Submodular Functions, Optimization, and Applications to Machine Learning

— Fall Quarter, Lecture 8 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/

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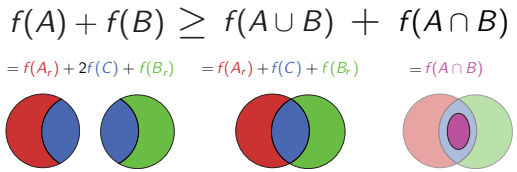
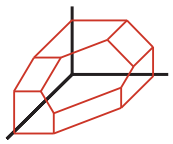
Oct 26th, 2020





Announcements, Assignments, and Reminders

- Homework 2.
- Reminder, all lectures are being recorded and posted to youtube. To get the links, see our announcements (`https://canvas.uw.edu/courses/1397085/announcements`).

Class Road Map - EE563

- L1(9/30): Motivation, Applications, Definitions, Properties
- L2(10/5): Sums concave(modular), uses (diversity/costs, feature selection), information theory
- L3(10/7): Monge, More Definitions, Graph and Combinatorial Examples,
- L4(10/12): Graph & Combinatorial Examples, Matrix Rank, Properties, Other Defs, Independence
- L5(10/14): Properties, Defs of Submodularity, Independence
- L6(10/19): Matroids, Matroid Examples, Matroid Rank,
- L7(10/21): Matroid Rank, More on Partition Matroid, Laminar Matroids, System of Distinct Reps, Transversals
- L8(10/26): Transversal Matroid, Matroid and representation, Dual Matroid, Other Matroid Properties, Combinatorial Geometries, Matroid and Greedy
- L9(10/28):
- L10(11/2):
- L11(11/4):
- L12(11/9):
- L–(11/11): Veterans Day, Holiday
- L13(11/16):
- L14(11/18):
- L15(11/23):
- L16(11/25):
- L17(11/30):
- L18(12/2):
- L19(12/7):
- L20(12/9): maximization.

Last day of instruction, Fri. Dec 11th. Finals Week: Dec 12-18, 2020

System of Distinct Representatives

- Let $(V, \mathcal{V})$ be a set system (i.e., $\mathcal{V} = (V_i : i \in I)$ where $V_i \subseteq V$ for all i), and I is an index set. Hence, $|I| = |\mathcal{V}|$.
- A family $(v_i : i \in I)$ with $v_i \in V$ is said to be a system of distinct representatives of $\mathcal{V}$ if $\exists$ a bijection $\pi : I \leftrightarrow I$ such that $v_i \in V_{\pi(i)}$ and $v_i \neq v_j$ for all $i \neq j$.
- In a system of distinct representatives, there is a requirement for the representatives to be distinct. We can re-state (and rename) this as a:

Definition 8.2.3 (transversal)

Given a set system $(V, \mathcal{V})$ and index set I for $\mathcal{V}$ as defined above, a set $T \subseteq V$ is a transversal of $\mathcal{V}$ if there is a bijection $\pi : T \leftrightarrow I$ such that

$$x \in V_{\pi(x)} \text{ for all } x \in T \tag{8.19}$$

- Note that due to $\pi : T \leftrightarrow I$ being a bijection, all of I and T are "covered" (so this makes things distinct automatically).

When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
- Given a set system $(V, \mathcal{V})$ with $\mathcal{V} = (V_i : i \in I)$, and $V_i \subseteq V$ for all i. Then, for any $J \subseteq I$, let

$$V(J) = \cup_{j \in J} V_j \tag{8.19}$$

so $|V(J)| : 2^I \to \mathbb{Z}_+$ is the set cover func. (we know is submodular).
- We have

Theorem 8.2.3 (Hall's theorem)

Given a set system $(V, \mathcal{V})$, the family of subsets $\mathcal{V} = (V_i : i \in I)$ has a transversal $(v_i : i \in I)$ iff for all $J \subseteq I$

$$|V(J)| \geq |J| \tag{8.20}$$

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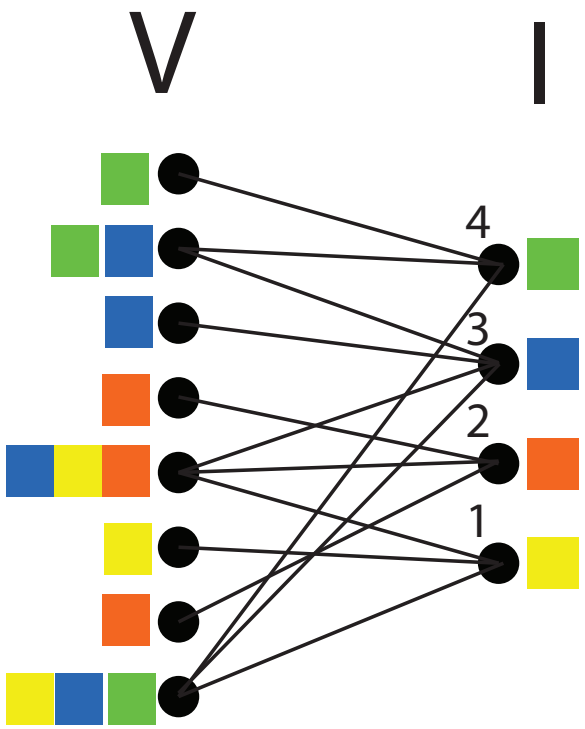
- Hall's theorem ($\forall J \subseteq I, |V(J)| \geq |J|$) as a bipartite graph.



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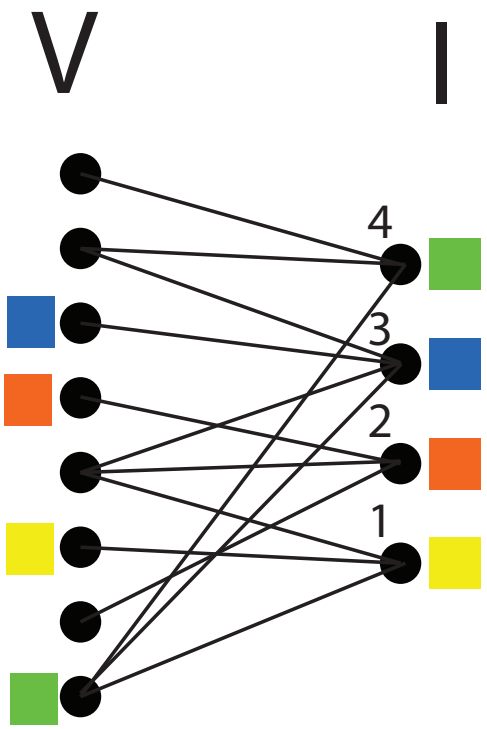

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so $|V(J)| : 2^I \to \mathbb{Z}_+$ is the set cover func. (we know is submodular).
- Moreover, we have

Theorem 8.2.4 (Rado's theorem (1942))

If $M = (V, r)$ is a matroid on V with rank function r, then the family of subsets $(V_i : i \in I)$ of V has a transversal $(v_i : i \in I)$ that is <u>independent in M</u> iff for all $J \subseteq I$

$$r(V(J)) \geq |J| \tag{8.21}$$

More general conditions for existence of transversals

Theorem 8.2.3 (Polymatroid transversal theorem)

If $\mathcal{V} = (V_i : i \in I)$ is a finite family of non-empty subsets of V, and $f : 2^V \to \mathbb{Z}_+$ is a non-negative, integral, monotone non-decreasing, and submodular function, then $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that

$$f(\cup_{i \in J}\{v_i\}) \geq |J| \text{ for all } J \subseteq I \tag{8.19}$$

if and only if

$$f(V(J)) \geq |J| \text{ for all } J \subseteq I \tag{8.20}$$

- Given Theorem **??**, we immediately get Theorem 8.2.3 by taking $f(S) = |S|$ for $S \subseteq V$.
- We get Theorem 8.2.4 by taking $f(S) = r(S)$ for $S \subseteq V$, the rank function of the matroid.

Transversal Matroid

Transversals, themselves, define a matroid.

Theorem 8.3.1

If $\mathcal{V}$ is a family of finite subsets of a ground set V, then the collection of partial transversals of $\mathcal{V}$ is the set of independent sets of a matroid $M = (V, \mathcal{V})$ on V.

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- This means that the transversals of $\mathcal{V}$ are the bases of matroid M.
- Therefore, all maximal partial transversals of $\mathcal{V}$ have the same cardinality!

Transversals and Bipartite Matchings

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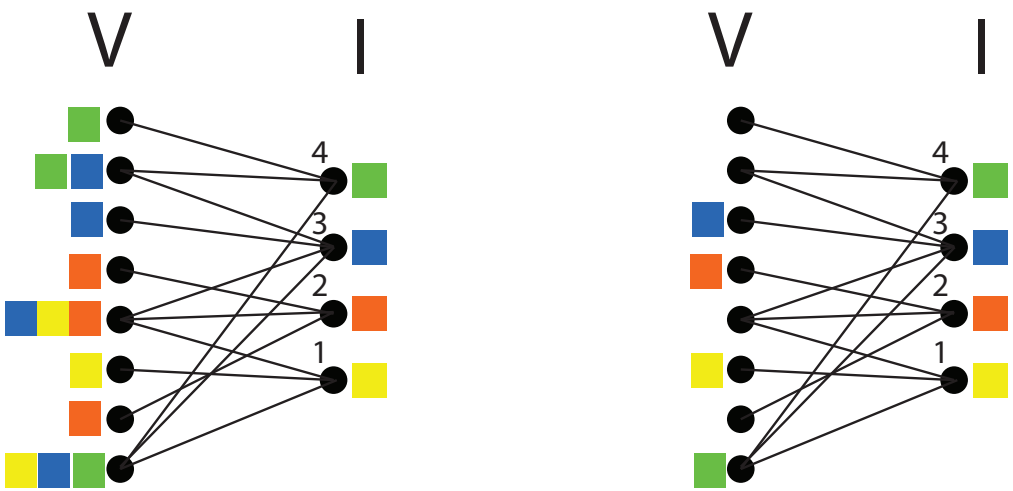

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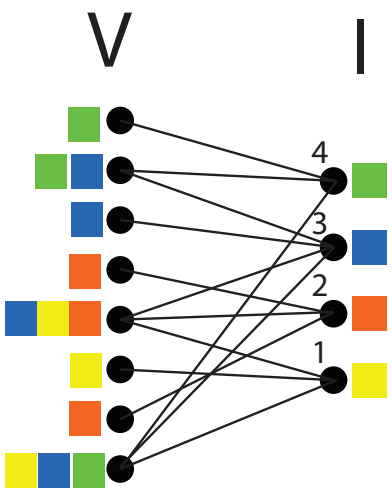

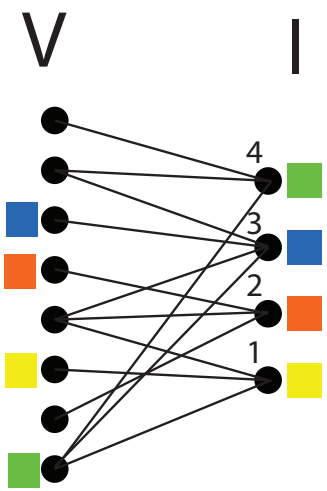

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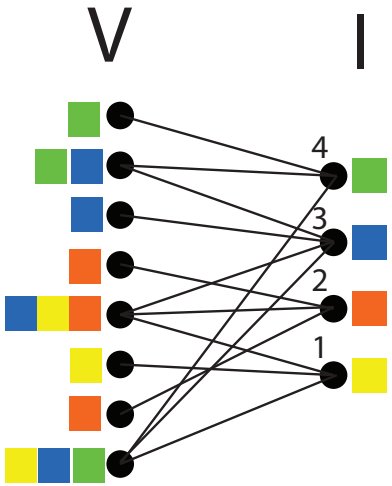
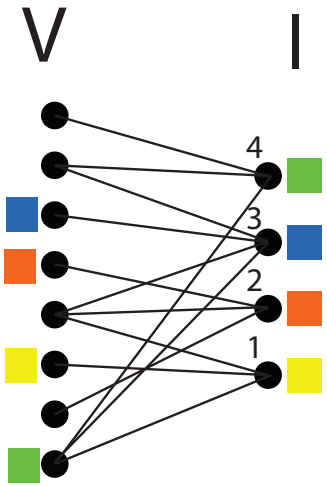
Lemma 8.3.2

A subset $T \subseteq V$ is a partial transversal of $\mathcal{V}$ iff there is a matching in (V, I, E) in which every edge has one endpoint in T (T matched into I).





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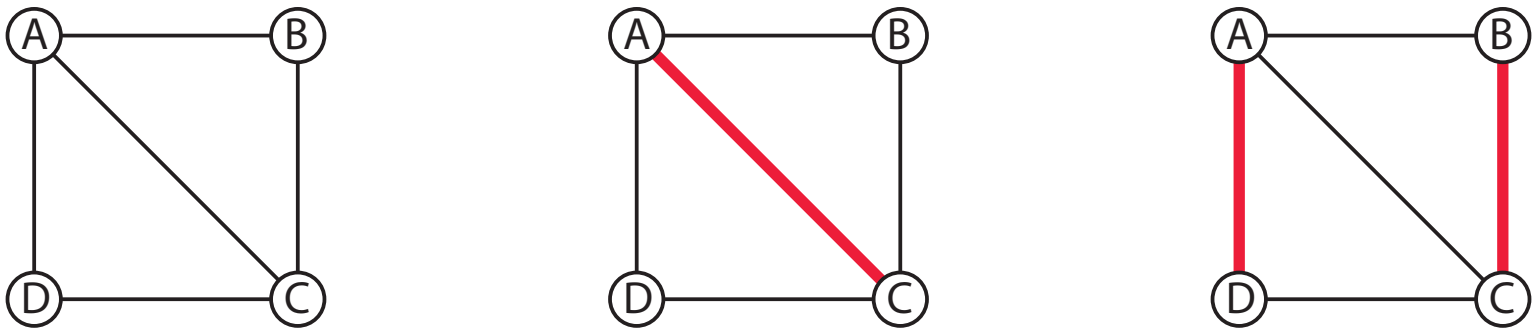

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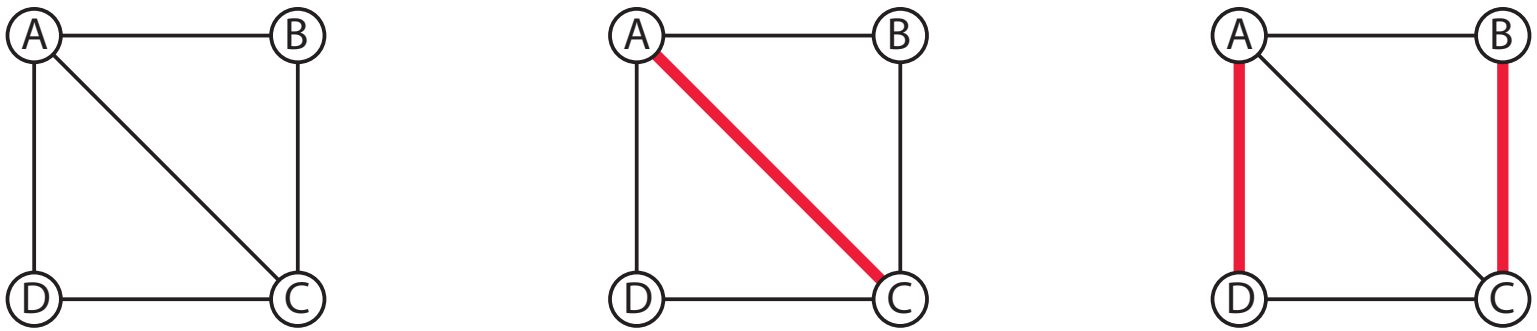

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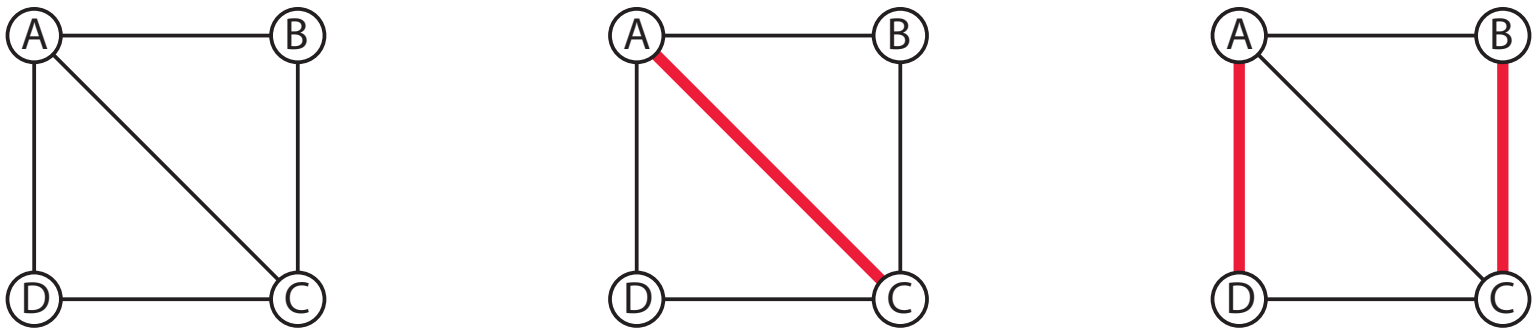


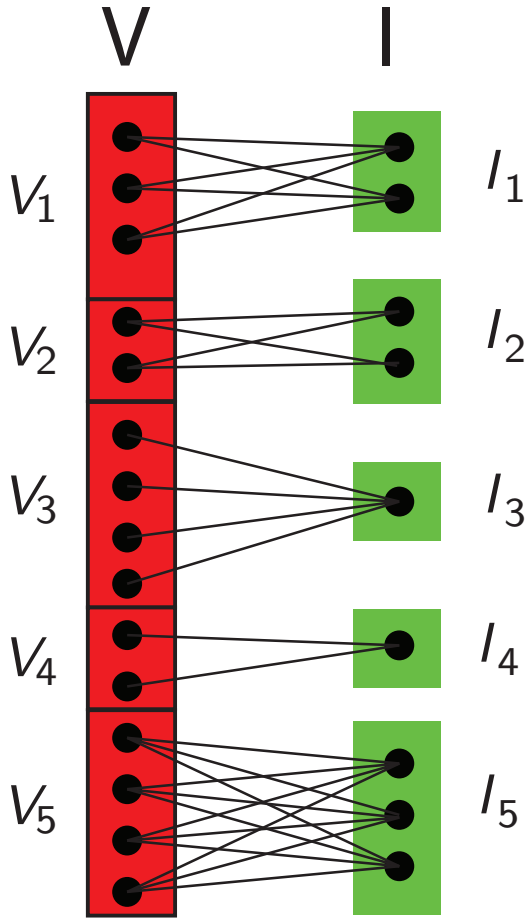
- $\{AC\}$ is a maximum matching, as is $\{AD, BC\}$, but they are not the same size.
- Let $\mathcal{M}$ be the set of matchings in an arbitrary graph $G = (V, E)$. Hence, $(E, \mathcal{M})$ is a set system. I1 holds since $\emptyset \in \mathcal{M}$. I2 also holds since if $M \in \mathcal{M}$ is a matching, then so is any $M' \subseteq M$. I3 doesn't hold (as seen above). Exercise: fully characterize the problem of finding the largest subset $\mathcal{M}' \subset \mathcal{M}$ of matchings so that $(E, \mathcal{M}')$ also satisfies I3?

Review

Next slide is from lecture 7.

Partition Matroid, rank as matching

- Example where $\ell = 5$, $(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3)$.



- Recall, $\Gamma : 2^V \to \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of X is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.
- Here, for $X \subseteq V$, we have $\Gamma(X) = \{i \in I : (v, i) \in E(G) \text{ and } v \in X\}$.
- For such a constructed bipartite graph, the rank function of a partition matroid is $r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i) =$ the maximum matching involving X.

Morphing Partition Matroid Rank

- Recall the partition matroid rank function. Note, $k_i = |I_i|$ in the bipartite graph representation, and since a matroid, w.l.o.g., $|V_i| \geq k_i$ (also, recall, $V(J) = \cup_{j \in J} V_j$).

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- Start with partition matroid rank function in the subsequent equations.

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$$= \sum_{i \in \{1,\dots,\ell\}} \min_{J_i \subseteq I_i} (|V(J_i) \cap A| + |I_i \setminus J_i|) \tag{8.5}$$

... Morphing Partition Matroid Rank

- Continuing,

$$r(A) = \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} \left(|V(J_i) \cap V(I_i) \cap A| - |I_i \cap J_i| + |I_i| \right) \tag{8.6}$$

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- In fact, this bottom (more general) expression is the expression for the rank of a transversal matroid.

Partial Transversals Are Independent Sets in a Matroid

In fact, we have

Theorem 8.3.3

Let $(V, \mathcal{V})$ where $\mathcal{V} = (V_1, V_2, \dots, V_\ell)$ be a subset system. Let $I = \{1, \dots, \ell\}$. Let $\mathcal{I}$ be the set of partial transversals of $\mathcal{V}$. Then $(V, \mathcal{I})$ is a matroid.

Proof.

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- We already saw that if T is a partial transversal of $\mathcal{V}$, and if $T' \subseteq T$, then T' is also a partial transversal. So (I2') holds.

$\square$

Partial Transversals Are Independent Sets in a Matroid

In fact, we have

Theorem 8.3.3

Let $(V, \mathcal{V})$ where $\mathcal{V} = (V_1, V_2, \ldots, V_\ell)$ be a subset system. Let $I = \{1, \ldots, \ell\}$. Let $\mathcal{I}$ be the set of partial transversals of $\mathcal{V}$. Then $(V, \mathcal{I})$ is a matroid.

Proof.

- We note that $\emptyset \in \mathcal{I}$ since the empty set is a transversal of the empty subfamily of $\mathcal{V}$, thus (I1') holds.
- We already saw that if T is a partial transversal of $\mathcal{V}$, and if $T' \subseteq T$, then T' is also a partial transversal. So (I2') holds.
- Suppose that T_1 and T_2 are partial transversals of $\mathcal{V}$ such that $|T_1| < |T_2|$. Exercise: show that (I3') holds.

$\square$

Transversal Matroid Rank

- Transversal matroid has rank

$$r(A) = \min_{J \subseteq I} \left(|V(J) \cap A| - |J| + |I| \right) \tag{8.10}$$

$$= \min_{J \subseteq I} m_J(I) \tag{8.11}$$

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- Therefore, this function is submodular.
- Note that it is a minimum over a set of modular functions in I. Is this true in general? Exercise:
- Exercise: Can you identify a set of sufficient properties over a set of modular functions $m_i : V \to \mathbb{R}_+$ so that $f(A) = \min_i m_i(A)$ is submodular? Can you identify both necessary and sufficient conditions?

Matroid loops

- A circuit in a matroids is well defined, a subset $A \subseteq E$ is circuit if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

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- Note, we also say that two elements s, t are said to be parallel if $\{s, t\}$ is a circuit (e.g., in a matrix, two column vectors, one of which is a scalar multiple of the other).

Representable

Definition 8.4.1 (Matroid isomorphism)

Two matroids M_1 and M_2 respectively on ground sets V_1 and V_2 are isomorphic if there is a bijection $\pi : V_1 \to V_2$ which preserves independence (equivalently, rank, circuits, and so on).

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- Let $\mathbb{F}$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $\mathbb{F}$, such as a Galois field GF(p) where p is prime (such as GF(2)), but not $\mathbb{Z}$). Succinctly: A field is a set with $+$, $*$, closure, associativity, commutativity, and additive and multiplicative identities and inverses.

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- We can more generally define matroids on a field.

Definition 8.4.2 (linear matroids on a field)

Let $\mathbf{X}$ be an $n \times m$ matrix and $E = \{1, \ldots, m\}$, where $\mathbf{X}_{ij} \in \mathbb{F}$ for some field, and let $\mathcal{I}$ be the set of subsets of E such that the columns of $\mathbf{X}$ are linearly independent over $\mathbb{F}$.

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- We can more generally define matroids on a field.

Definition 8.4.3 (representable (as a linear matroid))

Any matroid isomorphic to a linear matroid on a field is called representable over $\mathbb{F}$

Representability of Transversal Matroids

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- In particular:

Theorem 8.4.4

Transversal matroids are representable over all finite fields of sufficiently large cardinality, and are representable over any infinite field.

Converse: Representability of Transversal Matroids

The converse is not true, however.

Example 8.4.5

Let $V = \{1, 2, 3, 4, 5, 6\}$ be a ground set and let $M = (V, \mathcal{I})$ be a set system where $\mathcal{I}$ is all subsets of V of cardinality ≤ 2 except for the pairs $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$.

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- It can be shown that this is a matroid and is representable.
- However, this matroid is not isomorphic to any transversal matroid.

Review from Lecture 6

The next frame comes from lecture 6.

Matroids, other definitions using matroid rank $r : 2^V \to \mathbb{Z}_+$

Definition 8.5.3 (closed/flat/subspace)

A subset $A \subseteq E$ is closed (equivalently, a flat or a subspace) of matroid M if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

Definition: A hyperplane is a flat of rank $r(M) - 1$.

Definition 8.5.4 (closure)

Given $A \subseteq E$, the closure (or span) of A, is defined by $\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}$.

Therefore, a closed set A has $\text{span}(A) = A$, and the span of a set is closed.

Definition 8.5.5 (circuit)

A subset $A \subseteq E$ is circuit or a cycle if it is an <u>inclusionwise-minimal dependent set</u> (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

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- V is always trivially spanning.
- Consider the terminology: "spanning tree in a graph", comes from spanning in a matroid sense.

Dual of a Matroid

- Given a matroid $M = (V, \mathcal{I})$, a dual matroid $M^* = (V, \mathcal{I}^*)$ can be defined on the same ground set V, but using a very different set of independent sets $\mathcal{I}^*$.

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- We define the set of sets $\mathcal{I}^*$ for M^* as follows:

$$\mathcal{I}^* = \{A \subseteq V : V \setminus A \text{ is a spanning set of } M\} \tag{8.12}$$

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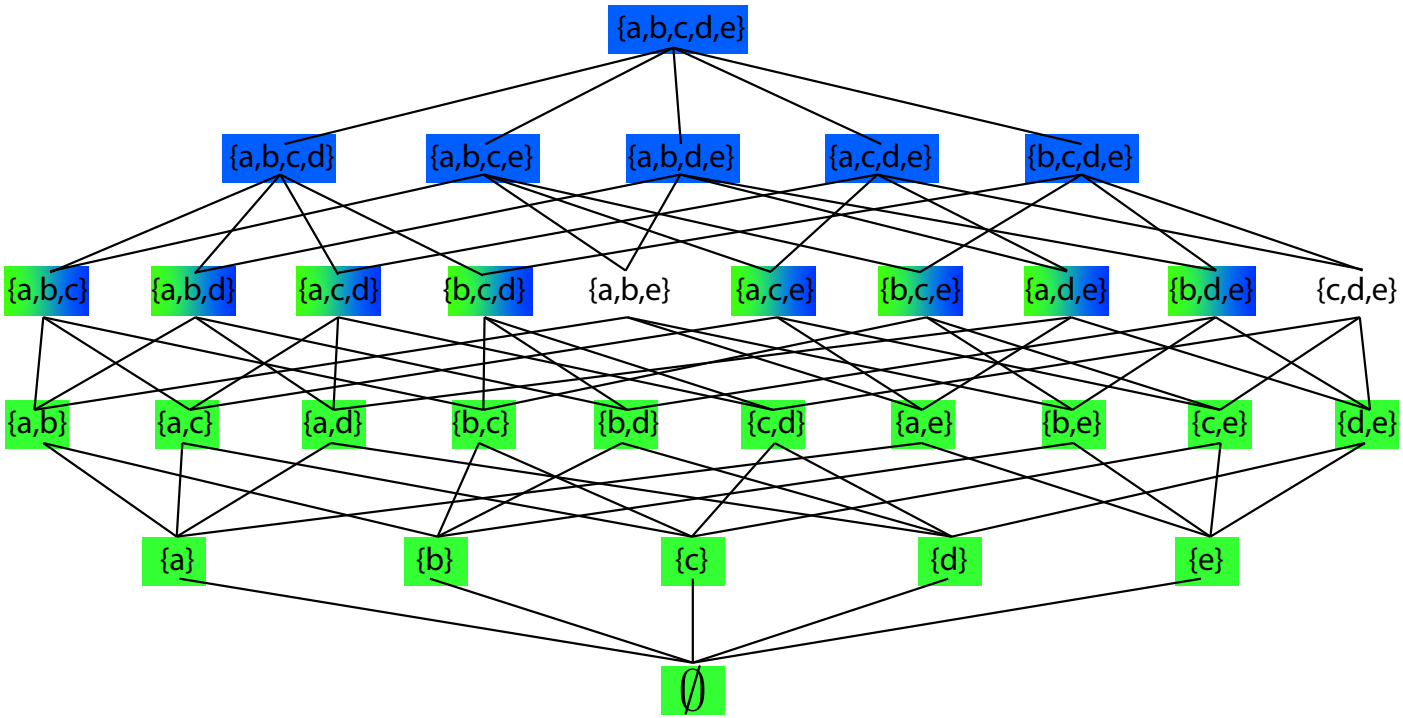
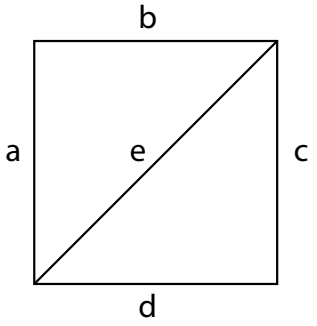
- Dual of the dual: Note, we have that $(M^*)^* = M$.

Dual of a Matroid: Visualization

Graphic matroid over edges $E = \{a, b, c, d, e\}$ for the graph on the right. Independent sets (green) and spanning sets (blue) are shown.





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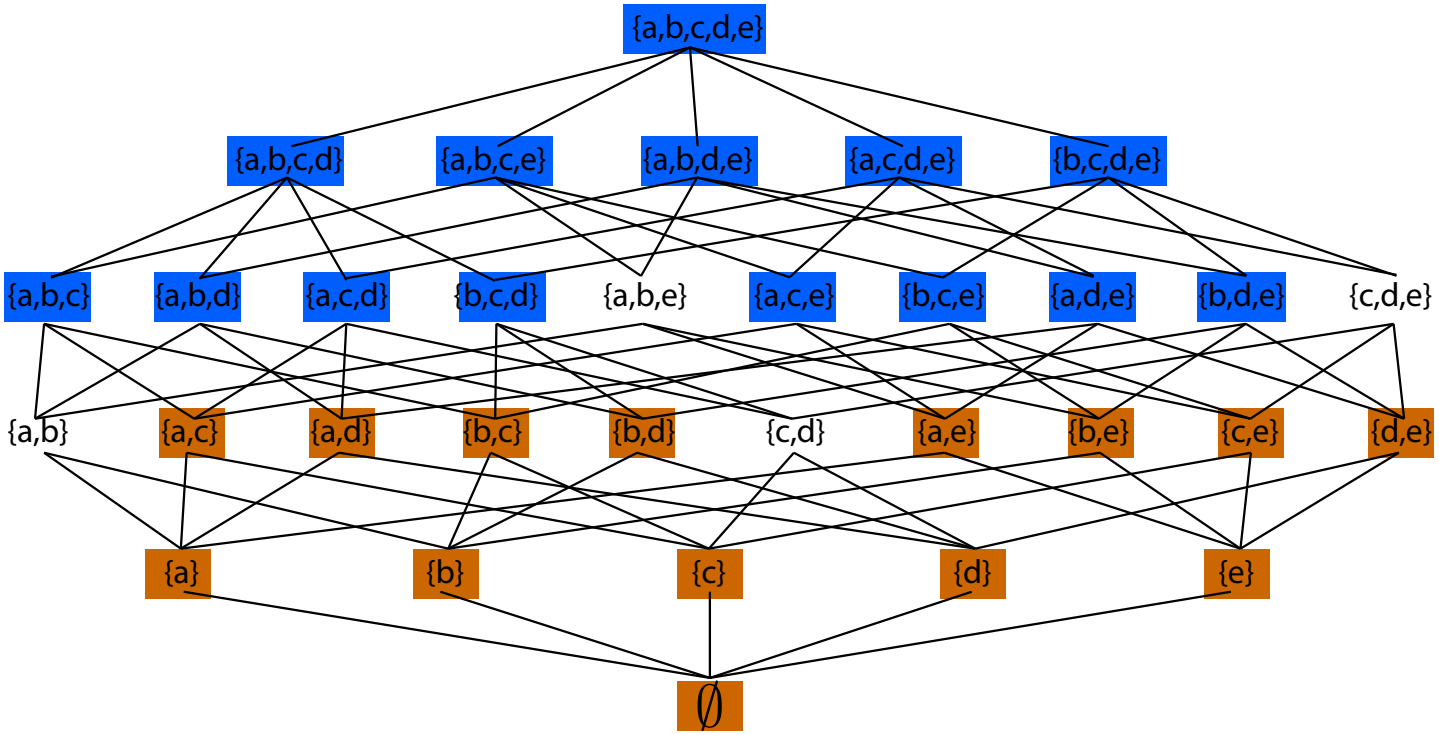
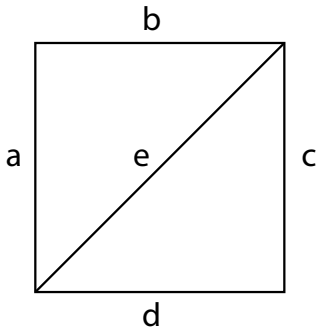
Graphic matroid over edges $E = \{a, b, c, d, e\}$ for the graph on the right. Spanning sets of M are blue. Complement of spanning sets of M (independent sets of M^*) are orange.





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Theorem 8.5.3 (Dual matroid bases)

Let $M = (V, \mathcal{I})$ be a matroid and $\mathcal{B}(M)$ be the set of bases of M. Then define

$$\mathcal{B}^*(M) = \{V \setminus B : B \in \mathcal{B}(M)\}. \tag{8.15}$$

Then $\mathcal{B}^*(M)$ is the set of basis of M^* (that is, $\mathcal{B}^*(M) = \mathcal{B}(M^*)$.

An exercise in duality Terminology

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3. *X is a hyperplane in M iff $V \setminus X$ is a cocircuit in M (circuit in M^*).*
4. *X is a circuit in M iff $V \setminus X$ is a cohyperplane in M (hyperplane in M^*).*

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- Recall, in cycle matroid, a spanning set of G is any set of edges that are incident to all nodes (i.e., any superset of a spanning forest), a minimal spanning set is a spanning tree (or forest), and a circuit has a nice visual interpretation (a cycle in the graph).
- A cut in a graph G is a set of edges, the removal of which increases the number of connected components. I.e., $X \subseteq E(G)$ is a cut in G if $k(G) < k(G \setminus X)$.

Example duality: graphic matroid

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- Recall, in cycle matroid, a spanning set of G is any set of edges that are incident to all nodes (i.e., any superset of a spanning forest), a minimal spanning set is a spanning tree (or forest), and a circuit has a nice visual interpretation (a cycle in the graph).
- A cut in a graph G is a set of edges, the removal of which increases the number of connected components. I.e., $X \subseteq E(G)$ is a cut in G if $k(G) < k(G \setminus X)$.
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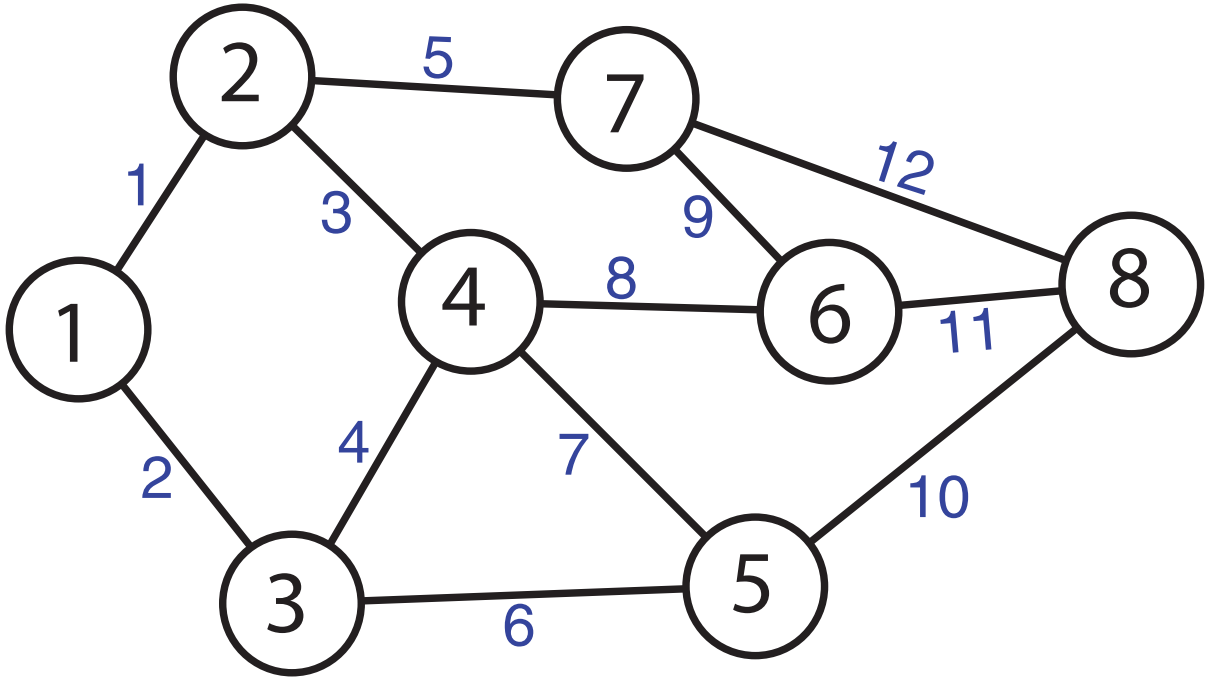
A graph G



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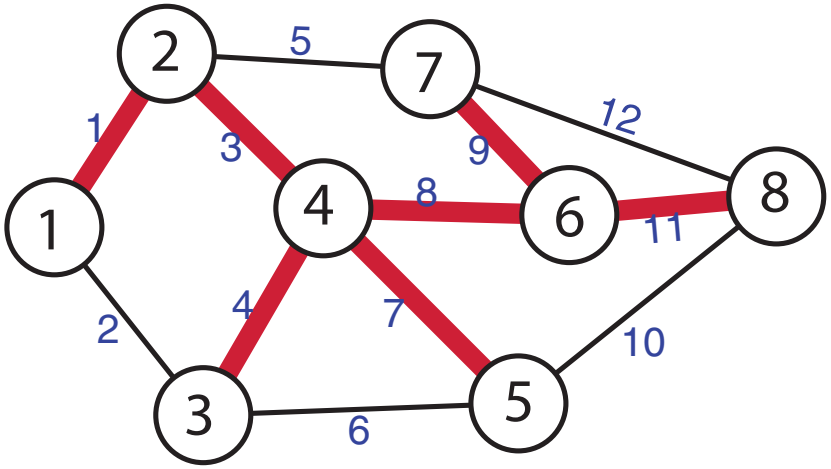
Minimally spanning in M (and thus a base (maximally independent) in M)



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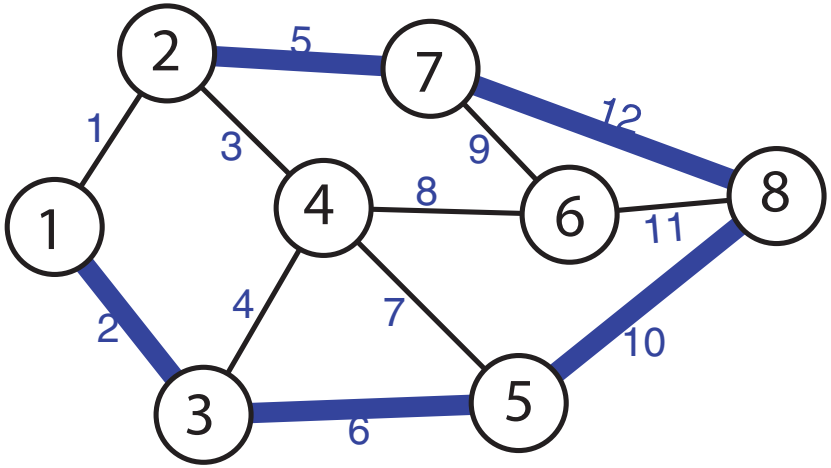

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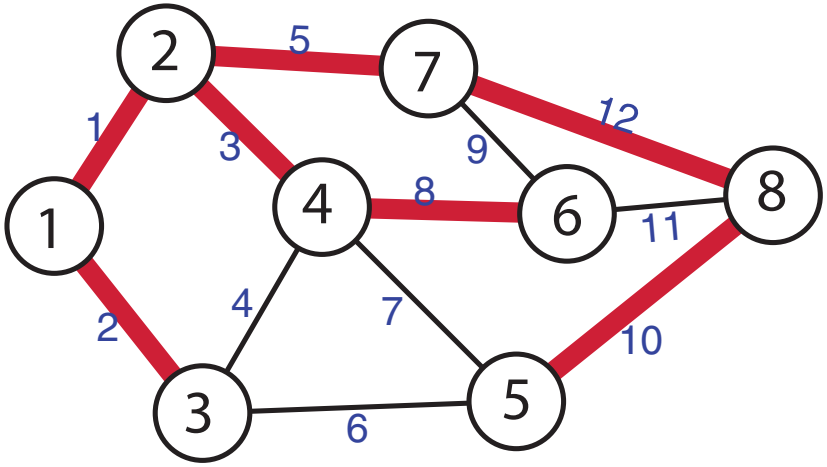

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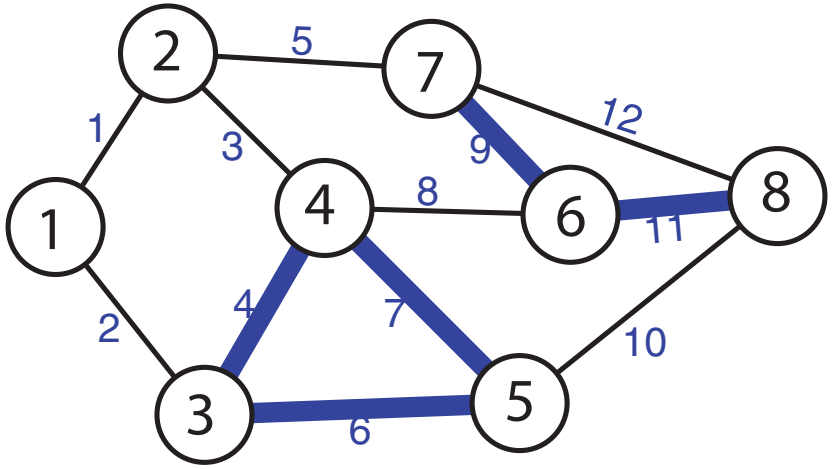

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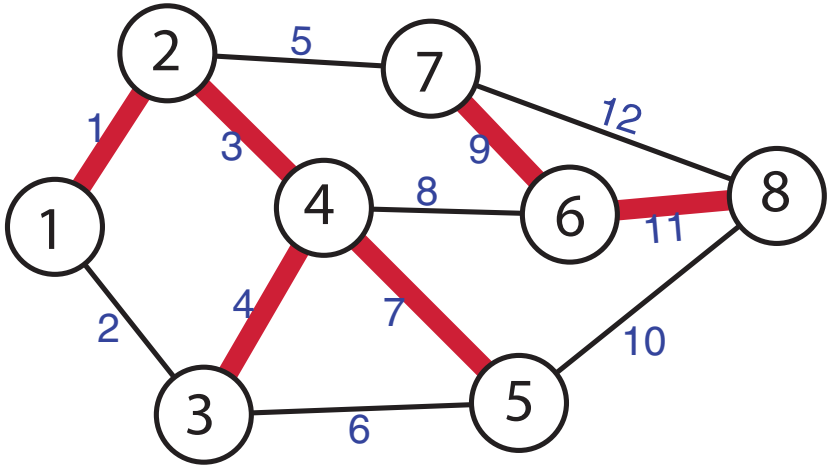
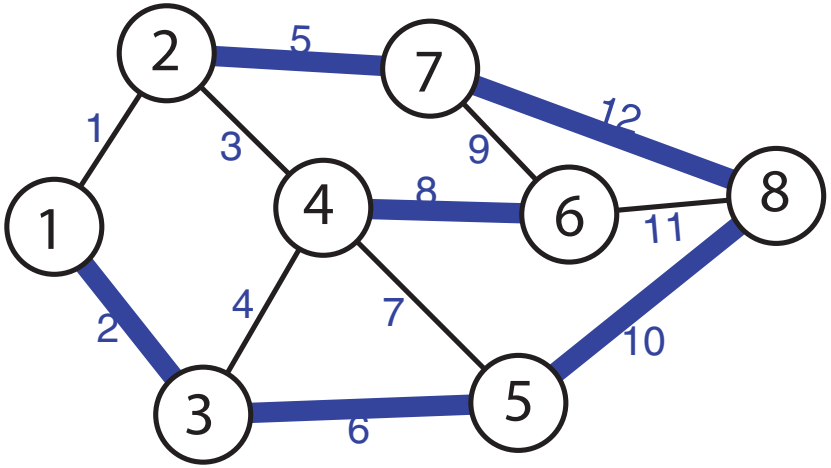
Independent but not spanning in M, and not closed in M.



Dependent in M* (contains a cocycle, is a nonminimal cut)



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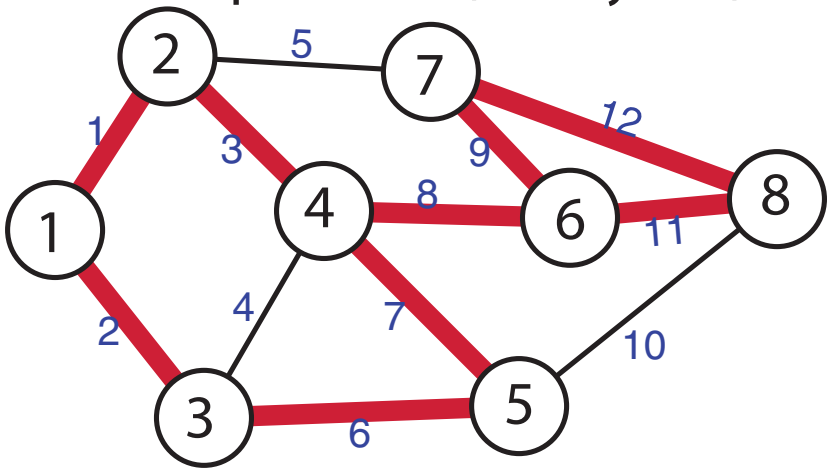
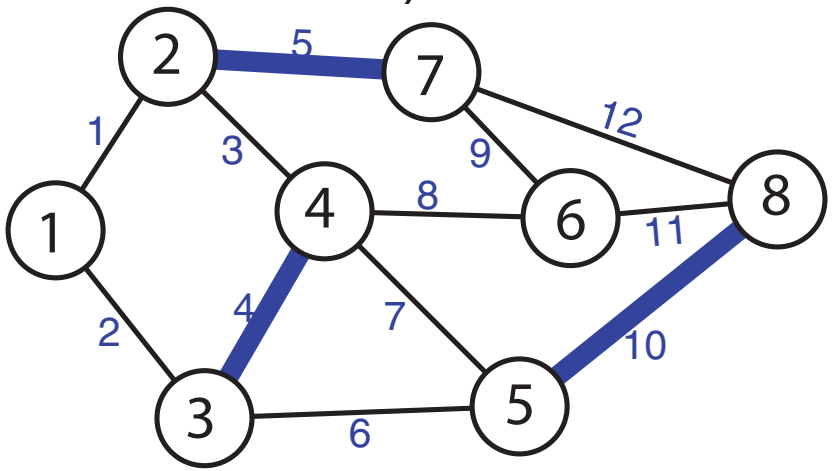
Spanning in M, but not a base, and not independent (has cycles)



Independent in M* (does not contain a cut)



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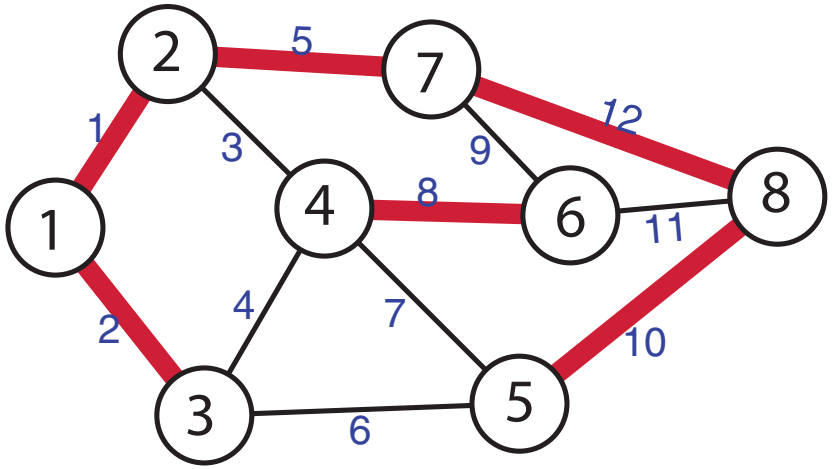

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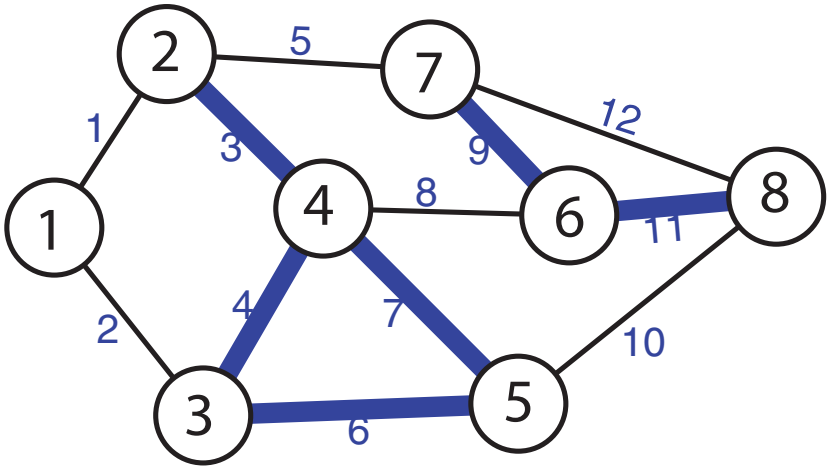

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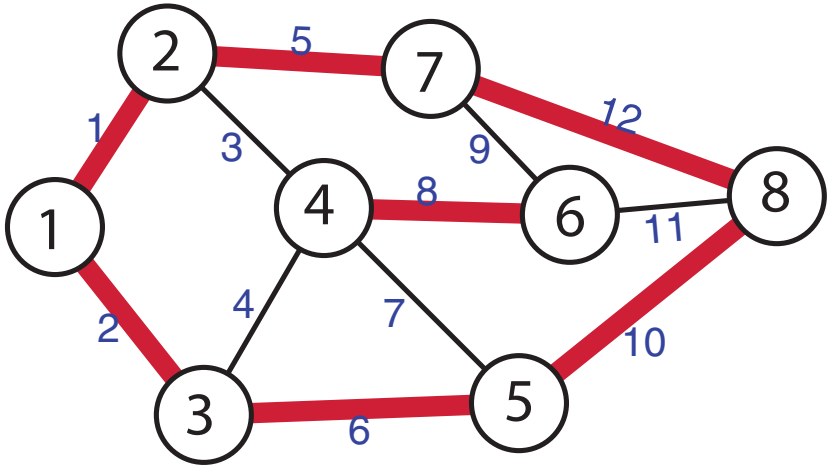
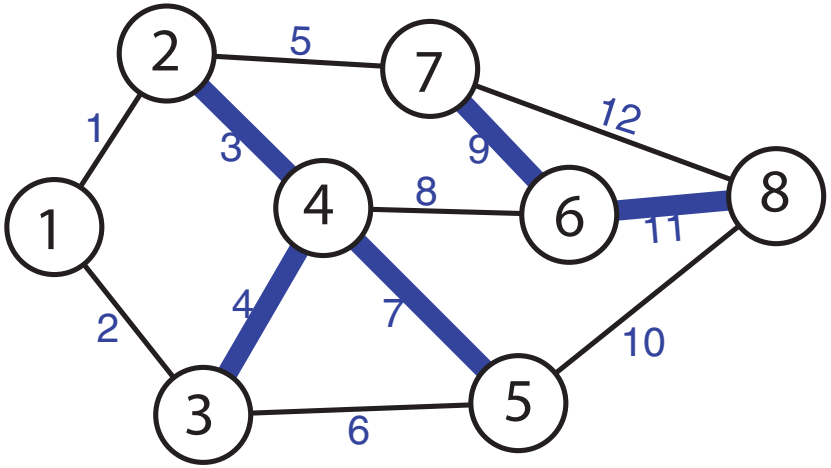
A hyperplane in M, dependent but not spanning in M



A cycle in M* (minimally dependent in M*, a cocycle, or a minimal cut)



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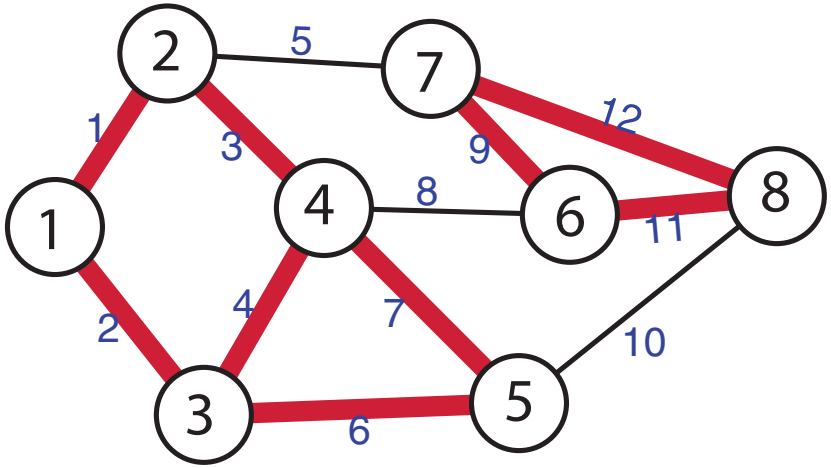

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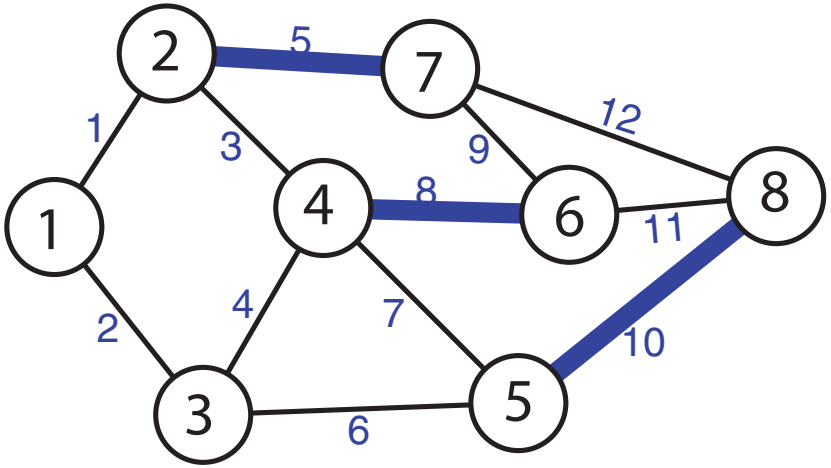

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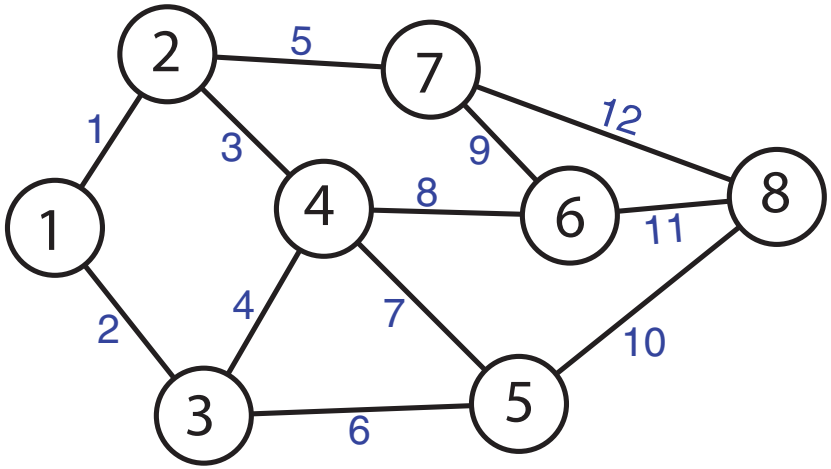
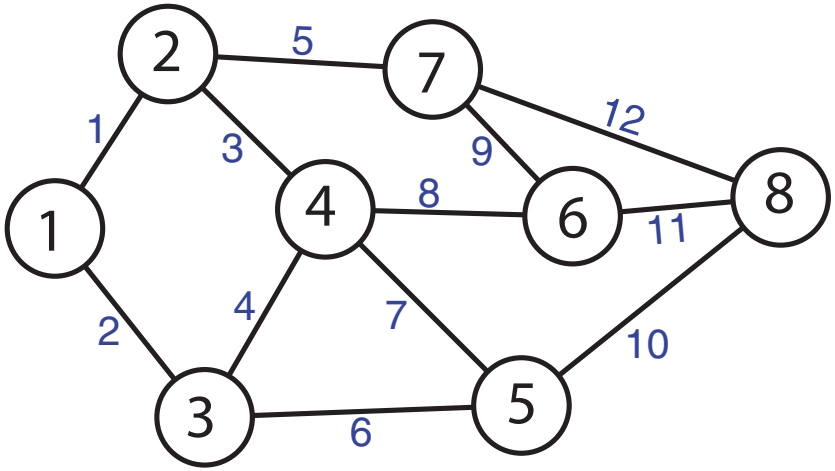
Cycle Matroid - independent sets have no cycles.



Cocycle matroid, independent sets contain no cuts.



The dual of a matroid is (indeed) a matroid

Theorem 8.5.5

Given matroid $M = (V, \mathcal{I})$, let $M^* = (V, \mathcal{I}^*)$ be as previously defined. Then M^* is a matroid.

Proof.

- Since $V \setminus \emptyset$ is spanning in primal, clearly $\emptyset \in \mathcal{I}^*$, so (I1') holds.

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- Also, if $I \subseteq J \in \mathcal{I}^*$, then clearly also $I \in \mathcal{I}^*$ since if $V \setminus J$ is spanning in M, so must $V \setminus I$. Therefore, (I2') holds.
- Next, given $I, J \in \mathcal{I}^*$ with $|I| < |J|$, it must be the case that $\bar{I} = V \setminus I$ and $\bar{J} = V \setminus J$ are both spanning in M with $|\bar{I}| > |\bar{J}|$.

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- Since $B_{\bar{J}} \setminus I \subseteq V \setminus I$, and $B_{\bar{J}} \setminus I$ is independent in M, we can choose the base $B_{\bar{I}}$ of M s.t. $B_{\bar{J}} \setminus I \subseteq B_{\bar{I}} \subseteq V \setminus I$.

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- Since $B_{\bar{J}}$ and J are disjoint, we have both: 1) $B_{\bar{J}} \setminus I$ and $J \setminus I$ are disjoint; and 2) $B_{\bar{J}} \cap I \subseteq I \setminus J$. Also note, $B_{\bar{I}}$ and I are disjoint.

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- Now $J \setminus I \not\subseteq B_{\bar{I}}$, since otherwise (i.e., assuming $J \setminus I \subseteq B_{\bar{I}}$):

$$|B_{\bar{J}}| = |B_{\bar{J}} \cap I| + |B_{\bar{J}} \setminus I| \tag{8.16}$$
$$\leq |I \setminus J| + |B_{\bar{J}} \setminus I| \tag{8.17}$$
$$< |J \setminus I| + |B_{\bar{J}} \setminus I| \leq |B_{\bar{I}}| \tag{8.18}$$

which is a contradiction. *The last inequality on the right follows since $J \setminus I \subseteq B_{\bar{I}}$ (by assumption) and $B_{\bar{J}} \setminus I \subseteq B_{\bar{I}}$ implies that $(J \setminus I) \cup (B_{\bar{J}} \setminus I) \subseteq B_{\bar{I}}$, but since J and $B_{\bar{J}}$ are disjoint, we have that $|J \setminus I| + |B_{\bar{J}} \setminus I| \leq |B_{\bar{I}}|$.*

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- Therefore, $J \setminus I \not\subseteq B_{\bar{I}}$, and there is a $v \in J \setminus I$ s.t. $v \notin B_{\bar{I}}$.
- So $B_{\bar{I}}$ is disjoint with $I \cup \{v\}$, means $B_{\bar{I}} \subseteq V \setminus (I \cup \{v\})$, or $V \setminus (I \cup \{v\})$ is spanning in M, and therefore $I \cup \{v\} \in \mathcal{I}^*$.

□

Matroid Duals and Representability

Theorem 8.5.6

Let M be an $\mathbb{F}$-representable matroid (i.e., one that can be represented by a finite sized matrix over field $\mathbb{F}$). Then M^* is also $\mathbb{F}$-representable.

Hence, for matroids as general as matric matroids, duality does not extend the space of matroids that can be used.

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Hence, for matroids as general as matric matroids, duality does not extend the space of matroids that can be used.

Theorem 8.5.7

Let M be a graphic matroid (i.e., one that can be represented by a graph $G = (V, E)$). Then M^* is not necessarily also graphic.

Hence, for graphic matroids, duality can increase the space and power of matroids, and since they are based on a graph, they are relatively easy to use: 1) all cuts are dependent sets; 2) minimal cuts are cycles; 3) bases of a cut are any one edge removed from minimal cuts; 4) independent sets are edges that are not cuts (minimal or otherwise); 5) bases of matroid are maximal non-cuts (non-cut containing edge sets).

Dual Matroid Rank

Theorem 8.5.8

The rank function r_{M^*} of the dual matroid M^* may be specified in terms of the rank r_M in matroid M as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \tag{8.19}$$

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2. *I.e., $|X|$ is modular, complement $f(V \setminus X)$ is submodular if f is submodular, $r_M(V)$ is a constant, and summing submodular functions and a constant preserves submodularity.*

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- Non-negativity integral follows since $|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V)$. *The right inequality follows since r_M is submodular.*

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- Non-negativity integral follows since $|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V)$.
- Monotone non-decreasing follows since, as X increases by one, $|X|$ always increases by 1, while $r_M(V \setminus X)$ decreases by one or zero.

Dual Matroid Rank

Theorem 8.5.8

The rank function r_{M^*} of the dual matroid M^* may be specified in terms of the rank r_M in matroid M as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \tag{8.19}$$

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2.
- Non-negativity integral follows since $|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V)$.
- Monotone non-decreasing follows since, as X increases by one, $|X|$ always increases by 1, while $r_M(V \setminus X)$ decreases by one or zero.
- Therefore, r_{M^*} is the rank function of a matroid. That it is the dual matroid rank function is shown in the next proof.

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A set X is independent in (V, r_{M^*}) if and only if

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But a subset X is independent in M^* only if $V \setminus X$ is spanning in M (by the definition of the dual matroid). □

Matroid restriction/deletion

- Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then

$$\mathcal{I}_Y = \{Z : Z \subseteq Y, Z \in \mathcal{I}\} \tag{8.22}$$

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- Hence, $M|Y = M \setminus (V \setminus Y)$, and $M|(V \setminus X) = M \setminus X$.
- The rank function is of the same form. I.e., $r_Y : 2^Y \to \mathbb{Z}_+$, where $r_Y(Z) = r(Z)$ for $Z \subseteq Y$, $Y = V \setminus X$.

Matroid contraction M/Z

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- A minor of a matroid is any matroid obtained via a series of deletions and contractions of some matroid.
- In fact, it is the case $M/Z = (M^* \setminus Z)^*$ (Exercise: show why).

Matroid Intersection

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This is an instance of the convolution of two submodular functions, f_1 and f_2 that, evaluated at $Y \subseteq V$, is written as:

$$(f_1 * f_2)(Y) = \min_{X \subseteq Y} \Big(f_1(X) + f_2(Y \setminus X) \Big) \tag{8.27}$$

Convolution and Hall's Theorem

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- So Hall's theorem can be expressed as convolution. Exercise: define $g(A) = [\Gamma(\cdot) * |\cdot|](A)$, prove that g is submodular.
- Note, in general, convolution of two submodular functions does not preserve submodularity (but in certain special cases it does).

Matroid Union

Definition 8.6.2

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, $\ldots$, $M_k = (V_k, \mathcal{I}_k)$ be matroids. We define the union of matroids as
$M_1 \vee M_2 \vee \cdots \vee M_k = (V_1 \uplus V_2 \uplus \cdots \uplus V_k, \mathcal{I}_1 \vee \mathcal{I}_2 \vee \cdots \vee \mathcal{I}_k)$, where

$$I_1 \vee \mathcal{I}_2 \vee \cdots \vee \mathcal{I}_k = \{I_1 \uplus I_2 \uplus \cdots \uplus I_k | I_1 \in \mathcal{I}_1, \ldots, I_k \in \mathcal{I}_k\} \tag{8.28}$$

Note $A \uplus B$ designates the disjoint union of A and B.

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Theorem 8.6.3

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, $\ldots$, $M_k = (V_k, \mathcal{I}_k)$ be matroids, with rank functions $r_1, \ldots, r_k$. Then the union of these matroids is still a matroid, having rank function

$$r(Y) = \min_{X \subseteq Y} \Big(|Y \setminus X| + r_1(X \cap V_1) + \cdots + r_k(X \cap V_k) \Big) \tag{8.29}$$

for any $Y \subseteq V_1 \uplus \ldots V_2 \uplus \cdots \uplus V_k$.

Exercise: Matroid Union, and Matroid duality

Exercise: Fully characterize $M \vee M^*$.

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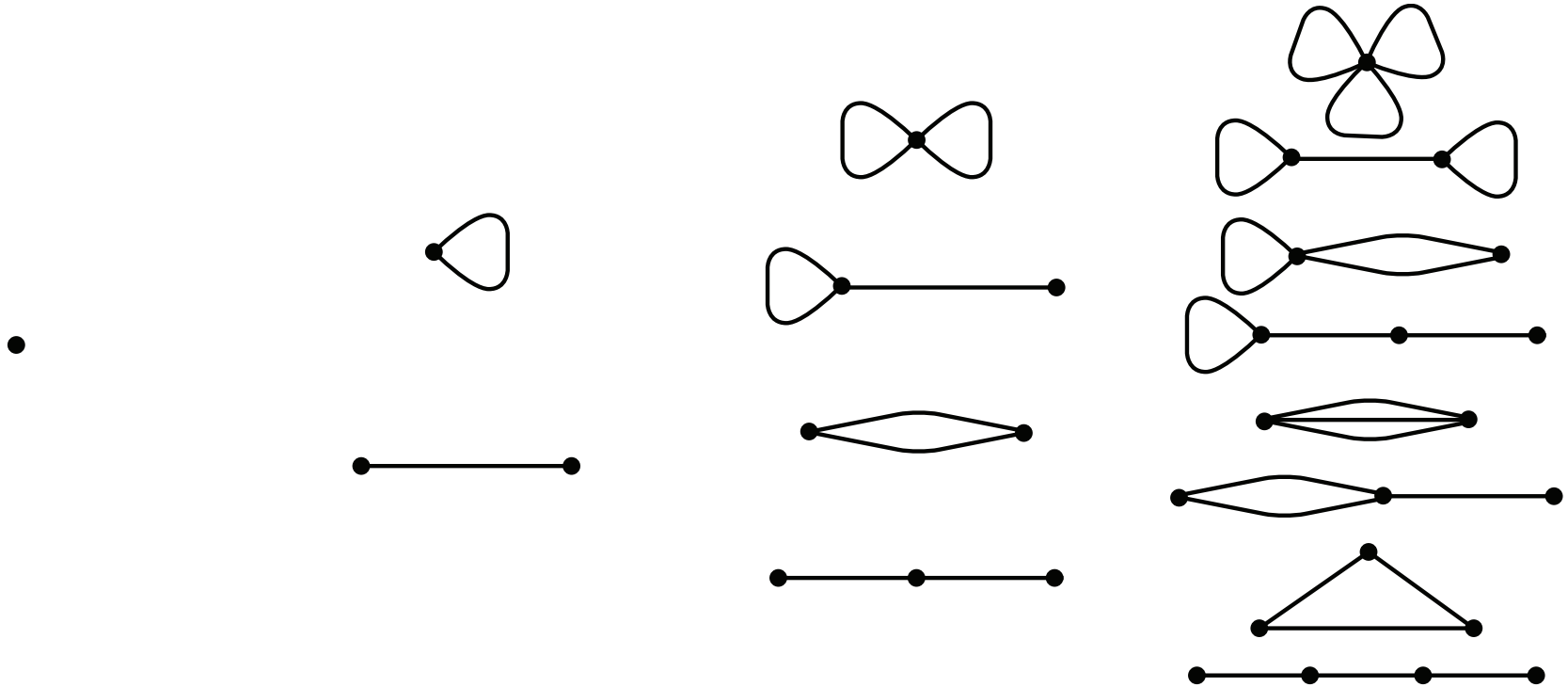

(a) The only matroid with zero elements.

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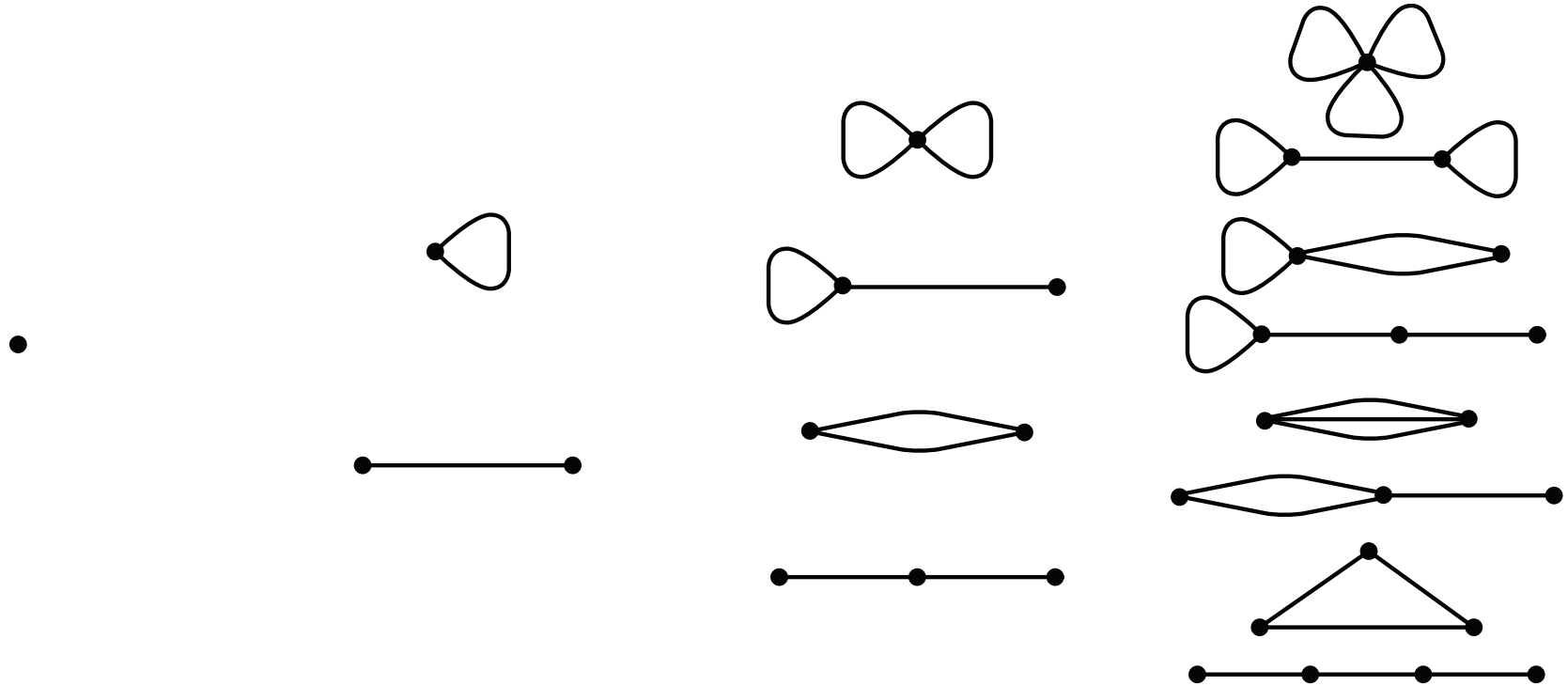

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- This is a nice way to visualize matroids with very low ground set sizes. What about matroids that are low rank but with many elements?

Affine Matroids

- Given an $n \times m$ matrix with entries over some field $\mathbb{F}$, we say that a subset $S \subseteq \{1, \ldots, m\}$ of indices (with corresponding column vectors $\{v_i : i \in S\}$, with $|S| = k \leq m$) is affinely dependent if $m \geq 1$ and there exists elements $\{a_1, \ldots, a_k\} \in \mathbb{F}$, not all zero with $\sum_{i=1}^{k} a_i = 0$, such that $\sum_{i=1}^{k} a_i v_i = 0$.

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Exercise: prove this.

Euclidean Representation of Low-rank Matroids

- Consider the affine matroid with $n \times m = 2 \times 6$ matrix on the field $\mathbb{F} = \mathbb{R}$, and let the elements be $\{(0,0),(1,0),(2,0),(0,1),(0,2),(1,1)\}$.

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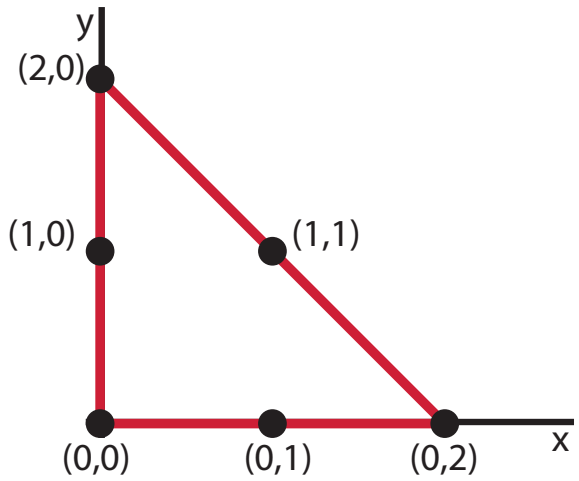

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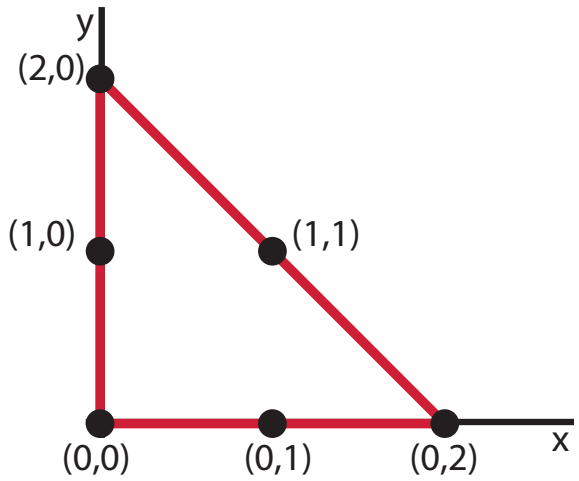

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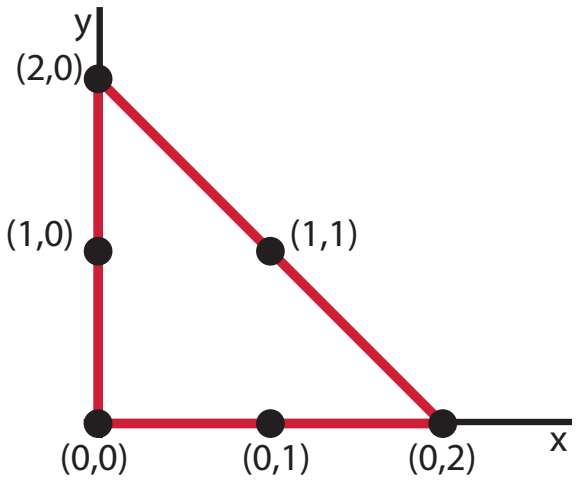

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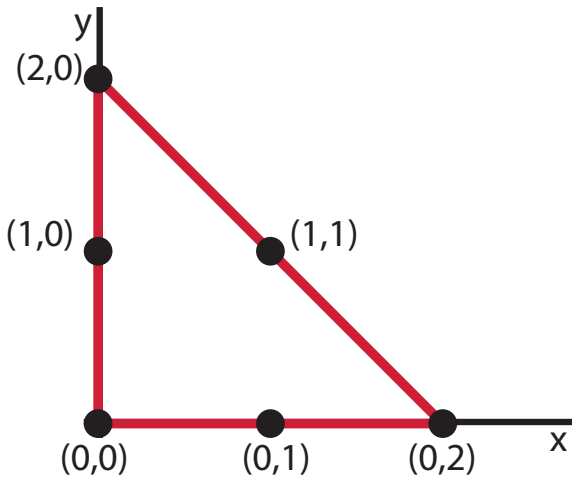

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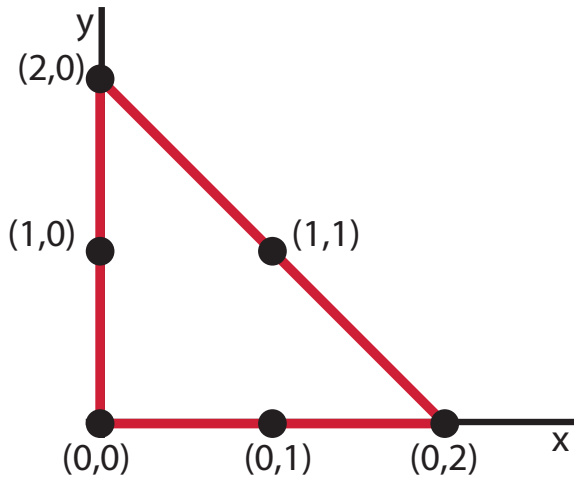

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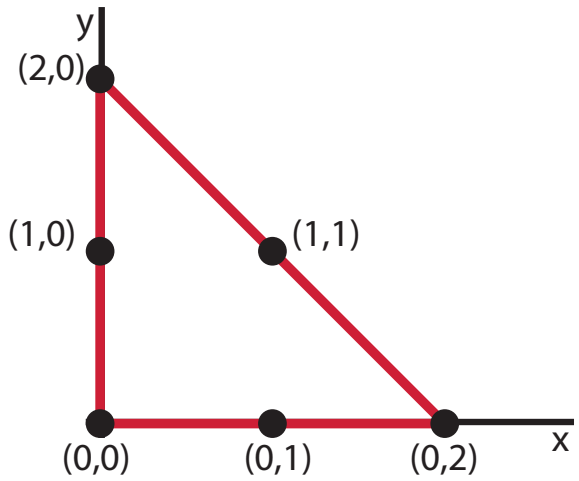
- Dependent sets consist of all subsets with ≥ 4 elements (rank 3), or 3 collinear elements (rank 2). Any two points have rank 2.



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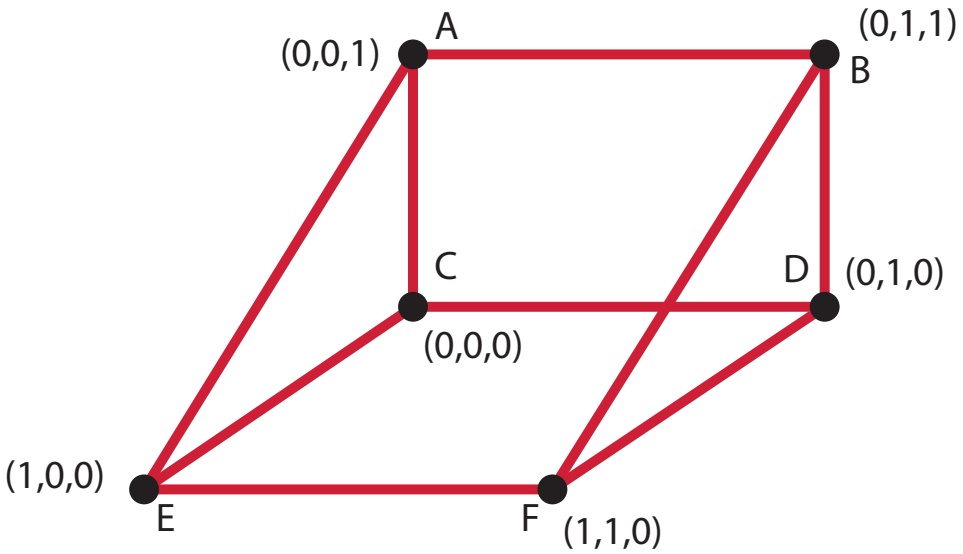

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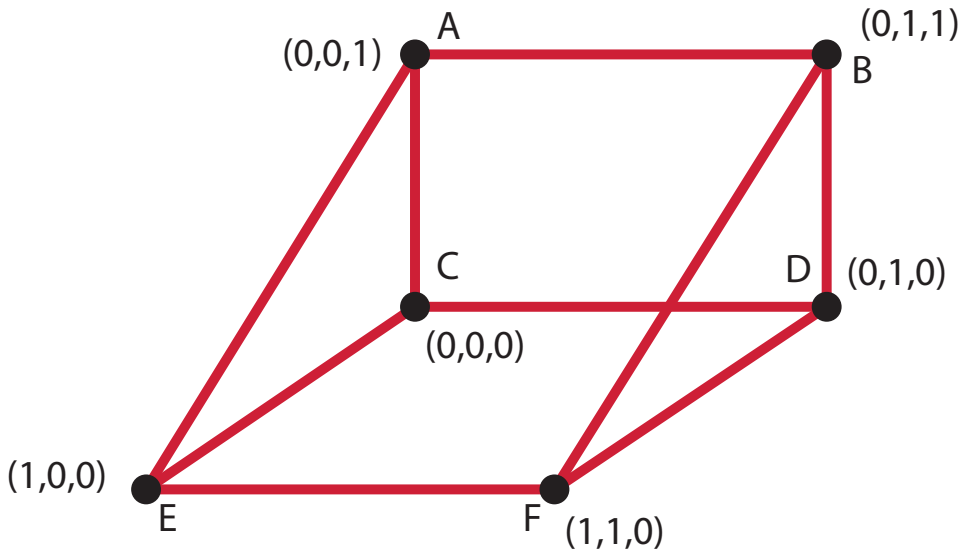

- All sets of 5 points are dependent. The only other sets of dependent points are coplanar ones of size 4. Namely: $\{(0,0,0),(0,1,0),(1,1,0),(1,0,0)\}$, $\{(0,0,0),(0,0,1),(0,1,1),(0,1,0)\}$, and $\{(0,0,1),(0,1,1),(1,1,0),(1,0,0)\}$.

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Theorem 8.7.2

Any matroid of rank $m \leq 4$ can be represented by an affine matroid in $\mathbb{R}^{m-1}$.

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- (see Oxley 2011 for more details).

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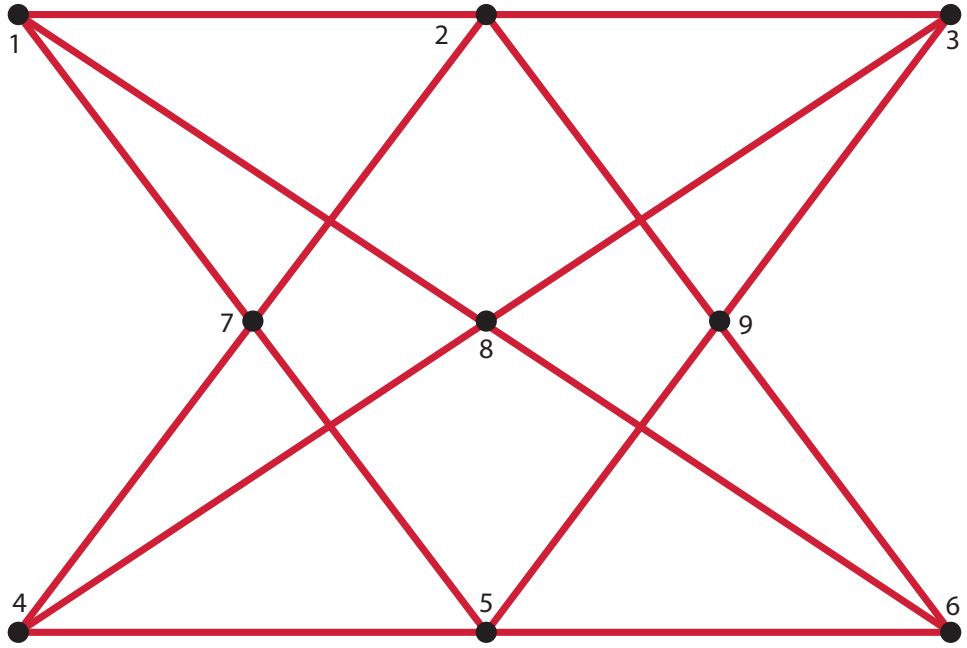

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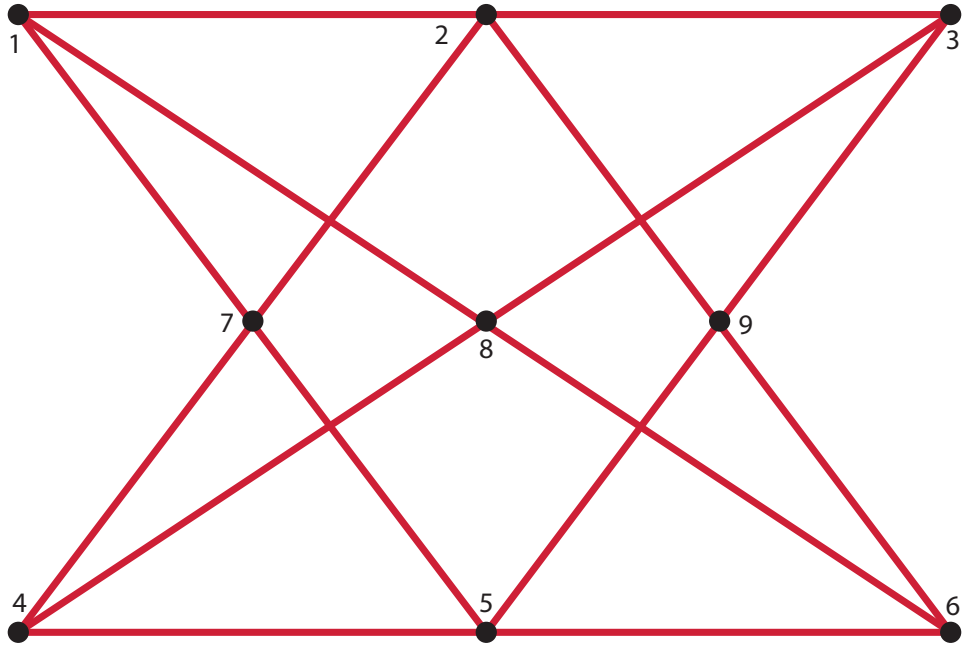

- Called the non-Pappus matroid. Has rank three, but any matric matroid with the above dependencies would require that $\{7, 8, 9\}$ is dependent, hence requiring an additional line in the above.

Euclidean Representation of Low-rank Matroids: A test

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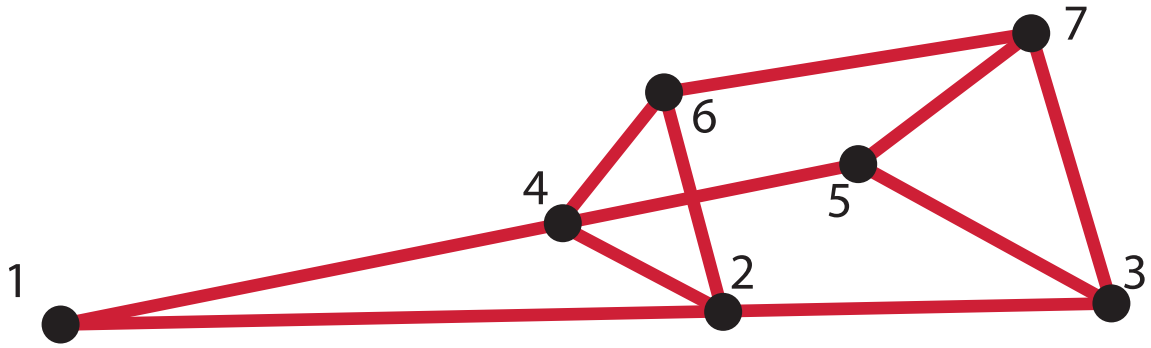

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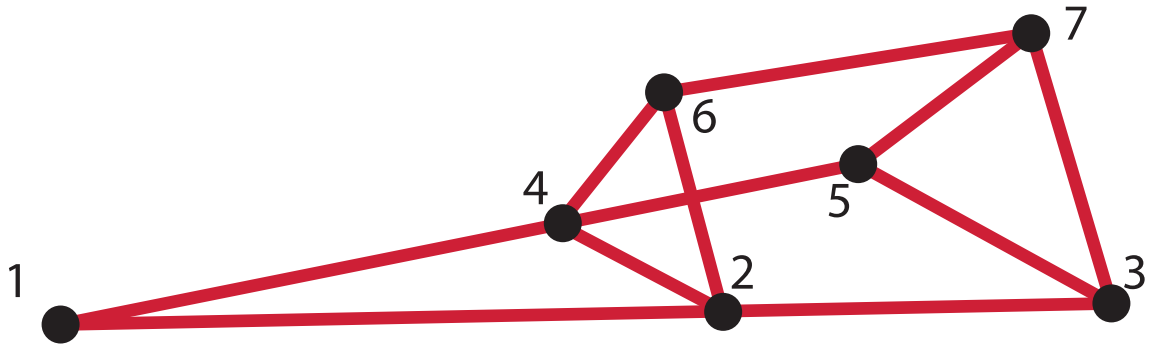

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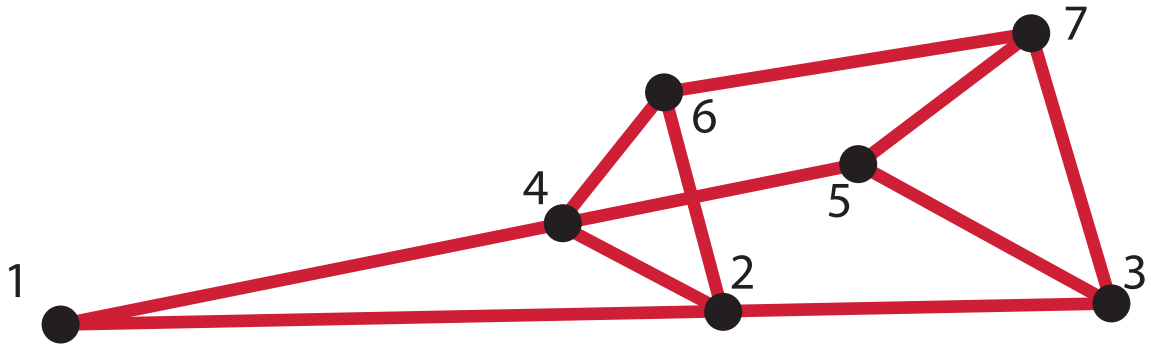

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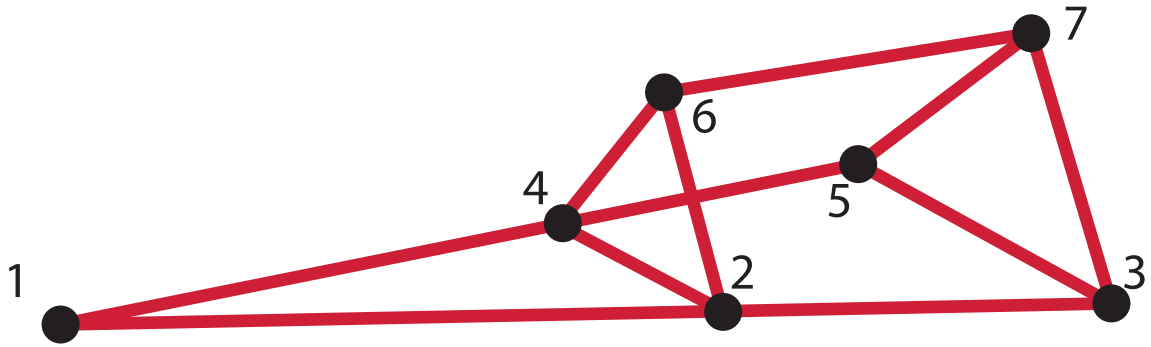

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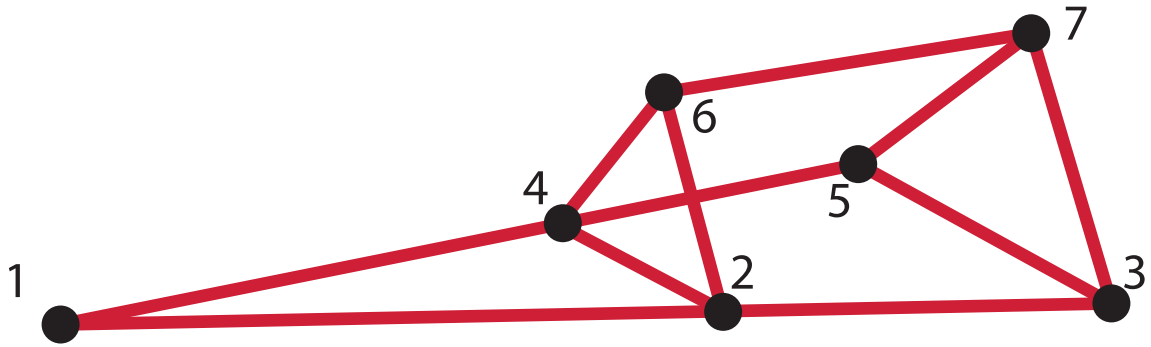

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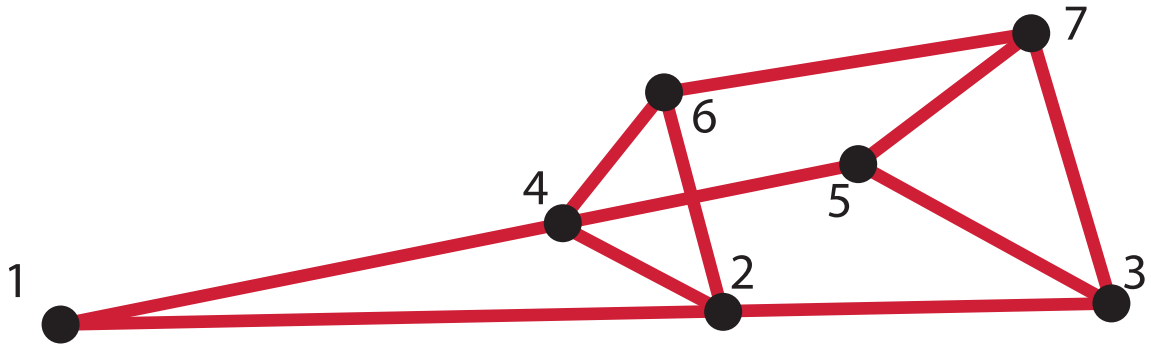

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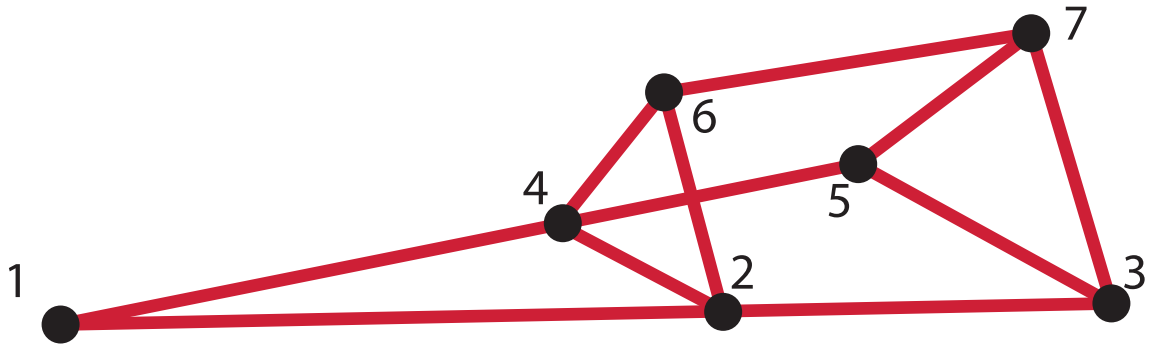

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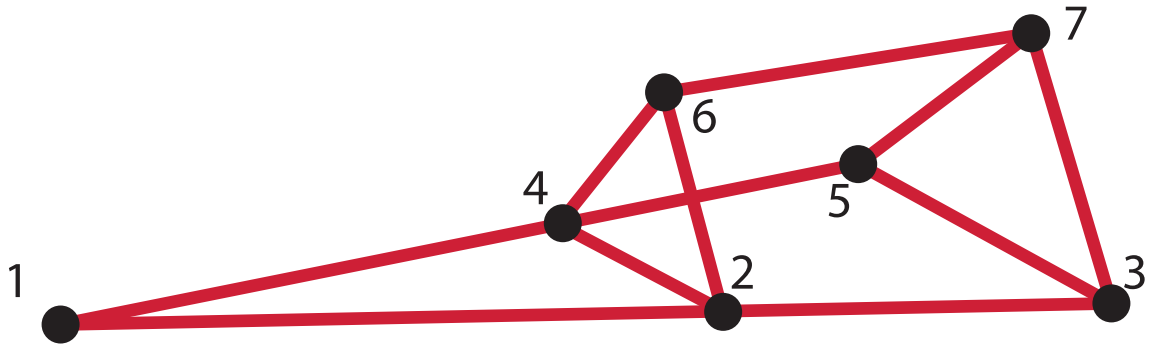

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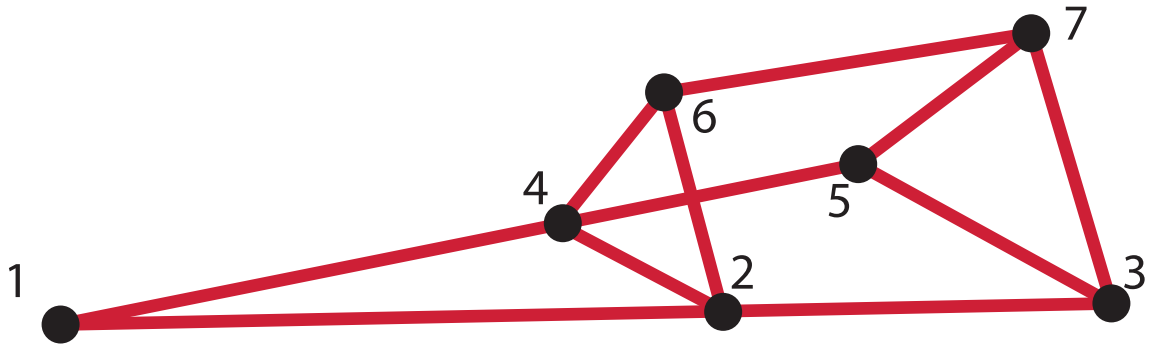

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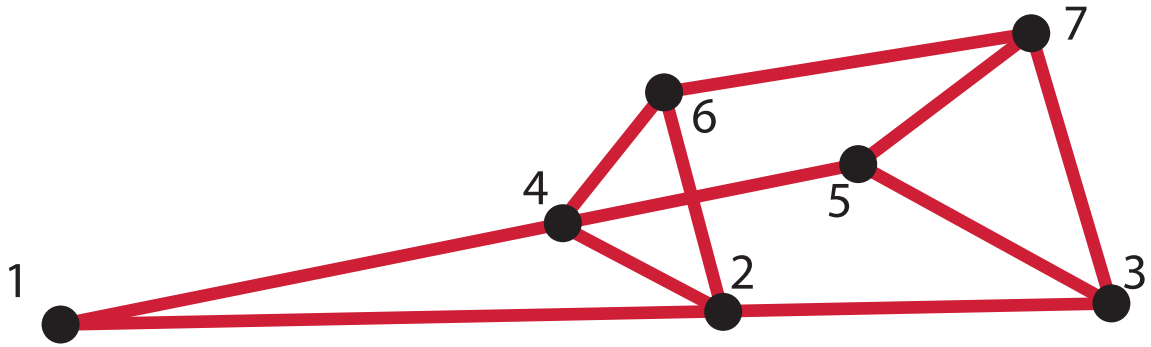

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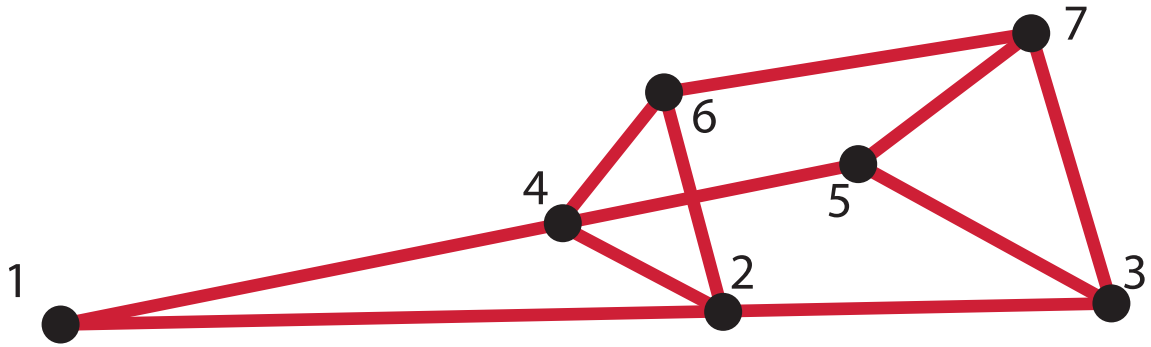

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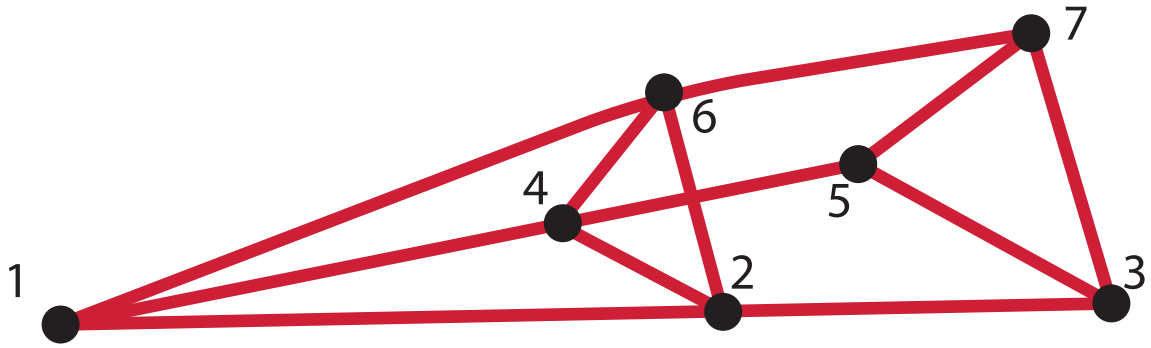

- If we extend the line from 6-7 to 1, then is it a matroid?
- Hence, not all 2D or 3D graphs of points and lines are matroids.

Matroid?

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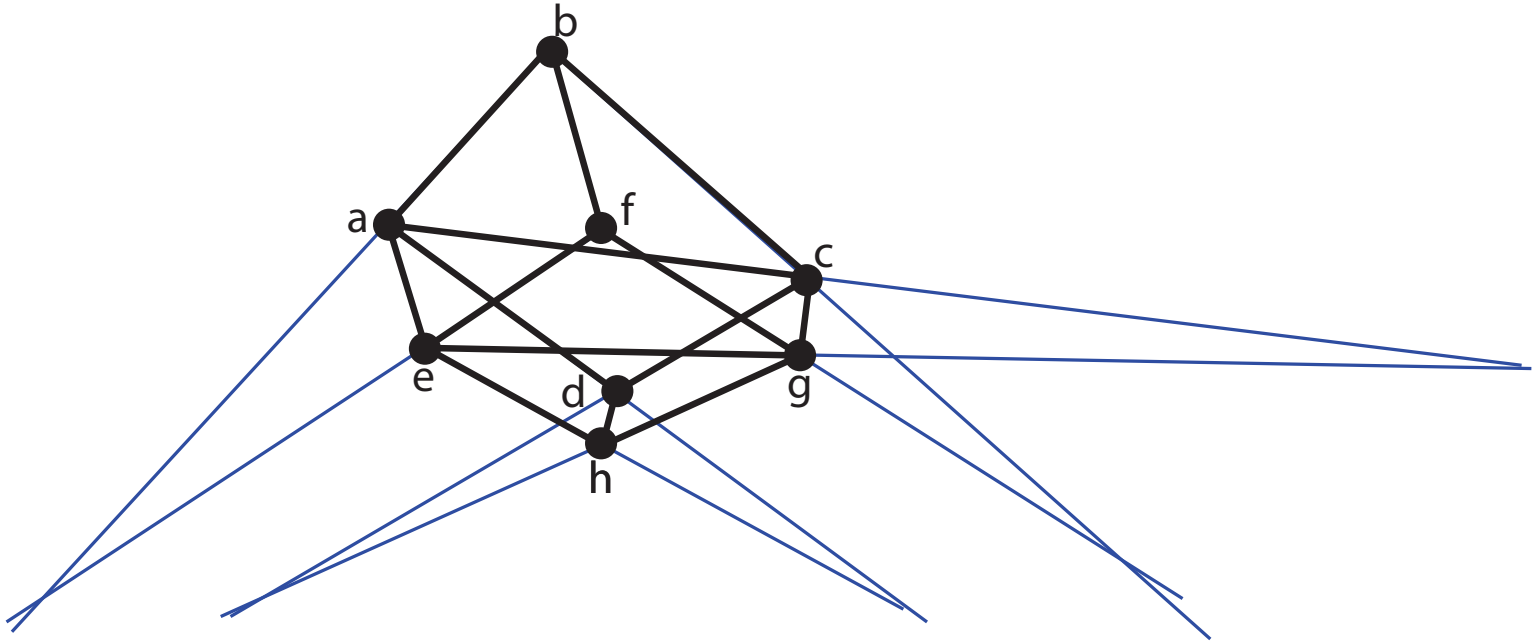

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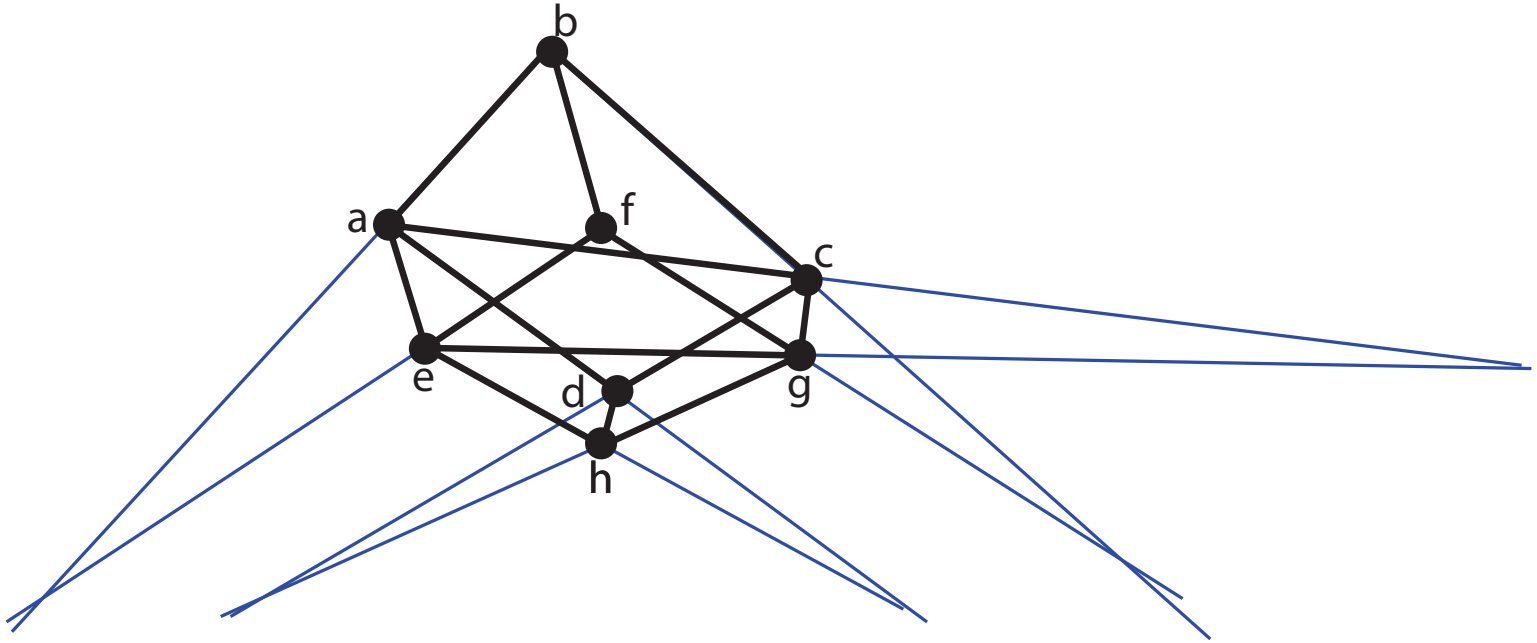

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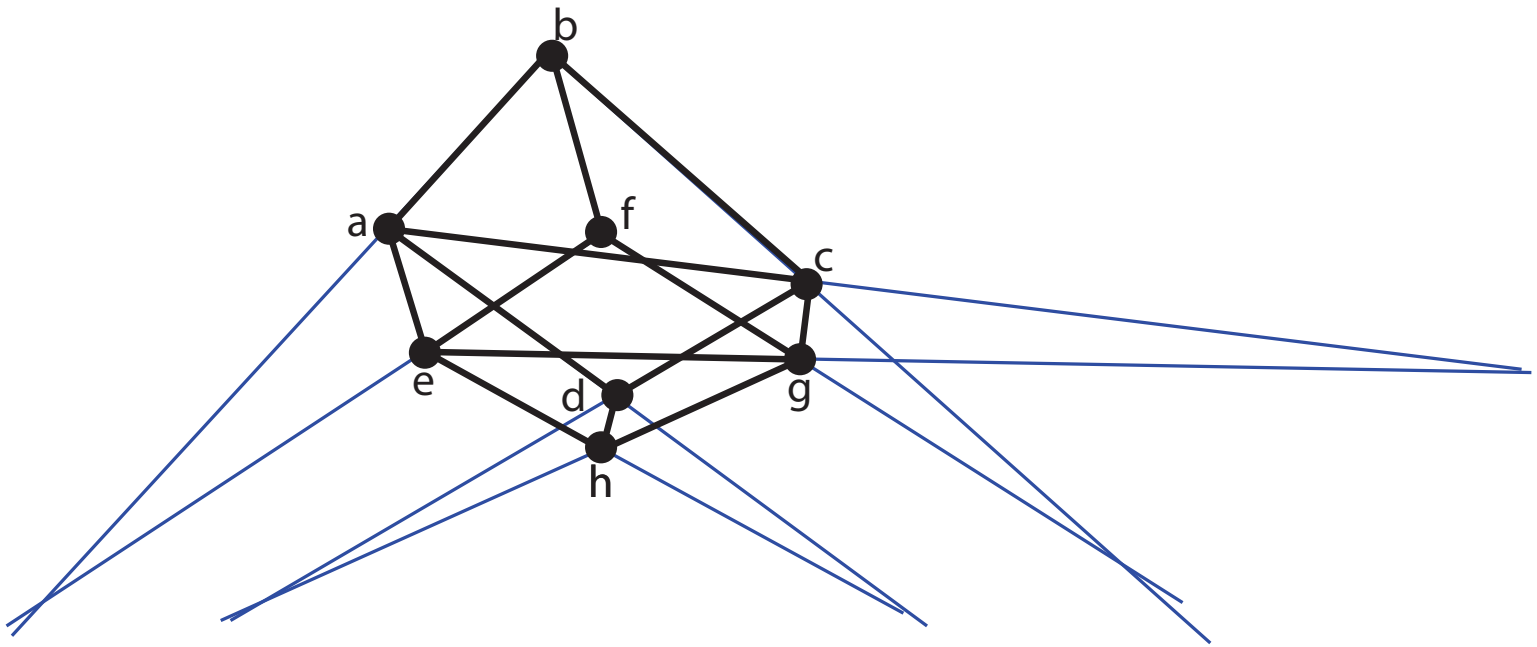

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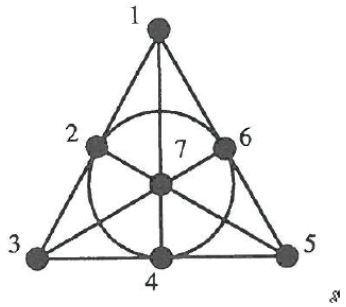
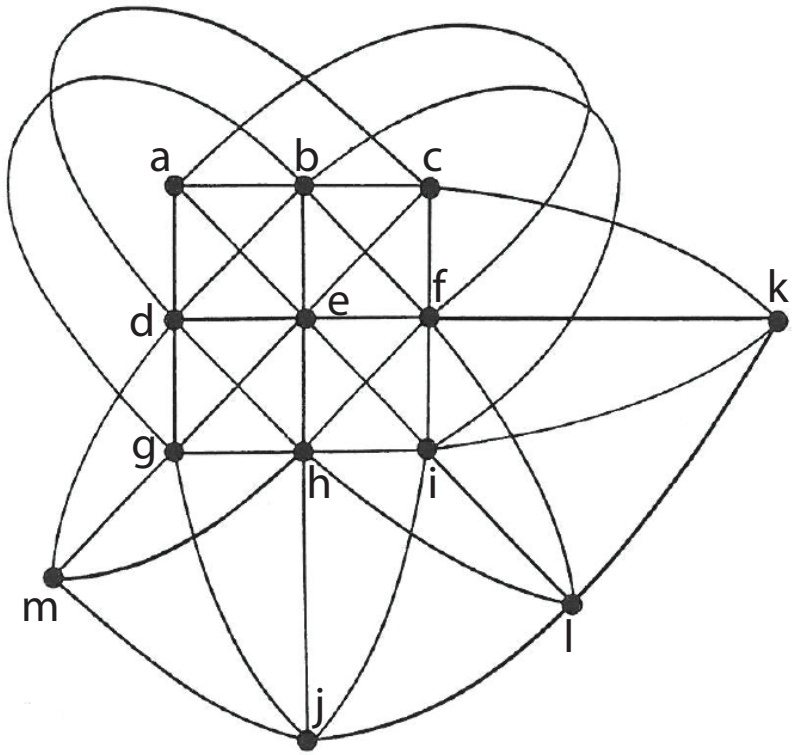
- Exercise: Is this a matroid? Exercise: If so, is it representable?

Projective Geometries: Other Examples

- Other examples can be more complex, consider the following two matroids (from Oxley, 2011):





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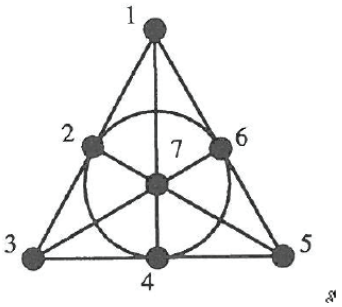

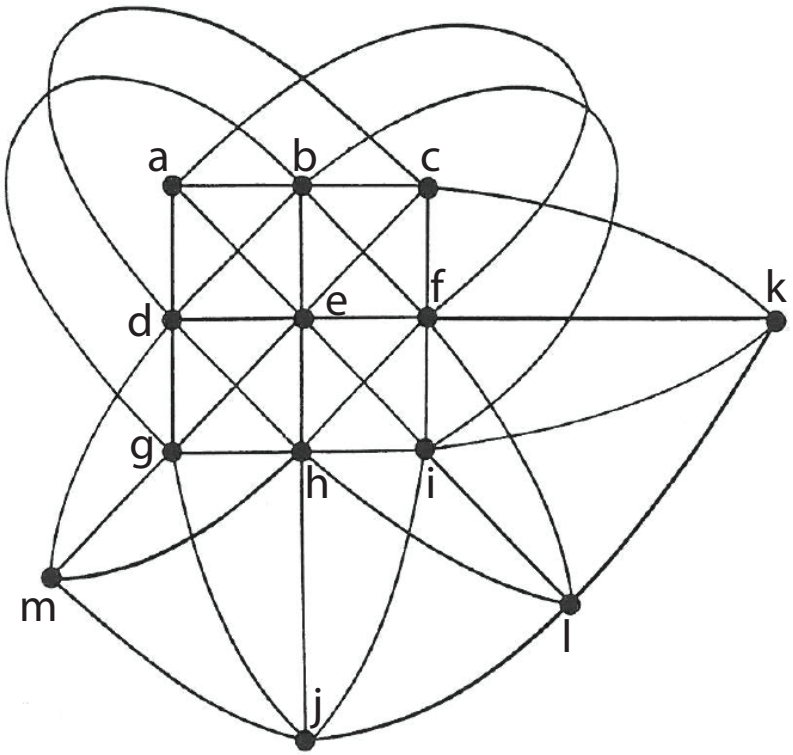

- Right: a matroid (and a 2D depiction of a geometry) over the field $\mathrm{GF}(3) = \{0, 1, 2\}$ mod 3 and is "coordinatizable" in $\mathrm{GF}(3)^3$.

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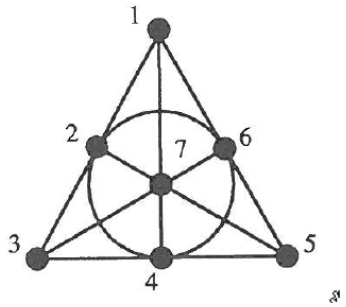

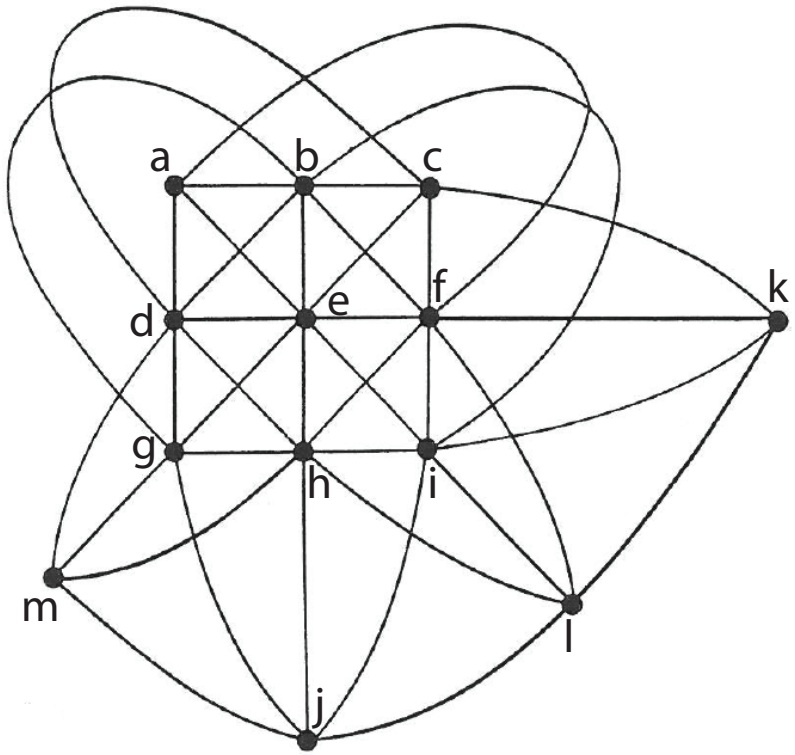

- Right: a matroid (and a 2D depiction of a geometry) over the field GF$(3) = \{0, 1, 2\}$ mod 3 and is "coordinatizable" in GF$(3)^3$.
- Hence, lines (in 2D) which are rank 2 sets may be curved; planes (in 3D) can be twisted.

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- Not all matroids are linear (i.e., matric) matroids.
- Matroids can be seen as related to projective geometries (and are sometimes called combinatorial geometries).
- Exists much research on different subclasses of matroids, and if/when they are contained in (or isomorphic to) each other.

Matroid Further Reading

- "Matroids: A Geometric Introduction", Gordon and McNulty, 2012.
- "The Coming of the Matroids", William Cunningham, 2012 (a nice history)
- Welsh, "Matroid Theory", 1975.
- Oxley, "Matroid Theory", 1992 (and 2011) (perhaps best "single source" on matroids right now).
- Crapo & Rota, "On the Foundations of Combinatorial Theory: Combinatorial Geometries", 1970 (while this is old, it is very readable).
- Lawler, "Combinatorial Optimization: Networks and Matroids", 1976.
- Schrijver, "Combinatorial Optimization", 2003

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- Greedy is good since it can be made to run very fast $O(n \log n)$.
- Often, however, greedy is heuristic (it might work well in practice, but worst-case performance can be unboundedly poor).
- We will next see that the greedy algorithm working optimally is a defining property of a matroid, and is also a defining property of a polymatroid function.

Matroid and the greedy algorithm

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Algorithm 1: The Matroid Greedy Algorithm

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while $\exists v \in E \setminus X$ *s.t.* $X \cup \{v\} \in \mathcal{I}$ **do**
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Theorem 8.8.1

Let $(E, \mathcal{I})$ be an independence system. Then the pair $(E, \mathcal{I})$ is a matroid if and only if for each weight function $w \in \mathcal{R}_+^E$, Algorithm 1 above leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.

Review from Lecture 6

- The next slide is from Lecture 6.

Matroids by bases

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

Theorem 8.8.3 (Matroid (by bases))

Let E be a set and $\mathcal{B}$ be a nonempty collection of subsets of E. Then the following are equivalent.

1. *$\mathcal{B}$ is the collection of bases of a matroid;*
2. *if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.*
3. *If $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.*

Properties 2 and 3 are called "exchange properties."
Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.

Matroid and the greedy algorithm

proof of Theorem 8.8.1.

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- Assume $(E, \mathcal{I})$ is a matroid and $w : E \to \mathcal{R}_+$ is given.
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- A is a base of M, and let $B = (b_1, \ldots, b_r)$ be <u>any</u> another base of M with elements also ordered decreasing by weight, so $w(b_1) \geq w(b_2) \geq \cdots \geq w(b_r)$.

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- We next show that not only is $w(A) \geq w(B)$ but that $w(a_i) \geq w(b_i)$ for all i.

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Matroid and the greedy algorithm

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- Assume otherwise, and let k be the first (smallest) integer such that $w(a_k) < w(b_k)$. Hence $w(a_j) \geq w(b_j)$ for $j < k$.

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- Since $|A_{k-1}| < |B_k|$, there exists a $b_i \in B_k \setminus A_{k-1}$ where $A_{k-1} \cup \{b_i\} \in \mathcal{I}$ for some $1 \leq i \leq k$.

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- But $w(b_i) \geq w(b_k) > w(a_k)$, and so the greedy algorithm would have chosen b_i rather than a_k, contradicting what greedy does.

$\square$

Matroid and the greedy algorithm

converse proof of Theorem 8.8.1.

- Given an independence system $(E, \mathcal{I})$, suppose the greedy algorithm leads to an independent set of max weight for every non-negative weight function. We'll show $(E, \mathcal{I})$ is a matroid.

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- Let $I, J \in \mathcal{I}$ with $|I| < |J|$. Suppose to the contrary, that $I \cup \{z\} \notin \mathcal{I}$ for all $z \in J \setminus I$.
- Define the following modular weight function w on E, and define $k = |I|$.

$$w(v) = \begin{cases} k+2 & \text{if } v \in I, \\ k+1 & \text{if } v \in J \setminus I, \\ 0 & \text{if } v \in E \setminus (I \cup J) \end{cases} \tag{8.30}$$

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- Now greedy will, after k iterations, recover I, but it cannot choose any element in $J \setminus I$ by assumption. Thus, greedy chooses a set of weight $k(k+2) = w(I)$.

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  so J has strictly larger weight but is still independent, contradicting greedy's optimality.
- Therefore, there must be a $z \in J \setminus I$ such that $I \cup \{z\} \in \mathcal{I}$, and since I and J are arbitrary, $(E, \mathcal{I})$ must be a matroid.

□

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- We don't need non-negativity, we can use any $w \in \mathbb{R}^E$ and keep going until we have a base.
- If we stop at a negative value, we'll once again get a maximum weight independent set.
- Exercise: what if we keep going until a base even if we encounter negative values?
- We can instead do as small as possible thus giving us a minimum weight independent set/base.

Summary of Important (for us) Matroid Definitions

Given an independence system, matroids are defined equivalently by any of the following:

- All maximally independent sets have the same size.
- A monotone non-decreasing submodular integral rank function with unit increments.
- The greedy algorithm achieves the maximum weight independent set for all weight functions.