# Submodular Functions, Optimization, and Applications to Machine Learning

## — Spring Quarter, Lecture 11 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/


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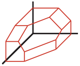
$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

$= f(A_r) + 2f(C) + f(B_r)$ $\quad = f(A_r) + f(C) + f(B_r)$ $\quad = f(A \cap B)$



# Cumulative Outstanding Reading

- Read chapters 1 and 2, and sections 3.1-3.2 from Fujishige's book.
- Good references for today: Schrijver-2003, Oxley-1992/2011, Welsh-1973, Goemans-2010, Cunningham-1984, Edmonds-1969.

# Announcements, Assignments, and Reminders

- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

# Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, Combinatorial Geometries
- L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
- L9: From Matroid Polytopes to Polymatroids.
- L10: Polymatroids and Submodularity
- L11: More properties of polymatroids, SFM special cases
- L12: polymatroid properties, extreme points polymatroids,
- L13: sat, dep, supp, exchange capacity, examples
- L14: Lattice theory: partially ordered sets; lattices; distributive, modular, submodular, and boolean lattices; ideals and join irreducibles.
- L15: Supp, Base polytope, polymatroids and entropic Venn diagrams, exchange capacity,
- L16: proof that minimum norm point yields min of submodular function, and the lattice of minimizers of a submodular function, Lovasz extension
- L17: Lovasz extension, Choquet Integration, more properties/examples of Lovasz extension, convex minimization and SFM.
- L18: Lovasz extension examples and structured convex norms, The Min-Norm Point Algorithm detailed.
- L19: symmetric submodular function minimization, maximizing monotone submodular function w. card constraints.
- L20: maximizing monotone submodular function w. other constraints, non-monotone maximization.

Finals Week: June 9th-13th, 2014.

# A polymatroid function's polyhedron is a polymatroid.

## Theorem 11.2.4

*Let $f$ be a polymatroid function defined on subsets of $E$. For any $x \in \mathbb{R}_+^E$, and any $P_f^+$-basis $y^x \in \mathbb{R}_+^E$ of $x$, the component sum of $y^x$ is*

$$y^x(E) = \textit{rank}(x) = \max\left(y(E) : y \leq x, y \in P_f^+\right) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \quad (11.34)$$

*As a consequence, $P_f^+$ is a polymatroid, since r.h.s. is constant w.r.t. $y^x$.*

By taking $B = \text{supp}(x)$ (so elements $E \setminus B$ are zero in $x$), and for $b \in B$, $x(b)$ is big enough, the r.h.s. min has solution $A^* = E \setminus B$. We recover submodular function from the polymatroid polyhedron via the following:

$$f(B) = \max\left\{y(B) : y \in P_f^+\right\} \quad (11.35)$$

In fact, we will ultimately see a number of important consequences of this theorem (other than just that $P_f^+$ is a polymatroid)

# Join $\vee$ and meet $\wedge$ for $x, y \in \mathbb{R}_+^E$

- For $x, y \in \mathbb{R}_+^E$, define vectors $x \wedge y \in \mathbb{R}_+^E$ and $x \vee y \in \mathbb{R}_+^E$ such that, for all $e \in E$

$$(x \vee y)(e) = \max(x(e), y(e)) \tag{11.18}$$

$$(x \wedge y)(e) = \min(x(e), y(e)) \tag{11.19}$$

Hence,

$$x \vee y \triangleq \Big(\max\big(x(e_1), y(e_1)\big), \max\big(x(e_2), y(e_2)\big), \dots, \max\big(x(e_n), y(e_n)\big)\Big)$$

and similarly

$$x \wedge y \triangleq \Big(\min\big(x(e_1), y(e_1)\big), \min\big(x(e_2), y(e_2)\big), \dots, \min\big(x(e_n), y(e_n)\big)\Big)$$

- From this, we can define things like an lattices, and other constructs.

# Vector rank, rank$(x)$, is submodular

- Recall that the matroid rank function is submodular.
- The vector rank function rank$(x)$ also satisfies a form of submodularity.

**Theorem 11.2.1 (vector rank and submodularity)**

*Let $P$ be a polymatroid polytope. The vector rank function rank : $\mathbb{R}_+^E \to \mathbb{R}$ with rank$(x) = \max\left(y(E) : y \leq x, y \in P\right)$ satisfies, for all $u, v \in \mathbb{R}_+^E$*

$$\text{rank}(u) + \text{rank}(v) \geq \text{rank}(u \vee v) + \text{rank}(u \wedge v) \tag{11.18}$$

# A polymatroid is a polymatroid function's polytope

- So, when $f$ is a polymatroid function, $P_f^+$ is a polymatroid.
- Is it the case that, conversely, for any polymatroid $P$, there is an associated polymatroidal function $f$ such that $P = P_f^+$?

## Theorem 11.2.1

*For any polymatroid $P$ (compact subset of $\mathbb{R}_+^E$, zero containing, down-monotone, and $\forall x \in \mathbb{R}_+^E$ any maximal independent subvector $y \leq x$ has same component sum $y(E) = \text{rank}(x)$), there is a polymatroid function $f : 2^E \to \mathbb{R}$ (normalized, monotone non-decreasing, submodular) such that $P = P_f^+$ where $P_f^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\}$.*

# Tight sets $\mathcal{D}(y)$ are closed, and max tight set $\mathrm{sat}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{A : A \subseteq E,\ y(A) = f(A)\} \tag{11.18}$$

## Theorem 11.2.1

*For any $y \in P_f^+$, with $f$ a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.*

## Proof.

We have already proven this as part of Theorem 9.4.5 □

Also recall the definition of $\mathrm{sat}(y)$, the maximal set of tight elements relative to $y \in \mathbb{R}_+^E$.

$$\mathrm{sat}(y) \stackrel{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\} \tag{11.19}$$

# A word on terminology & notation

- Recall how a matroid is sometimes given as $(E, r)$ where $r$ is the rank function.

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# A word on terminology & notation

- Recall how a matroid is sometimes given as $(E, r)$ where $r$ is the rank function.
- We mention also that the term "polymatroid" is sometimes not used for the polytope itself, but instead but for the pair $(E, f)$,
- But now we see that $(E, f)$ is equivalent to a polymatroid polytope, so this is sensible.

# Where are we going with this?

- Consider the right hand side of Theorem 9.4.5:
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- We are going to study this problem, and approaches that address it, as part of our ultimate goal which is to present strategies for submodular function minimization (that we will ultimately get to, in near future lectures).

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- We are going to study this problem, and approaches that address it, as part of our ultimate goal which is to present strategies for submodular function minimization (that we will ultimately get to, in near future lectures).
- As a bit of a hint on what's to come, recall that we can write it as: $x(E) + \min\left(f(A) - x(A) : A \subseteq E\right)$ where $f$ is a polymatroid function.

# Another Interesting Fact: Matroids from polymatroid functions

**Theorem 11.3.1**

*Given integral polymatroid function $f$, let $(E, \mathcal{F})$ be a set system with ground set $E$ and set of subsets $\mathcal{F}$ such that*

$$\forall F \in \mathcal{F}, \ \ \forall \emptyset \subset S \subseteq F, |S| \leq f(S) \tag{11.1}$$

*Then $M = (E, \mathcal{F})$ is a matroid.*

**Proof.**

Exercise □

And its rank function is Exercise.

# Matroid instance of Theorem 9.4.5

- Considering Theorem 9.4.5, the matroid case is now a special case, where we have that:

**Corollary 11.3.2**

*We have that:*

$$\max \{y(E) : y \in P_{\textit{ind. set}}(M), y \leq x\} = \min \{r_M(A) + x(E \setminus A) : A \subseteq E\} \tag{11.2}$$

where $r_M$ is the matroid rank function of some matroid.

# Most violated inequality problem in matroid polytope case

- Consider

$$P_r^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r_M(A), \forall A \subseteq E\} \tag{11.3}$$

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- This corresponds to $\min\{r_M(A) + x(E \setminus A) : A \subseteq E\}$ since $x$ is modular and $x(E \setminus A) = x(E) - x(A)$.
- More importantly, $\min\{r_M(A) + x(E \setminus A) : A \subseteq E\}$ a form of submodular function minimization, namely $\min\{r_M(A) - x(A) : A \subseteq E\}$ for a submodular function consisting of a difference of matroid rank and modular (so no longer necessarily monotone, nor positive).

# Problem to Solve

In particular, we will solve the following problem:

- Given a matroid $M = (E, \mathcal{I})$ along with an independence testing oracle (i.e., for any $A \subseteq E$, tells us if $A \in \mathcal{I}$ or not), and a vector $x \in \mathcal{R}_+^E$;

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- find: a maximizing $y \in P_{\text{ind. set}}$ with $y \leq x$, and moreover (as a byproduct of the algorithm), express $y$ as a convex combination of incidence vectors of independent sets in $M$, and also return a set $A \subseteq E$ that satisfies $y(E) = r_M(A) + x(E \setminus A)$. *Of course, by Theorem 9.4.5, for any such $y$ we must have that $y(E) \leq r(A) + x(E \setminus A)$.*

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- By Theorem 9.4.5, the existence of such an $A$ will certify that $y(E)$ is maximal in $P_{\text{ind. set}}$, $A$ is minimal in terms of $f(A) \stackrel{\text{def}}{=} r_M(A) - x(A)$ (thus most violated).

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- This can also be used to test membership in $P_{\text{ind. set}}$ (i.e., if $y = x$) depending on the sign of $f$ at $A$.
- This will also run in polynomial time.

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- We keep a family of independent sets $(I_i : i \in J)$ and coefficients $(\lambda_i : i \in J)$ such that $\sum_{i \in J} \lambda_i = 1$ and $y = \sum_{i \in J} \lambda_i \mathbf{1}_{I_i}$.

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- and the way these additions are done is via solutions to a max-flow problem in an associated flow-graph (which we'll describe).
- Each update will, of course, ensure that $y \in P_{\text{ind. set}}$, but also we'll keep $y \leq x$.
- It's going to take us a few lectures to fully develop this algorithm, so please keep in mind of the overall goal.

# Bipartite Matching

- Consider a bipartite graph $G = (V, F, E)$ where left nodes are $V$, right nodes are $F$, and $E \subseteq V \times F$ are the only edges.

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- An $A \subseteq E$ is an augmenting path if it is an alternating path between two unmatched vertices.

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- The algorithm becomes:

**Algorithm 8.1:** Alternating Path Bipartite Matching

Let $A$ be an arbitrary (including empty) matching in $G = (V, F, E)$ ;
**while** *There exists an augmenting path* $S$ *in* $G$ **do**
  $A \leftarrow A \ominus S$ ;

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**Algorithm 8.1:** Alternating Path Bipartite Matching

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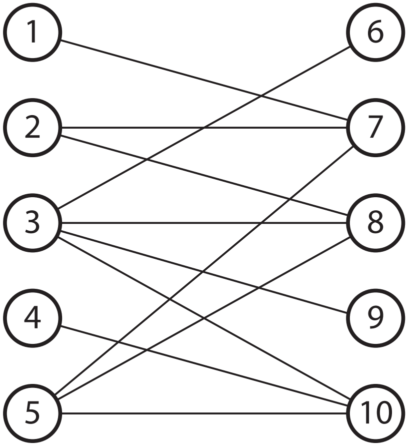
Let A be an arbitrary (including empty) matching in G = (V, F, E) ;
while There exists an augmenting path S in G do
    A ← A ⊖ S ;
```

- This can easily be made to run in $O(m^2 n)$, where $|V| = m$, $|F| = n$, $m \leq n$, but it can be made to run much faster as well (see Schrijver-2003).

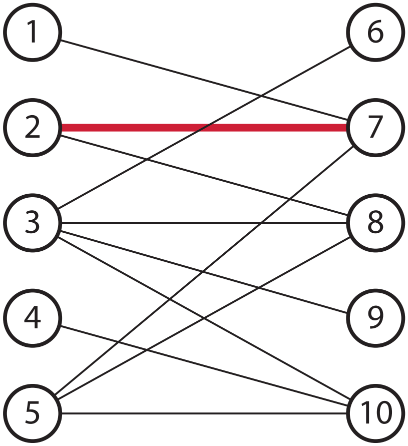
# Bipartite Matching Example

Consider the following bipartite graph $G = (V, F, E)$ with $|V| = |F| = 5$. Any edge is an augmenting path since it will adjoin two unmatched vertices.



# Bipartite Matching Example

Any edge, not intersecting nodes adjacent to current matching is an augmenting path.



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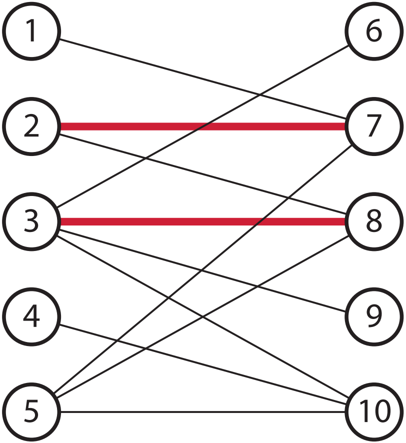
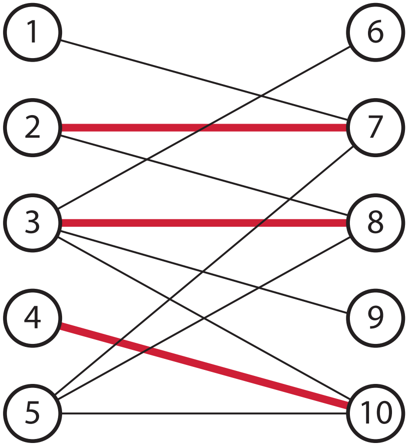


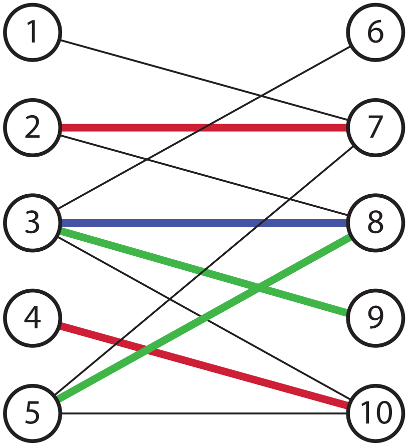
# Bipartite Matching Example

No possible further single edge addition at this point. We need a multi-edge augmenting path if it exists.



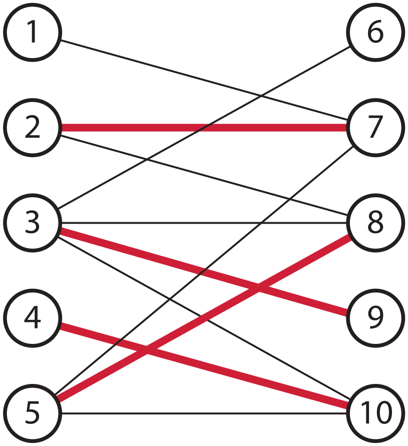
# Bipartite Matching Example

Augmenting path is green and blue edges (blue is already in matching, green is new).



# Bipartite Matching Example

Removing blue from matching and adding green leads to higher cardinality matching.



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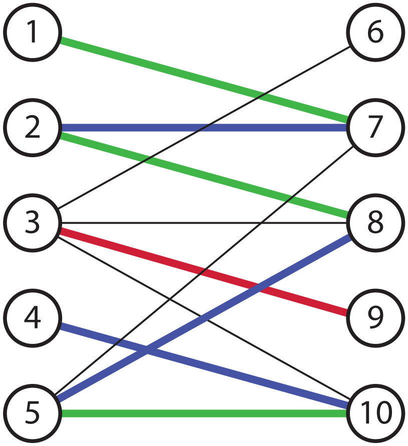
At this point, resulting alternating path is not augmenting, since it is not between two unmatched vertices (and no augmenting path is possible).



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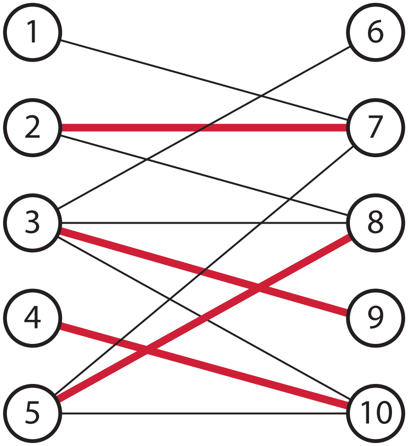
At this point, resulting alternating path is not augmenting, since it is not between two unmatched vertices (and no augmenting path is possible). At this point, matching is maximum cardinality.



# Review

- The next slide is from lecture 7 and the one after from lecture 5.

# Matroid Intersection

- Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$.
- While $(V, \mathcal{I}_1 \cap \mathcal{I}_2)$ is typically not a matroid (Exercise: show graphical example.), we might be interested in finding the maximum size common independent set. That is, find $\max |X|$ such that both $X \in \mathcal{I}_1$ and $X \in \mathcal{I}_2$.

## Theorem 11.5.5

*Let $M_1$ and $M_2$ be given as above, with rank functions $r_1$ and $r_2$. Then the size of the maximum size set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by*

$$(r_1 * r_2)(V) \triangleq \min_{X \subseteq V} \Big( r_1(X) + r_2(V \setminus X) \Big) \tag{11.7}$$

This is an instance of the convolution of two submodular functions, $f_1$ and $f_2$ that, evaluated at $Y \subseteq V$, is written as:

$$(f_1 * f_2)(Y) = \min_{X \subseteq Y} \Big( f_1(X) + f_2(Y \setminus X) \Big) \tag{11.8}$$

# Partition Matroid

- Let $V$ be our ground set.
- Let $V = V_1 \cup V_2 \cup \dots \cup V_\ell$ be a partition of $V$ into $\ell$ blocks (i.e., disjoint sets). Define a set of subsets of $V$ as

$$\mathcal{I} = \{X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \dots, \ell\}. \tag{11.3}$$

where $k_1, \dots, k_\ell$ are fixed parameters, $k_i \geq 0$. Then $M = (V, \mathcal{I})$ is a matroid.

- Note that a $k$-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$.
- Parameters associated with a partition matroid: $\ell$ and $k_1, k_2, \dots, k_\ell$ although often the $k_i$'s are all the same.
- We'll show that property (I3') in Def **??** holds. If $X, Y \in \mathcal{I}$ with $|Y| > |X|$, then there must be at least one $i$ with $|Y \cap V_i| > |X \cap V_i|$. Therefore, adding one element $e \in V_i \cap (Y \setminus X)$ to $X$ won't break independence.

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- Therefore, a matching in $G$ is simultaneously independent in both $M_V$ and $M_F$ and finding the maximum matching is finding the maximum cardinality set independent in both matroids.

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- Therefore, a matching in $G$ is simultaneously independent in both $M_V$ and $M_F$ and finding the maximum matching is finding the maximum cardinality set independent in both matroids.
- For the bipartite graph case, therefore, this can be solved in polynomial time.

# Matroid Intersection and Network Communication

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- This is again a matroid intersection problem.

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- Let $M_1$ be the cycle matroid on $G'$.

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- Let $M_1$ be the cycle matroid on $G'$.
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- Then a Hamiltonian cycle exists iff there is an $n$-element intersection of $M_1$, $M_2$, and $M_3$.

# Matroid Intersection and TSP

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- Since TSP is NP-complete, we obviously can't solve matroid intersections of 3 more matroids, unless P=NP.
- But bipartite graph example gives us hope for 2 matroids, and also ideas for an algorithm ...

# Recall from Lecture 5: Matroids by circuits

A set is independent if and only if it contains no circuit. Therefore, it is not surprising that circuits can also characterize a matroid.

### Theorem 11.5.1

*Matroid (by circuits) Let $E$ be a set and $\mathcal{C}$ be a collection of nonempty subsets of $E$, such that no two sets in $\mathcal{C}$ are contained in each other. Then the following are equivalent.*

1. *(C1) $\mathcal{C}$ is the collection of circuits of a matroid;*
2. *(C2) if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$;*
3. *(C3) if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, and $y \in C \setminus C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$ containing $y$;*

# Fundamental circuits in matroids

## Lemma 11.5.2

*Let* $I \in \mathcal{I}(M)$, *and* $e \in E$, *then* $I \cup \{e\}$ *contains at most one circuit in* $M$.

## Proof.

$\square$

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- Suppose, to the contrary, that there are two distinct circuits $C_1, C_2$ such that $C_1 \cup C_2 \subseteq I \cup \{e\}$.

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In general, let $C(I, e)$ be the unique circuit associated with $I \cup \{e\}$ (commonly called the fundamental circuit in $M$ w.r.t. $I$ and $e$).

# Matroid Intersection Algorithm Idea

- Consider two matroids $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ and start with any $I \in \mathcal{I}_1 \cap \mathcal{I}_2$.

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- Then $\text{span}_1(I) = \text{span}_1(I - v_2 + v_3)$.
- Moreover, since $I + v_1 \in \mathcal{I}_1$, $v_1 \notin \text{span}_1(I)$, so $\text{span}_1(I + v_1) = \text{span}_1(I + v_1 - v_2 + v_3)$.

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- But $I + v_1 - v_2 + v_3$ might not be independent in $M_2$ again, so need to find an $v_4 \in C_2(I + v_1 - v_2, v_3)$, $v_4 \in I$ to remove, and so on.

# Matroid Intersection Algorithm Idea

- Hopefully (eventually) we'll find an odd length sequence $S = (v_1, v_2, \dots, v_s)$ such that we will be independent in both $M_1$ and $M_2$ and thus be one greater in size than $I$.

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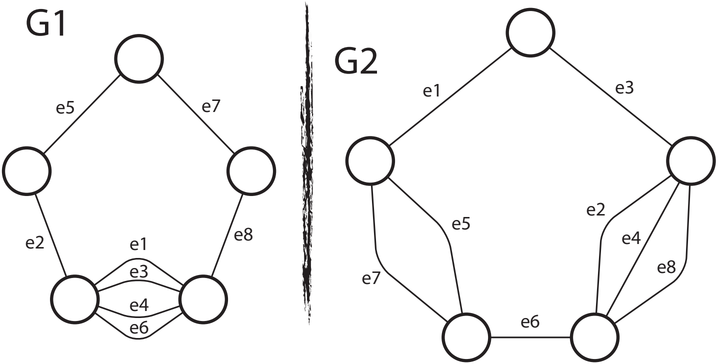
- Hopefully (eventually) we'll find an odd length sequence $S = (v_1, v_2, \dots, v_s)$ such that we will be independent in both $M_1$ and $M_2$ and thus be one greater in size than $I$.
- We will have $v_i \notin I$ for $i$ odd (will be shown in blue), and will have $v_i \in I$ for $i$ even (will be shown in green), while $v \in I \setminus S$ will be shown in red.
- We then replace $I$ with $I \ominus S$ (quite analogous to the bipartite matching case), and start again.

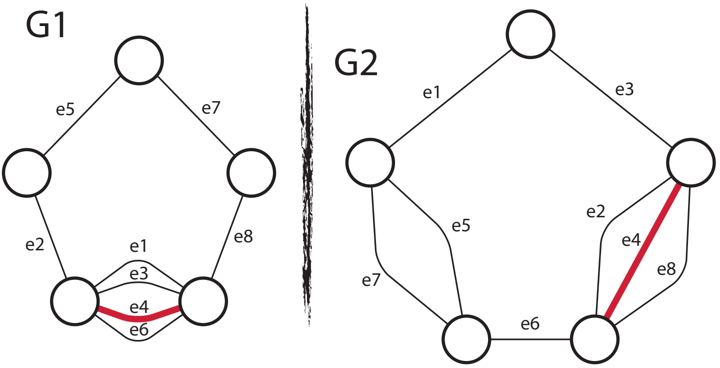
# Graphic Matroid Intersection Example

Consider the following two graph $G_1 = (V_1, E)$ and $G_2 = (V_2, E)$ and corresponding matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$. Any edge is independent in both (an augmenting "sequence") since a single edge can't create a circuit starting at $I = \emptyset$. We start with $e_4$.



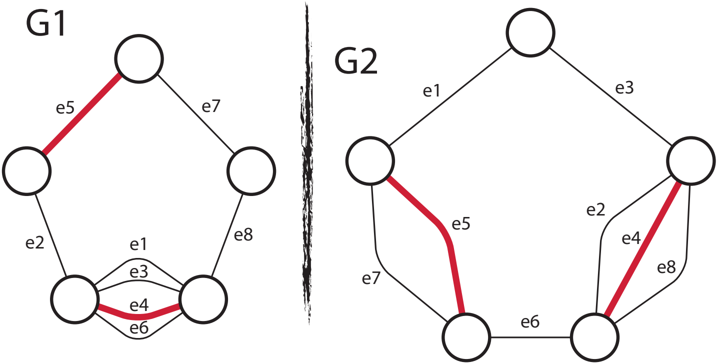
# Graphic Matroid Intersection Example

Setting $I \leftarrow e_4$ with edge $e_4$ creates a circuit neither in $M_1$ nor $M_2$. We can add another single edge w/o creating a circuit in either matroid.



# Graphic Matroid Intersection Example

$e_5 \in E - \text{span}_1(\{e_4\})$. Then, after $I \leftarrow I + e_5$, (i.e., when $I = \{e_4, e_5\}$) we're still independent in $M_2$, but no further single edge additions possible w/o creating a circuit (why?).



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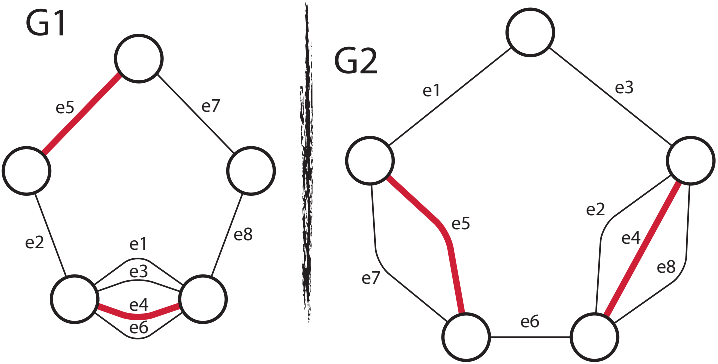
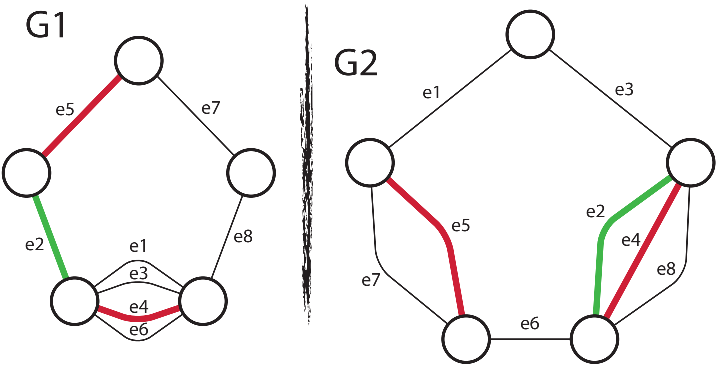


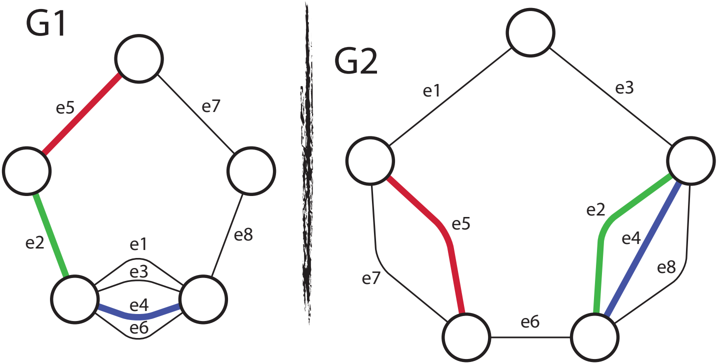
# Graphic Matroid Intersection Example

Augmenting sequence is green and blue edges (blue is already in $I$, green is new). We choose $e_2 \in E - \text{span}_1(I)$, but now $I + e_2$ is not independent in $M_2$.



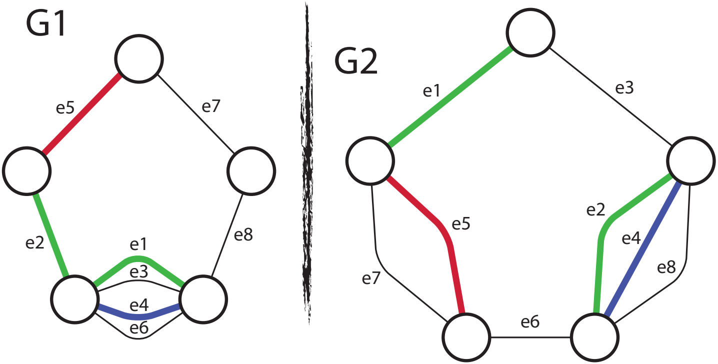
# Graphic Matroid Intersection Example

So there must exist $C_2(I, e_2)$. We choose $e_4 \in C_2(I, e_2)$ to remove.



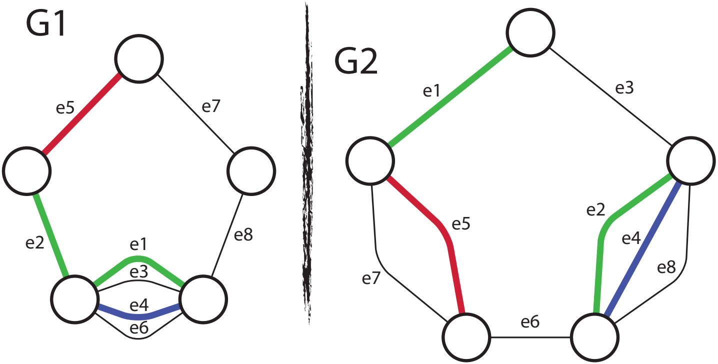
# Graphic Matroid Intersection Example

Next, we choose $e_1 \in \mathrm{span}_1(I) - \mathrm{span}_1(I - e_4)$ to add.



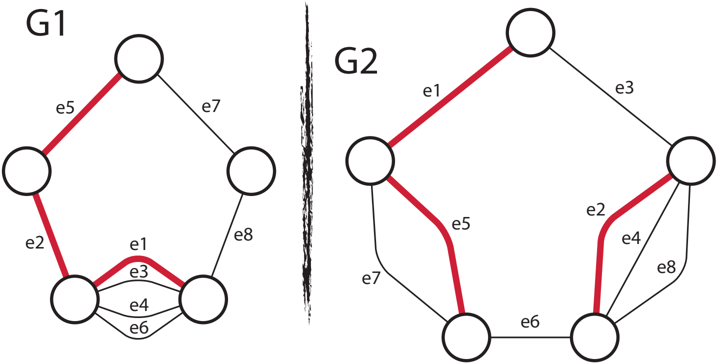
# Graphic Matroid Intersection Example

Next, we choose $e_1 \in \text{span}_1(I) - \text{span}_1(I - e_4)$ to add. In this case, we not only have $\text{span}_1(I + e_2) = \text{span}_1(I + e_2 - e_4 + e_1)$, but we also have that $(I + e_2 - e_4) + e_1 \in \mathcal{I}_2$.



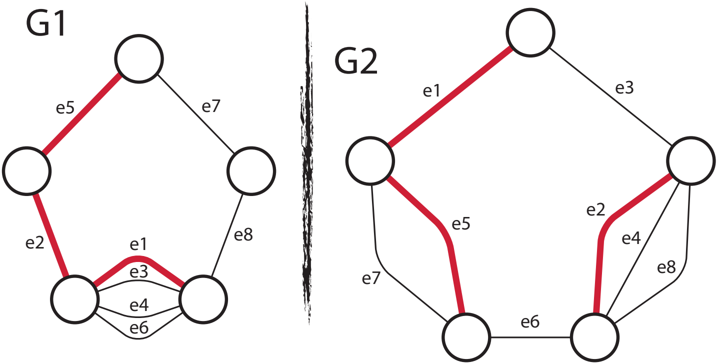
# Graphic Matroid Intersection Example

Removing blue and adding green leads to higher cardinality independent set in both matroids. This corresponds to doing $I \leftarrow I \ominus S$ where $S = (e_2, e_4, e_1)$ and $I = \{e_4, e_5\}$.



# Graphic Matroid Intersection Example

At this point, are any further augmenting sequences possible? Exercise.



# Alternating and Augmenting Sequences

- Let $I$ be an intersection of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ (i.e., $I \in \mathcal{I}_1 \cap \mathcal{I}_2$).

## Alternating and Augmenting Sequences

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- Let $S = (e_1, e_2, \dots, e_s)$ be a sequence of distinct elements, where $e_i \in E - I$ for $i$ odd, and $e_i \in I$ for $i$ even, and let $S_i = (e_1, e_2, \dots, e_i)$. We say that $S$ is an alternating sequence w.r.t. $I$ if the following are true.

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  1. $I + e_1 \in \mathcal{I}_1$
  2. For all even $i$, $\text{span}_2(I \ominus S_i) = \text{span}_2(I)$ which implies that $I \ominus S_i \in \mathcal{I}_2$.

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  3. For all odd $i$, $\text{span}_1(I \ominus S_i) = \text{span}_1(I + e_1)$, and therefore $I \ominus S_i \in \mathcal{I}_1$.

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  3. For all odd $i$, $\text{span}_1(I \ominus S_i) = \text{span}_1(I + e_1)$, and therefore $I \ominus S_i \in \mathcal{I}_1$.
- Lastly, if also, $|S| = s$ is odd, and $I \ominus S \in \mathcal{I}_2$, then $S$ is called an augmenting sequence w.r.t. $I$.

# Alternating and Augmenting Sequences

- If $I$ admits an augmenting sequence $S$, then the above argument shows that $I \ominus S$ is independent in $M_1$, independent in $M_2$, and also we have that $|I| + 1 = |I \ominus S|$.

# Alternating and Augmenting Sequences

- If $I$ admits an augmenting sequence $S$, then the above argument shows that $I \ominus S$ is independent in $M_1$, independent in $M_2$, and also we have that $|I| + 1 = |I \ominus S|$.
- Thus, by finding augmenting sequences, we can increase the size of the matroid intersection until we stop. Moreover, we have:

**Proposition 11.5.3**

*If there is an augmenting sequence, then the intersection is not maximum.*

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- Thus, by finding augmenting sequences, we can increase the size of the matroid intersection until we stop. Moreover, we have:

**Proposition 11.5.3**

*If there is an augmenting sequence, then the intersection is not maximum.*

- We next wish to show that, if the intersection is not maximum, then there is an augmenting sequence.

# Border graphs

- We construct an auxiliary directed bipartite graph (Border graph) $B(I) = (E \setminus I, I, Z)$, relative to the current $I$, that will help us with this problem. The graph has only directed edges from $E \setminus I$ to $I$, or from $I$ back to $E \setminus I$.

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- Left-going edges: For each $e_i \in \text{span}_1(I) \setminus I$, create $e_i \leftarrow e_j$ directed edge $(e_j, e_i) \in Z$ from all $e_j \in C_1(I, e_i) \setminus \{e_i\}$. Note $e_j \in I$ and $e_i \in E \setminus I$.

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- If $e_i \notin \text{span}_1(I)$, then $e_i$ has in-degree zero (a source).

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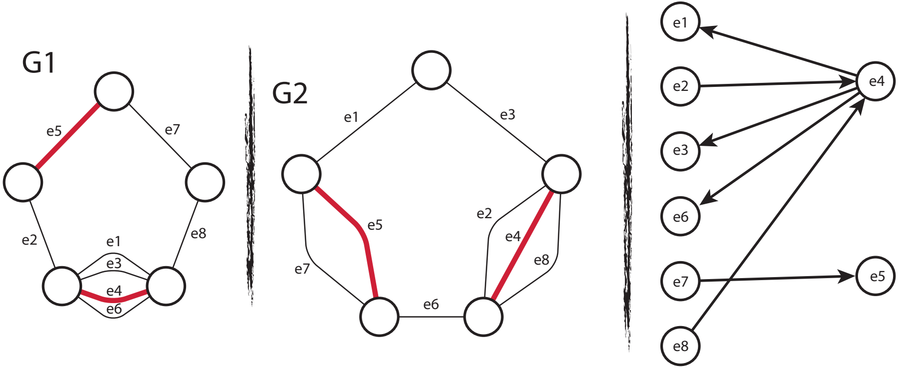
- If $e_i \notin \text{span}_2(I)$, then $e_i$ has out-degree zero (a sink).

# Border graph Example




- $\{e_2, e_7, e_8\}$ are sources and $\{e_1, e_3, e_6\}$ are sinks. $I = \{e_4, e_5\}$. $\text{span}_1(I) \setminus I = \{e_1, e_3, e_6\}$ and $\text{span}_2(I) \setminus I = \{e_7, e_2, e_8\}$

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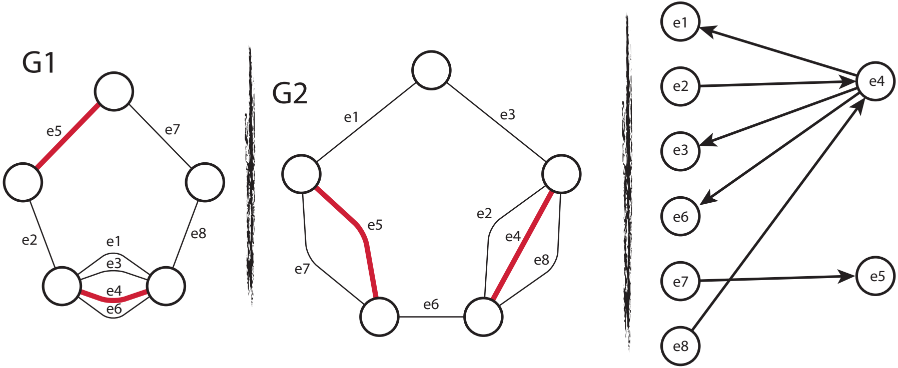


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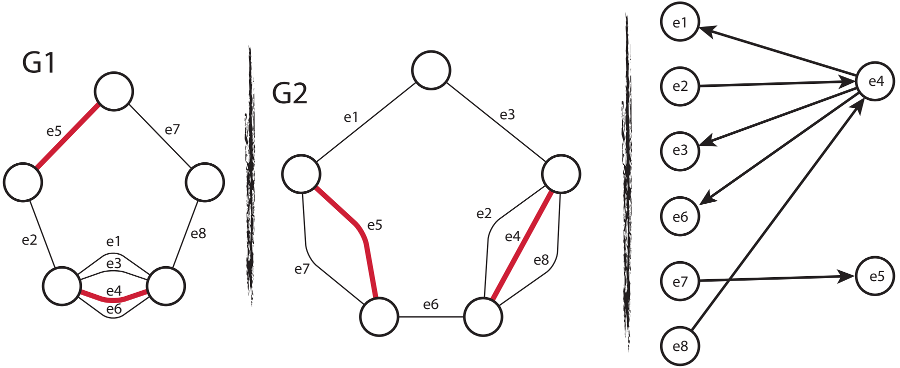
- $C_1(I, e_1) \setminus \{e_1\} = C_1(I, e_3) \setminus \{e_3\} = C_1(I, e_6) \setminus \{e_6\} = e_4$.

# Border graph Example

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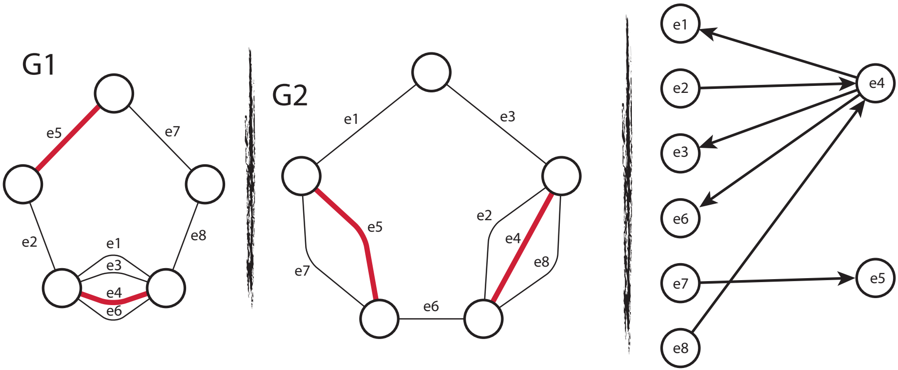


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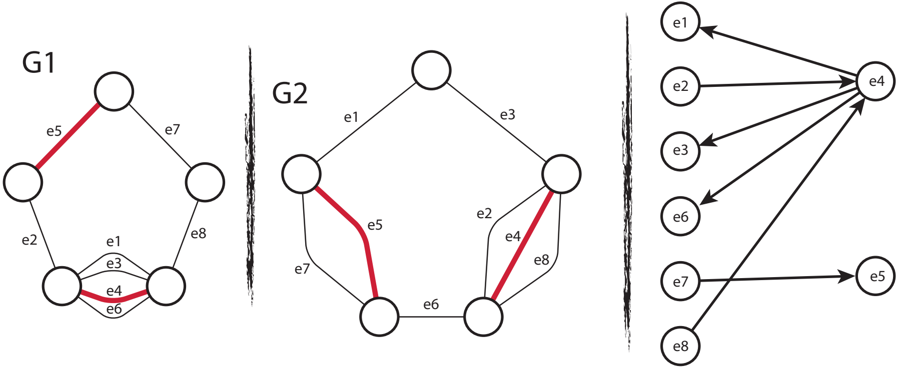
- Augmenting sequences are $(e_2, e_4, e_1)$, $(e_2, e_4, e_3)$, and $(e_2, e_4, e_6)$, all dipaths in the Border graph.

# Border graph Example

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# Identifying Augmenting Sequences

**Lemma 11.5.4**

*If $S$ is a source-sink path in $B(I)$, and there is no shorter source-sink path between the same source and sink (i.e., there are no short-cuts), then $S$ is an augmenting sequence w.r.t. $I$.*

# Identifying Augmenting Sequences

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## Lemma 11.5.5

*Let $I$ and $J$ be matroid intersections of $M_1$ and $M_2$ such that $|I| + 1 = |J|$. Then there exists a source-sink path $S$ in $B(I)$ where $S \subseteq I \ominus J$.*

# Identifying Augmenting Sequences

**Theorem 11.5.6**

*Let* $I_p$ *and* $I_{p+1}$ *be intersections of* $M_1$ *and* $M_2$ *with* $p$ *and* $p+1$ *elements respectively. Then there exists an augmenting sequence* $S \subseteq I_p \ominus I_{p+1}$ *w.r.t.* $I_p$*.*

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**Theorem 11.5.7**

*An intersection is of maximum cardinality iff it admits no augmenting sequence.*

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**Theorem 11.5.8**

*For any intersection $I$, there exists a maximum cardinality intersection $I^*$ such that $\text{span}_1(I) \subseteq \text{span}_1(I^*)$ and $\text{span}_2(I) \subseteq \text{span}_2(I^*)$.*

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All this can be made to run in poly time.