Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 11 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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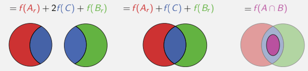
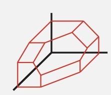
$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

$= f(A_r) + 2f(C) + f(B_r)$ $\quad = f(A_r) + f(C) + f(B_r)$ $\quad = f(A \cap B)$





Cumulative Outstanding Reading

- Read chapters 1 and 2, and sections 3.1-3.2 from Fujishige's book.
- Good references for today: Schrijver-2003, Oxley-1992/2011, Welsh-1973, Goemans-2010, Cunningham-1984, Edmonds-1969.

Announcements, Assignments, and Reminders

- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, Combinatorial Geometries
- L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
- L9: From Matroid Polytopes to Polymatroids.
- L10: Polymatroids and Submodularity
- L11: More properties of polymatroids, SFM special cases
- L12:
- L13:
- L14:
- L15:
- L16:
- L17:
- L18:
- L19:
- L20:

Finals Week: June 9th-13th, 2014.

A polymatroid function's polyhedron is a polymatroid.

Theorem 11.2.4

Let f be a polymatroid function defined on subsets of E. For any $x \in \mathbb{R}_+^E$, and any P_f^+-basis $y^x \in \mathbb{R}_+^E$ of x, the component sum of y^x is

$$y^x(E) = \textit{rank}(x) = \max\Big(y(E) : y \le x, y \in P_f^+\Big)$$
$$= \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \qquad (11.34)$$

As a consequence, P_f^+ is a polymatroid, since r.h.s. is constant w.r.t. y^x.

By taking $B = \mathrm{supp}(x)$ (so elements $E \setminus B$ are zero in x), and for $b \in B$, $x(b)$ is big enough, the r.h.s. min has solution $A^* = E \setminus B$. We recover submodular function from the polymatroid polyhedron via the following:

$$f(B) = \max\Big\{y(B) : y \in P_f^+\Big\} \qquad (11.35)$$

In fact, we will ultimately see a number of important consequences of this theorem (other than just that P_f^+ is a polymatroid)

Join $\vee$ and meet $\wedge$ for $x, y \in \mathbb{R}_+^E$

- For $x, y \in \mathbb{R}_+^E$, define vectors $x \wedge y \in \mathbb{R}_+^E$ and $x \vee y \in \mathbb{R}_+^E$ such that, for all $e \in E$

$$(x \vee y)(e) = \max(x(e), y(e)) \qquad (11.18)$$
$$(x \wedge y)(e) = \min(x(e), y(e)) \qquad (11.19)$$

Hence,

$$x \vee y \triangleq \bigg(\max\Big(x(e_1), y(e_1)\Big), \max\Big(x(e_2), y(e_2)\Big), \dots, \max\Big(x(e_n), y(e_n)\Big)\bigg)$$

and similarly

$$x \wedge y \triangleq \bigg(\min\Big(x(e_1), y(e_1)\Big), \min\Big(x(e_2), y(e_2)\Big), \dots, \min\Big(x(e_n), y(e_n)\Big)\bigg)$$

- From this, we can define things like an lattices, and other constructs.

Vector rank, rank(x), is submodular

- Recall that the matroid rank function is submodular.
- The vector rank function rank(x) also satisfies a form of submodularity.

Theorem 11.2.1 (vector rank and submodularity)

Let P be a polymatroid polytope. The vector rank function rank : $\mathbb{R}_+^E \to \mathbb{R}$ with rank$(x) = \max\left(y(E) : y \leq x, y \in P\right)$ satisfies, for all $u, v \in \mathbb{R}_+^E$

$$\text{rank}(u) + \text{rank}(v) \geq \text{rank}(u \vee v) + \text{rank}(u \wedge v) \tag{11.18}$$

A polymatroid is a polymatroid function's polytope

- So, when f is a polymatroid function, P_f^+ is a polymatroid.
- Is it the case that, conversely, for any polymatroid P, there is an associated polymatroidal function f such that $P = P_f^+$?

Theorem 11.2.1

For any polymatroid P (compact subset of $\mathbb{R}_+^E$, zero containing, down-monotone, and $\forall x \in \mathbb{R}_+^E$ any maximal independent subvector $y \leq x$ has same component sum $y(E) = \text{rank}(x)$), there is a polymatroid function $f : 2^E \to \mathbb{R}$ (normalized, monotone non-decreasing, submodular) such that $P = P_f^+$ where $P_f^+ = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\right\}$.

First, a bit on $\mathcal{D}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{A : A \subseteq E,\ y(A) = f(A)\} \tag{11.18}$$

Theorem 11.2.1

For any $y \in P_f^+$, with f a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

Proof.

We have already proven this as part of Theorem 9.4.5 □

Also recall the definition of $\text{sat}(y)$, the maximal set of tight elements relative to $y \in \mathbb{R}_+^E$.

$$\text{sat}(y) \stackrel{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\} \tag{11.19}$$

A word on terminology & notation

- Recall how a matroid is sometimes given as (E, r) where r is the rank function.
- We mention also that the term "polymatroid" is sometimes not used for the polytope itself, but instead but for the pair (E, f),
- But now we see that (E, f) is equivalent to a polymatroid polytope, so this is sensible.

Where are we going with this?

- Consider the right hand side of Theorem 9.4.5: $\min(x(A) + f(E \setminus A) : A \subseteq E)$
- We are going to study this problem, and approaches that address it, as part of our ultimate goal which is to present strategies for submodular function minimization (that we will ultimately get to, in near future lectures).
- As a bit of a hint on what's to come, recall that we can write it as: $x(E) + \min(f(A) - x(A) : A \subseteq E)$ where f is a polymatroid function.

Another Interesting Fact: Matroids from polymatroid functions

Theorem 11.3.1

Given integral polymatroid function f, let $(E, \mathcal{F})$ be a set system with ground set E and set of subsets $\mathcal{F}$ such that

$$\forall F \in \mathcal{F}, \ \ \forall \emptyset \subset S \subseteq F, |S| \leq f(S) \tag{11.1}$$

Then $M = (E, \mathcal{F})$ is a matroid.

Proof.

Exercise □

And its rank function is Exercise.

Matroid instance of Theorem 9.4.5

- Considering Theorem 9.4.5, the matroid case is now a special case, where we have that:

Corollary 11.3.2

We have that:

$$\max\{y(E) : y \in P_{\text{ind. set}}(M), y \le x\} = \min\{r_M(A) + x(E \setminus A) : A \subseteq E\} \tag{11.2}$$

where r_M is the matroid rank function of some matroid.

Most violated inequality problem in matroid polytope case

- Consider

$$P_r^+ = \left\{x \in \mathbb{R}^E : x \ge 0, x(A) \le r_M(A), \forall A \subseteq E\right\} \tag{11.3}$$

- We saw before that $P_r^+ = P_{\text{ind. set}}$.
- Suppose we have any $x \in \mathbb{R}_+^E$ such that $x \notin P_r^+$, then one or more of the inequalities in Eq. (11.3) are violated.
- The most violated inequality when x is considered w.r.t. P_r^+ corresponds to the set A that maximizes $x(A) - r_M(A)$, i.e., $\max\{x(A) - r_M(A) : A \subseteq E\}$.
- This corresponds to $\min\{r_M(A) + x(E \setminus A) : A \subseteq E\}$ since x is modular and $x(E \setminus A) = x(E) - x(A)$.
- More importantly, $\min\{r_M(A) + x(E \setminus A) : A \subseteq E\}$ a form of submodular function minimization, namely $\min\{r_M(A) - x(A) : A \subseteq E\}$ for a submodular function consisting of a difference of matroid rank and modular (so no longer necessarily monotone, nor positive).

Problem to Solve

In particular, we will solve the following problem:

- Given a matroid $M = (E, \mathcal{I})$ along with an independence testing oracle (i.e., for any $A \subseteq E$, tells us if $A \in \mathcal{I}$ or not), and a vector $x \in \mathcal{R}_+^E$;
- find: a maximizing $y \in P_{\text{ind. set}}$ with $y \leq x$, and moreover (as a byproduct of the algorithm), express y as a convex combination of incidence vectors of independent sets in M, and also return a set $A \subseteq E$ that satisfies $y(E) = r_M(A) + x(E \setminus A)$. *Of course, by Theorem 9.4.5, for any such y we must have that $y(E) \leq r(A) + x(E \setminus A)$.*
- By Theorem 9.4.5, the existence of such an A will certify that $y(E)$ is maximal in $P_{\text{ind. set}}$, A is minimal in terms of $f(A) \stackrel{\text{def}}{=} r_M(A) - x(A)$ (thus most violated).
- This can also be used to test membership in $P_{\text{ind. set}}$ (i.e., if $y = x$) depending on the sign of f at A.
- This will also run in polynomial time.

Idea of the algorithm

- We build up y from the ground up.
- We keep a family of independent sets $(I_i : i \in J)$ and coefficients $(\lambda_i : i \in J)$ such that $\sum_{i \in J} \lambda_i = 1$ and $y = \sum_{i \in J} \lambda_i \mathbf{1}_{I_i}$.
- We gradually build up y by adding new independent sets (and augmenting J), adding to the existing independent sets, and adjusting coefficients.
- and the way these additions are done is via solutions to a max-flow problem in an associated flow-graph (which we'll describe).
- Each update will, of course, ensure that $y \in P_{\text{ind. set}}$, but also we'll keep $y \leq x$.
- It's going to take us a few lectures to fully develop this algorithm, so please keep in mind of the overall goal.

Bipartite Matching

- Consider a bipartite graph $G = (V, F, E)$ where left nodes are V, right nodes are F, and $E \subseteq V \times F$ are the only edges.
- A matching $A \subseteq E$ is a subset of edges such that no two edges are incident to the same vertex.
- A node j is matched in A if $(j, k) \in A$ for some $k \in F$, and otherwise j is called unmatched. Likewise for some $k \in F$.
- Given $A \subseteq E$, an alternating path S (relative to A) is an (undirected) path of unique edges that are alternatively in A and not in A. I.e., if $S = (e_1, e_2, \dots, e_s)$ is an alternating path, then $S_{1/2} \stackrel{\text{def}}{=} S \setminus A$ where $S_{1/2}$ is either the odd or the even elements of S.
- An $A \subseteq E$ is an augmenting path if it is an alternating path between two unmatched vertices.

Bipartite Matching

- Given a matching $A \subseteq E$ (which might be empty), we can increase the matching if we can find an augmenting path S.
- The updated matching becomes $A' = A \setminus S \cup S \setminus A = A \ominus S$, where $\ominus$ is the symmetric difference operator.
- The algorithm becomes:

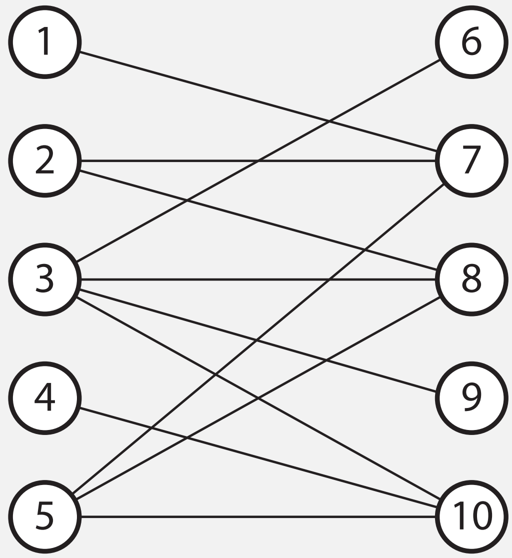
Algorithm 8.1: Alternating Path Bipartite Matching

Let A be an arbitrary (including empty) matching in $G = (V, F, E)$;
while *There exists an augmenting path S in G* **do**
  $A \leftarrow A \ominus S$;

- This can easily be made to run in $O(m^2 n)$, where $|V| = m$, $|F| = n$, $m \leq n$, but it can be made to run much faster as well (see Schrijver-2003).

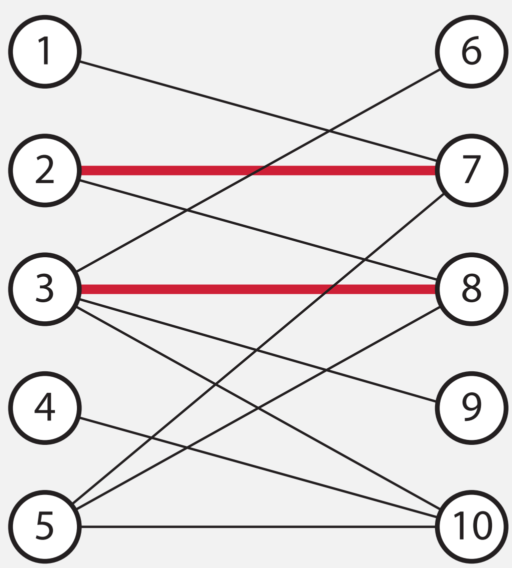
Bipartite Matching Example

Consider the following bipartite graph $G = (V, F, E)$ with $|V| = |F| = 5$. Any edge is an augmenting path since it will adjoin two unmatched vertices.



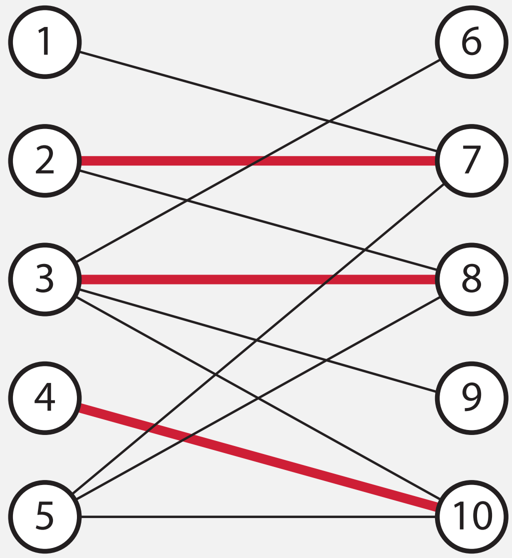
Bipartite Matching Example

Any edge, not intersecting nodes adjacent to current matching is an augmenting path.



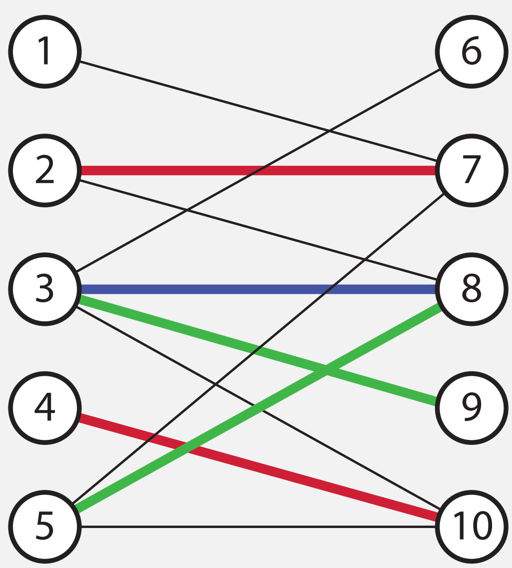
Bipartite Matching Example

No possible further single edge addition at this point. We need a multi-edge augmenting path if it exists.



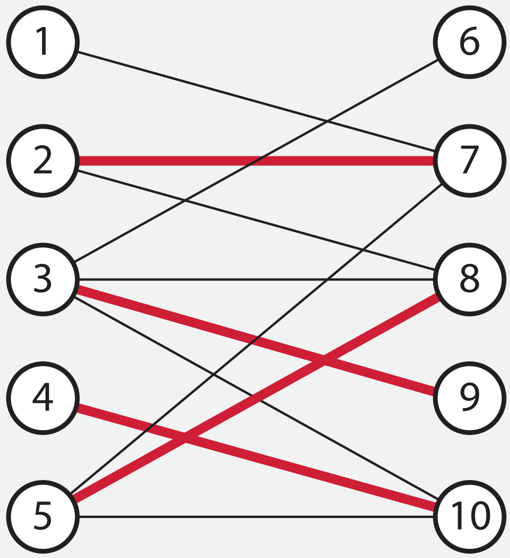
Bipartite Matching Example

Augmenting path is green and blue edges (blue is already in matching, green is new).



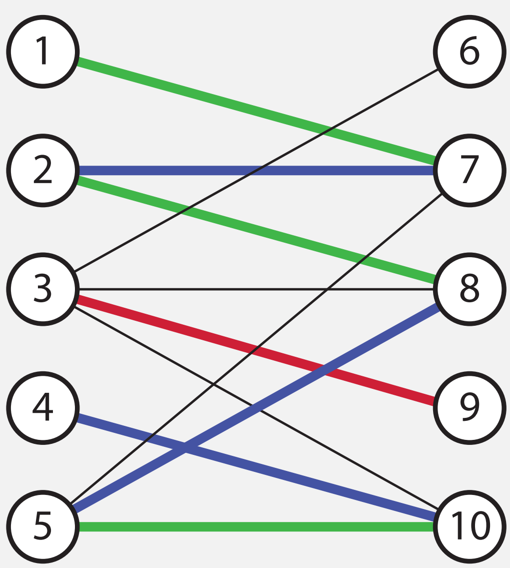
Bipartite Matching Example

Removing blue from matching and adding green leads to higher cardinality matching.



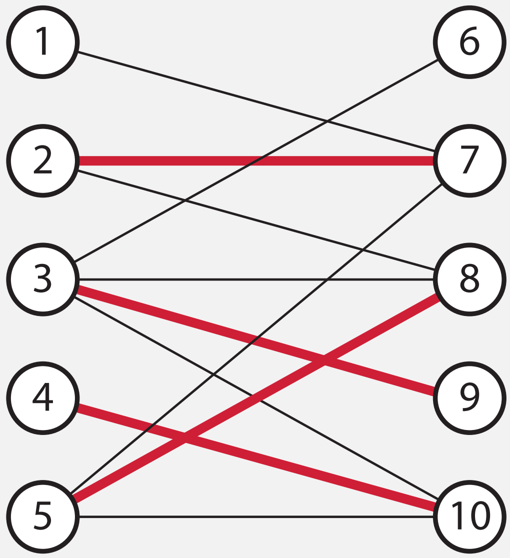
Bipartite Matching Example

At this point, resulting alternating path is not augmenting, since it is not between two unmatched vertices (and no augmenting path is possible).



Bipartite Matching Example

At this point, matching is maximum cardinality.



Review

- The next slide is from lecture 7 and the one after from lecture 5.

Matroid Intersection

- Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$.
- While $(V, \mathcal{I}_1 \cap \mathcal{I}_2)$ is typically not a matroid (Exercise: show graphical example.), we might be interested in finding the maximum size common independent set. That is, find $\max |X|$ such that both $X \in \mathcal{I}_1$ and $X \in \mathcal{I}_2$.

Theorem 11.5.5

Let M_1 and M_2 be given as above, with rank functions r_1 and r_2. Then the size of the maximum size set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by

$$(r_1 * r_2)(V) \triangleq \min_{X \subseteq V} \Big(r_1(X) + r_2(V \setminus X) \Big) \tag{11.7}$$

This is an instance of the convolution of two submodular functions, f_1 and f_2 that, evaluated at $Y \subseteq V$, is written as:

$$(f_1 * f_2)(Y) = \min_{X \subseteq Y} \Big(f_1(X) + f_2(Y \setminus X) \Big) \tag{11.8}$$

Partition Matroid

- Let V be our ground set.
- Let $V = V_1 \cup V_2 \cup \cdots \cup V_\ell$ be a partition of V into blocks or disjoint sets (disjoint union). Define a set of subsets of V as

$$\mathcal{I} = \{X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell\}. \tag{11.3}$$

  where $k_1, \ldots, k_\ell$ are fixed parameters, $k_i \geq 0$. Then $M = (V, \mathcal{I})$ is a matroid.
- Note that a k-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$.
- We'll show that property (I3') in Def **??** holds. If $X, Y \in \mathcal{I}$ with $|Y| > |X|$, then there must be at least one i with $|Y \cap V_i| > |X \cap V_i|$. Therefore, adding one element $e \in V_i \cap (Y \setminus X)$ to X won't break independence.

Matroid Intersection and Bipartite Matching

- Why might we want to do matroid intersection?
- Consider bipartite graph $G = (V, F, E)$. Define two partition matroids $M_V = (E, \mathcal{I}_V)$, and $M_F = (E, \mathcal{I}_F)$.
- Independence in each matroid corresponds to:
  1. $I \in \mathcal{I}_V$ if $|I \cap (V, f)| \leq 1$ for all $f \in F$,
  2. and $I \in \mathcal{I}_F$ if $|I \cap (v, F)| \leq 1$ for all $v \in V$.
- Therefore, a matching in G is simultaneously independent in both M_V and M_F and finding the maximum matching is finding the maximum cardinality set independent in both matroids.
- For the bipartite graph case, therefore, this can be solved in polynomial time.

Matroid Intersection and Network Communication

- Let $G_1 = (V_1, E)$ and $G_2 = (V_2, E)$ be two graphs on an isomorphic set of edges (lets just give them same names E).
- Consider two cycle matroids associated with these graphs $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$. They might be very different (e.g., an edge might be between two distinct nodes in G_1 but the same edge is a loop in multi-graph G_2.)
- We may wish to find the maximum size edge-induced subgraph that is still forest in both graphs (i.e., adding any edges will create a circuit in either M_1, M_2, or both).
- This is again a matroid intersection problem.

Matroid Intersection and TSP

- Definition: a Hamiltonian cycle is a cycle that passes through each node exactly once.
- Given directed graph G, goal is to find such a Hamiltonian cycle.
- From G with n nodes, create G' with $n+1$ nodes by duplicating (w.l.o.g.) a particular node $v_1 \in V(G)$ to v_1^+, v_1^-, and have all outgoing edges from v_1 come instead from v_1^+ and all edges incoming to v_1 go instead to v_1^-.
- Let M_1 be the cycle matroid on G'.
- Let M_2 be the partition matroid having as independent sets those that have no more than one edge leaving any node — i.e., $I \in \mathcal{I}(M_2)$ if $|I \cap \delta^+(v)| \leq 1$ for all $v \in V(G')$.
- Let M_3 be the partition matroid having as independent sets those that have no more than one edge entering any node — i.e., $I \in \mathcal{I}(M_3)$ if $|I \cap \delta^-(v)| \leq 1$ for all $v \in V(G')$.
- Then a Hamiltonian cycle exists iff there is an n-element intersection of M_1, M_2, and M_3.

- Since TSP is NP-complete, we obviously can't solve matroid intersections of 3 more matroids, unless P=NP.
- But bipartite graph example gives us hope for 2 matroids, and also ideas for an algorithm ...

Recall from Lecture 5: Matroids by circuits

A set is independent if and only if it contains no circuit. Therefore, it is not surprising that circuits can also characterize a matroid.

Theorem 11.5.1

Matroid (by circuits) Let E be a set and $\mathcal{C}$ be a collection of nonempty subsets of E, such that no two sets in $\mathcal{C}$ are contained in each other. Then the following are equivalent.

1. *(C1) $\mathcal{C}$ is the collection of circuits of a matroid;*
2. *(C2) if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$;*
3. *(C3) if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, and $y \in C \setminus C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$ containing y;*

Fundamental circuits in matroids

Lemma 11.5.2

Let $I \in \mathcal{I}(M)$, and $e \in E$, then $I \cup \{e\}$ contains at most one circuit in M.

Proof.

- Suppose, to the contrary, that there are two distinct circuits C_1, C_2 such that $C_1 \cup C_2 \subseteq I \cup \{e\}$.
- Then $e \in C_1 \cap C_2$, and by (C2), there is a circuit C_3 of M s.t. $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\} \subseteq I$
- This contradicts the independence of I.

□

In general, let $C(I, e)$ be the unique circuit associated with $I \cup \{e\}$ (commonly called the fundamental circuit in M w.r.t. I and e).

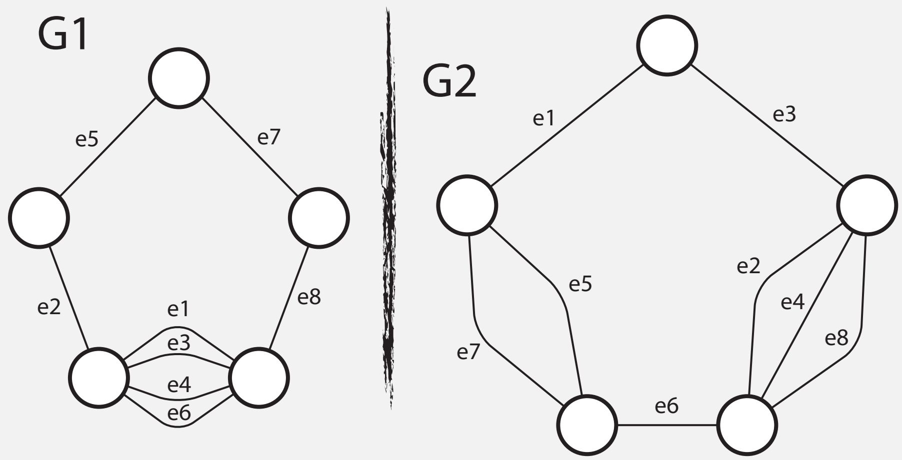
Matroid Intersection Algorithm Idea

- Consider two matroids $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ and start with any $I \in \mathcal{I}_1 \cap \mathcal{I}_2$.
- Consider some $v_1 \notin \operatorname{span}_1(I)$, so that $I + v_1 \in \mathcal{I}_1$.
- If $I + v_1 \in \mathcal{I}_2$, then v_1 is "augmenting", and we can augment I to $I + v_1$ and still be independent in both M_1 and M_2.
- If $I + v_1 \notin \mathcal{I}_2$, $\exists C_2(I, v_1)$ a circuit in M_2, and choosing $v_2 \in C_2(I, v_1)$ s.t. $v_2 \neq v_1$ leads to $I + v_1 - v_2$ which (because $\operatorname{span}_2(I) = \operatorname{span}_2(I + v_1 - v_2)$) is again independent in M_2. $I + v_1 - v_2$ is also independent in M_1. Note, $v_2 \in I$.
- Next choose a $v_3 \in \operatorname{span}_1(I) - \operatorname{span}_1(I - v_2)$ to recover what was lost in $I \cup \{v_1\}$ when we removed v_2 from it. Note, $v_3 \notin I$.
- Then $\operatorname{span}_1(I) = \operatorname{span}_1(I - v_2 + v_3)$.
- Moreover, since $I + v_1 \in \mathcal{I}_1$, $v_1 \notin \operatorname{span}_1(I)$, so $\operatorname{span}_1(I + v_1) = \operatorname{span}_1(I + v_1 - v_2 + v_3)$.
- But $I + v_1 - v_2 + v_3$ might not be independent in M_2 again, so need to find an $v_4 \in C_2(I + v_1 - v_2, v_3)$, $v_4 \in I$ to remove, and so on.

- Hopefully (eventually) we'll find an odd length sequence $S = (v_1, v_2, \dots, v_s)$ such that we will be independent in both M_1

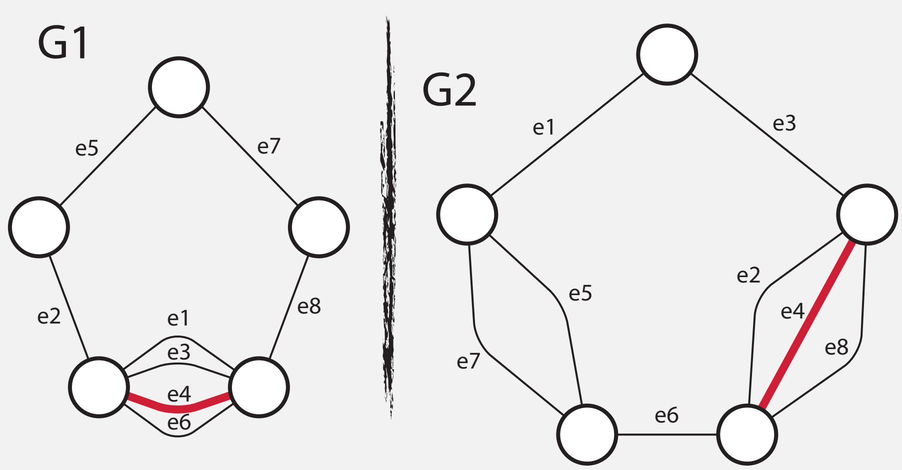
Graphic Matroid Intersection Example

Consider the following two graph $G_1 = (V_1, E)$ and $G_2 = (V_2, E)$ and corresponding matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$. Any edge is independent in both (an augmenting "sequence") since a single edge can't create a circuit starting at $I = \emptyset$. We start with e_4.



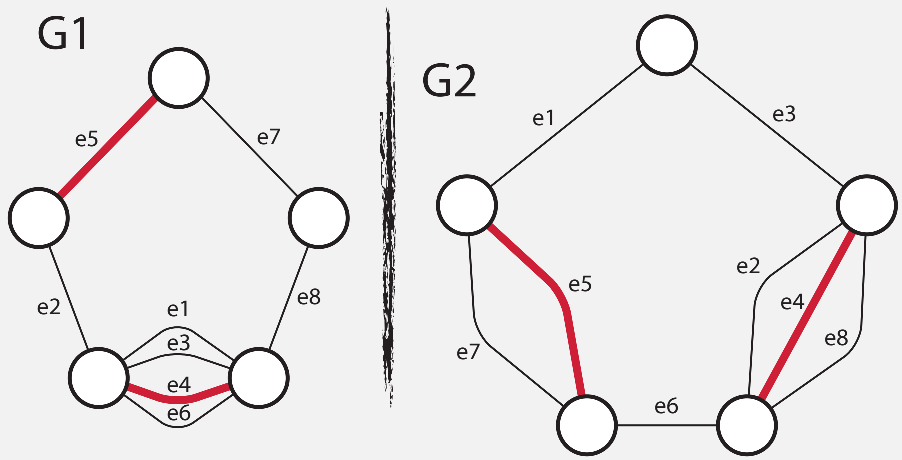
Graphic Matroid Intersection Example

Setting $I \leftarrow e_4$ with edge e_4 creates a circuit neither in M_1 nor M_2. We can add another single edge w/o creating a circuit in either matroid.



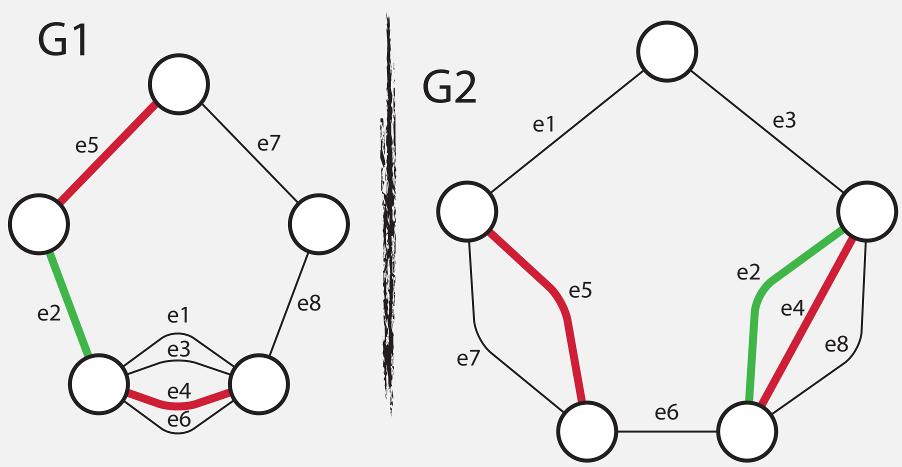
Graphic Matroid Intersection Example

$e_5 \in E - \text{span}_1(\{e_4\})$. Then, after $I \leftarrow I + e_5$, (i.e., when $I = \{e_4, e_5\}$) we're still independent in M_2, but no further single edge additions possible w/o creating a circuit (why?). We need a multi-edge "augmenting sequence" if it exists.



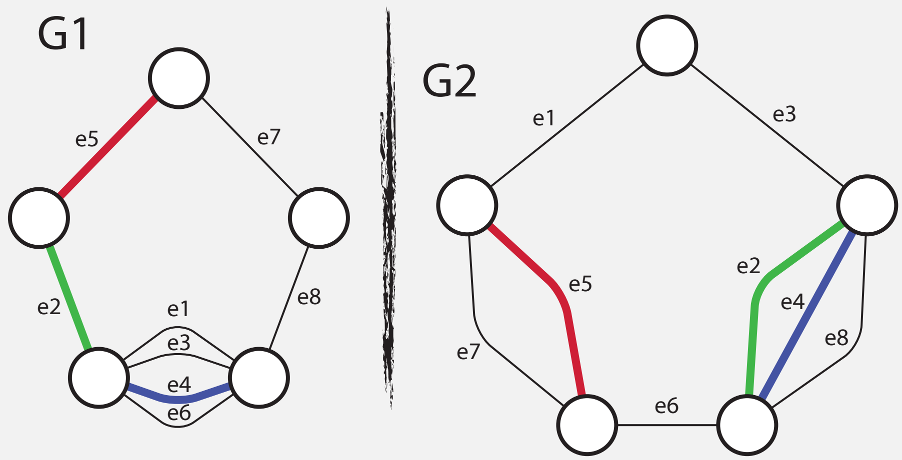
Graphic Matroid Intersection Example

Augmenting sequence is green and blue edges (blue is already in I, green is new). We choose $e_2 \in E - \text{span}_1(I)$, but now $I + e_2$ is not independent in M_2.



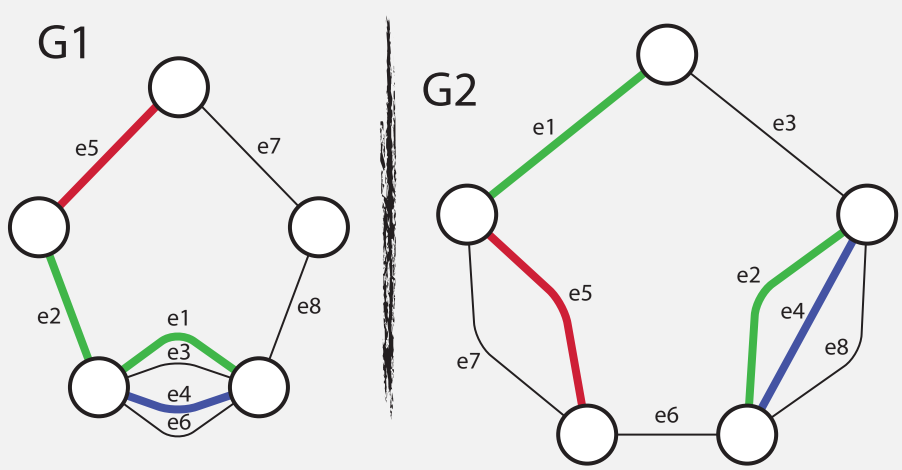
Graphic Matroid Intersection Example

So there must exist $C_2(I, e_2)$. We choose $e_4 \in C_2(I, e_2)$ to remove.



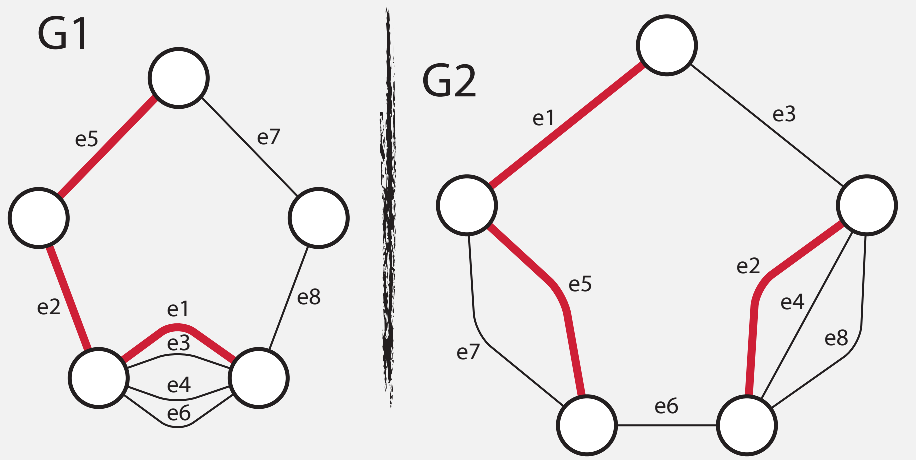
Graphic Matroid Intersection Example

Next, we choose $e_1 \in \operatorname{span}_1(I) - \operatorname{span}_1(I - e_4)$ to add. In this case, we not only have $\operatorname{span}_1(I + e_2) = \operatorname{span}_1(I + e_2 - e_4 + e_1)$, but we also have that $(I + e_2 - e_4) + e_1 \in \mathcal{I}_2$.



Graphic Matroid Intersection Example

Removing blue and adding green leads to higher cardinality independent set in both matroids. This corresponds to doing $I \leftarrow I \ominus S$ where $S = (e_2, e_4, e_1)$ and $I = \{e_4, e_5\}$.
At this point, are any further augmenting sequences possible? Exercise.



Alternating and Augmenting Sequences

- Let I be an intersection of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ (i.e., $I \in \mathcal{I}_1 \cap \mathcal{I}_2$).
- Let $S = (e_1, e_2, \dots, e_s)$ be a sequence of distinct elements, where $e_i \in E - I$ for i odd, and $e_i \in I$ for i even, and let $S_i = (e_1, e_2, \dots, e_i)$. We say that S is an alternating sequence w.r.t. I if the following are true.
  1. $I + e_1 \in \mathcal{I}_1$
  2. For all even i, $\text{span}_2(I \ominus S_i) = \text{span}_2(I)$ which implies that $I \ominus S_i \in \mathcal{I}_2$.
  3. For all odd i, $\text{span}_1(I \ominus S_i) = \text{span}_1(I + e_1)$, and therefore $I \ominus S_i \in \mathcal{I}_1$.
- Lastly, if also, $|S| = s$ is odd, and $I \ominus S \in \mathcal{I}_2$, then S is called an augmenting sequence w.r.t. I.

Alternating and Augmenting Sequences

- If I admits an augmenting sequence S, then the above argument shows that $I \ominus S$ is independent in M_1, independent in M_2, and also we have that $|I| + 1 = |I \ominus S|$.
- Thus, by finding augmenting sequences, we can increase the size of the matroid intersection until we stop. Moreover, we have:

Proposition 11.5.3

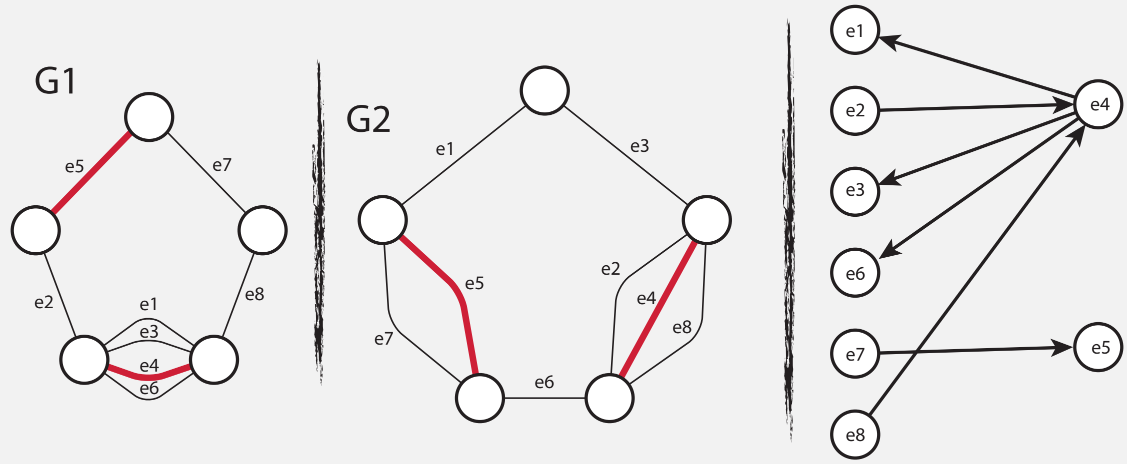
If there is an augmenting sequence, then the intersection is not maximum.

- We next wish to show that, if the intersection is not maximum, then there is an augmenting sequence.

Border graphs

- We construct an auxiliary directed bipartite graph (Border graph) $B(I) = (E \setminus I, I, Z)$, relative to the current I, that will help us with this problem. The graph has only directed edges from $E \setminus I$ to I, or from I back to $E \setminus I$.
- Left-going edges: For each $e_i \in \text{span}_1(I) \setminus I$, create $e_i \leftarrow e_j$ directed edge $(e_j, e_i) \in Z$ from all $e_j \in C_1(I, e_i) \setminus \{e_i\}$. Note $e_j \in I$ and $e_i \in E \setminus I$.
- If $e_i \notin \text{span}_1(I)$, then e_i has in-degree zero (a source).
- Right-going edges: For each $e_i \in \text{span}_2(I) \setminus I$, create $e_i \rightarrow e_j$ edge $(e_i, e_j) \in Z$ to all $e_j \in C_2(I, e_i) \setminus \{e_i\}$.
- If $e_i \notin \text{span}_2(I)$, then e_i has out-degree zero (a sink).

Border graph Example



- $\{e_2, e_7, e_8\}$ are sources and $\{e_1, e_3, e_6\}$ are sinks. $I = \{e_4, e_5\}$. $\text{span}_1(I) \setminus I = \{e_1, e_3, e_6\}$ and $\text{span}_2(I) \setminus I = \{e_7, e_2, e_8\}$
- $C_1(I, e_1) \setminus \{e_1\} = C_1(I, e_3) \setminus \{e_3\} = C_1(I, e_6) \setminus \{e_6\} = e_4$.
- $C_2(I, e_7) \setminus \{e_7\} = e_5$, $C_2(I, e_2) \setminus \{e_2\} = C_2(I, e_8) \setminus \{e_8\} = e_4$.
- Augmenting sequences are (e_2, e_4, e_1), (e_2, e_4, e_3), and (e_2, e_4, e_6), all dipaths in the Border graph. Exercise: Are there others?

Identifying Augmenting Sequences

Lemma 11.5.4

If S is a source-sink path in $B(I)$, and there is no shorter source-sink path between the same source and sink (i.e., there are no short-cuts), then S is an augmenting sequence w.r.t. I.

Lemma 11.5.5

Let I and J be matroid intersections of M_1 and M_2 such that $|I| + 1 = |J|$. Then there exists a source-sink path S in $B(I)$ where $S \subseteq I \ominus J$.

Identifying Augmenting Sequences

Theorem 11.5.6

Let I_p and I_{p+1} be intersections of M_1 and M_2 with p and $p+1$ elements respectively. Then there exists an augmenting sequence $S \subseteq I_p \ominus I_{p+1}$ w.r.t. I_p.

Theorem 11.5.7

An intersection is of maximum cardinality iff it admits no augmenting sequence.

Theorem 11.5.8

For any intersection I, there exists a maximum cardinality intersection I^* such that $\text{span}_1(I) \subseteq \text{span}_1(I^*)$ and $\text{span}_2(I) \subseteq \text{span}_2(I^*)$.

All this can be made to run in poly time.