Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture 11 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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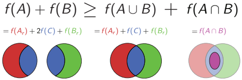

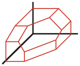

Cumulative Outstanding Reading

- Read chapters 1 and 2, and sections 3.1-3.2 from Fujishige's book.
- Good references for today: Schrijver-2003, Oxley-1992/2011, Welsh-1973, Goemans-2010, Cunningham-1984, Edmonds-1969.

Announcements, Assignments, and Reminders

- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, Combinatorial Geometries
- L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
- L9: From Matroid Polytopes to Polymatroids.
- L10: Polymatroids and Submodularity
- L11: More properties of polymatroids, SFM special cases
- L12:
- L13:
- L14:
- L15:
- L16:
- L17:
- L18:
- L19:
- L20:

Finals Week: June 9th-13th, 2014.

A polymatroid function's polyhedron is a polymatroid.

Theorem 11.2.4

Let f be a polymatroid function defined on subsets of E. For any $x \in \mathbb{R}_+^E$, and any P_f^+-basis $y^x \in \mathbb{R}_+^E$ of x, the component sum of y^x is

$$y^x(E) = \text{rank}(x) = \max\left(y(E) : y \leq x, y \in P_f^+\right)$$
$$= \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \quad (11.34)$$

As a consequence, P_f^+ is a polymatroid, since r.h.s. is constant w.r.t. y^x.

By taking $B = \text{supp}(x)$ (so elements $E \setminus B$ are zero in x), and for $b \in B$, $x(b)$ is big enough, the r.h.s. min has solution $A^* = E \setminus B$. We recover submodular function from the polymatroid polyhedron via the following:

$$f(B) = \max\left\{y(B) : y \in P_f^+\right\} \quad (11.35)$$

In fact, we will ultimately see a number of important consequences of this theorem (other than just that P_f^+ is a polymatroid)

Join $\vee$ and meet $\wedge$ for $x, y \in \mathbb{R}_+^E$

- For $x, y \in \mathbb{R}_+^E$, define vectors $x \wedge y \in \mathbb{R}_+^E$ and $x \vee y \in \mathbb{R}_+^E$ such that, for all $e \in E$

$$(x \vee y)(e) = \max(x(e), y(e)) \tag{11.18}$$

$$(x \wedge y)(e) = \min(x(e), y(e)) \tag{11.19}$$

Hence,

$$x \vee y \triangleq \Big(\max\big(x(e_1), y(e_1)\big), \max\big(x(e_2), y(e_2)\big), \dots, \max\big(x(e_n), y(e_n)\big)\Big)$$

and similarly

$$x \wedge y \triangleq \Big(\min\big(x(e_1), y(e_1)\big), \min\big(x(e_2), y(e_2)\big), \dots, \min\big(x(e_n), y(e_n)\big)\Big)$$

- From this, we can define things like an lattices, and other constructs.

Vector rank, rank(x), is submodular

- Recall that the matroid rank function is submodular.
- The vector rank function rank(x) also satisfies a form of submodularity.

Theorem 11.2.1 (vector rank and submodularity)

Let P be a polymatroid polytope. The vector rank function rank $: \mathbb{R}_+^E \to \mathbb{R}$ *with rank*$(x) = \max\left(y(E) : y \leq x, y \in P\right)$ *satisfies, for all* $u, v \in \mathbb{R}_+^E$

$$\text{rank}(u) + \text{rank}(v) \geq \text{rank}(u \vee v) + \text{rank}(u \wedge v) \tag{11.18}$$

A polymatroid is a polymatroid function's polytope

- So, when f is a polymatroid function, P_f^+ is a polymatroid.
- Is it the case that, conversely, for any polymatroid P, there is an associated polymatroidal function f such that $P = P_f^+$?

Theorem 11.2.1

For any polymatroid P (compact subset of $\mathbb{R}_+^E$, zero containing, down-monotone, and $\forall x \in \mathbb{R}_+^E$ any maximal independent subvector $y \leq x$ has same component sum $y(E) = \text{rank}(x)$), there is a polymatroid function $f : 2^E \to \mathbb{R}$ (normalized, monotone non-decreasing, submodular) such that $P = P_f^+$ where $P_f^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\}$.

First, a bit on $\mathcal{D}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{A : A \subseteq E,\ y(A) = f(A)\} \tag{11.18}$$

Theorem 11.2.1

For any $y \in P_f^+$, with f a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

Proof.

We have already proven this as part of Theorem 9.4.5 □

Also recall the definition of $\mathrm{sat}(y)$, the maximal set of tight elements relative to $y \in \mathbb{R}_+^E$.

$$\mathrm{sat}(y) \stackrel{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\} \tag{11.19}$$

A word on terminology & notation

- Recall how a matroid is sometimes given as (E, r) where r is the rank function.

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A word on terminology & notation

- Recall how a matroid is sometimes given as (E, r) where r is the rank function.
- We mention also that the term "polymatroid" is sometimes not used for the polytope itself, but instead but for the pair (E, f),
- But now we see that (E, f) is equivalent to a polymatroid polytope, so this is sensible.

(E, P)

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- Consider the right hand side of Theorem 9.4.5:
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- We are going to study this problem, and approaches that address it, as part of our ultimate goal which is to present strategies for submodular function minimization (that we will ultimately get to, in near future lectures).

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- We are going to study this problem, and approaches that address it, as part of our ultimate goal which is to present strategies for submodular function minimization (that we will ultimately get to, in near future lectures).
- As a bit of a hint on what's to come, recall that we can write it as: $x(E) + \min\left(f(A) - x(A) : A \subseteq E\right)$ where f is a polymatroid function.

Another Interesting Fact: Matroids from polymatroid functions

Theorem 11.3.1

Given integral polymatroid function f, let $(E, \mathcal{F})$ be a set system with ground set E and set of subsets $\mathcal{F}$ such that

$$\forall F \in \mathcal{F}, \;\; \forall \emptyset \subset S \subseteq F, |S| \leq f(S) \tag{11.1}$$

Then $M = (E, \mathcal{F})$ is a matroid.

Proof.

Exercise □

And its rank function is Exercise.

Matroid instance of Theorem 9.4.5

- Considering Theorem 9.4.5, the matroid case is now a special case, where we have that:

Corollary 11.3.2

We have that:

$$\max\{y(E) : y \in P_{\textit{ind. set}}(M), y \leq x\} = \min\{r_M(A) + x(E \setminus A) : A \subseteq E\} \tag{11.2}$$

where r_M is the matroid rank function of some matroid.

Most violated inequality problem in matroid polytope case

- Consider

$$P_r^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r_M(A), \forall A \subseteq E\} \tag{11.3}$$

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- This corresponds to $\min\{r_M(A) + x(E \setminus A) : A \subseteq E\}$ since x is modular and $x(E \setminus A) = x(E) - x(A)$.
- More importantly, $\min\{r_M(A) + x(E \setminus A) : A \subseteq E\}$ a form of submodular function minimization, namely $\min\{r_M(A) - x(A) : A \subseteq E\}$ for a submodular function consisting of a difference of matroid rank and modular (so no longer necessarily monotone, nor positive).

Problem to Solve

In particular, we will solve the following problem:

- Given a matroid $M = (E, \mathcal{I})$ along with an independence testing oracle (i.e., for any $A \subseteq E$, tells us if $A \in \mathcal{I}$ or not), and a vector $x \in \mathcal{R}_+^E$;

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- find: a maximizing $y \in P_{\text{ind. set}}$ with $y \leq x$, and moreover (as a byproduct of the algorithm), express y as a convex combination of incidence vectors of independent sets in M, and also return a set $A \subseteq E$ that satisfies $y(E) = r_M(A) + x(E \setminus A)$. *Of course, by Theorem 9.4.5, for any such y we must have that $y(E) \leq r(A) + x(E \setminus A)$.*

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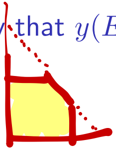
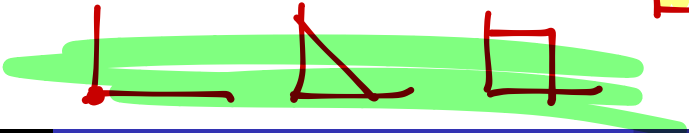
- By Theorem 9.4.5, the existence of such an A will certify that $y(E)$ is maximal in $P_{\text{ind. set}}$, A is minimal in terms of $f(A) \overset{\text{def}}{=} r_M(A) - x(A)$ (thus most violated).





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- This can also be used to test membership in $P_{\text{ind. set}}$ (i.e., if $y = x$) depending on the sign of f at A.
- This will also run in polynomial time.

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- and the way these additions are done is via solutions to a max-flow problem in an associated flow-graph (which we'll describe).
- Each update will, of course, ensure that $y \in P_{\text{ind. set}}$, but also we'll keep $y \leq x$.
- It's going to take us a few lectures to fully develop this algorithm, so please keep in mind of the overall goal.

Bipartite Matching

- Consider a bipartite graph $G = (V, F, E)$ where left nodes are V, right nodes are F, and $E \subseteq V \times F$ are the only edges.

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- Given $A \subseteq E$, an alternating path S (relative to A) is an (undirected) path of unique edges that are alternatively in A and not in A. I.e., if $S = (e_1, e_2, \dots, e_s)$ is an alternating path, then $S_{1/2} \stackrel{\text{def}}{=} S \setminus A$ where $S_{1/2}$ is either the odd or the even elements of S.

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- An $A \subseteq E$ is an augmenting path if it is an alternating path between two unmatched vertices.

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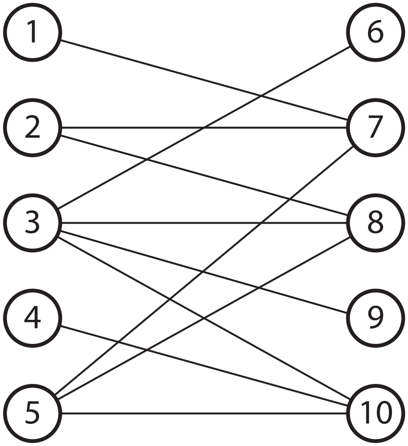
- This can easily be made to run in $O(m^2 n)$, where $|V| = m$, $|F| = n$, $m \leq n$, but it can be made to run much faster as well (see Schrijver-2003).

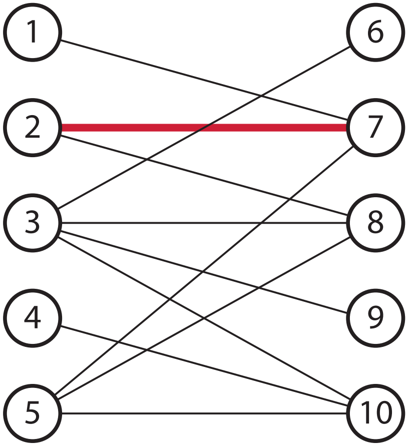
Bipartite Matching Example

Consider the following bipartite graph $G = (V, F, E)$ with $|V| = |F| = 5$. Any edge is an augmenting path since it will adjoin two unmatched vertices.



Bipartite Matching Example

Any edge, not intersecting nodes adjacent to current matching is an augmenting path.



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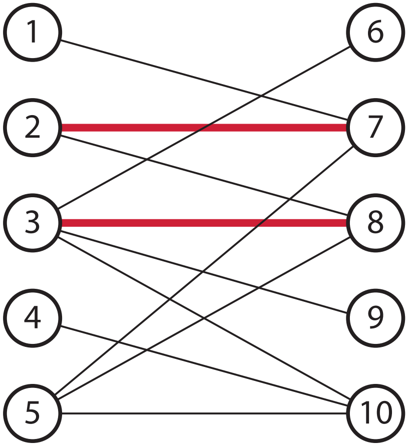
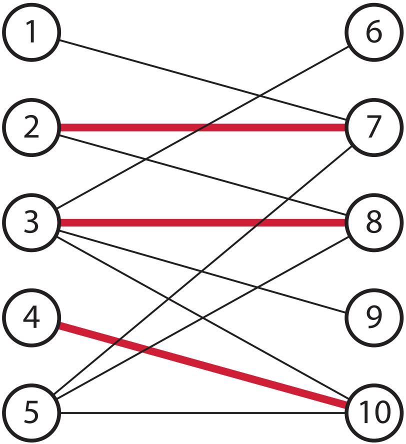


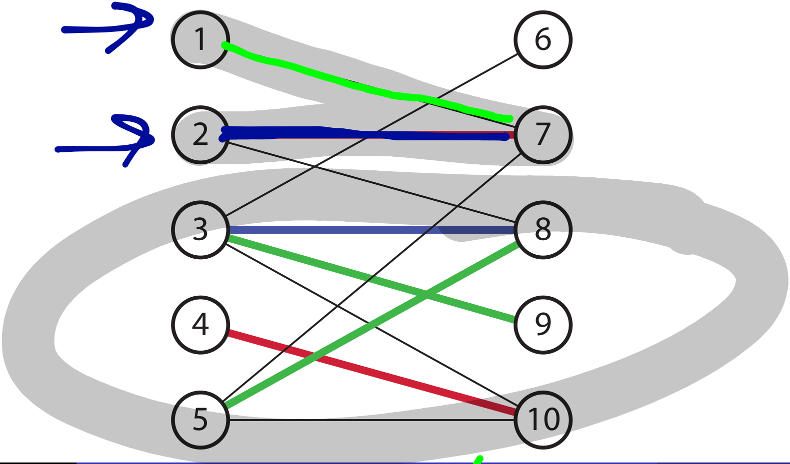
Bipartite Matching Example

No possible further single edge addition at this point. We need a multi-edge augmenting path if it exists.



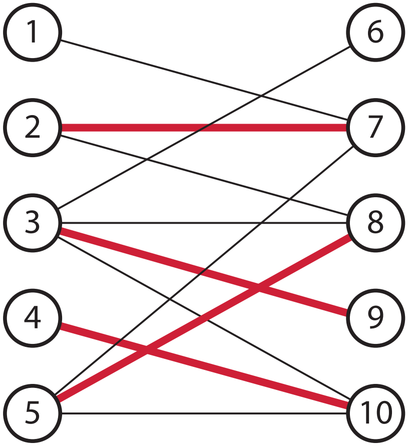
Bipartite Matching Example

Augmenting path is green and blue edges (blue is already in matching, green is new).



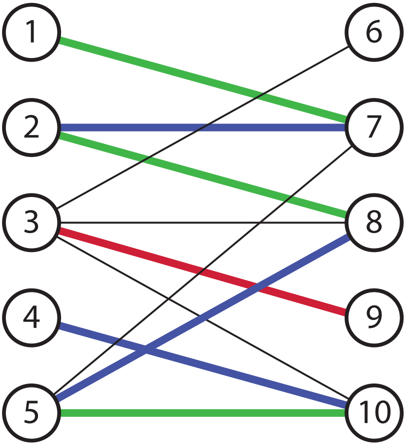
Bipartite Matching Example

Removing blue from matching and adding green leads to higher cardinality matching.



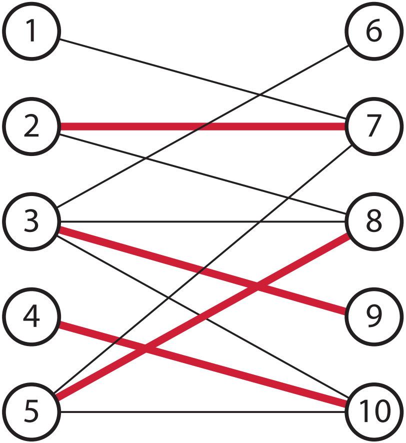
Bipartite Matching Example

At this point, resulting alternating path is not augmenting, since it is not between two unmatched vertices (and no augmenting path is possible).



Bipartite Matching Example

At this point, resulting alternating path is not augmenting, since it is not between two unmatched vertices (and no augmenting path is possible). At this point, matching is maximum cardinality.



Review

- The next slide is from lecture 7 and the one after from lecture 5.

Matroid Intersection

- Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$.
- While $(V, \mathcal{I}_1 \cap \mathcal{I}_2)$ is typically not a matroid (Exercise: show graphical example.), we might be interested in finding the maximum size common independent set. That is, find $\max |X|$ such that both $X \in \mathcal{I}_1$ and $X \in \mathcal{I}_2$.

Theorem 11.5.5

Let M_1 and M_2 be given as above, with rank functions r_1 and r_2. Then the size of the maximum size set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by

$$(r_1 * r_2)(V) \triangleq \min_{X \subseteq V} \Big(r_1(X) + r_2(V \setminus X) \Big) \tag{11.7}$$

This is an instance of the convolution of two submodular functions, f_1 and f_2 that, evaluated at $Y \subseteq V$, is written as:

$$(f_1 * f_2)(Y) = \min_{X \subseteq Y} \Big(f_1(X) + f_2(Y \setminus X) \Big) \tag{11.8}$$

Partition Matroid

- Let V be our ground set.
- Let $V = V_1 \cup V_2 \cup \dots \cup V_\ell$ be a partition of V into blocks or disjoint sets (disjoint union). Define a set of subsets of V as

$$\mathcal{I} = \{X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \dots, \ell\}. \tag{11.3}$$

  where $k_1, \dots, k_\ell$ are fixed parameters, $k_i \geq 0$. Then $M = (V, \mathcal{I})$ is a matroid.
- Note that a k-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$.
- We'll show that property (I3') in Def **??** holds. If $X, Y \in \mathcal{I}$ with $|Y| > |X|$, then there must be at least one i with $|Y \cap V_i| > |X \cap V_i|$. Therefore, adding one element $e \in V_i \cap (Y \setminus X)$ to X won't break independence.

Matroid Intersection and Bipartite Matching

- Why might we want to do matroid intersection?

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- Consider bipartite graph $G = (V, F, E)$. Define two partition matroids $M_V = (E, \mathcal{I}_V)$, and $M_F = (E, \mathcal{I}_F)$.

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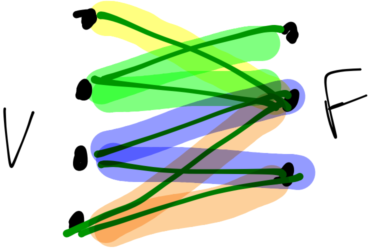
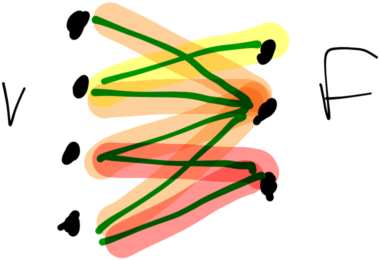
Partition based on left vertices.

partition of E corresponding to right vertices





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- Therefore, a matching in G is simultaneously independent in both M_V and M_F and finding the maximum matching is finding the maximum cardinality set independent in both matroids.

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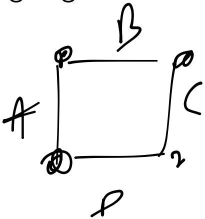
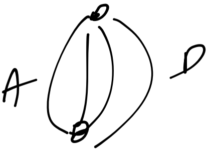
- Therefore, a matching in G is simultaneously independent in both M_V and M_F and finding the maximum matching is finding the maximum cardinality set independent in both matroids.
- For the bipartite graph case, therefore, this can be solved in polynomial time.

Matroid Intersection and Network Communication

- Let $G_1 = (V_1, E)$ and $G_2 = (V_2, E)$ be two graphs on the same underlying edges.





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- We may wish to find the maximum size edge-induced subgraph that is still forest in both graphs (i.e., adding any edges will create a circuit in either M_1, M_2, or both).
- This is again a matroid intersection problem.

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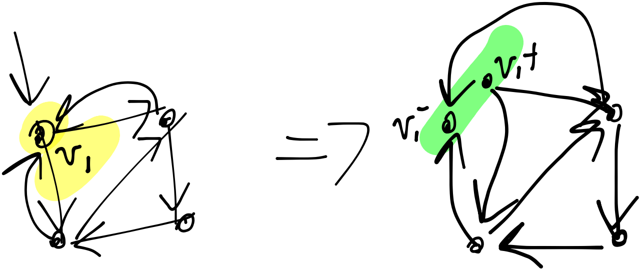
- From G with n nodes, create G' with $n+1$ nodes by duplicating (w.l.o.g.) a particular node $v_1 \in V(G)$ to v_1^+, v_1^-, and have all outgoing edges from v_1 come instead from v_1^+ and all edges incoming to v_1 go instead to v_1^-.



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- Let M_1 be the cycle matroid on G'.
- Let M_2 be the partition matroid having as independent sets those that have no more than one edge leaving any node — i.e., $I \in \mathcal{I}(M_2)$ if $|I \cap \delta^+(v)| \leq 1$ for all $v \in V(G')$.

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- Then a Hamiltonian cycle exists iff there is an n-element intersection of M_1, M_2, and M_3.

Matroid Intersection and TSP

- Since TSP is NP-complete, we obviously can't solve matroid intersections of 3 more matroids, unless P=NP.

Matroid Intersection and TSP

- Since TSP is NP-complete, we obviously can't solve matroid intersections of 3 more matroids, unless P=NP.
- But bipartite graph example gives us hope for 2 matroids, and also ideas for an algorithm ...

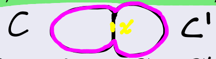
Recall from Lecture 5: Matroids by circuits

A set is independent if and only if it contains no circuit. Therefore, it is not surprising that circuits can also characterize a matroid.

Theorem 11.5.1

Matroid (by circuits) Let E be a set and $\mathcal{C}$ be a collection of nonempty subsets of E, such that no two sets in $\mathcal{C}$ are contained in each other. Then the following are equivalent.

1. *(C1) $\mathcal{C}$ is the collection of circuits of a matroid;*
2. *(C2) if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$;*



3. *(C3) if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, and $y \in C \setminus C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$ containing y;*

Fundamental circuits in matroids

Lemma 11.5.2

Let $I \in \mathcal{I}(M)$, and $e \in E$, then $I \cup \{e\}$ contains at most one circuit in M.

Proof.

$\square$

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□

In general, let $C(I, e)$ be the unique circuit associated with $I \cup \{e\}$ (commonly called the fundamental circuit in M w.r.t. I and e).

Matroid Intersection Algorithm Idea

- Consider two matroids $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ and start with any $I \in \mathcal{I}_1 \cap \mathcal{I}_2$.

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- If $I + v_1 \notin \mathcal{I}_2$, $\exists C_2(I, v_1)$ a circuit in M_2, and choosing $v_2 \in C_2(I, v_1)$ s.t. $v_2 \neq v_1$ leads to $I + v_1 - v_2$ which (because $\text{span}_2(I) = \text{span}_2(I + v_1 - v_2)$) is again independent in M_2. $I + v_1 - v_2$ is also independent in M_1. $\Leftarrow$ subdivision.

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- Then $\text{span}_1(I) = \text{span}_1(I - v_2 + v_3)$.

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- Then $\operatorname{span}_1(I) = \operatorname{span}_1(I - v_2 + v_3)$.
- Moreover, since $I + v_1 \in \mathcal{I}_1$, $v_1 \notin \operatorname{span}_1(I)$, so $\operatorname{span}_1(I + v_1) = \operatorname{span}_1(I + v_1 - v_2 + v_3)$.

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- But $I + v_1 - v_2 + v_3$ might not be independent in M_2 again, so we need to find an $v_4 \in C_2(I + v_1 - v_2, v_3)$ to remove, and so on.

Matroid Intersection Algorithm Idea

- Hopefully (eventually) we'll find an odd length sequence $S = (v_1, v_2, \ldots, v_s)$ such that we will be independent in both M_1 and M_2 and thus be one greater in size than I.

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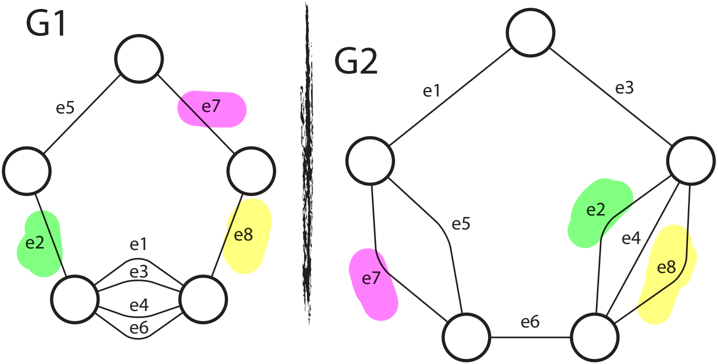
- Hopefully (eventually) we'll find an odd length sequence $S = (v_1, v_2, \ldots, v_s)$ such that we will be independent in both M_1 and M_2 and thus be one greater in size than I.
- We will have $v_i \notin I$ for i odd (will be shown in blue), and will have $v_i \in I$ for i even (will be shown in green), while $v \in I \setminus S$ will be shown in red.
- We then replace I with $I \ominus S$ (quite analogous to the bipartite matching case), and start again.

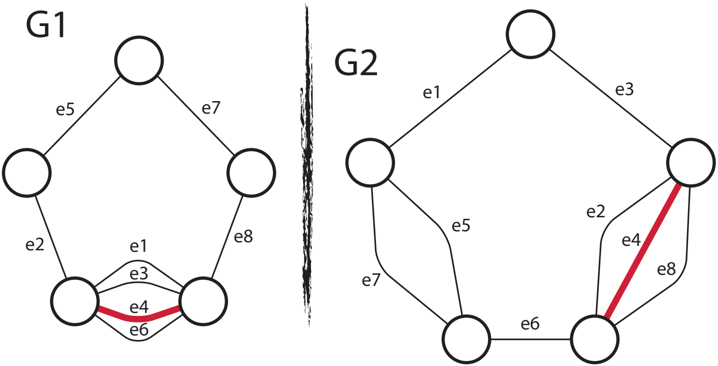
Graphic Matroid Intersection Example

Consider the following two graph $G_1 = (V_1, E)$ and $G_2 = (V_2, E)$ and corresponding matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$. Any edge is independent in both (an augmenting "sequence") since a single edge can't create a circuit starting at $I = \emptyset$. We start with e_4.



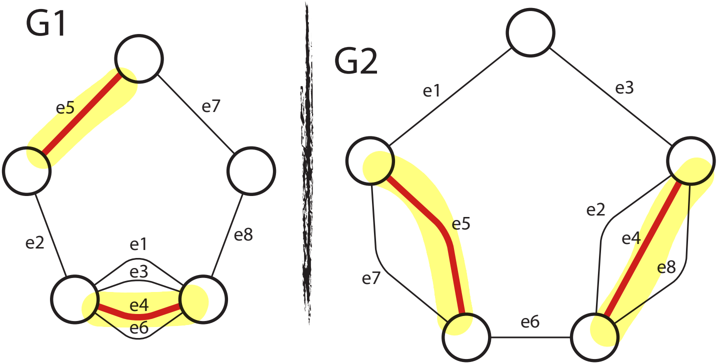
Graphic Matroid Intersection Example

Setting $I \leftarrow e_4$ with edge e_4 creates a circuit neither in M_1 nor M_2. We can add another single edge w/o creating a circuit in either matroid.



Graphic Matroid Intersection Example

$e_5 \in E - \text{span}_1(\{e_4\})$. Then, after $I \leftarrow I + e_5$, (i.e., when $I = \{e_4, e_5\}$) we're still independent in M_2, but no further single edge additions possible w/o creating a circuit (why?).



Graphic Matroid Intersection Example

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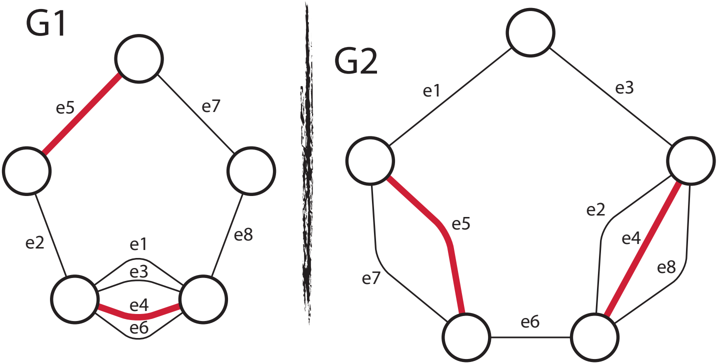
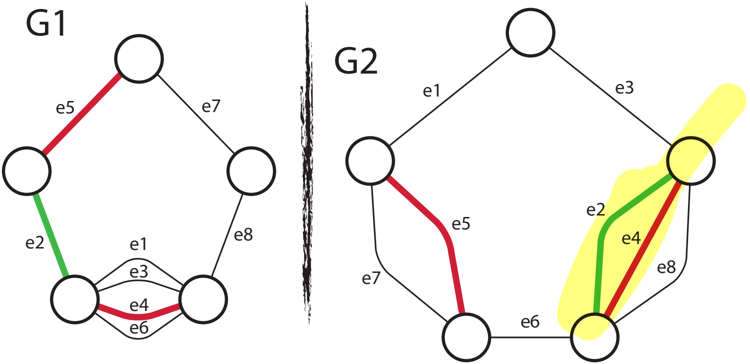


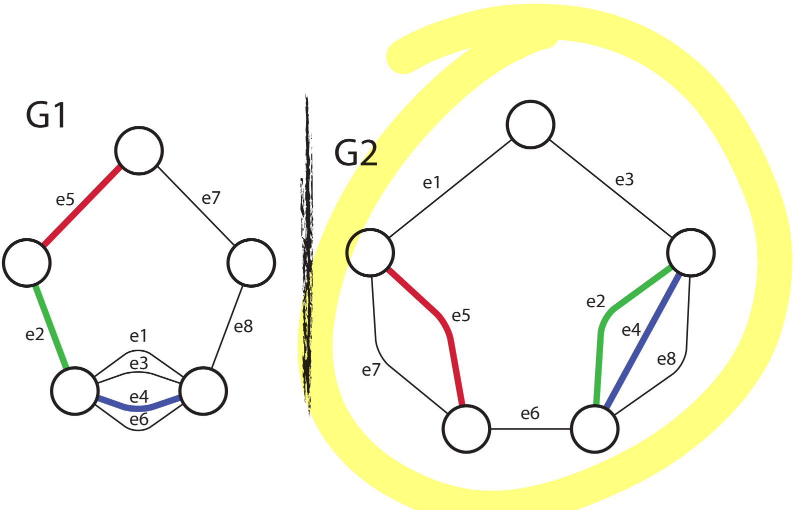
Graphic Matroid Intersection Example

Augmenting sequence is green and blue edges (blue is already in I, green is new). We choose $e_2 \in E - \text{span}_1(I)$, but now $I + e_2$ is not independent in M_2.



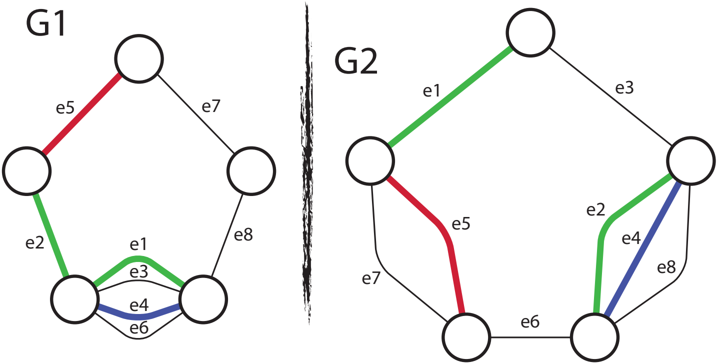
Graphic Matroid Intersection Example

So there must exist $C_2(I, e_2)$. We choose $e_4 \in C_2(I, e_2)$ to remove.



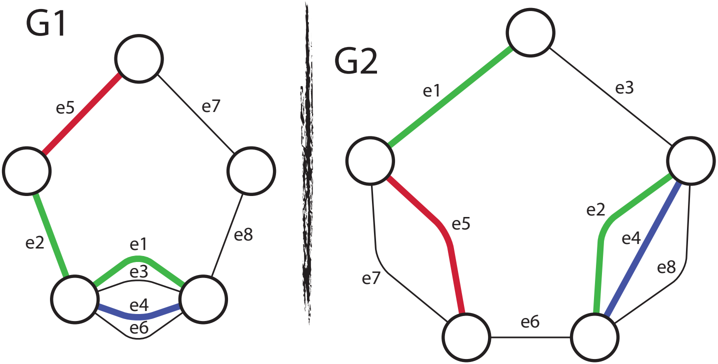
Graphic Matroid Intersection Example

Next, we choose $e_1 \in \text{span}_1(I) - \text{span}_1(I - e_4)$ to add.



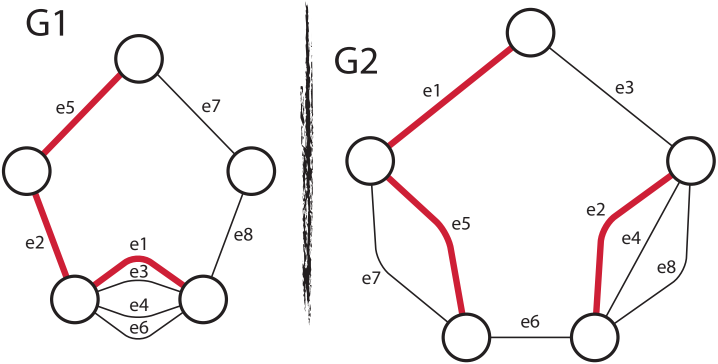
Graphic Matroid Intersection Example

Next, we choose $e_1 \in \operatorname{span}_1(I) - \operatorname{span}_1(I - e_4)$ to add. In this case, we not only have $\operatorname{span}_1(I + e_2) = \operatorname{span}_1(I + e_2 - e_4 + e_1)$, but we also have that $(I + e_2 - e_4) + e_1 \in \mathcal{I}_2$.



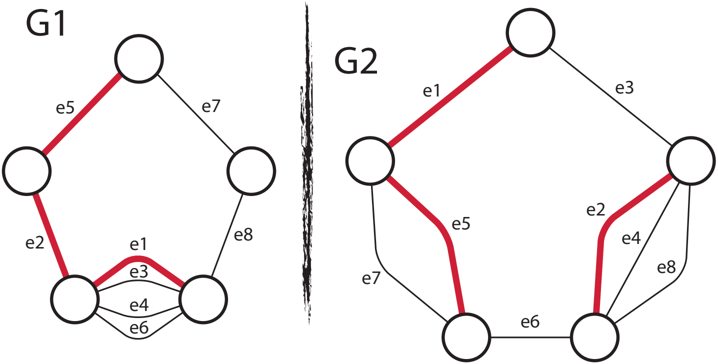
Graphic Matroid Intersection Example

Removing blue and adding green leads to higher cardinality independent set in both matroids. This corresponds to doing $I \leftarrow I \ominus S$ where $S = (e_2, e_4, e_1)$ and $I = \{e_4, e_5\}$.



Graphic Matroid Intersection Example

At this point, are any further augmenting sequences possible? Exercise.



Alternating and Augmenting Sequences

- Let I be an intersection of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ (i.e., $I \in \mathcal{I}_1 \cap \mathcal{I}_2$).

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- Let $S = (e_1, e_2, \ldots, e_s)$ be a sequence of distinct elements, where $e_i \in E - I$ for i odd, and $e_i \in I$ for i even, and let $S_i = (e_1, e_2, \ldots, e_i)$. We say that S is an alternating sequence w.r.t. I if the following are true.

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    1. $I + e_1 \in \mathcal{I}_1$

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  1. $I + e_1 \in \mathcal{I}_1$
  2. For all even i, $\text{span}_2(I \ominus S_i) = \text{span}_2(I)$ which implies that $I \ominus S_i \in \mathcal{I}_2$.

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  3. For all odd i, $\text{span}_1(I \ominus S_i) = \text{span}_1(I + e_1)$, and therefore $I \ominus S_i \in \mathcal{I}_1$.

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    3. For all odd i, $\text{span}_1(I \ominus S_i) = \text{span}_1(I + e_1)$, and therefore $I \ominus S_i \in \mathcal{I}_1$.
- Lastly, if also, $|S| = s$ is odd, and $I \ominus S \in \mathcal{I}_2$, then S is called an augmenting sequence w.r.t. I.

Alternating and Augmenting Sequences

- If I admits an augmenting sequence S, then the above argument shows that $I \ominus S$ is independent in M_1, independent in M_2, and also we have that $|I| + 1 = |I \ominus S|$.

Alternating and Augmenting Sequences

- If I admits an augmenting sequence S, then the above argument shows that $I \ominus S$ is independent in M_1, independent in M_2, and also we have that $|I| + 1 = |I \ominus S|$.
- Thus, by finding augmenting sequences, we can increase the size of the matroid intersection until we stop. Moreover, we have:

Proposition 11.5.3

If there is an augmenting sequence, then the intersection is not maximum.

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If there is an augmenting sequence, then the intersection is not maximum.

- We next wish to show that, if the intersection is not maximum, then there is an augmenting sequence.

Border graphs

- We construct an auxiliary directed bipartite graph (Border graph) $B(I) = (E \setminus I, I, Z)$, relative to the current I, that will help us with this problem. The graph has only directed edges from $E \setminus I$ to I, or from I back to $E \setminus I$.

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- Left-going edges: For each $e_i \in \text{span}_1(I) \setminus I$, create $e_i \leftarrow e_j$ directed edge $(e_j, e_i) \in Z$ from all $e_j \in C_1(I, e_i) \setminus \{e_i\}$. Note $e_j \in I$ and $e_i \in E \setminus I$.

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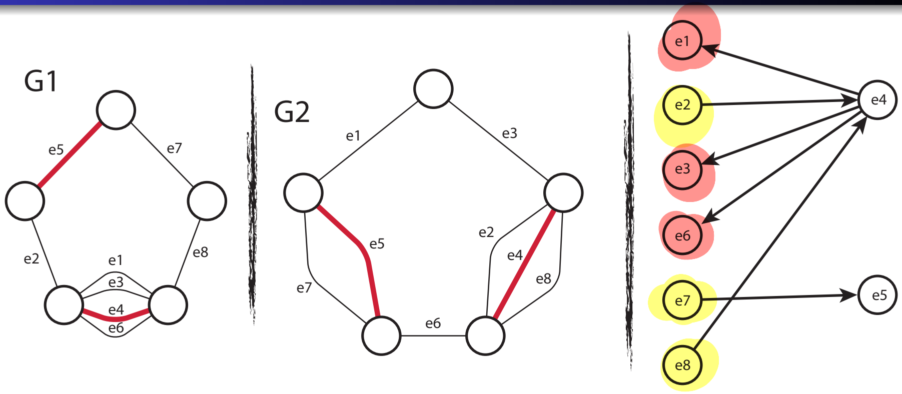
Border graph Example



- $\{e_2, e_7, e_8\}$ are sources and $\{e_1, e_3, e_6\}$ are sinks. $I = \{e_4, e_5\}$. $\text{span}_1(I) \setminus I = \{e_1, e_3, e_6\}$ and $\text{span}_2(I) \setminus I = \{e_7, e_2, e_8\}$

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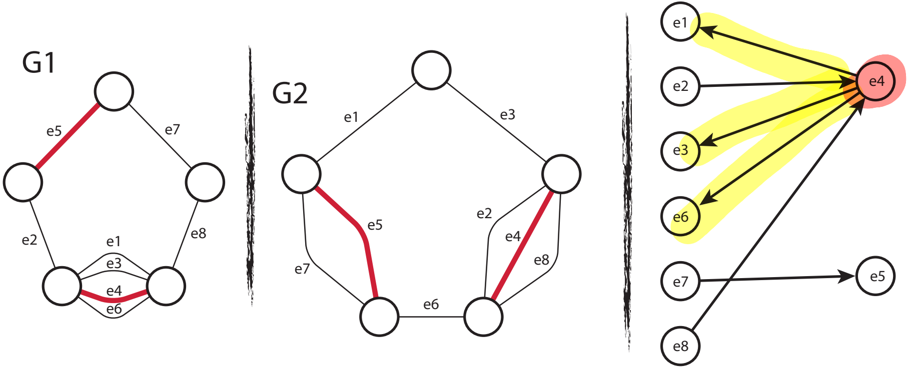

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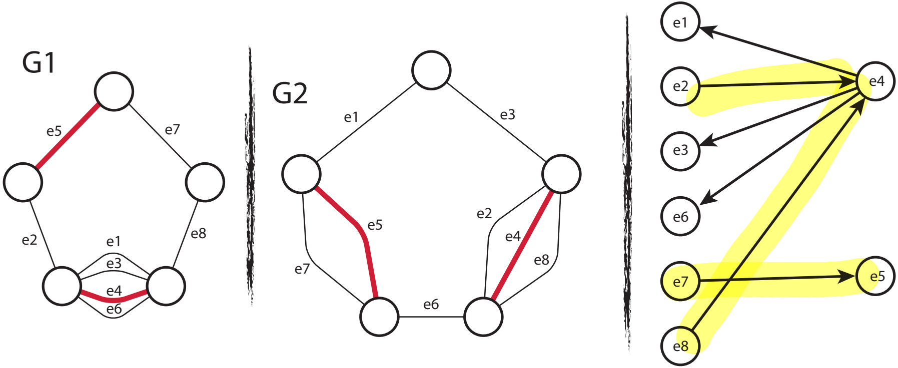
Border graph Example



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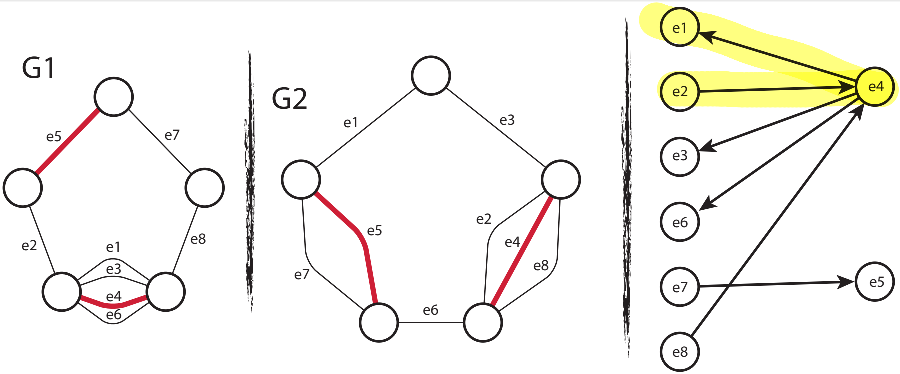
Border graph Example



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- Augmenting sequences are (e_2, e_4, e_1), (e_2, e_4, e_3), and (e_2, e_4, e_6), all dipaths in the Border graph.

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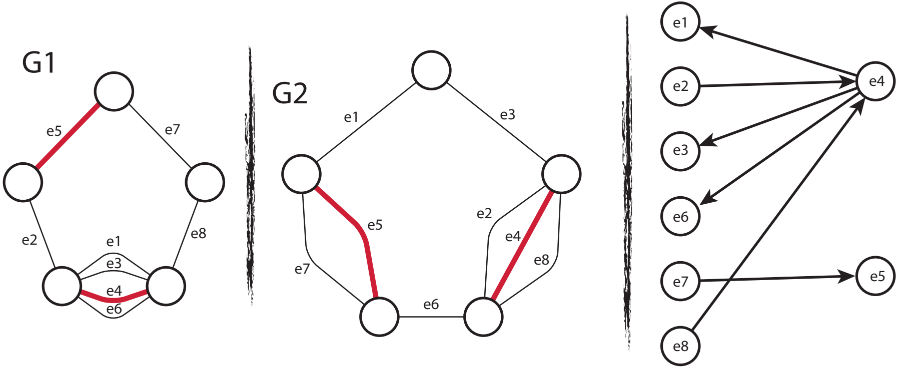

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- Augmenting sequences are (e_2, e_4, e_1), (e_2, e_4, e_3), and (e_2, e_4, e_6), all dipaths in the Border graph. Exercise: Are there others?

Identifying Augmenting Sequences

Lemma 11.5.4

If S is a source-sink path in $B(I)$, and there is no shorter source-sink path between the same source and sink (i.e., there are no short-cuts), then S is an augmenting sequence w.r.t. I.

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Lemma 11.5.5

Let I and J be matroid intersections of M_1 and M_2 such that $|I| + 1 = |J|$. Then there exists a source-sink path S in $B(I)$ where $S \subseteq I \ominus J$.

Identifying Augmenting Sequences

Theorem 11.5.6

Let I_p *and* I_{p+1} *be intersections of* M_1 *and* M_2 *with* p *and* $p+1$ *elements respectively. Then there exists an augmenting sequence* $S \subseteq I_p \ominus I_{p+1}$ *w.r.t.* I_p.

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An intersection is of maximum cardinality iff it admits no augmenting sequence.

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For any intersection I, there exists a maximum cardinality intersection I^* such that $\text{span}_1(I) \subseteq \text{span}_1(I^*)$ and $\text{span}_2(I) \subseteq \text{span}_2(I^*)$.

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All this can be made to run in poly time.

Matroid Partition Problem

- Suppose $M_i = (E, \mathcal{I}_i)$ is a matroid and that we have k of them on the same ground set E.

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- We wish to, if possible, partition E into k blocks, $I_i, i \in \{1, 2, \dots, k\}$ where $I_i \in \mathcal{I}_i$.
- Moreover, we want partition to be lexicographically maximum, that is $|I_1|$ is maximum, $|I_2|$ is maximum given $|I_1|$, and so on.

Matroid Partition Problem

Theorem 11.6.1

Let M_i be a collection of k matroids as described. Then, a set $I \subseteq E$ can be partitioned into k subsets $I_i, i = 1 \dots k$ where $I_i \in \mathcal{I}_i$ is independent in matroid i, if and only if, for all $A \subseteq I$

$$|A| \leq \sum_{i=1}^{k} r_i(A) \tag{11.4}$$

where r_i is the rank function of M_i.

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- But considering vector of all ones $\mathbf{1} \in \mathbb{R}_+^E$, this is the same as

$$\frac{1}{k}|A| = \frac{1}{k}\mathbf{1}(A) \leq r(A) \;\; \forall A \subseteq E \tag{11.6}$$

Matroid Partition Problem

- Recall definition of matroid polytope

$$P_r^+ = \{y \in \mathbb{R}_+^E : y(A) \le r(A) \text{ for all } A \subseteq E\} \tag{11.7}$$

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- Recall definition of matroid polytope

$$P_r^+ = \{y \in \mathbb{R}_+^E : y(A) \leq r(A) \text{ for all } A \subseteq E\} \tag{11.7}$$

- Then we see that this special case of the matroid partition problem is just testing if $\frac{1}{k}\mathbf{1} \in P_r^+$, a problem of testing the membership in matroid polyhedra.

Review

- The next two slides from respectively from Lecture 9 and Lecture 8.

Polymatroidal polyhedron (or a "polymatroid")

Definition 11.7.4 (polymatroid)

A polymatroid is a compact set $P \subseteq \mathbb{R}_+^E$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For every $x \in \mathbb{R}_+^E$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any P-basis of x), has the same component sum $y(E)$

- Vectors within P (i.e., any $y \in P$) are called independent, and any vector outside of P is called dependent.
- Since all P-bases of x have the same component sum, if $\mathcal{B}_x$ is the set of P-bases of x, than $\text{rank}(x) = y(E)$ for any $y \in \mathcal{B}_x$.

Maximum weight independent set via greedy weighted rank

Theorem 11.7.6

Let $M = (V, \mathcal{I})$ be a matroid, with rank function r, then for any weight function $w \in \mathbb{R}_+^V$, there exists a chain of sets $U_1 \subset U_2 \subset \cdots \subset U_n \subseteq V$ such that

$$\max \{w(I) | I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i) \tag{11.19}$$

where $\lambda_i \geq 0$ satisfy

$$w = \sum_{i=1}^{n} \lambda_i \mathbf{1}_{U_i} \tag{11.20}$$

Polymatroidal polyhedron and greedy

- Let $(E, \mathcal{I})$ be a set system and $w \in \mathbb{R}_+^E$ be a weight vector.

Polymatroidal polyhedron and greedy

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- For a matroid, we saw that set system $(E, \mathcal{I})$ is a matroid iff for each weight function $w \in \mathbb{R}_+^E$, the greedy algorithm leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.

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- Stated succinctly, considering $\max \{w(I) : I \in \mathcal{I}\}$, then $(E, \mathcal{I})$ is a matroid iff greedy works for this maximization.

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- Can we also characterize a polymatroid in this way?
- That is, if we consider $\max \left\{ wx : x \in P_f^+ \right\}$, where P_f^+ represents the "independent vectors", is it the case that P_f^+ is a polymatroid iff greedy works for this maximization?

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- Can we also characterize a polymatroid in this way?
- That is, if we consider $\max \left\{wx : x \in P_f^+\right\}$, where P_f^+ represents the "independent vectors", is it the case that P_f^+ is a polymatroid iff greedy works for this maximization?
- Can we even relax things so that $w \in \mathbb{R}^E$?

Polymatroidal polyhedron and greedy

- What is the greedy solution in this setting?

Polymatroidal polyhedron and greedy

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- Sort elements of E w.r.t. w so that, w.l.o.g. $E = (e_1, e_2, \dots, e_m)$ with $w(e_1) \geq w(e_2) \geq \dots \geq w(e_m)$.

Polymatroidal polyhedron and greedy

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- Let $k+1$ be the first point (if any) at which we are non-positive, i.e., $w(e_k) > 0$ and $0 \geq w(e_{k+1})$.
  That is, we have

$$w(e_1) \geq w(e_2) \geq \cdots \geq w(e_k) > 0 \geq w(e_{k+1}) \geq \cdots \geq w(e_m) \tag{11.8}$$

Polymatroidal polyhedron and greedy

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- Let $k+1$ be the first point (if any) at which we are non-positive, i.e., $w(e_k) > 0$ and $0 \geq w(e_{k+1})$.
- Next define partial accumulated sets E_i, for $i = 0 \dots m$, we have w.r.t. the above sorted order:

$$E_i \stackrel{\text{def}}{=} \{e_1, e_2, \dots e_i\} \tag{11.9}$$

(note $E_0 = \emptyset$, $f(E_0) = 0$, and E and E_i is always sorted w.r.t w).

Polymatroidal polyhedron and greedy

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(note $E_0 = \emptyset$, $f(E_0) = 0$, and $\underline{E \text{ and } E_i \text{ is always sorted w.r.t } w}$).
- The greedy solution is the vector $x \in \mathbb{R}_+^E$ with elements defined as:

$$x(e_1) \stackrel{\text{def}}{=} f(E_1) = f(e_1) = f(e_1|E_0) = f(e_1|\emptyset) \tag{11.10}$$

$$x(e_i) \stackrel{\text{def}}{=} f(E_i) - f(E_{i-1}) = f(e_i|E_{i-1}) \text{ for } i = 2 \ldots k \tag{11.11}$$

$$x(e_i) \stackrel{\text{def}}{=} 0 \text{ for } i = k+1 \ldots m = |E| \tag{11.12}$$

Some Intuition: greedy and gain

- Note $x(e_i) = f(e_i|E_{i-1}) \leq f(e_i|E')$ for any $E' \subseteq E_{i-1}$

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- Hence, for the largest value of w (namely $w(e_1)$), we use for $x(e_1)$ the largest possible gain value of e_1 (namely $f(e_1|\emptyset) \geq f(e_1|A)$ for any $A \subseteq E \setminus \{e_1\}$).

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- For the next largest value of w (namely $w(e_2)$), we use for $x(e_2)$ the next largest gain value of e_2 (namely $f(e_2|e_1)$). This still ensures (as we will soon see in Theorem 11.7.1) that the resulting $x \in P_f$.

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- Note $x(e_i) = f(e_i|E_{i-1}) \leq f(e_i|E')$ for any $E' \subseteq E_{i-1}$
- So $x(e_1) = f(e_1)$ and this corresponds to $w(e_1) \geq w(e_i)$ for all $i \neq 1$.
- Hence, for the largest value of w (namely $w(e_1)$), we use for $x(e_1)$ the largest possible gain value of e_1 (namely $f(e_1|\emptyset) \geq f(e_1|A)$ for any $A \subseteq E \setminus \{e_1\}$).
- For the next largest value of w (namely $w(e_2)$), we use for $x(e_2)$ the next largest gain value of e_2 (namely $f(e_2|e_1)$). This still ensures (as we will soon see in Theorem 11.7.1) that the resulting $x \in P_f$.
- This process continues, using the next largest possible gain of e_i for $x(e_i)$ while ensuring we do not leave the polytope, given the values we've already chosen for $x(e_{i'})$ for $i' < i$.

Polymatroidal polyhedron and greedy

Theorem 11.7.1

The vector $x \in \mathbb{R}_+^E$ as previously defined using the greedy algorithm maximizes wx over P_f^+.

Proof.

Polymatroidal polyhedron and greedy

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- Consider the LP strong duality equation:

$$\max(wx : x \in P_f^+) = \min\Big(\sum_{A \subseteq E} y_A f(A) : y \in \mathbb{R}_+^{2^E}, \sum_{A \subseteq E} y_A \mathbf{1}_A \geq w\Big) \tag{11.13}$$

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- Define the following vector $y \in \mathbb{R}_+^{2^E}$ as

$$y_{E_i} \leftarrow w(e_i) - w(e_{i+1}) \text{ for } i = 1 \ldots (m-1), \tag{11.14}$$

$$y_E \leftarrow w(e_m), \text{ and} \tag{11.15}$$

$$y_A \leftarrow 0 \text{ otherwise} \tag{11.16}$$

Polymatroidal polyhedron and greedy

Proof.

- We first will see that greedy $x \in P_f^+$ (that is $x(A) \leq f(A), \forall A$).

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Polymatroidal polyhedron and greedy

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- Order $A = (a_1, a_2, \dots, a_k)$ based on order $(e_1, e_2, \dots, e_m)$.

		a_1		a_2	a_3			a_4		a_5	$\dots$	
e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	$\dots$	e_m

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- Define $e^{-1} : E \to \{1, \ldots, m\}$ so that $e^{-1}(e_i) = i$.
  This means that $\forall j \leq k$

$$\{a_1, a_2, \ldots, a_j\} \subseteq \left\{e_1, e_2, \ldots, e_{e^{-1}(a_j)}\right\} \tag{11.17}$$

and

$$\{a_1, a_2, \ldots, a_{j-1}\} \subseteq \left\{e_1, e_2, \ldots, e_{e^{-1}(a_j)-1}\right\} \tag{11.18}$$

Also recall matlab notation: $a_{1:j} \equiv \{a_1, a_2, \ldots, a_j\}$.
E.g., with $j = 4$ *we get* $e^{-1}(a_4) = 9$, *and*

$$\{a_1, a_2, a_3, a_4\} \subseteq \{e_1, e_2, \ldots, e_9\} \tag{11.19}$$

Polymatroidal polyhedron and greedy

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- Define $e^{-1} : E \to \{1, \ldots, m\}$ so that $e^{-1}(e_i) = i$.
- Then, we have $x \in P_f^+$ since:

$$f(A) = \sum_{i=1}^{k} f(a_i | a_{1:i-1}) \tag{11.17}$$

$$\geq \sum_{i=1}^{k} f(a_i | e_{1:e^{-1}(a_i)-1}) \tag{11.18}$$

$$= \sum_{a \in A} f(a | e_{1:e^{-1}(a)-1}) = x(A) \tag{11.19}$$

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Polymatroidal polyhedron and greedy

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- Next, y is also feasible for the dual constraints in Eq. 11.13 since:

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Polymatroidal polyhedron and greedy

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Polymatroidal polyhedron and greedy

Proof.

- Next, y is also feasible for the dual constraints in Eq. 11.13 since:
- clearly, $y \geq 0$;
- also, considering y component wise, for any i, we have that

$$\sum_{A: e_i \in A} y_A = \sum_{j \geq i} y_{E_j} = \sum_{j=i}^{m-1} (w(e_j) - w(e_{j+1})) + w(e_m) = w(e_i)$$

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Polymatroidal polyhedron and greedy

Proof.

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- clearly, $y \geq 0$;
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- Now optimality for x and y follows from strong duality, i.e.:

$$wx = \sum_{e \in E} w(e)x(e) = \sum_{e \in E} w(e) f(e_i | E_{i-1}) = \sum_{i=1}^{m} w(e_i)\Big(f(E_i) - f(E_{i-1})\Big)$$

$$= \sum_{i=1}^{n-1} f(E_i)\Big(w(e_i) - w(e_{i+1})\Big) + f(E)w(e_m) = \sum_{A \subseteq E} y_A f(A)$$

...

Polymatroidal polyhedron and greedy

Proof.

- The penultimate equality (in Eq. 11.2) follows via Abel summation:

$$wx = \sum_{i=1}^{m} w_i x_i \tag{11.20}$$

$$= \sum_{i=1}^{m} w_i \Big(f(E_i) - f(E_{i-1}) \Big) \tag{11.21}$$

$$= \sum_{i=1}^{m} w_i f(E_i) - \sum_{i=1}^{m-1} w_{i+1} f(E_i) \tag{11.22}$$

$$= w_m f(E_m) + \sum_{i=1}^{m-1} (w_i - w_{i+1}) f(E_i) \tag{11.23}$$

□

Polymatroidal polyhedron and greedy

Theorem 11.7.1

Conversely, suppose P_f^+ is a polytope of form $P_f^+ = \{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\}$, then the greedy solution to $\max(wx : x \in P)$ is optimum only if f is submodular.

Proof.

- Order elements of E arbitrarily as $(e_1, e_2, \ldots, e_m)$ and define $E_i = (e_1, e_2, \ldots, e_i)$.

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- For $1 \le p \le q \le m$, define $A = \{e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_p\} = E_p$ and $B = \{e_1, e_2, \ldots, e_k, e_{p+1}, \ldots, e_q\} = E_k \cup (E_q \setminus E_p)$

Polymatroidal polyhedron and greedy

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- Note, then we have $A \cap B = \{e_1, \dots, e_k\} = E_k$, and $A \cup B = E_q$.

Polymatroidal polyhedron and greedy

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- Note, then we have $A \cap B = \{e_1, \dots, e_k\} = E_k$, and $A \cup B = E_q$.
- Define $w \in \{0,1\}^m$ as:

$$w \stackrel{\text{def}}{=} \sum_{i=1}^{q} \mathbf{1}_{e_i} = \mathbf{1}_{A \cup B} \tag{11.24}$$

Polymatroidal polyhedron and greedy

Theorem 11.7.1

Conversely, suppose P_f^+ *is a polytope of form* $P_f^+ = \{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\}$*, then the greedy solution to* $\max(wx : x \in P)$ *is optimum only if* f *is submodular.*

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- For $1 \leq p \leq q \leq m$, define $A = \{e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_p\} = E_p$ and $B = \{e_1, e_2, \ldots, e_k, e_{p+1}, \ldots, e_q\} = E_k \cup (E_q \setminus E_p)$
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- Define $w \in \{0,1\}^m$ as:

$$w \stackrel{\text{def}}{=} \sum_{i=1}^{q} \mathbf{1}_{e_i} = \mathbf{1}_{A \cup B} \tag{11.24}$$

- Suppose optimum solution x is given by the greedy procedure.

Polymatroidal polyhedron and greedy

Proof.

- Then

$$\sum_{i=1}^{k} x_i = f(E_1) + \sum_{i=2}^{k} (f(E_i) - f(E_{i-1})) = f(E_k) = f(A \cap B) \tag{11.25}$$

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Polymatroidal polyhedron and greedy

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- and

$$\sum_{i=1}^{p} x_i = f(E_1) + \sum_{i=2}^{p}(f(E_i) - f(E_{i-1})) = f(E_p) = f(A) \tag{11.26}$$

...

Polymatroidal polyhedron and greedy

Proof.

- Then

$$\sum_{i=1}^{k} x_i = f(E_1) + \sum_{i=2}^{k}(f(E_i) - f(E_{i-1})) = f(E_k) = f(A \cap B) \tag{11.25}$$

- and

$$\sum_{i=1}^{p} x_i = f(E_1) + \sum_{i=2}^{p}(f(E_i) - f(E_{i-1})) = f(E_p) = f(A) \tag{11.26}$$

- and

$$\sum_{i=1}^{q} x_i = f(E_1) + \sum_{i=2}^{q}(f(E_i) - f(E_{i-1})) = f(E_q) = f(A \cup B) \tag{11.27}$$

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Polymatroidal polyhedron and greedy

Proof.

- Thus, we have

$$x(B) = \sum_{i \in 1,\ldots,k,p+1,\ldots,q} x_i = \sum_{i : e_i \in B} x_i = f(A \cup B) + f(A \cap B) - f(A) \tag{11.28}$$

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- Thus,

$$x(B) = f(A \cup B) + f(A \cap B) - f(A) = \sum_{i : e_i \in B} x_i \leq f(B) \tag{11.29}$$

ensuring the submodularity of f, since A and B are arbitrary.

□

Polymatroidal polyhedron and greedy

- Thus, restating the above results into a single complete theorem, we have a result very similar to what we saw for matroids (i.e., Theorem 8.6.1)

Theorem 11.7.1

If $f : 2^E \to \mathbb{R}_+$ is given, and P is a polytope in $\mathbb{R}_+^E$ of the form $P = \left\{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\right\}$, then the greedy solution to the problem $\max(wx : x \in P)$ is $\forall w$ optimum iff f is monotone non-decreasing submodular (i.e., iff P is a polymatroid).

Multiple Polytopes associated with arbitrary f

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  Note that due to constraint $x(\emptyset) \leq f(\emptyset)$, we must have $f(\emptyset) \geq 0$ since if not (i.e., if $f(\emptyset) < 0$), then P_f^+ doesn't exist.
  Another form of normalization can do is:

$$f'(A) = \begin{cases} f(A) & \textit{if } A \neq \emptyset \\ 0 & \textit{if } A = \emptyset \end{cases} \tag{11.30}$$

  This preserves submodularity due to $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$, and if $A \cap B = \emptyset$ then r.h.s. only gets smaller when $f(\emptyset) \geq 0$.

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- We can define several polytopes:

$$P_f = \left\{x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E\right\} \tag{11.30}$$

$$P_f^+ = P_f \cap \left\{x \in \mathbb{R}^E : x \geq 0\right\} \tag{11.31}$$

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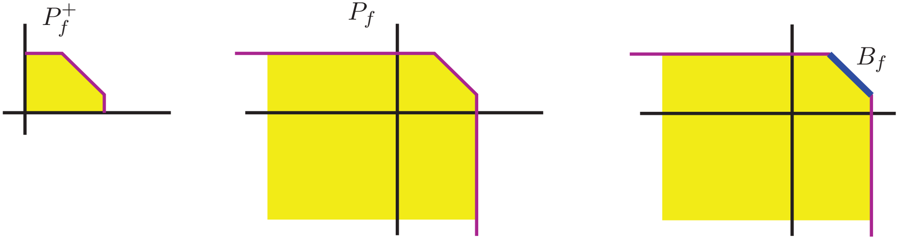
- B_f is called the base polytope

Multiple Polytopes associated with f



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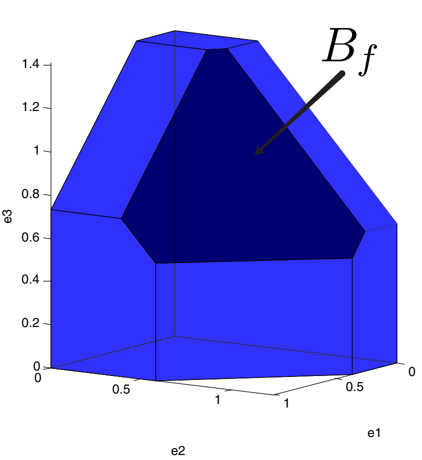
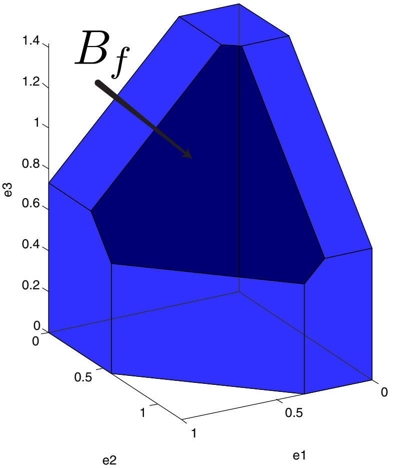
Base Polytope in 3D





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A polymatroid function's polyhedron is a polymatroid.

Theorem 11.8.1

Let f be a submodular function defined on subsets of E. For any $x \in \mathbb{R}^E$, we have:

$$\max\left(y(E) : y \leq x, y \in P_f\right) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \quad (11.38)$$

If we take x to be zero, we get:

Corollary 11.8.2

Let f be a submodular function defined on subsets of E. $x \in \mathbb{R}^E$, we have:

$$\max\left(y(E) : y \leq 0, y \in P_f\right) = \min\left(f(A) : A \subseteq E\right) \quad (11.39)$$

Greedy and P_f

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- In fact, we next will see that the greedy x is a vertex of B_f.

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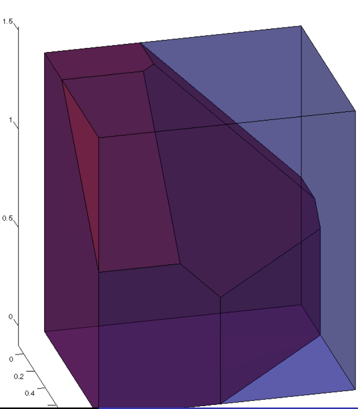

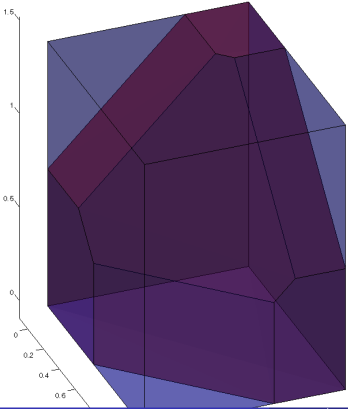

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$$B_f = P_f \cap \left\{x \in \mathbb{R}_+^E : x(E) = f(E)\right\} \tag{11.40}$$

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- We formalize this next:

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- Given any arbitrary order of $E = (e_1, e_2, \ldots, e_m)$, define $E_i = (e_1, e_2, \ldots, e_i)$.

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- An extreme point of P_f is a point that is not a convex combination of two other distinct points in P_f. Equivalently, an extreme point corresponds to setting certain inequalities in the specification of P_f to be equalities, so that there is a unique single point solution.

Polymatroid extreme points

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For a given ordering $E = (e_1, \ldots, e_m)$ of E and a given E_i and x generated by E_i using the greedy procedure, then x is an extreme point of P_f

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Proof.

- We already saw that $x \in P_f$ (Theorem 11.7.1).
- To show that x is an extreme point of P_f, note that it is the unique solution of the following system of equations

$$x(E_j) = f(E_j) \text{ for } 1 \leq j \leq i \leq m \tag{11.44}$$

$$x(e) = 0 \text{ for } e \in E \setminus E_i \tag{11.45}$$

There are $i \leq m$ equations and $i \leq m$ unknowns, and simple Gaussian elimination gives us back the x constructed via the Greedy algorithm!!

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- Also, since $x \in P_f$, for each i, we see that,

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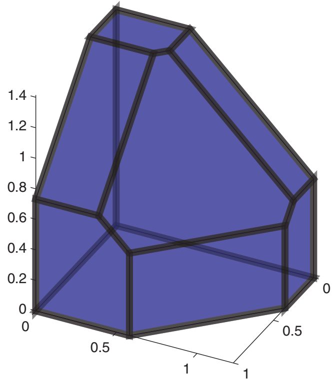
- Thus, the greedy procedure provides a modular function lower bound on f that is tight on all points E_i in the order. This can be useful in its own right.

Polymatroid extreme points

some examples



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- Moreover, we have (and will ultimately prove)

Corollary 11.9.2

If x is an extreme point of P_f and $B \subseteq E$ is given such that $\text{supp}(x) = \{e \in E : x(e) \neq 0\} \subseteq B \subseteq \cup(A : x(A) = f(A)) = \text{sat}(x)$, *then x is generated using greedy by some ordering of B.*

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- Note, $\text{sat}(x) = \text{cl}(x) = \cup(A : x(A) = f(A))$ is also called the closure of x (recall that sets A such that $x(A) = f(A)$ are called tight, and such sets are closed under union and intersection, as seen in Lecture 8, Theorem **??**)

Polymatroid extreme points

- Moreover, we have (and will ultimately prove)

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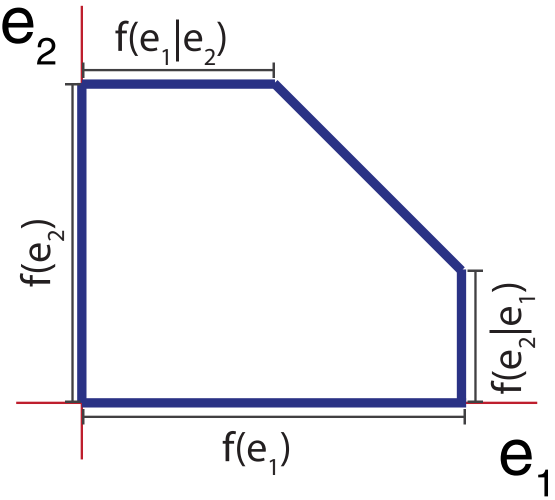
- For arbitrary x, $\text{supp}(x)$ is not necessarily tight, but for an extreme point, $\text{supp}(x)$ is.

Polymatroid with labeled edge lengths

- Recall $f(e|A) = f(A+e) - f(A)$
- Notice how submodularity, $f(e|B) \leq f(e|A)$ for $A \subseteq B$, defines the shape of the polytope.
- In fact, we have strictness here $f(e|B) < f(e|A)$ for $A \subset B$.
- Also, consider how the greedy algorithm proceeds along the edges of the polytope.



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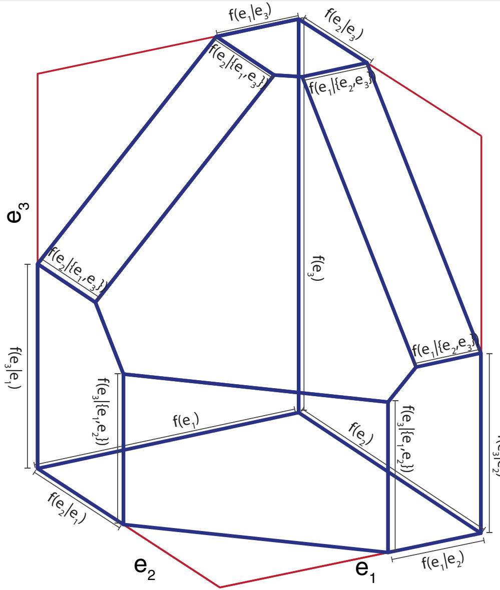

Intuition: why greedy works with polymatroids

- Given w, the goal is to find $x = (x(e_1), x(e_2))$ that maximizes $x^\intercal w = x(e_1)w(e_1) + x(e_2)w(e_2)$.
- If $w(e_2) > w(e_1)$ the upper extreme point indicated maximizes $x^\intercal w$ over $x \in P_f^+$.
- If $w(e_2) < w(e_1)$ the lower extreme point indicated maximizes $x^\intercal w$ over $x \in P_f^+$.