mathcal{D}(x, e)\}$.
- Analogously, we can define the "0" element of this sub-lattice as $\text{dry}(x, e) \stackrel{\text{def}}{=} \bigcap \{A : A \in \mathcal{D}(x, e)\}$.
- We can see $\text{dry}(x, e)$ as the elements that are necessary for e-containing tightness, with $e \in \text{sat}(x)$.
- That is, we can view $\text{dry}(x, e)$ as

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- But actually, $\text{dry}(x, e) = \text{dep}(x, e)$, so we have derived another expression for $\text{dep}(x, e)$ in Eq. (12.88).

Dependence Function and Fundamental Matroid Circuit

- Now, let $(E, \mathcal{I}) = (E, r)$ be a matroid, and let $I \in \mathcal{I}$ giving $\mathbf{1}_I \in P_r$. Let $e \in \text{sat}(\mathbf{1}_I) = \text{span}(I) = \text{closure}(I)$.

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- Then there is a unique minimal $A \ni e$ with $|I \cap A| = r(A)$.
- Thus, $\text{dep}(\mathbf{1}_I, e)$ must be a circuit since if it included more than a circuit, it would not be minimal in this sense.

Dependence Function and Fundamental Matroid Circuit

- Therefore, when $e \in \text{sat}(\mathbf{1}_I) \setminus I$, then $\text{dep}(\mathbf{1}_I, e) = C(I, e)$ where $C(I, e)$ is the unique circuit contained in $I + e$ in a matroid (the fundamental circuit of e and I that we encountered before).

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- We are thus free to take subsets of I as A, all of which must contain e, but all of which have rank equal to size.
- Also note: in general for $x \in P_f$ and $e \in \text{sat}(x)$, we have $\text{dep}(x, e)$ is tight by definition.

Summary of sat, and dep

- For $x \in P_f$, $\text{sat}(x)$ (span, closure) is the maximal saturated (x-tight) set w.r.t. x. I.e., $\text{sat}(x) = \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\}$. That is,

$$\text{cl}(x) \stackrel{\text{def}}{=} \text{sat}(x) \triangleq \bigcup \{A : A \in \mathcal{D}(x)\} \tag{12.91}$$

$$= \bigcup \{A : A \subseteq E, x(A) = f(A)\} \tag{12.92}$$

$$= \{e : e \in E, \forall \alpha > 0, x + \alpha \mathbf{1}_e \notin P_f\} \tag{12.93}$$

- For $e \in \text{sat}(x)$, $\text{dep}(x, e)$ (fundamental circuit) is the minimal (common) saturated (x-tight) set w.r.t. x containing e. That is,

$$\text{dep}(x, e) = \begin{cases} \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases}$$

$$= \left\{e' : \exists \alpha > 0, \text{ s.t. } x + \alpha(\mathbf{1}_e - \mathbf{1}_{e'}) \in P_f\right\} \tag{12.94}$$

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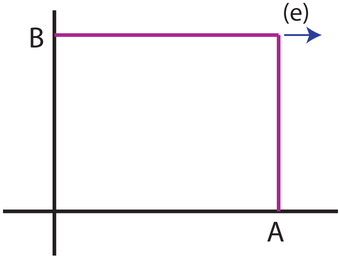
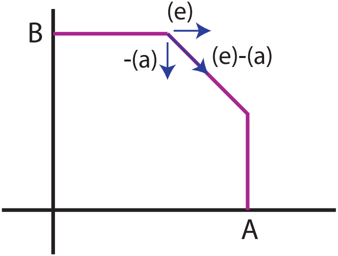
- But, analogous to the circuit case, is there an exchange property for $\text{dep}(x, e)$ in the form of vector movement restriction?
- We might expect the vector $\text{dep}(x, e)$ property to take the form: a positive move in the e-direction stays within P_f^+ only if we simultaneously take a negative move in one of the $\text{dep}(x, e)$ directions.

Dependence Function and exchange in 2D

- Viewable in 2D, we have for $A, B \subseteq E$, $A \cap B = \emptyset$:





Left: $A \cap \text{dep}(x, e) = \emptyset$, and we can't move further in (e) direction, and moving in any negative $a \in A$ direction doesn't change that. Notice no dependence between (e) and any element in A.

Right: $A \subseteq \text{dep}(x, e)$, and we can't move further in the (e) direction, but we can move further in (e) direction by moving in some $a \in A$ negative direction. Notice dependence between (e) and elements in A.

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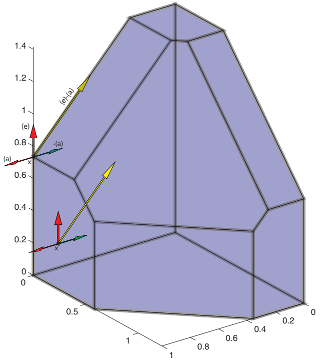

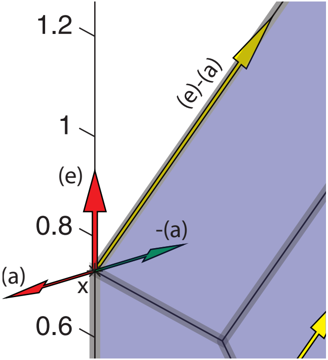

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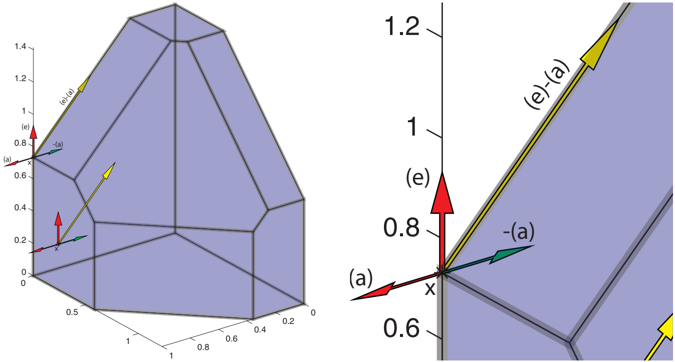

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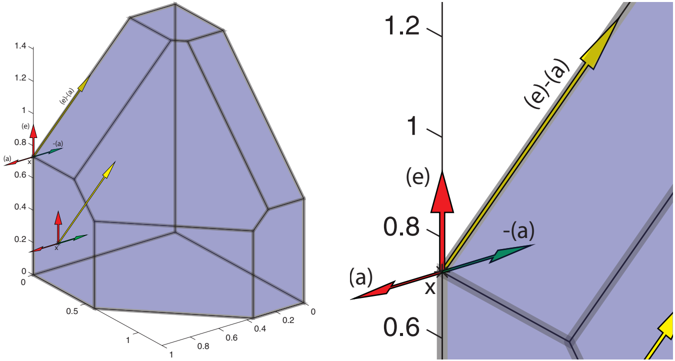

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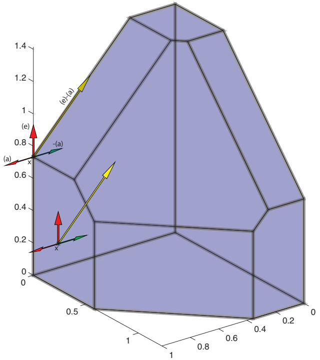

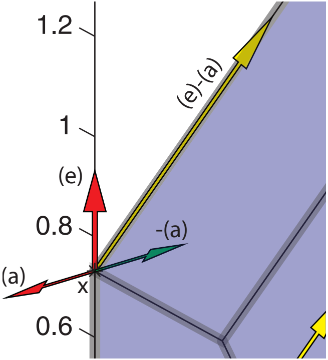

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- We next show this formally ...

dep and exchange derived

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dep and exchange derived

- The derivation for $\text{dep}(x, e)$ involves turning a strict inequality into a non-strict one with a strict explicit slack variable α:

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- Now, $1_e(A) - \mathbf{1}_{e'}(A) = 0$ if either $\{e, e'\} \subseteq A$, or $\{e, e'\} \cap A = \emptyset$.

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- Now, $1_e(A) - \mathbf{1}_{e'}(A) = 0$ if either $\{e, e'\} \subseteq A$, or $\{e, e'\} \cap A = \emptyset$.
- Also, if $e' \in A$ but $e \notin A$, then $x(A) + \alpha(\mathbf{1}_e(A) - \mathbf{1}_{e'}(A)) = x(A) - \alpha \leq f(A)$ since $x \in P_f$.

dep and exchange derived

- thus, we get the same in the above if we remove the constraint $A \not\ni e', e \in A$, that is we get

$$\text{dep}(x, e) = \{e' : \exists \alpha > 0, \text{ s.t. } x(A) + \alpha(\mathbf{1}_e(A) - \mathbf{1}_{e'}(A)) \leq f(A), \forall A\} \quad (12.103)$$

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- Compare with original, the minimal element of $\mathcal{D}(x, e)$, with $e \in \text{sat}(x)$:

$$\text{dep}(x, e) = \begin{cases} \bigcap \{A : e \in A \subseteq E, x(A) = f(A)\} & \text{if } e \in \text{sat}(x) \\ \emptyset & \text{else} \end{cases} \tag{12.105}$$