Submodular Functions, Optimization, and Applications to Machine Learning

— Spring Quarter, Lecture ~~9~~ 10 —

http://j.ee.washington.edu/~bilmes/classes/ee596b_spring_2014/

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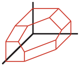
$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

$= f(A_r) + 2f(C) + f(B_r)$ $\quad = f(A_r) + f(C) + f(B_r)$ $\quad = f(A \cap B)$



Cumulative Outstanding Reading

- Read chapters 1 and 2, and sections 3.1-3.2 from Fujishige's book.
- Good references for today: Schrijver-2003, Oxley-1992/2011, Welsh-1973, Goemans-2010, Cunningham-1984, Edmonds-1969.

Announcements, Assignments, and Reminders

- Weekly Office Hours: Wednesdays, 5:00-5:50, or by skype or google hangout (email me).

Class Road Map - IT-I

- L1 (3/31): Motivation, Applications, & Basic Definitions
- L2: (4/2): Applications, Basic Definitions, Properties
- L3: More examples and properties (e.g., closure properties), and examples, spanning trees
- L4: proofs of equivalent definitions, independence, start matroids
- L5: matroids, basic definitions and examples
- L6: More on matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation
- L7: Dual Matroids, other matroid properties, Combinatorial Geometries
- L8: Combinatorial Geometries, matroids and greedy, Polyhedra, Matroid Polytopes,
- L9: From Matroid Polytopes to Polymatroids.
- L10: Polymatroids and Submodularity
- L11:
- L12:
- L13:
- L14:
- L15:
- L16:
- L17:
- L18:
- L19:
- L20:

Finals Week: June 9th-13th, 2014.

Maximum weight independent set via greedy weighted rank

Theorem 9.2.6

Let $M = (V, \mathcal{I})$ *be a matroid, with rank function* r*, then for any weight function* $w \in \mathbb{R}_+^V$*, there exists a chain of sets* $U_1 \subset U_2 \subset \cdots \subset U_n \subseteq V$ *such that*

$$\max \{w(I) | I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i) \tag{9.19}$$

where $\lambda_i \geq 0$ *satisfy*

$$w = \sum_{i=1}^{n} \lambda_i \mathbf{1}_{U_i} \tag{9.20}$$

Polytope Equivalence (Summarizing the above)

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I$.
- Taking the convex hull, we get the independent set polytope, that is

$$P_{\text{ind. set}} = \text{conv}\left\{\cup_{I \in \mathcal{I}} \{\mathbf{1}_I\}\right\} \tag{9.12}$$

- Now take the rank function r of M, and define the following polyhedron:

$$P_r^+ = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\right\} \tag{9.13}$$

Theorem 9.2.2

$$P_r^+ = P_{\textit{ind. set}} \tag{9.14}$$

P-basis of x given compact set $P \subseteq \mathbb{R}_+^E$

Definition 9.2.4 (subvector)

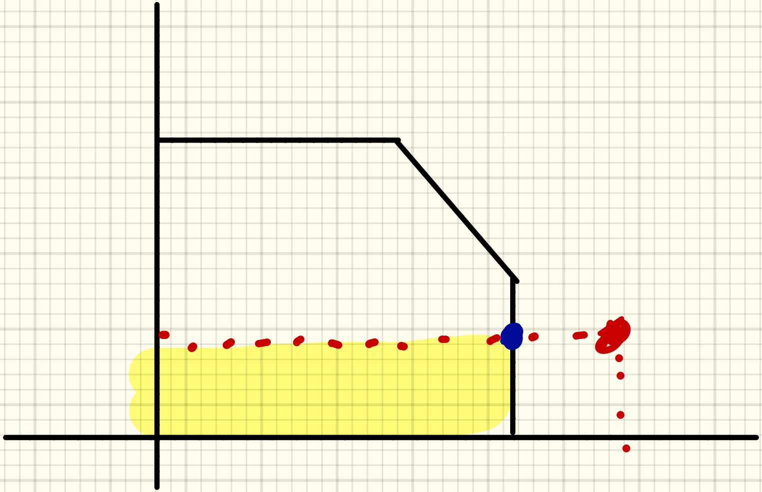
y is a subvector of x if $y \leq x$ (meaning $y(e) \leq x(e)$ for all $e \in E$).

Definition 9.2.5 (P-basis)

Given a compact set $P \subseteq \mathcal{R}_+^E$, for any $x \in \mathbb{R}_+^E$, a subvector y of x is called a P-basis of x if y maximal in P.
In other words, y is a P-basis of x if y is a maximal P-contained subvector of x.

Here, by y being "maximal", we mean that there exists no $z > y$ (more precisely, no $z \geq y + \epsilon \mathbf{1}_e$ for some $e \in E$ and $\epsilon > 0$) having the properties of y (the properties of y being: in P, and a subvector of x).
In still other words: y is a P-basis of x if:

1. $y \leq x$ (y is a subvector of x); and
2. $y \in P$ and $y + \epsilon \mathbf{1}_e \notin P$ for all $e \in E$ where $y(e) < x(e)$ and $\forall \epsilon > 0$ (y is maximal P-contained).



A vector form of rank

- Recall the definition of rank from a matroid $M = (E, \mathcal{I})$.

$$\text{rank}(A) = \max\{|I| : I \subseteq A, I \in \mathcal{I}\} \tag{9.25}$$

- vector rank: Given a compact set $P \subseteq \mathcal{R}_+^E$, we can define a form of "vector rank" relative to this P in the following way: Given an $x \in \mathbb{R}^E$, we define the vector rank, relative to P, as:

$$\text{rank}(x) = \max\left(y(E) : y \leq x, y \in P\right) \tag{9.26}$$

where $y \leq x$ is componentwise inequality ($y_i \leq x_i, \forall i$).

- If $\mathcal{B}_x$ is the set of P-bases of x, than $\text{rank}(x) = \max_{y \in \mathcal{B}_x} y(E)$.
- If $x \in P$, then $\text{rank}(x) = x(E)$ (x is its own unique self P-basis).
- In general, this might be hard to compute and/or have ill-defined properties. We next look at an object that restrains and cultivates this form of rank.

Polymatroidal polyhedron (or a "polymatroid")

Definition 9.2.4 (polymatroid)

A polymatroid is a compact set $P \subseteq \mathbb{R}_+^E$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For every $x \in \mathbb{R}_+^E$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any P-basis of x), has the same component sum $y(E)$

- Vectors within P (i.e., any $y \in P$) are called independent, and any vector outside of P is called dependent.
- Since all P-bases of x have the same component sum, if $\mathcal{B}_x$ is the set of P-bases of x, than $\text{rank}(x) = y(E)$ for any $y \in \mathcal{B}_x$.

Matroid and Polymatroid: side-by-side

A Matroid is:

1. a set system $(E, \mathcal{I})$
2. empty-set containing $\emptyset \in \mathcal{I}$
3. down closed, $\emptyset \subseteq I' \subseteq I \in \mathcal{I} \Rightarrow I' \in \mathcal{I}$.
4. any maximal set I in $\mathcal{I}$, bounded by another set A, has the same matroid rank (any maximal independent subset $I \subseteq A$ has same size $|I|$).

A Polymatroid is:

1. a compact set $P \subseteq \mathbb{R}_+^E$
2. zero containing, $\mathbf{0} \in P$
3. down monotone, $0 \leq y \leq x \in P \Rightarrow y \in P$
4. any maximal vector y in P, bounded by another vector x, has the same vector rank (any maximal independent subvector $y \leq x$ has same sum $y(E)$).

.

Polymatroid function and its polyhedron.

Definition 9.2.4

A polymatroid function is a real-valued function f defined on subsets of E which is normalized, non-decreasing, and submodular. That is we have

1. $f(\emptyset) = 0$ (normalized)
2. $f(A) \leq f(B)$ for any $A \subseteq B \subseteq E$ (monotone non-decreasing)
3. $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$ for any $A, B \subseteq E$ (submodular)

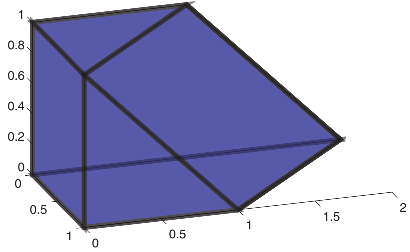
We can define the polyhedron P_f^+ associated with a polymatroid function as follows

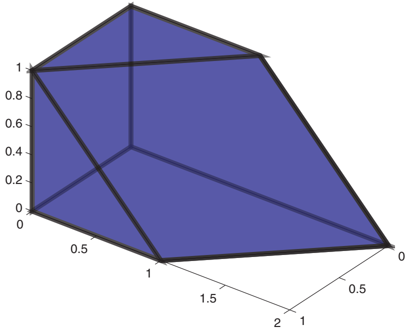
$$P_f^+ = \left\{ y \in \mathbb{R}_+^E : y(A) \leq f(A) \text{ for all } A \subseteq E \right\} \tag{9.25}$$

$$= \left\{ y \in \mathbb{R}^E : y \geq 0, y(A) \leq f(A) \text{ for all } A \subseteq E \right\} \tag{9.26}$$

Associated polyhedron with a polymatroid function

- Consider the asymmetric graph cut function on the simple chain graph $v_1 - v_2 - v_3$. That is, $f(S) = |\{(v,s) \in E(G) : v \in V, s \in S\}|$ is count of any edges within S or between S and $V \setminus S$, so that $\delta(S) = f(S) + f(V \setminus S) - f(V)$ is the standard graph cut.
- Observe: P_f^+ (at two views):

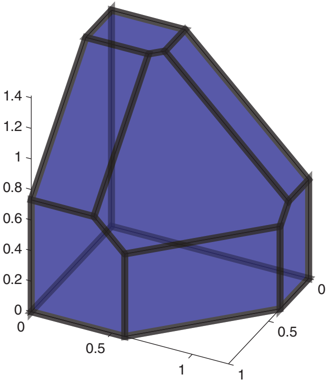


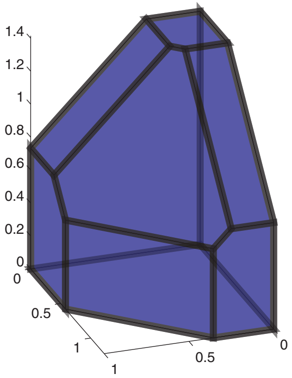


- which axis is which?

Associated polyhedron with a polymatroid function

- Consider modular function $w : V \to \mathbb{R}_+$ as $w = (1, 1.5, 2)^\intercal$, and then the submodular function $f(S) = \sqrt{w(S)}$.
- Observe: P_f^+ (at two views):





- which axis is which?

A polymatroid vs. a polymatroid function's polyhedron

- Summarizing the above, we have:
  - Given a polymatroid function f, its associated polytope is given as

$$P_f^+ = \{y \in \mathbb{R}_+^E : y(A) \le f(A) \text{ for all } A \subseteq E\} \quad (9.34)$$

P

  - We also have the definition of a polymatroidal polytope (compact subset, zero containing, down-monotone, and $\forall x$ any maximal independent subvector $y \le x$ has same component sum $y(E)$).
- Is there any relationship between these two polytopes?
- In the next theorem, we show that any P_f^+-basis has the same component sum, when f is a polymatroid function, and P_f^+ satisfies the other properties so that P_f^+ is a polymatroid.

A polymatroid function's polyhedron is a polymatroid.

Theorem 9.2.4

Let f be a polymatroid function defined on subsets of E. For any $x \in \mathbb{R}_+^E$, and any P_f^+-basis $y^x \in \mathbb{R}_+^E$ of x, the component sum of y^x is

$$y^x(E) = \text{rank}(x) = \max\left(y(E) : y \leq x, y \in P_f^+\right)$$
$$= \boxed{\min\left(x(A) + f(E \setminus A) : A \subseteq E\right)} \quad (9.34)$$

As a consequence, P_f^+ is a polymatroid, since r.h.s. is constant w.r.t. y^x.

By taking $B = \text{supp}(x)$ (so elements $E \setminus B$ are zero in x), and for $b \in B$, $x(b)$ is big enough, the r.h.s. min has solution $A^* = E \setminus B$. We recover submodular function from the polymatroid polyhedron via the following:

$$f(B) = \max\left\{y(B) : y \in P_f^+\right\} \quad (9.35)$$

In fact, we will ultimately see a number of important consequences of this theorem (other than just that P_f^+ is a polymatroid)

A polymatroid function's polyhedron is a polymatroid.

Proof.

- Clearly $0 \in P_f^+$ since f is non-negative.

...

A polymatroid function's polyhedron is a polymatroid.

Proof.

- Clearly $0 \in P_f^+$ since f is non-negative.
- Also, for any $y \in P_f^+$ then any $x <= y$ is also such that $x \in P_f^+$. So, P_f^+ is down-monotone.

$0 \leq x(A) \leq y(A) \leq f(A) \quad \forall A$

...

A polymatroid function's polyhedron is a polymatroid.

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- Also, for any $y \in P_f^+$ then any $x <= y$ is also such that $x \in P_f^+$. So, P_f^+ is down-monotone.
- Now suppose that we are given an $x \in \mathbb{R}_+^E$, and maximal $y^x \in P_f^+$ with $y^x \leq x$ (i.e., y^x is a P_f^+-basis of x).

...

A polymatroid function's polyhedron is a polymatroid.

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- Now suppose that we are given an $x \in \mathbb{R}_+^E$, and maximal $y^x \in P_f^+$ with $y^x \leq x$ (i.e., y^x is a P_f^+-basis of x).
- Goal is to show that any such y^x has $y^x(E) =$ const, dependent only on x and also f (which defines the polytope) but not dependent on y^x, the particular P-basis.

...

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- Goal is to show that any such y^x has $y^x(E) =$ const, dependent only on x and also f (which defines the polytope) but not dependent on y^x, the particular P-basis.
- Doing so will thus establish that P_f^+ is a polymatroid.

...

A polymatroid function's polyhedron is a polymatroid.

$x(E \setminus A) = x(E) - x(A)$

... proof continued.

- First trivial case: could have $y^x = x$, which happens if $x(A) \leq f(A), \forall A \subseteq E$ (i.e., $x \in P_f^+$ strictly). In such case,

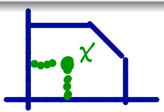
$$\min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \quad (9.1)$$

$$= x(E) + \min\left(f(E \setminus A) - x(E \setminus A) : A \subseteq E\right) \quad (9.2)$$

$$= x(E) + \min\left(f(A) - x(A) : A \subseteq E\right) \quad (9.3)$$

$$= x(E) + 0 \quad (9.4)$$

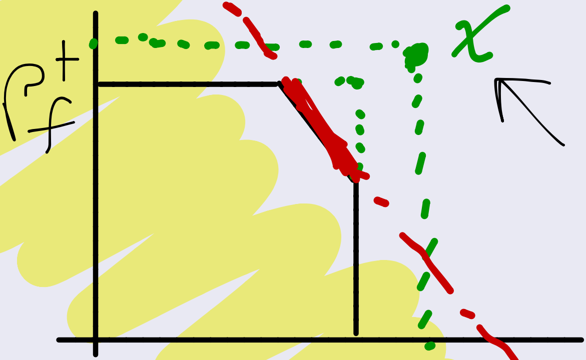
...



A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- 2nd trivial case is when $x(A) > f(A), \forall A \subseteq E$ (i.e., $x \notin P_f^+$ strictly),



$y^x(E) = f(E)$

$x(E) \leq f(E)$

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- 2nd trivial case is when $x(A) > f(A), \forall A \subseteq E$ (i.e., $x \notin P_f^+$ strictly),
- Then for any order $(a_1, a_2, \ldots)$ of the elements and $A_i \triangleq (a_1, a_2, \ldots, a_i)$, we have $x(a_i) \geq f(a_i) \geq f(a_i|A_{i-1})$, the second inequality by submodularity.

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

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- Then for any order $(a_1, a_2, \dots)$ of the elements and $A_i \triangleq (a_1, a_2, \dots, a_i)$, we have $x(a_i) \geq f(a_i) \geq f(a_i | A_{i-1})$, the second inequality by submodularity.
- This gives

$\sum_{i=1}^{|A|} \left(f(a_i | A_{i-1}) - x(a_i) \right)$

$$\min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{9.5}$$
$$= x(E) + \min\left(f(A) - x(A) : A \subseteq E\right) \tag{9.6}$$
$$= x(E) + \min\left(\sum_i f(a_i | A_{i-1}) - \sum_i x(a_i) : A \subseteq E\right) \tag{9.7}$$
$$= x(E) + f(E) - x(E) = f(E) \tag{9.8}$$

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- Assume neither trivial case. Because $y^x \in P_f^+$, we have that $y^x(A) \leq f(A)$ for all $A \subseteq E$.

. . .

A polymatroid function's polyhedron is a polymatroid.

... proof continued.

- Assume neither trivial case. Because $y^x \in P_f^+$, we have that $y^x(A) \le f(A)$ for all $A \subseteq E$.
- We show that the constant is given by

$$y^x(E) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{9.9}$$

...

A polymatroid function's polyhedron is a polymatroid.

... proof continued.

- Assume neither trivial case. Because $y^x \in P_f^+$, we have that $y^x(A) \leq f(A)$ for all $A \subseteq E$.
- We show that the constant is given by

$$y^x(E) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{9.9}$$

- For any P_f^+-basis y^x of x, and any $A \subseteq E$, we have that

$y^x \leq x$

$y^x(A) \leq f(A)$ $\forall A$

$$y^x(E) = y^x(A) + y^x(E \setminus A) \tag{9.10}$$
$$\leq x(A) + f(E \setminus A). \tag{9.11}$$

This follows since $y^x \leq x$ and since $y^x \in P_f^+$.

...

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

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- We show that the constant is given by

$$y^x(E) = \min \left(x(A) + f(E \setminus A) : A \subseteq E \right) \tag{9.9}$$

- For any P_f^+-basis y^x of x, and any $A \subseteq E$, we have that

$$y^x(E) = y^x(A) + y^x(E \setminus A) \tag{9.10}$$
$$\leq x(A) + f(E \setminus A). \tag{9.11}$$

This follows since $y^x \leq x$ and since $y^x \in P_f^+$.

- Given one A where equality holds, the above min result follows.

...

A polymatroid function's polyhedron is a polymatroid.

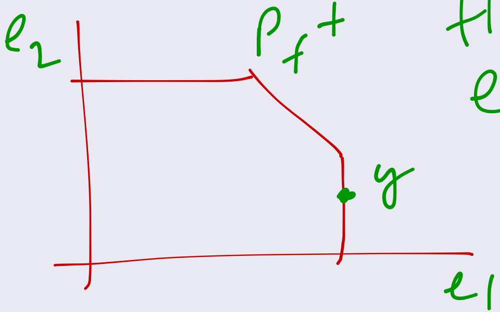
. . . proof continued.

- For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets B, C is again tight, since

$$f(B) + f(C)$$

$f(e_1) = y(e_1)$

e_1



. . .

A polymatroid function's polyhedron is a polymatroid.

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- For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets B, C is again tight, since

$$f(B) + f(C) = y(B) + y(C) \tag{9.12}$$

. . .

A polymatroid function's polyhedron is a polymatroid.

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$$f(B) + f(C) = y(B) + y(C) \tag{9.12}$$
$$= y(B \cap C) + y(B \cup C) \tag{9.13}$$

. . .

A polymatroid function's polyhedron is a polymatroid.

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- For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets B, C is again tight, since

$$f(B) + f(C) = y(B) + y(C) \tag{9.12}$$
$$= y(B \cap C) + y(B \cup C) \tag{9.13}$$
$$\leq f(B \cap C) + f(B \cup C) \tag{9.14}$$

. . .

A polymatroid function's polyhedron is a polymatroid.

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- For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets B, C is again tight, since

$$f(B) + f(C) = y(B) + y(C) \tag{9.12}$$
$$= y(B \cap C) + y(B \cup C) \tag{9.13}$$
$$\leq f(B \cap C) + f(B \cup C) \tag{9.14}$$
$$\leq f(B) + f(C) \tag{9.15}$$

. . .

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets B, C is again tight, since

$$
\begin{aligned}
f(B) + f(C) &= y(B) + y(C) && (9.12)\\
&= y(B \cap C) + y(B \cup C) && (9.13)\\
&\leq f(B \cap C) + f(B \cup C) && (9.14)\\
&\leq f(B) + f(C) && (9.15)
\end{aligned}
$$

which requires equality everywhere above.

. . .

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

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$$f(B) + f(C) = y(B) + y(C) \tag{9.12}$$
$$= y(B \cap C) + y(B \cup C) \tag{9.13}$$
$$\leq f(B \cap C) + f(B \cup C) \tag{9.14}$$
$$\leq f(B) + f(C) \tag{9.15}$$

  which requires equality everywhere above.

- Because $y(B) \leq f(B), \forall B$, this means $y(B \cap C) = f(B \cap C)$ and $y(B \cup C) = f(B \cup C)$, so both also are tight.

. . .

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets B, C is again tight, since

$$f(B) + f(C) = y(B) + y(C) \tag{9.12}$$
$$= y(B \cap C) + y(B \cup C) \tag{9.13}$$
$$\leq f(B \cap C) + f(B \cup C) \tag{9.14}$$
$$\leq f(B) + f(C) \tag{9.15}$$

which requires equality everywhere above.

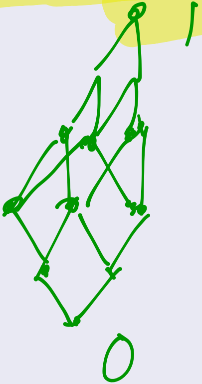
- Because $y(B) \leq f(B), \forall B$, this means $y(B \cap C) = f(B \cap C)$ and $y(B \cup C) = f(B \cup C)$, so both also are tight.
- For $y \in P_f^+$, it will be ultimately useful to define this lattice family of tight sets: $\mathcal{D}(y) \triangleq \{A : A \subseteq E,\ y(A) = f(A)\}$.

. . .

A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- Also, define $\text{sat}(y) \stackrel{\text{def}}{=} \bigcup\{T : T \in \mathcal{D}(y)\}$



A polymatroid function's polyhedron is a polymatroid.

. . . proof continued.

- Also, define $\text{sat}(y) \stackrel{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\}$
- Consider again a P_f^+-basis y^x (so maximal).

A polymatroid function's polyhedron is a polymatroid.

...proof continued.

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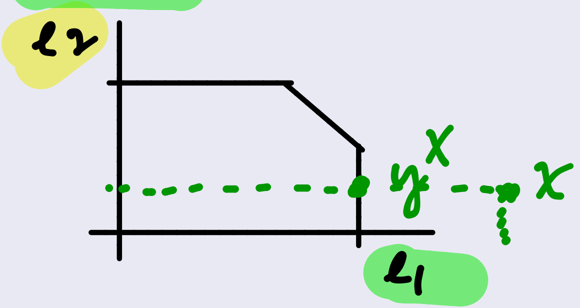

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- Hence, we have

y^x

$$y^x(E) = y^x(A) + y^x(E \setminus A) = x(A) + f(E \setminus A) \tag{9.16}$$

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$$y(E) = y(A) + y(E \setminus A) = x(A) + f(E \setminus A) \tag{9.16}$$

- So we identified the A to be the elements that are non-tight, and achieved the min, as desired.

□

A polymatroid is a polymatroid function's polytope

- So, when f is a polymatroid function, P_f^+ is a polymatroid.

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Theorem 9.3.1

For any polymatroid P (compact subset of $\mathbb{R}_+^E$, zero containing, down-monotone, and $\forall x \in \mathbb{R}_+^E$ any maximal independent subvector $y \leq x$ has same component sum $y(E) = \text{rank}(x)$), there is a polymatroid function $f : 2^E \to \mathbb{R}$ (normalized, monotone non-decreasing, submodular) such that $P = P_f^+$ where $P_f^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\}$.

First, a bit on $\mathcal{D}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{A : A \subseteq E,\ y(A) = f(A)\} \tag{9.17}$$

Theorem 9.3.2

For any $y \in P_f^+$, with f a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

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Proof.

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Also recall the definition of $\mathrm{sat}(y)$, the maximal set of tight elements relative to $y \in \mathbb{R}_+^E$.

$$\mathrm{sat}(y) \stackrel{\mathrm{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\} \tag{9.18}$$

Next, a bit on rank(x), join and meet for $x, y \in \mathbb{R}_+^E$

- For $x, y \in \mathbb{R}_+^E$, define vectors $x \wedge y \in \mathbb{R}_+^E$ and $x \vee y \in \mathbb{R}_+^E$ such that, for all $e \in E$

$$(x \vee y)(e) = \max(x(e), y(e)) \tag{9.19}$$

$$(x \wedge y)(e) = \min(x(e), y(e)) \tag{9.20}$$

Hence, $\vee$ $\cup$

$$x \vee y = (\max(x(e_1), y(e_1)), \max(x(e_2), y(e_2)), \dots, \max(x(e_n), y(e_n)))$$

and similarly

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- From this, we can define things like an lattices, and other constructs.

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- The vector rank function rank(x) also satisfies a form of submodularity.

Theorem 9.3.3 (vector rank and submodularity)

Let P be a polymatroid polytope. The vector rank function rank : $\mathbb{R}_+^E \to \mathbb{R}$ with rank$(x) = \max\left(y(E) : y \leq x, y \in P\right)$ satisfies, for all $u, v \in \mathbb{R}_+^E$

$$\text{rank}(u) + \text{rank}(v) \geq \text{rank}(u \vee v) + \text{rank}(u \wedge v) \tag{9.21}$$

Next, a bit on rank(x)

Proof of Theorem 9.3.3.

- Let a be a P-basis of $u \wedge v$, so rank$(u \wedge v) = a(E)$.

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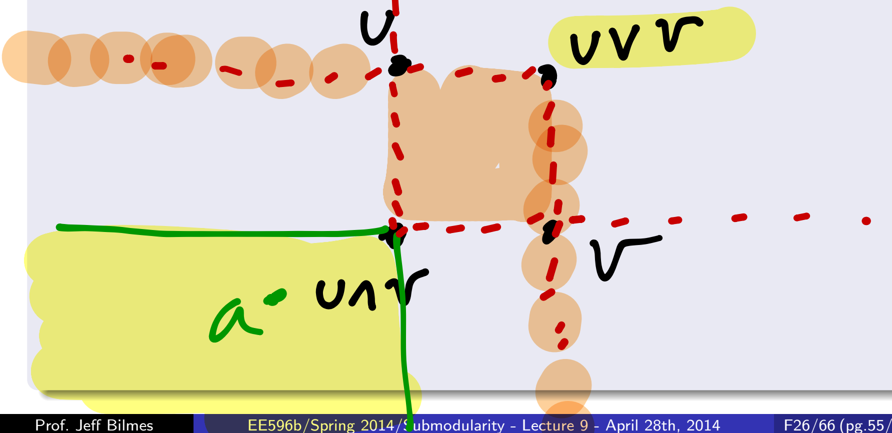

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  Therefore, $a = b \wedge (u \wedge v)$.

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$$a + b \tag{9.22}$$

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- Since $a = b \wedge (u \wedge v)$ and since $b \le u \vee v$, we get

$$a + b = b + b \wedge u \wedge v = b \wedge u + b \wedge v \tag{9.22}$$

To see this, consider each case where either b is the minimum, or u is minimum with $b \le v$, or v is minimum with $b \le u$.

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- But $b \wedge u$ and $b \wedge v$ are independent subvectors of u and v respectively, so $(b \wedge u)(E) \leq \text{rank}(u)$ and $(b \wedge v)(E) \leq \text{rank}(v)$.

$\square$

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- Hence,

$$\text{rank}(u \wedge v) + \text{rank}(u \vee v)$$

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$$= (b \wedge u)(E) + (b \wedge v)(E) \tag{9.24}$$

$$\leq \text{rank}(u) + \text{rank}(v) \tag{9.25}$$

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A polymatroid function's polyhedron vs. a polymatroid.

- Note the remarkable similarity between the proof of Theorem 9.3.3 and the proof of Theorem 5.5.1 that the standard matroid rank function is submodular.

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- Next, we prove Theorem 9.3.1, that any polymatroid polytope P has a polymatroid function f such that $P = P_f^+$.
- Given this result, we can conclude that a polymatroid is really an extremely natural polyhedral generalization of a matroid. This was all realized by Jack Edmonds in the mid 1960s (and published in 1969 in his landmark paper "Submodular Functions, Matroids, and Certain Polyhedra").

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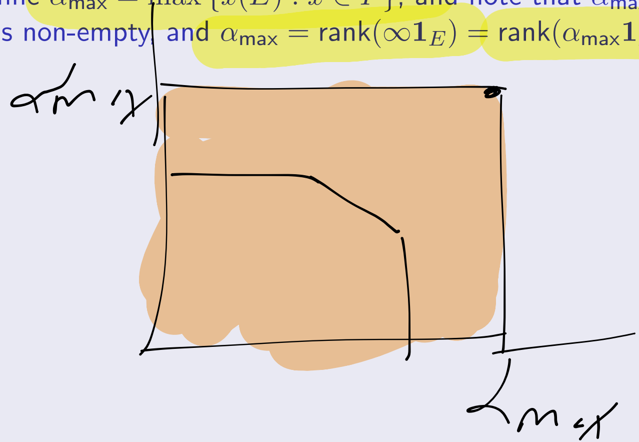

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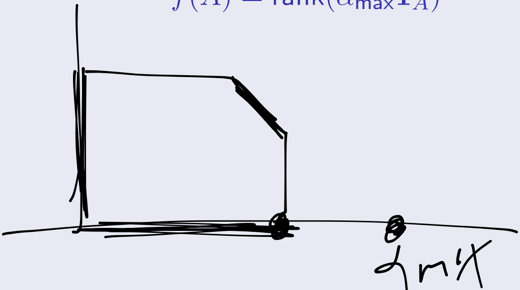
$$f(A) \triangleq \text{rank}(\alpha_{\text{max}} \mathbf{1}_A) \tag{9.26}$$



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$$\begin{aligned} f(A) + f(B) &= \text{rank}(\alpha_{\text{max}} \mathbf{1}_A) + \text{rank}(\alpha_{\text{max}} \mathbf{1}_B) && (9.27) \\ &\geq \text{rank}(\alpha_{\text{max}} \mathbf{1}_A \vee \alpha_{\text{max}} \mathbf{1}_B) + \text{rank}(\alpha_{\text{max}} \mathbf{1}_A \wedge \alpha_{\text{max}} \mathbf{1}_B) && (9.28) \end{aligned}$$

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- We are given a polymatroid P.
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- Hence, for any $x \in P$, $x(e) \leq \alpha_{\text{max}}, \forall e \in E$.
- Define a function $f : 2^V \to \mathbb{R}$ as, for any $A \subseteq E$,

$$f(A) \triangleq \text{rank}(\alpha_{\text{max}} \mathbf{1}_A) \tag{9.26}$$

- Then f is submodular since

$$f(A) + f(B) = \text{rank}(\alpha_{\text{max}} \mathbf{1}_A) + \text{rank}(\alpha_{\text{max}} \mathbf{1}_B) \tag{9.27}$$

$$\geq \text{rank}(\alpha_{\text{max}} \mathbf{1}_A \vee \alpha_{\text{max}} \mathbf{1}_B) + \text{rank}(\alpha_{\text{max}} \mathbf{1}_A \wedge \alpha_{\text{max}} \mathbf{1}_B) \tag{9.28}$$

$$= \text{rank}(\alpha_{\text{max}} \mathbf{1}_{A \cup B}) + \text{rank}(\alpha_{\text{max}} \mathbf{1}_{A \cap B}) \tag{9.29}$$

$$= f(A \cup B) + f(A \cap B) \tag{9.30}$$

...

Proof of Theorem 9.3.1

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$$P_f^+ = \{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\} \tag{9.31}$$

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- Given an $x \in P$, then for any $A \subseteq E$,

$$x(A) \le \underbrace{\max\{z(E) : z \in P, z \le \alpha_{\max}\mathbf{1}_A\} = \text{rank}(\alpha_{\max}\mathbf{1}_A)} = f(A),$$

$x(e) \le \alpha_{max} \quad \forall e \in E$

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- Hence, $P \subseteq P_f^+$.

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- Hence, $P \subseteq P_f^+$.
- We will next show that $P_f^+ \subseteq P$ to complete the proof.

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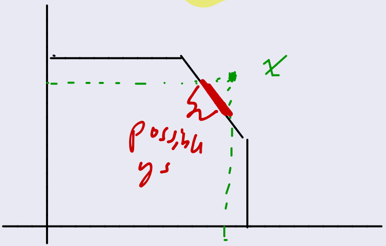
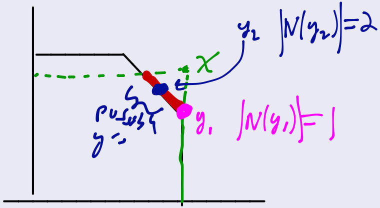
$$N(y) = \{e \in E : y(e) < x(e)\} \tag{9.32}$$





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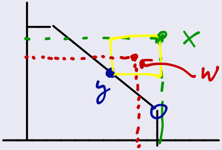
- Choose w between y and x, so that

$$y \leq w \triangleq (y + x)/2 \leq x \tag{9.33}$$

so y is also a P-basis of w.



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- Hence, $\text{rank}(x) = \text{rank}(w)$, and the set of P-bases of w are also P-bases of x.

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- For any $A \subseteq E$, define $x_A \in \mathbb{R}_+^E$ as

$$x_A(e) = \begin{cases} x(e) & \text{if } e \in A \\ 0 & \text{else} \end{cases} \tag{9.34}$$

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- Thus, $y \wedge x_{N(y)}$ is not a P-basis of $w \wedge x_{N(y)}$ since, over $N(y)$, it is neither tight at w nor tight at the rank (i.e., not a maximal independent subvector on $N(y)$).

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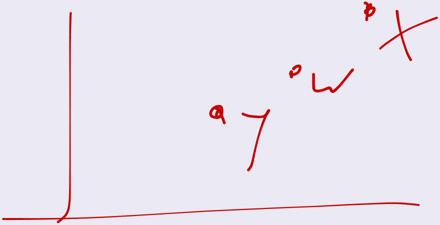

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- This contradiction means that we must have had $x \in P$.
- Therefore, $P_f^+ = P$.

□

More on polymatroids

Theorem 9.3.4

A polymatroid can equivalently be defined as a pair (E, P) where E is a finite ground set and $P \subseteq R_+^E$ is a compact non-empty set of independent vectors such that

1. *every subvector of an independent vector is independent (if $x \in P$ and $y \leq x$ then $y \in P$, i.e., down closed)*

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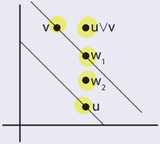

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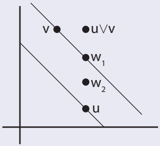


Corollary 9.3.5

The independent vectors of a polymatroid form a convex polyhedron in $\mathbb{R}_+^E$.

Review

- The next slide comes from lecture 5.

Matroids by bases

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

Theorem 9.3.1 (Matroid (by bases))

Let E be a set and $\mathcal{B}$ be a nonempty collection of subsets of E. Then the following are equivalent.

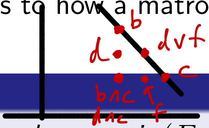
1. *$\mathcal{B}$ is the collection of bases of a matroid;*
2. *if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.*
3. *If $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.*

Properties 2 and 3 are called "exchange properties."

Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.

More on polymatroids

For any compact set P, b is a base of P if it is a maximal subvector within P. Recall the bases of matroids. In fact, we can define a polymatroid via vector bases (analogous to how a matroid can be defined via matroid bases).

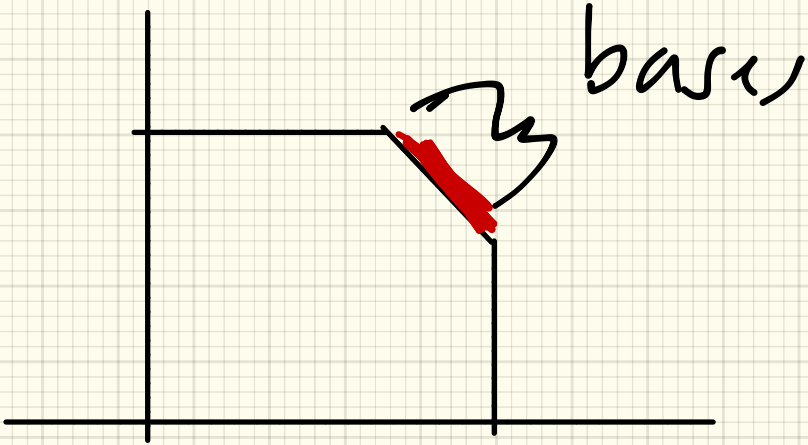


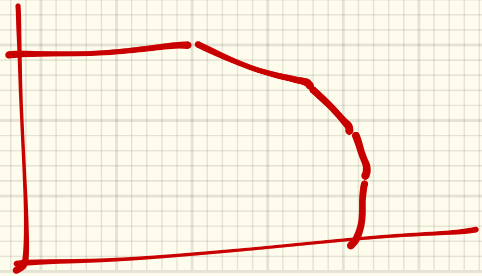
Theorem 9.3.6

A polymatroid can equivalently be defined as a pair (E, P) where E is a finite ground set and $P \subseteq R_+^E$ is a compact non-empty set of independent vectors such that

1. *every subvector of an independent vector is independent (if $x \in P$ and $y \leq x$ then $y \in P$, i.e., down closed)*
2. *if b, c are bases of P and d is such that $b \wedge c < d < b$, then there exists an f, with $d \wedge c < f \leq c$ such that $d \vee f$ is a base of P*
3. *All of the bases of P have the same rank.*

Note, all three of the above are required for a polymatroid (a matroid analogy would require the equivalent of only the first two).


base



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also, a word on terminology

- Recall how a matroid is sometimes given as (E, r) where r is the rank function.
- We mention also that the term "polymatroid" is sometimes not used for the polytope itself, but instead but for the pair (E, f),
- But now we see that (E, f) is equivalent to a polymatroid polytope, so this is sensible.

Where are we going with this?

- Consider the right hand side of Theorem **??**: $\min \left(x(A) + f(E \setminus A) : A \subseteq E \right)$

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- We are going to study this problem, and approaches that address it, as part of our ultimate goal which is to present strategies for submodular function minimization (that we will ultimately get to, in near future lectures).
- As a bit of a hint on what's to come, note that we can write it as: $x(E) + \min\left(f(A) - x(A) : A \subseteq E\right)$ where f is a polymatroid function.

Another Interesting Fact: Matroids from polymatroid functions

Theorem 9.3.7

Given integral polymatroid function f, let $(E, \mathcal{F})$ be a set system with ground set E and set of subsets $\mathcal{F}$ such that

$$\forall F \in \mathcal{F}, \;\; \forall \emptyset \subset S \subseteq F, |S| \leq f(S) \tag{9.37}$$

Then $M = (E, \mathcal{F})$ is a matroid.

Proof.

Exercise □

And its rank function is Exercise.

Matroid instance of Theorem ??

- Considering Theorem **??**, the matroid case is now a special case, where we have that:

Corollary 9.3.8

We have that:

$$\max\{y(E) : y \in P_{\textit{ind. set}}(M), y \leq x\} = \min\{r_M(A) + x(E \setminus A) : A \subseteq E\} \tag{9.38}$$

where r_M is the matroid rank function of some matroid.

Most violated inequality problem in matroid polytope case

- Consider

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- Suppose we have any $x \in \mathbb{R}_+^E$ such that $x \notin P_r^+$.
- The most violated inequality when x is considered w.r.t. P_r^+ corresponds to the set A that maximizes $x(A) - r_M(A)$, i.e., $\max\{x(A) - r_M(A) : A \subseteq E\}$.

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- This corresponds to $\min\{r_M(A) + x(E \setminus A) : A \subseteq E\}$ since x is modular and $x(E \setminus A) = x(E) - x(A)$.

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- This corresponds to $\min\{r_M(A) + x(E \setminus A) : A \subseteq E\}$ since x is modular and $x(E \setminus A) = x(E) - x(A)$.
- More importantly, $\min\{r_M(A) + x(E \setminus A) : A \subseteq E\}$ a form of submodular function minimization, namely $\min\{r_M(A) - x(A) : A \subseteq E\}$ for a submodular function consisting of a difference of matroid rank and modular (so no longer necessarily monotone, nor positive).

Problem to Solve

In particular, we will solve the following problem:

- Given a matroid $M = (E, \mathcal{I})$ along with an independence testing oracle (i.e., for any $A \subseteq E$, tells us if $A \in \mathcal{I}$ or not), and a vector $x \in \mathcal{R}_+^E$;

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- find: a maximizing $y \in P_{\text{ind. set}}$ with $y \leq x$, and moreover (as a byproduct of the algorithm), express y as a convex combination of incidence vectors of independent sets in M, and also return a set $A \subseteq E$ that satisfies $y(E) = r_M(A) + x(E \setminus A)$. *Of course, for any such y we must have that $y(E) \leq r(A) + x(E \setminus A)$.*

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- This will also run in polynomial time.

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- Each update will, of course, ensure that $y \in P_{\text{ind. set}}$, but also we'll keep $y \le x$.
- It's going to take us a few lectures to fully develop this algorithm, so please keep in mind of the overall goal.

Bipartite Matching

- Consider a bipartite graph $G = (V, F, E)$ where left nodes are V, right nodes are F, and $E \subseteq V \times F$ are the only edges.

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- Given $A \subseteq E$, an alternating path S (relative to A) is an (undirected) path of unique edges that are alternatively in A and not in A. I.e., if $S = (e_1, e_2, \ldots, e_s)$ is an alternating path, then $S_{1/2} \stackrel{\text{def}}{=} S \setminus A$ where $S_{1/2}$ is either the odd or the even elements of S.

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Algorithm 8.1: Alternating Path Bipartite Matching

Let A be an arbitrary (including empty) matching in $G = (V, F, E)$;
while *There exists an augmenting path* S *in* G **do**
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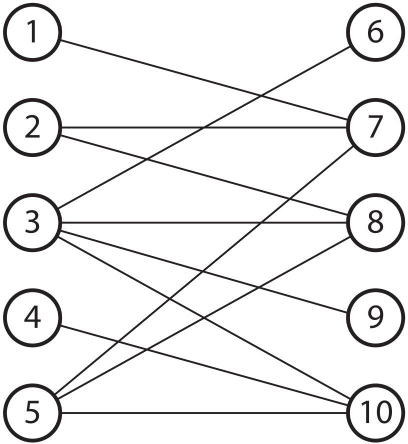
Algorithm 8.1: Alternating Path Bipartite Matching

1 Let A be an arbitrary (including empty) matching in $G = (V, F, E)$;
2 **while** *There exists an augmenting path S in G* **do**
3     $A \leftarrow A \ominus S$;

- This can easily be made to run in $O(m^2 n)$, where $|V| = m$, $|F| = n$, $m \leq n$, but it can be made to run much faster as well (see Schrijver-2003).

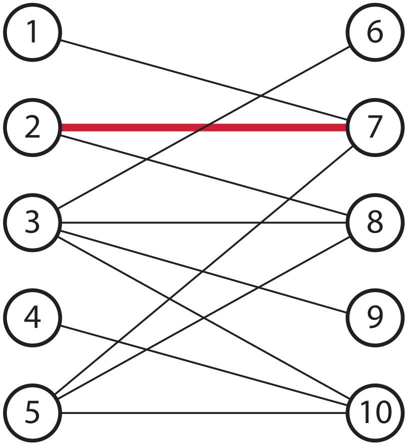
Bipartite Matching Example

Consider the following bipartite graph $G = (V, F, E)$ with $|V| = |F| = 5$. Any edge is an augmenting path since it will adjoin two unmatched vertices.



Bipartite Matching Example

Any edge, not intersecting nodes adjacent to current matching is an augmenting path.



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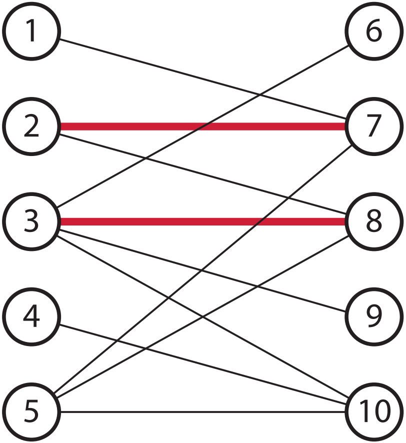
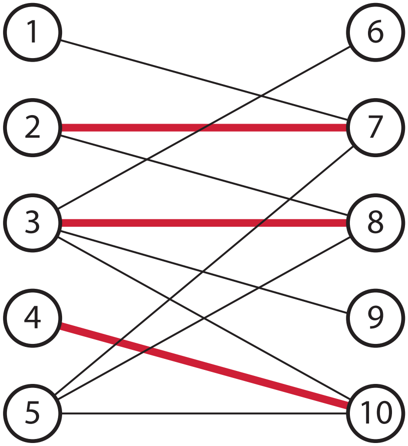


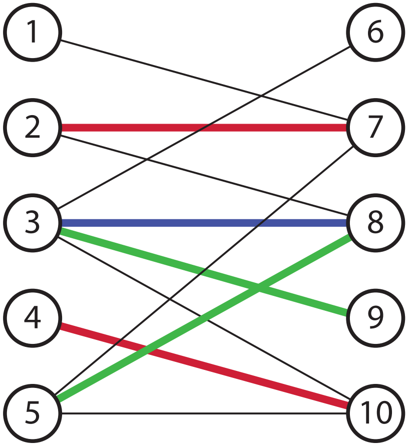
Bipartite Matching Example

No possible further single edge addition at this point. We need a multi-edge augmenting path if it exists.



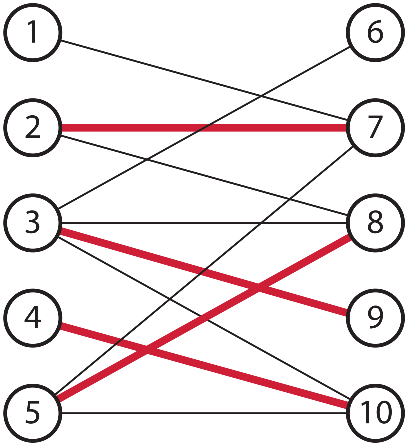
Bipartite Matching Example

Augmenting path is green and blue edges (blue is already in matching, green is new).



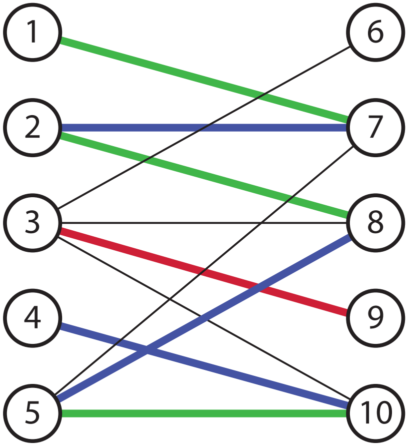
Bipartite Matching Example

Removing blue from matching and adding green leads to higher cardinality matching.



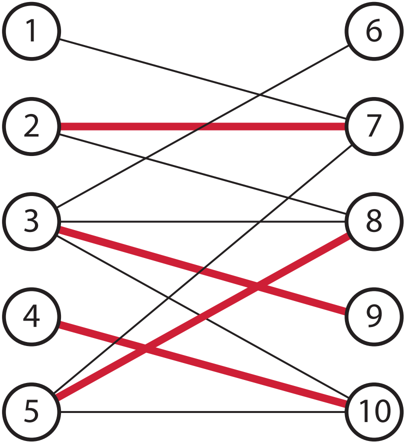
Bipartite Matching Example

At this point, resulting alternating path is not augmenting, since it is not between two unmatched vertices (and no augmenting path is possible).



Bipartite Matching Example

At this point, resulting alternating path is not augmenting, since it is not between two unmatched vertices (and no augmenting path is possible). At this point, matching is maximum cardinality.



Review

- The next slide is from lecture 7 and the one after from lecture 4.

Matroid Intersection

- Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$.
- While $(V, \mathcal{I}_1 \cap \mathcal{I}_2)$ is typically not a matroid (Exercise: show graphical example.), we might be interested in finding the maximum size common independent set. That is, find $\max |X|$ such that both $X \in \mathcal{I}_1$ and $X \in \mathcal{I}_2$.

Theorem 9.5.5

Let M_1 and M_2 be given as above, with rank functions r_1 and r_2. Then the size of the maximum size set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by

$$(r_1 * r_2)(V) \triangleq \min_{X \subseteq V} \Big(r_1(X) + r_2(V \setminus X) \Big) \tag{9.7}$$

This is an instance of the convolution of two submodular functions, f_1 and f_2 that, evaluated at $Y \subseteq V$, is written as:

$$(f_1 * f_2)(Y) = \min_{X \subseteq Y} \Big(f_1(X) + f_2(Y \setminus X) \Big) \tag{9.8}$$

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- Let $V = V_1 \cup V_2 \cup \cdots \cup V_\ell$ be a partition of V into blocks or disjoint sets (disjoint union). Define a set of subsets of V as

$$\mathcal{I} = \{X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \dots, \ell\}. \tag{9.3}$$

where $k_1, \dots, k_\ell$ are fixed parameters, $k_i \geq 0$. Then $M = (V, \mathcal{I})$ is a matroid.

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- Note that a k-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$.
- We'll show that property (I3') in Def **??** holds. If $X, Y \in \mathcal{I}$ with $|Y| > |X|$, then there must be at least one i with $|Y \cap V_i| > |X \cap V_i|$. Therefore, adding one element $e \in V_i \cap (Y \setminus X)$ to X won't break independence.

Matroid Intersection and Bipartite Matching

- Why might we want to do matroid intersection?

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- Therefore, a matching in G is simultaneously independent in both M_V and M_F and finding the maximum matching is finding the maximum cardinality set independent in both matroids.
- For the bipartite graph case, therefore, this can be solved in polynomial time.

Matroid Intersection and Network Communication

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- This is again a matroid intersection problem.

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- Let M_1 be the cycle matroid on G'.
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- Then a Hamiltonian cycle exists iff there is an n-element intersection of M_1, M_2, and M_3.

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- Since TSP is NP-complete, we obviously can't solve matroid intersections of 3 more matroids, unless P=NP.
- But bipartite graph example gives us hope for 2 matroids, and also ideas for an algorithm ...

Recall from Lecture 5: Matroids by circuits

A set is independent if and only if it contains no circuit. Therefore, it is not surprising that circuits can also characterize a matroid.

Theorem 9.5.1

Matroid (by circuits) Let E be a set and $\mathcal{C}$ be a collection of nonempty subsets of E, such that no two sets in $\mathcal{C}$ are contained in each other. Then the following are equivalent.

1. *(C1) $\mathcal{C}$ is the collection of circuits of a matroid;*
2. *(C2) if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$;*
3. *(C3) if $C, C' \in \mathcal{C}$, and $x \in C \cap C'$, and $y \in C \setminus C'$, then $(C \cup C') \setminus \{x\}$ contains a set in $\mathcal{C}$ containing y;*

Circuits

Lemma 9.5.2

Let $I \in \mathcal{I}(M)$, and $e \in E$, then $I \cup \{e\}$ contains at most one circuit in M.

Proof.

- Suppose, to the contrary, that there are two distinct circuits C_1, C_2 such that $C_1 \cup C_2 \subseteq I \cup \{e\}$.

□

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- Then $e \in C_1 \cap C_2$, and by (C2), there is a circuit C_3 of M s.t. $C_3 \subseteq (C_1 \cup C_2) \setminus \{e\} \subseteq I$

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Circuits

Lemma 9.5.2

Let $I \in \mathcal{I}(M)$, and $e \in E$, then $I \cup \{e\}$ contains at most one circuit in M.

Proof.

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- This contradicts the independence of I.

$\square$

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In general, let $C(I, e)$ be the unique circuit associated with $I \cup \{e\}$ (commonly called the fundamental circuit in M w.r.t. I and e).

Matroid Intersection Algorithm Idea

- Consider two matroids $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ and start with any $I \in \mathcal{I}_1 \cap \mathcal{I}_2$.

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- Then $\text{span}_1(I) = \text{span}_1(I - v_2 + v_3)$.

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- Then $\operatorname{span}_1(I) = \operatorname{span}_1(I - v_2 + v_3)$.
- Moreover, since $I + v_1 \in \mathcal{I}_1$, $v_1 \notin \operatorname{span}_1(I)$, so $\operatorname{span}_1(I + v_1) = \operatorname{span}_1(I + v_1 - v_2 + v_3)$.

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- Then $\text{span}_1(I) = \text{span}_1(I - v_2 + v_3)$.
- Moreover, since $I + v_1 \in \mathcal{I}_1$, $v_1 \notin \text{span}_1(I)$, so $\text{span}_1(I + v_1) = \text{span}_1(I + v_1 - v_2 + v_3)$.
- But $I + v_1 - v_2 + v_3$ might not be independent in M_2 again, so we need to find an $v_4 \in C_2(I + v_1 - v_2, v_3)$ to remove, and so on.

Matroid Intersection Algorithm Idea

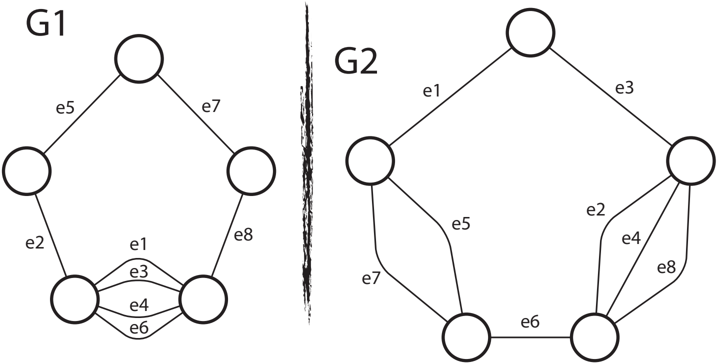
- Hopefully (eventually) we'll find an odd length sequence $S = (v_1, v_2, \dots, v_s)$ such that we will be independent in both M_1 andM_2 and thus be one greater in size than I.

Matroid Intersection Algorithm Idea

- Hopefully (eventually) we'll find an odd length sequence $S = (v_1, v_2, \dots, v_s)$ such that we will be independent in both M_1 andM_2 and thus be one greater in size than I.
- We then replace I with $I \ominus S$ (quite analogous to the bipartite matching case), and start again.

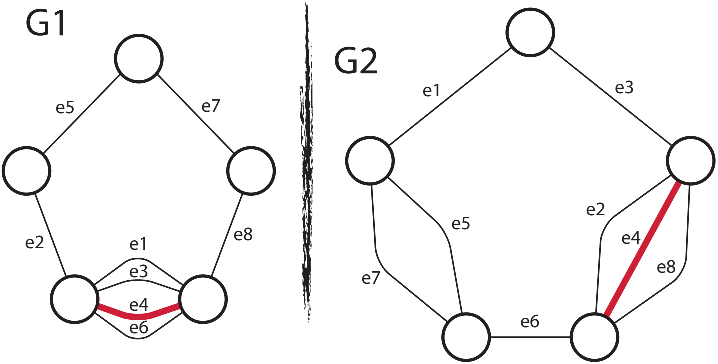
Graphic Matroid Intersection Example

Consider the following two graph $G_1 = (V_1, E)$ and $G_2 = (V_2, E)$ and corresponding matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$. Any edge is independent in both (an augmenting "sequence") since a single edge can't create a circuit starting at $I = \emptyset$. We start with e_4.



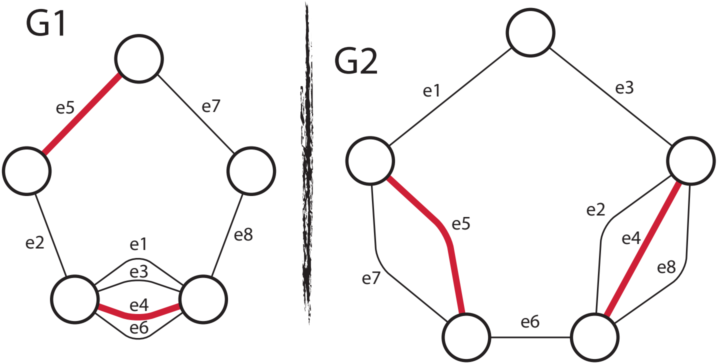
Graphic Matroid Intersection Example

Adding edge $I \leftarrow I + e_4$ creates a circuit neither in M_1 nor M_2. We can add another single edge w/o creating a circuit in either matroid.



Graphic Matroid Intersection Example

$e_5 \in E - \text{span}_1(\{e_4\})$. Then, after $I \leftarrow I + e_5$, (i.e., when $I = \{e_4, e_5\}$) we're still independent in M_2, but no further single edge additions possible w/o creating a circuit (why?).



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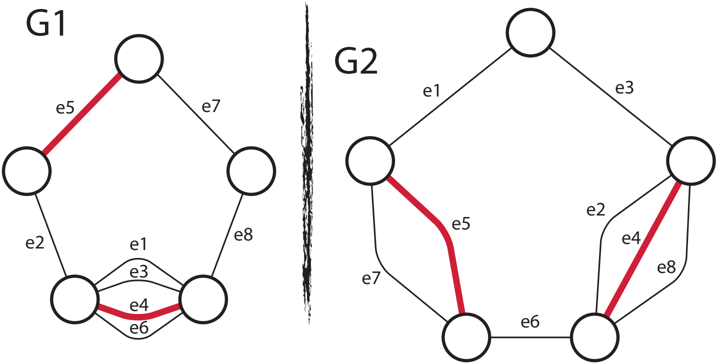
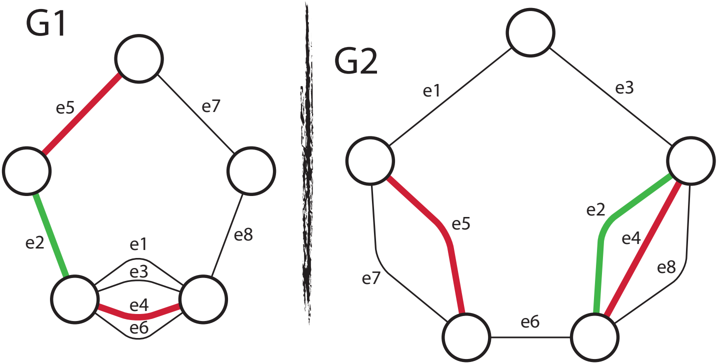
$e_5 \in E - \text{span}_1(\{e_4\})$. Then, after $I \leftarrow I + e_5$, (i.e., when $I = \{e_4, e_5\}$) we're still independent in M_2, but no further single edge additions possible w/o creating a circuit (why?). We need a multi-edge "augmenting sequence" if it exists.



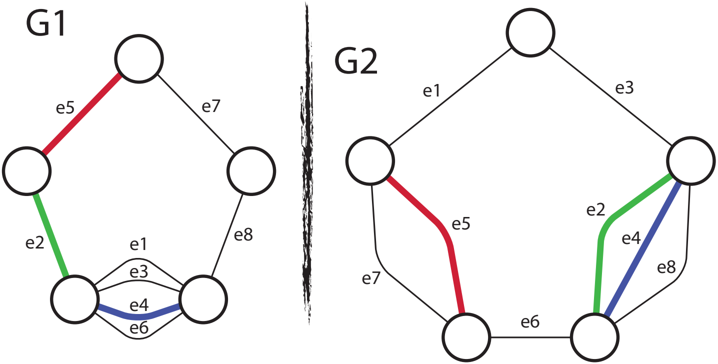
Graphic Matroid Intersection Example

Augmenting sequence is green and blue edges (blue is already in I, green is new). We choose $e_2 \in E - \mathrm{span}_1(I)$, but now $I + e_2$ is not independent in M_2.



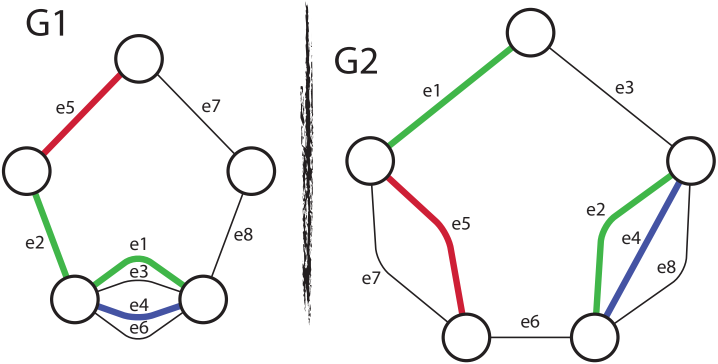
Graphic Matroid Intersection Example

So there must exist $C_2(I, e_2)$. We choose $e_4 \in C_2(I, e_2)$ to remove.



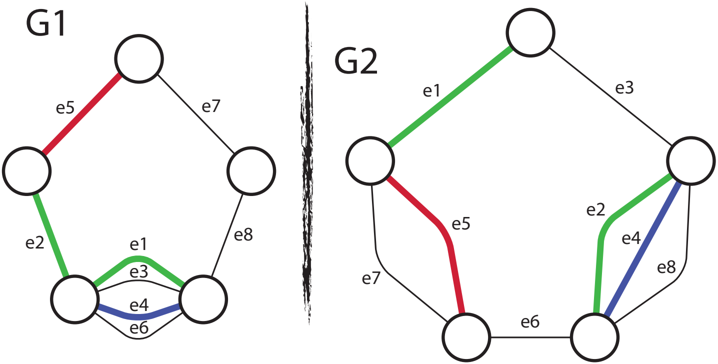
Graphic Matroid Intersection Example

Next, we choose $e_1 \in \text{span}_1(I) - \text{span}_1(I - e_4)$ to add.



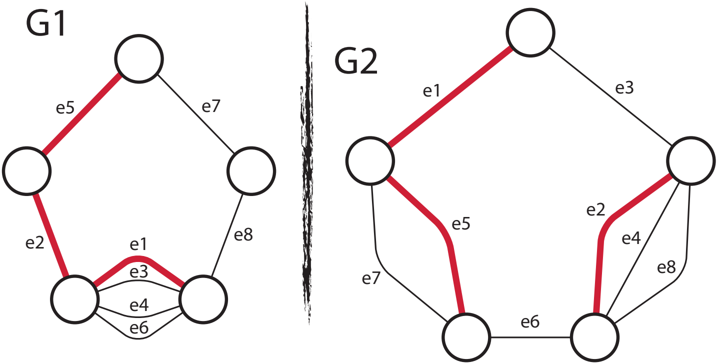
Graphic Matroid Intersection Example

Next, we choose $e_1 \in \text{span}_1(I) - \text{span}_1(I - e_4)$ to add. In this case, we not only have $\text{span}_1(I + e_2) = \text{span}_1(I + e_2 - e_4 + e_1)$, but we also have that $(I + e_2 - e_4) + e_1 \in \mathcal{I}_2$.



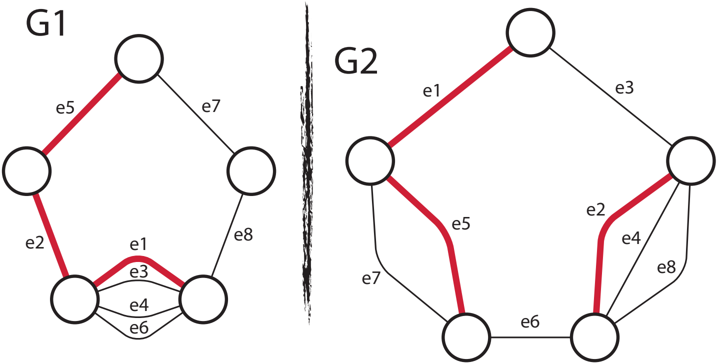
Graphic Matroid Intersection Example

Removing blue and adding green leads to higher cardinality independent set in both matroids. This corresponds to doing $I \leftarrow I \ominus S$ where $S = (e_2, e_4, e_1)$ and $I = \{e_4, e_5\}$.



Graphic Matroid Intersection Example

At this point, are any further augmenting sequences possible? Exercise.



Alternating and Augmenting Sequences

- Let I be an intersection of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ (i.e., $I \in \mathcal{I}_1 \cap \mathcal{I}_2$).

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- Let $S = (e_1, e_2, \dots, e_s)$ be a sequence of distinct elements, where $e_i \in E - I$ for i odd, and $e_i \in I$ for i even, and let $S_i = (e_1, e_2, \dots, e_i)$. We say that S is an alternating sequence w.r.t. I if the following are true.

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  1. $I + e_1 \in \mathcal{I}_1$
  2. For all even i, $\text{span}_2(I \ominus S_i) = \text{span}_2(I)$ which implies that $I \ominus S_i \in \mathcal{I}_2$.

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  2. For all even i, $\operatorname{span}_2(I \ominus S_i) = \operatorname{span}_2(I)$ which implies that $I \ominus S_i \in \mathcal{I}_2$.
  3. For all odd i, $\operatorname{span}_1(I \ominus S_i) = \operatorname{span}_1(I + e_1)$, and therefore $I \ominus S_i \in \mathcal{I}_1$.

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  3. For all odd i, $\operatorname{span}_1(I \ominus S_i) = \operatorname{span}_1(I + e_1)$, and therefore $I \ominus S_i \in \mathcal{I}_1$.
- Lastly, if also, $|S| = s$ is odd, and $I \ominus S \in \mathcal{I}_2$, then S is called an augmenting sequence w.r.t. I.

Alternating and Augmenting Sequences

- If I admits an augmenting sequence S, then the above argument shows that $I \ominus S$ is independent in M_1, independent in M_2, and also we have that $|I| + 1 = |I \ominus S|$.

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- Thus, by finding augmenting sequences, we can increase the size of the matroid intersection until we stop. Moreover (and this next thing should be a theorem), if there is an augmenting sequence, then the intersection is not maximum.
- We next wish to show that, if the intersection is not maximum, then there is an augmenting sequence.

Border graphs

- We construct an auxiliary directed bipartite graph (Border graph) $B(I) = (E \setminus I, I, Z)$, relative to the current I, that will help us with this problem. The graph has only directed edges from $E \setminus I$ to I, or from I back to $E \setminus I$.

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- Left-going edges: For each $e_i \in \text{span}_1(I) \setminus I$, create $e_i \leftarrow e_j$ directed edge $(e_j, e_i) \in Z$ for any $e_j \in C_1(I, e_i) \setminus \{e_i\}$. Note $e_j \in I$ and $e_i \in E \setminus I$.

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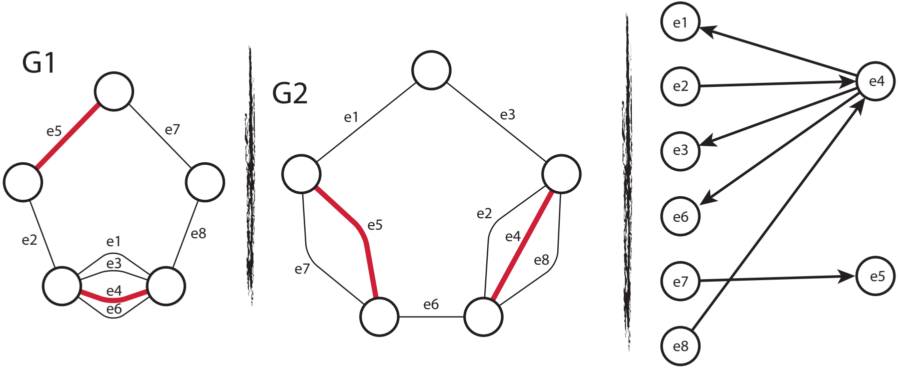
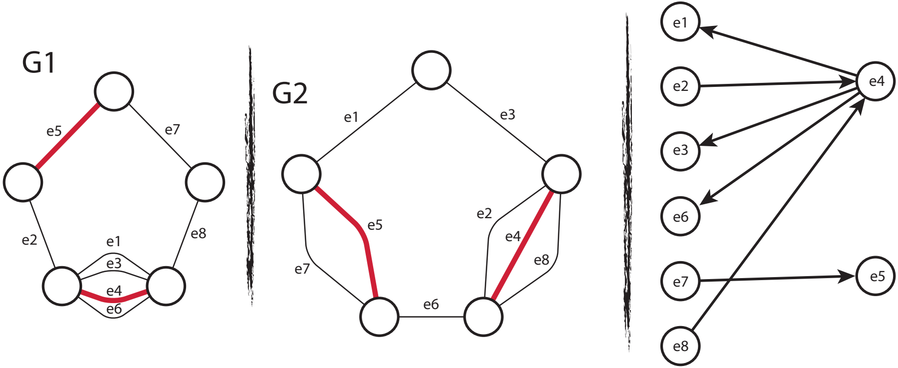
- Right-going edges: For each $e_i \in \text{span}_2(I) \setminus I$, create $e_i \rightarrow e_j$ edge $(e_i, e_j) \in Z$ for any $e_j \in C_2(I, e_i) \setminus \{e_i\}$.
- If $e_i \notin \text{span}_2(I)$, then e_i has out-degree zero (a sink).

Border graph Example



- $\{e_2, e_7, e_8\}$ are sources and $\{e_1, e_3, e_6\}$ are sinks. $\mathrm{span}_1(I) \setminus I = \{e_1, e_3, e_6\}$ and $\mathrm{span}_2(I) \setminus I = \{e_7, e_2, e_8\}$

Border graph Example



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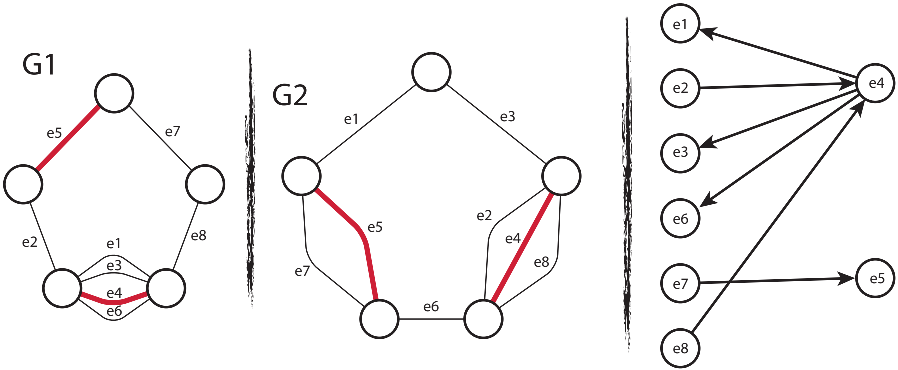
- Augmenting sequences are (e_2, e_4, e_1), (e_2, e_4, e_3), and (e_2, e_4, e_6), all of which are dipaths in the Border graph.

Border graph Example



- $\{e_2, e_7, e_8\}$ are sources and $\{e_1, e_3, e_6\}$ are sinks. $\text{span}_1(I) \setminus I = \{e_1, e_3, e_6\}$ and $\text{span}_2(I) \setminus I = \{e_7, e_2, e_8\}$
- Augmenting sequences are (e_2, e_4, e_1), (e_2, e_4, e_3), and (e_2, e_4, e_6), all of which are dipaths in the Border graph.
- Are there others?

Identifying Augmenting Sequences

Lemma 9.5.3

If S is a source-sink path in $B(I)$, and there is no shorter source-sink path between the same source and sink (i.e., there are no short-cuts), then S is an augmenting sequence w.r.t. I.

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Lemma 9.5.4

Let I and J be intersections such that $|I| + 1 = |J|$. Then there exists a source-sink path S in $B(I)$ where $S \subseteq I \ominus J$.

Identifying Augmenting Sequences

Theorem 9.5.5

Let I_p and I_{p+1} be intersections of M_1 and M_2 with p and $p+1$ elements respectively. Then there exists an augmenting sequence $S \subseteq I_p \ominus I_{p+1}$ w.r.t. I_p.

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Theorem 9.5.7

For any intersection I, there exists a maximum cardinality intersection I^* such that $\text{span}_1(I) \subseteq \text{span}_1(I^*)$ and $\text{span}_2(I) \subseteq \text{span}_2(I^*)$.

Matroid Partition Problem

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- We wish to, if possible, partition E into k blocks, $I_i, i \in \{1, 2, \dots, k\}$ where $I_i \in \mathcal{I}_i$.
- Moreover, we want partition to be lexicographically maximum, that is $|I_1|$ is maximum, $|I_2|$ is maximum given $|I_1|$, and so on.

Matroid Partition Problem

Theorem 9.6.1

Let M_i be a collection of k matroids as described. Then, a set $I \subseteq E$ can be partitioned into k subsets $I_i, i = 1 \ldots k$ where $I_i \in \mathcal{I}_i$ is independent in matroid i, if and only if, for all $A \subseteq I$

$$|A| \leq \sum_{i=1}^{k} r_i(A) \tag{9.40}$$

where r_i is the rank function of M_i.

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- But considering vector of all ones $\mathbf{1} \in \mathbb{R}_+^E$, this is the same as

$$\frac{1}{k}\mathbf{1}(A) \leq r(A) \;\; \forall A \subseteq E \tag{9.42}$$

Matroid Partition Problem

- Recall definition of matroid polytope

$$P_r^+ = \{y \in \mathbb{R}_+^E : y(A) \le r(A) \text{ for all } A \subseteq E\} \tag{9.43}$$

Matroid Partition Problem

- Recall definition of matroid polytope

$$P_r^+ = \left\{ y \in \mathbb{R}_+^E : y(A) \le r(A) \text{ for all } A \subseteq E \right\} \tag{9.43}$$

- Then we see that this special case of the matroid partition problem is just testing if $\frac{1}{k}\mathbf{1} \in P_r^+$, a problem of testing the membership in matroid polyhedra.