Submodular Functions, Optimization, and Applications to Machine Learning
— Fall Quarter, Lecture 8 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/

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\[ f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \]

\[ -f(A) + 2f(C) + f(B) \]

\[ -f(A) + f(C) + f(B) \]

\[ -f(A \cap B) \]
Announcements, Assignments, and Reminders

- Homework 2.
- Reminder, all lectures are being recorded and posted to youtube. To get the links, see our announcements (https://canvas.uw.edu/courses/1397085/announcements).
Class Road Map - EE563

L1(9/30): Motivation, Applications, Definitions, Properties
L2(10/5): Sums concave(modular), uses (diversity/costs, feature selection), information theory
L3(10/7): Monge, More Definitions, Graph and Combinatorial Examples,
L4(10/12): Graph & Combinatorial Examples, Matrix Rank, Properties, OtherDefs, Independence
L5(10/14): Properties,Defs of Submodularity, Independence
L6(10/19): Matroids, Matroid Examples, Matroid Rank,
L8(10/26): Transversal Matroid, Matroid and representation, Dual Matroid, Other Matroid Properties, Combinatorial Geometries, Matroid and Greedy
L9(10/28):
L10(11/2):
L11(11/4):
L12(11/9):
L13(11/16):
L14(11/18):
L15(11/23):
L16(11/25):
L17(11/30):
L18(12/2):
L19(12/7):
L20(12/9): maximization.

Last day of instruction, Fri. Dec 11th. Finals Week: Dec 12-18, 2020
System of Distinct Representatives

- Let \((V, \mathcal{V})\) be a set system (i.e., \(\mathcal{V} = (V_i : i \in I)\) where \(V_i \subseteq V\) for all \(i\)), and \(I\) is an index set. Hence, \(|I| = |\mathcal{V}|\).
- A family \((v_i : i \in I)\) with \(v_i \in V\) is said to be a system of distinct representatives of \(\mathcal{V}\) if \(\exists\) a bijection \(\pi : I \leftrightarrow I\) such that \(v_i \in V_{\pi(i)}\) and \(v_i \neq v_j\) for all \(i \neq j\).
- In a system of distinct representatives, there is a requirement for the representatives to be distinct. We can re-state (and rename) this as a:

**Definition 8.2.3 (transversal)**

Given a set system \((V, \mathcal{V})\) and index set \(I\) for \(\mathcal{V}\) as defined above, a set \(T \subseteq V\) is a transversal of \(\mathcal{V}\) if there is a bijection \(\pi : T \leftrightarrow I\) such that

\[
x \in V_{\pi(x)} \text{ for all } x \in T
\]

(8.19)

- Note that due to \(\pi : T \leftrightarrow I\) being a bijection, all of \(I\) and \(T\) are “covered” (so this makes things distinct automatically).
When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?

- Given a set system \((V, \mathcal{V})\) with \(\mathcal{V} = (V_i : i \in I)\), and \(V_i \subseteq V\) for all \(i\). Then, for any \(J \subseteq I\), let

\[
V(J) = \bigcup_{j \in J} V_j
\]  

(8.19)

so \(|V(J)| : 2^I \rightarrow \mathbb{Z}_+\) is the set cover func. (we know is submodular).

- We have

**Theorem 8.2.3 (Hall’s theorem)**

*Given a set system \((V, \mathcal{V})\), the family of subsets \(\mathcal{V} = (V_i : i \in I)\) has a transversal \((v_i : i \in I)\) iff for all \(J \subseteq I\)*

\[
|V(J)| \geq |J|
\]  

(8.20)
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- Hall's theorem \((\forall J \subseteq I, |V(J)| \geq |J|)\) as a bipartite graph.
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Then, for any \(J \subseteq I\), let

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V(J) = \bigcup_{j \in J} V_j
\] (8.19)

so \(|V(J)| : 2^I \rightarrow \mathbb{Z}_+\) is the set cover func. (we know is submodular).

- Moreover, we have

**Theorem 8.2.4 (Rado’s theorem (1942))**

If \(M = (V, r)\) is a matroid on \(V\) with rank function \(r\), then the family of subsets \((V_i : i \in I)\) of \(V\) has a transversal \((v_i : i \in I)\) that is independent in \(M\) iff for all \(J \subseteq I\)

\[
r(V(J)) \geq |J|
\] (8.21)
More general conditions for existence of transversals

Theorem 8.2.3 (Polymatroid transversal theorem)

If $\mathcal{V} = (V_i : i \in I)$ is a finite family of non-empty subsets of $V$, and $f : 2^V \rightarrow \mathbb{Z}^+$ is a non-negative, integral, monotone non-decreasing, and submodular function, then $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that

$$f(\bigcup_{i \in J} \{v_i\}) \geq |J| \text{ for all } J \subseteq I \quad (8.19)$$

if and only if

$$f(V(J)) \geq |J| \text{ for all } J \subseteq I \quad (8.20)$$

- Given Theorem ??, we immediately get Theorem 8.2.3 by taking $f(S) = |S|$ for $S \subseteq V$.
- We get Theorem 8.2.4 by taking $f(S) = r(S)$ for $S \subseteq V$, the rank function of the matroid.
Transversals, themselves, define a matroid.

**Theorem 8.3.1**

*If \( V \) is a family of finite subsets of a ground set \( V \), then the collection of partial transversals of \( V \) is the set of independent sets of a matroid \( M = (V, \mathcal{V}) \) on \( V \).*
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- This means that the transversals of \( \mathcal{V} \) are the bases of matroid \( M \).
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- This means that the transversals of $\mathcal{V}$ are the bases of matroid $M$.
- Therefore, all maximal partial transversals of $\mathcal{V}$ have the same cardinality!
Transversals and Bipartite Matchings

- Transversals correspond exactly to matchings in bipartite graphs.
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- Given a set system \((V, \mathcal{V})\), with \(\mathcal{V} = (V_i : i \in I)\), we can define a bipartite graph \(G = (V, I, E)\) associated with \(\mathcal{V}\) that has edge set \(\{(v, i) : v \in V, i \in I, v \in V_i\}\).
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- A matching in this graph is a set of edges no two of which that have a common endpoint.
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Transversals and Bipartite Matchings

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- A matching in this graph is a set of edges no two of which have a common endpoint. In fact, we easily have:

**Lemma 8.3.2**

A subset \(T \subseteq V\) is a partial transversal of \(\mathcal{V}\) iff there is a matching in \((V, I, E)\) in which every edge has one endpoint in \(T\) (\(T\) matched into \(I\)).
Arbitrary Matchings and Matroids?

- Are arbitrary matchings matroids?
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- Consider the following graph (left), and two max-matchings (two right instances)

\[
\begin{array}{c}
\text{A} & \text{B} \\
\text{D} & \text{C} \\
\end{array}
\quad
\begin{array}{c}
\text{A} & \text{B} \\
\text{D} & \text{C} \\
\end{array}
\quad
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\]
Arbitrary Matchings and Matroids?

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- Consider the following graph (left), and two max-matchings (two right instances)

\[ \{AC\} \text{ is a maximum matching, as is } \{AD, BC\}, \text{ but they are not the same size.} \]
Are arbitrary matchings matroids?

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Let \( \mathcal{M} \) be the set of matchings in an arbitrary graph \( G = (V, E) \). Hence, \((E, M)\) is a set system. I1 holds since \( \emptyset \in \mathcal{M} \). I2 also holds since if \( M \in \mathcal{M} \) is a matching, then so is any \( M' \subseteq M \). I3 doesn't hold (as seen above). Exercise: fully characterize the problem of finding the largest subset \( \mathcal{M'} \subseteq \mathcal{M} \) of matchings so that \((E, \mathcal{M'})\) also satisfies I3?
Next slide is from lecture 7.
Partition Matroid, rank as matching

- Example where $\ell = 5$, $(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3)$.

- Recall, $\Gamma : 2^V \rightarrow \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of $X$ is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.

- Here, for $X \subseteq V$, we have $\Gamma(X) = \{i \in I : (v, i) \in E(G) \text{ and } v \in X\}$.

- For such a constructed bipartite graph, the rank function of a partition matroid is $r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i)$, which is the maximum matching involving $X$. 

\begin{itemize}
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  \item For such a constructed bipartite graph, the rank function of a partition matroid is $r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i)$, which is the maximum matching involving $X$. 
\end{itemize}
Morphing Partition Matroid Rank

Recall the partition matroid rank function. Note, $k_i = |I_i|$ in the bipartite graph representation, and since a matroid, w.l.o.g., $|V_i| \geq k_i$ (also, recall, $V(J) = \bigcup_{j \in J} V_j$).
Recall the partition matroid rank function. Note, $k_i = |I_i|$ in the bipartite graph representation, and since a matroid, w.l.o.g., $|V_i| \geq k_i$ (also, recall, $V(J) = \cup_{j \in J} V_j$).

Start with partition matroid rank function in the subsequent equations.

$$r(A) = \sum_{i \in \{1, \ldots, \ell\}} \min(|A \cap V_i|, k_i)$$

(8.1)
Recall the partition matroid rank function. Note, \( k_i = |I_i| \) in the bipartite graph representation, and since a matroid, w.l.o.g., \( |V_i| \geq k_i \) (also, recall, \( V(J) = \bigcup_{j \in J} V_j \)).

Start with partition matroid rank function in the subsequent equations.

\[
r(A) = \sum_{i \in \{1, \ldots, \ell\}} \min(|A \cap V_i|, k_i) \tag{8.1}
\]

\[
= \sum_{i=1}^\ell \min(|A \cap V(I_i)|, |I_i|) \tag{8.2}
\]
Recall the partition matroid rank function. Note, \( k_i = |I_i| \) in the bipartite graph representation, and since a matroid, w.l.o.g., \( |V_i| \geq k_i \) (also, recall, \( V(J) = \bigcup_{j \in J} V_j \)).

Start with partition matroid rank function in the subsequent equations.

\[
\begin{align*}
    r(A) &= \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i) \\
    &= \sum_{i=1}^{\ell} \min(|A \cap V(I_i)|, |I_i|) \\
    &= \sum_{i=1}^{\ell} \min \left( \begin{array}{l}
        |A \cap V(I_i)|, \\
        0
    \end{array} \right) + |I_i \setminus J_i|
\end{align*}
\]
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= \sum_{i \in \{1, \ldots, \ell\}} \min_{J_i \in \{\emptyset, I_i\}} \left( \begin{cases} |A \cap V(I_i)| & \text{if } J_i \neq \emptyset \\ 0 & \text{if } J_i = \emptyset \end{cases} \right) + |I_i \setminus J_i| \tag{8.3}
\]
\[
= \sum_{i \in \{1, \ldots, \ell\}} \min_{J_i \subseteq I_i} \left( \begin{cases} |A \cap V(I_i)| & \text{if } J_i \neq \emptyset \\ 0 & \text{if } J_i = \emptyset \end{cases} \right) + |I_i \setminus J_i| \tag{8.4}
\]
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Start with partition matroid rank function in the subsequent equations.

\[
r(A) = \sum_{i \in \{1, \ldots, \ell\}} \min(|A \cap V_i|, k_i) = \ell \sum_{i=1} \min(|A \cap V(I_i)|, |I_i|) = \sum_{i=1} \min_{J_i \in \{\emptyset, I_i\}} \left( \begin{cases} |A \cap V(I_i)| & \text{if } J_i \neq \emptyset \\ 0 & \text{if } J_i = \emptyset \end{cases} \right) + |I_i \setminus J_i| \] (8.1)

\[
= \sum_{i=1} \min_{J_i \subseteq I_i} \left( \begin{cases} |A \cap V(I_i)| & \text{if } J_i \neq \emptyset \\ 0 & \text{if } J_i = \emptyset \end{cases} \right) + |I_i \setminus J_i| \] (8.2)

\[
= \sum_{i=1} \min_{J_i \subseteq I_i} (|V(J_i) \cap A| + |I_i \setminus J_i|) \] (8.3)
Continuing,

$$r(A) = \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} \left( |V(J_i) \cap V(I_i) \cap A| - |I_i \cap J_i| + |I_i| \right)$$  \hspace{1cm} (8.6)
Continuing,

\[
\begin{align*}
    r(A) &= \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} (|V(J_i) \cap V(I_i) \cap A| - |I_i \cap J_i| + |I_i|) \quad (8.6) \\
    &= \min_{J \subseteq I} \left( \sum_{i=1}^{\ell} |V(J) \cap V(I_i) \cap A| - |I_i \cap J| + |I_i| \right) \quad (8.7)
\end{align*}
\]
Continuing,

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= \min_{J \subseteq I} (|V(J) \cap V(I) \cap A| - |J| + |I|) \quad (8.8)
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Continuing,

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Continuing,

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&= \min_{J \subseteq I} (|V(J) \cap A| - |J| + |I|)
\end{align*}
\]

In fact, this bottom (more general) expression is the expression for the rank of a transversal matroid.
In fact, we have

**Theorem 8.3.3**

Let \((V, \mathcal{V})\) where \(\mathcal{V} = (V_1, V_2, \ldots, V_\ell)\) be a subset system. Let \(I = \{1, \ldots, \ell\}\). Let \(\mathcal{I}\) be the set of partial transversals of \(\mathcal{V}\). Then \((V, \mathcal{I})\) is a matroid.

**Proof.**

We note that \(\emptyset \in \mathcal{I}\) since the empty set is a transversal of the empty subfamily of \(\mathcal{V}\), thus (I1') holds.

We already saw that if \(T\) is a partial transversal of \(\mathcal{V}\), and if \(T' \subseteq T\), then \(T'\) is also a partial transversal. So (I2') holds.

Suppose that \(T_1\) and \(T_2\) are partial transversals of \(\mathcal{V}\) such that \(|T_1| < |T_2|\). Exercise: show that (I3') holds.
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**Theorem 8.3.3**

Let $(V, \mathcal{V})$ where $\mathcal{V} = (V_1, V_2, \ldots, V_\ell)$ be a subset system. Let $I = \{1, \ldots, \ell\}$. Let $\mathcal{I}$ be the set of partial transversals of $\mathcal{V}$. Then $(V, \mathcal{I})$ is a matroid.

**Proof.**

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Partial Transversals Are Independent Sets in a Matroid

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Transversal Matroid Rank

- Transversal matroid has rank

\[ r(A) = \min_{J \subseteq I} (|V(J) \cap A| - |J| + |I|) \]  \hspace{1cm} (8.10)

\[ = \min_{J \subseteq I} m_J(I) \]  \hspace{1cm} (8.11)

Therefore, this function is submodular.

Note that it is a minimum over a set of modular functions in \( I \). Is this true in general?

Exercise: Can you identify a set of sufficient properties over a set of modular functions \( m_i: V \rightarrow \mathbb{R}^+ \) so that \( f(A) = \min_i m_i(A) \) is submodular? Can you identify both necessary and sufficient conditions?
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Exercise: Can you identify a set of sufficient properties over a set of modular functions \( m_i : V \rightarrow \mathbb{R}_+ \) so that \( f(A) = \min_i m_i(A) \) is submodular? Can you identify both necessary and sufficient conditions?
Matroid loops

- A circuit in a matroids is well defined, a subset $A \subseteq E$ is circuit if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).
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- Such an $\{a\}$ is called a loop.
- In a matric (i.e., linear) matroid, the only such loop is the value 0, as all non-zero vectors have rank 1. The 0 can appear $\geq 1$ time with different indices, as can a self loop in a graph appear on different nodes.
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In a matric (i.e., linear) matroid, the only such loop is the value 0, as all non-zero vectors have rank 1. The 0 can appear $\geq 1$ time with different indices, as can a self loop in a graph appear on different nodes.

Note, we also say that two elements $s, t$ are said to be parallel if $\{s, t\}$ is a circuit (e.g., in a matrix, two column vectors, one of which is a scalar multiple of the other).
Definition 8.4.1 (Matroid isomorphism)

Two matroids $M_1$ and $M_2$ respectively on ground sets $V_1$ and $V_2$ are isomorphic if there is a bijection $\pi : V_1 \rightarrow V_2$ which preserves independence (equivalently, rank, circuits, and so on).
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- Let \( \mathbb{F} \) be any field (such as \( \mathbb{R}, \mathbb{Q} \), or some finite field \( \mathbb{F} \), such as a Galois field \( \text{GF}(p) \) where \( p \) is prime (such as \( \text{GF}(2) \)), but not \( \mathbb{Z} \)).
- Succinctly: A field is a set with +, *, closure, associativity, commutativity, and additive and multiplicative identities and inverses.
**Definition 8.4.1 (Matroid isomorphism)**

Two matroids $M_1$ and $M_2$ respectively on ground sets $V_1$ and $V_2$ are **isomorphic** if there is a bijection $\pi : V_1 \rightarrow V_2$ which preserves independence (equivalently, rank, circuits, and so on).

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- We can more generally define matroids on a field.
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- We can more generally define matroids on a field.

Definition 8.4.2 (linear matroids on a field)

Let $X$ be an $n \times m$ matrix and $E = \{1, \ldots, m\}$, where $X_{ij} \in \mathbb{F}$ for some field, and let $\mathcal{I}$ be the set of subsets of $E$ such that the columns of $X$ are linearly independent over $\mathbb{F}$. 
Definition 8.4.1 (Matroid isomorphism)

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- Let $F$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $F$, such as a Galois field $GF(p)$ where $p$ is prime (such as $GF(2)$), but not $\mathbb{Z}$). Succinctly: A field is a set with $+$, $\times$, closure, associativity, commutativity, and additive and multiplicative identities and inverses.
- We can more generally define matroids on a field.

Definition 8.4.3 (representable (as a linear matroid))

Any matroid isomorphic to a linear matroid on a field is called representable over $F$. 

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In particular:

**Theorem 8.4.4**

*Transversal matroids are representable over all finite fields of sufficiently large cardinality, and are representable over any infinite field.*
Converse: Representability of Transversal Matroids

The converse is not true, however.

Example 8.4.5

Let $V = \{1, 2, 3, 4, 5, 6\}$ be a ground set and let $M = (V, \mathcal{I})$ be a set system where $\mathcal{I}$ is all subsets of $V$ of cardinality $\leq 2$ except for the pairs $\{1, 2\}, \{3, 4\}, \{5, 6\}$.
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**Example 8.4.5**

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- It can be shown that this is a matroid and is representable.
- However, this matroid is not isomorphic to any transversal matroid.
Review from Lecture 6

The next frame comes from lecture 6.
Definition 8.5.3 (closed/flat/subspace)

A subset $A \subseteq E$ is closed (equivalently, a flat or a subspace) of matroid $M$ if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

Definition: A hyperplane is a flat of rank $r(M) - 1$.

Definition 8.5.4 (closure)

Given $A \subseteq E$, the closure (or span) of $A$, is defined by

$$\text{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}.$$ 

Therefore, a closed set $A$ has $\text{span}(A) = A$, and the span of a set is closed.

Definition 8.5.5 (circuit)

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise-minimal dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).
Spanning Sets

We have the following definitions:

Definition 8.5.1 (spanning set of a set)
Given a matroid \( M = (V, I) \), and a set \( Y \subseteq V \), then any set \( X \subseteq Y \) such that \( r(X) = r(Y) \) is called a spanning set of \( Y \).

Definition 8.5.2 (spanning set of a matroid)
Given a matroid \( M = (V, I) \), any set \( A \subseteq V \) such that \( r(A) = r(V) \) is called a spanning set of the matroid.

A base of a matroid is a minimal spanning set (and it is independent) but supersets of a base are also spanning.

\( V \) is always trivially spanning.

Consider the terminology: “spanning tree in a graph”, comes from spanning in a matroid sense.
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Given a matroid $\mathcal{M} = (V, \mathcal{I})$, and a set $Y \subseteq V$, then any set $X \subseteq Y$ such that $r(X) = r(Y)$ is called a **spanning set** of $Y$. 

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Given a matroid \( \mathcal{M} = (V, \mathcal{I}) \), any set \( A \subseteq V \) such that \( r(A) = r(V) \) is called a **spanning set** of the matroid.

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- \( V \) is always trivially spanning.

- Consider the terminology: “spanning tree in a graph”, comes from spanning in a matroid sense.
Dual of a Matroid

- Given a matroid $M = (V, \mathcal{I})$, a dual matroid $M^* = (V, \mathcal{I}^*)$ can be defined on the same ground set $V$, but using a very different set of independent sets $\mathcal{I}^*$. 

\[
\mathcal{I}^* = \{ A \subseteq V : V \setminus A \text{ is a spanning set of } M \} 
\]

\[
\mathcal{I}^* = \{ V \setminus S : S \subseteq V \text{ is a spanning set of } M \} 
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i.e., $\mathcal{I}^*$ are complements of spanning sets of $M$. That is, a set $A \subseteq V$ is independent in the dual matroid $M^*$ (i.e., $A \in \mathcal{I}^*$) if $A$'s complement is spanning in $M$ (residual $V \setminus A$ must contain a base in $M$).

Dual of the dual: Note, we have that $(M^*)^* = M$. 

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We define the set of sets $\mathcal{I}^*$ for $M^*$ as follows:

$$\mathcal{I}^* = \{ A \subseteq V : V \setminus A \text{ is a spanning set of } M \} \quad (8.12)$$

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  i.e., $\mathcal{I}^*$ are complements of spanning sets of $M$.
- That is, a set $A$ is independent in the dual matroid $M^*$ if removal of $A$ from $V$ does not decrease the rank in $M$:

  $\mathcal{I}^* = \{A \subseteq V : \text{rank}_M(V \setminus A) = \text{rank}_M(V)\}$ \hspace{1cm} (8.14)
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- In other words, a set \( A \subseteq V \) is independent in the dual \( M^* \) (i.e., \( A \in \mathcal{I}^* \)) if \( A \)'s complement is spanning in \( M \) (residual \( V \setminus A \) must contain a base in \( M \)).
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Graphic matroid over edges $E = \{a, b, c, d, e\}$ for the graph on the right. Independent sets (green) and spanning sets (blue) are shown.

Diagram showing the graphic matroid over edges $E = \{a, b, c, d, e\}$ for the graph on the right. Independent sets (green) and spanning sets (blue) are shown.
Graphic matroid over edges $E = \{a, b, c, d, e\}$ for the graph on the right. Spanning sets of $M$ are blue. Complement of spanning sets of $M$ (independent sets of $M^*$) are orange.
Dual of a Matroid: Bases

- The smallest spanning sets are bases.
The smallest spanning sets are bases. Hence, a base $B$ of $M$ (where $B = V \setminus B^*$ is as small as possible while still spanning) is the complement of a base $B^*$ of $M^*$ (where $B^* = V \setminus B$ is as large as possible while still being independent).
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In fact, we have that

**Theorem 8.5.3 (Dual matroid bases)**

Let $M = (V, I)$ be a matroid and $\mathcal{B}(M)$ be the set of bases of $M$. Then define

$$\mathcal{B}^*(M) = \{V \setminus B : B \in \mathcal{B}(M)\}. \quad (8.15)$$

Then $\mathcal{B}^*(M)$ is the set of basis of $M^*$ (that is, $\mathcal{B}^*(M) = \mathcal{B}(M^*)$).
An exercise in duality Terminology

- \( B^*(M) \), the bases of \( M^* \), are called cobases of \( M \).
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- $\mathcal{B}^*(M)$, the bases of $M^*$, are called cobases of $M$.
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**Proposition 8.5.4 (from Oxley 2011)**

Let $M = (V, I)$ be a matroid, and let $X \subseteq V$. Then
1. $X$ is independent in $M$ if and only if $V \setminus X$ is cospanning in $M$ (spanning in $M^*$).
2. $X$ is spanning in $M$ if and only if $V \setminus X$ is coindependent in $M$ (independent in $M^*$).
3. $X$ is a hyperplane in $M$ if and only if $V \setminus X$ is a cocircuit in $M$ (circuit in $M^*$).
4. $X$ is a circuit in $M$ if and only if $V \setminus X$ is a cohyperplane in $M$ (hyperplane in $M^*$).
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Using a graphic/cycle matroid, we can already see how dual matroid concepts demonstrates the extraordinary flexibility and power that a matroid can have.
Example duality: graphic matroid

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- Recall, in cycle matroid, a spanning set of $G$ is any set of edges that are incident to all nodes (i.e., any superset of a spanning forest), a minimal spanning set is a spanning tree (or forest), and a circuit has a nice visual interpretation (a cycle in the graph).
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- A **minimal cut** in $G$ is a cut $X \subseteq E(G)$ such that $X \setminus \{x\}$ is not a cut for any $x \in X$. 
Example duality: graphic matroid

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- Recall, in cycle matroid, a spanning set of $G$ is any set of edges that are incident to all nodes (i.e., any superset of a spanning forest), a minimal spanning set is a spanning tree (or forest), and a circuit has a nice visual interpretation (a cycle in the graph).
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- A minimal cut in $G$ is a cut $X \subseteq E(G)$ such that $X \setminus \{x\}$ is not a cut for any $x \in X$.
- A cocycle (cocircuit) in a graphic matroid is a minimal graph cut.
Example duality: graphic matroid

- Using a graphic/cycle matroid, we can already see how dual matroid concepts demonstrates the extraordinary flexibility and power that a matroid can have.
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- A cocycle (cocircuit) in a graphic matroid is a minimal graph cut.

- A mincut is a circuit in the dual “cocycle” (or “cut”) matroid.

- All dependent sets in a cocycle matroid are cuts (i.e., a dependent set is a minimal cut or contains one).
Example: cocycle matroid (sometimes “cut matroid”) 

- The dual of the cycle matroid is called the cocycle matroid. Recall, 
  \[ I^* = \{ A \subseteq V : V \setminus A \text{ is a spanning set of } M \} \]
Example: cocycle matroid (sometimes “cut matroid”)

- The dual of the cycle matroid is called the cocycle matroid. Recall, 
  \[ \mathcal{I}^* = \{ A \subseteq V : V \setminus A \text{ is a spanning set of } M \} \]

- \( \mathcal{I}^* \) consists of all sets of edges the complement of which contains a spanning tree — i.e., an independent set can’t consist of edges that, if removed, would render the graph non-spanning.
Example: cocycle matroid (sometimes “cut matroid”)

- The dual of the cycle matroid is called the cocycle matroid. Recall, \( \mathcal{I}^* = \{ A \subseteq V : V \setminus A \text{ is a spanning set of } M \} \)
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A graph G
The dual of the cycle matroid is called the cocycle matroid. Recall, \( I^* = \{ A \subseteq V : V \setminus A \text{ is a spanning set of } M \} \)

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Minimally spanning in \( M \) (and thus a base (maximally independent) in \( M \))

Maximally independent in \( M^* \) (thus a base, minimally spanning, in \( M^* \))
Example: cocycle matroid (sometimes “cut matroid”)

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Minimally spanning in $M$ (and thus a base (maximally independent) in $M$)   
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Example: cocycle matroid (sometimes "cut matroid")

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Independent but not spanning in \( M \), and not closed in \( M \).

Dependent in \( M^* \) (contains a cocycle, is a nonminimal cut)
Example: cocycle matroid (sometimes “cut matroid”)

- The dual of the cycle matroid is called the cocycle matroid. Recall, \( \mathcal{I}^* = \{ A \subseteq V : V \setminus A \text{ is a spanning set of } M \} \)

- \( \mathcal{I}^* \) consists of all sets of edges the complement of which contains a spanning tree — i.e., an independent set can’t consist of edges that, if removed, would render the graph non-spanning.

Spanning in \( M \), but not a base, and not independent (has cycles)

Independent in \( M^* \) (does not contain a cut)
Example: cocycle matroid (sometimes "cut matroid")

- The dual of the cycle matroid is called the cocycle matroid. Recall, 
  \[ I^* = \{ A \subseteq V : V \setminus A \text{ is a spanning set of } M \} \]

- \( I^* \) consists of all sets of edges the complement of which contains a spanning tree — i.e., an independent set can’t consist of edges that, if removed, would render the graph non-spanning.

Independent but not spanning in \( M \), and not closed in \( M \).

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A hyperplane in $M$, dependent but not spanning in $M$

A cycle in $M^*$ (minimally dependent in $M^*$, a cocycle, or a minimal cut)
Example: cocycle matroid (sometimes “cut matroid”)

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Cycle Matroid - independent sets have no cycles.

Cocycle matroid, independent sets contain no cuts.
The dual of a matroid is (indeed) a matroid

**Theorem 8.5.5**

Given matroid \( M = (V, \mathcal{I}) \), let \( M^* = (V, \mathcal{I}^*) \) be as previously defined. Then \( M^* \) is a matroid.

**Proof.**

- Since \( V \setminus \emptyset \) is spanning in primal, clearly \( \emptyset \in \mathcal{I}^* \), so (I1') holds.
The dual of a matroid is (indeed) a matroid

**Theorem 8.5.5**

Given matroid $M = (V, \mathcal{I})$, let $M^* = (V, \mathcal{I}^*)$ be as previously defined. Then $M^*$ is a matroid.

**Proof.**

- Since $V \setminus \emptyset$ is spanning in primal, clearly $\emptyset \in \mathcal{I}^*$, so (I1') holds.

- Also, if $I \subseteq J \in \mathcal{I}^*$, then clearly also $I \in \mathcal{I}^*$ since if $V \setminus J$ is spanning in $M$, so must $V \setminus I$. Therefore, (I2') holds.

- Next, given $I, J \in \mathcal{I}^*$ with $|I| < |J|$, it must be the case that $\bar{I} = V \setminus I$ and $\bar{J} = V \setminus J$ are both spanning in $M$ with $|\bar{I}| > |\bar{J}|$. 

...
The dual of a matroid is (indeed) a matroid

**Theorem 8.5.5**

Given matroid \( M = (V, \mathcal{I}) \), let \( M^* = (V, \mathcal{I}^*) \) be as previously defined. Then \( M^* \) is a matroid.

**Proof.**

Consider \( I, J \in \mathcal{I}^* \) with \(|I| < |J|\). We need to show that there is some member \( v \in J \setminus I \) such that \( I + v \) is independent in \( M^* \), which means that \( V \setminus (I + v) = (V \setminus I) \setminus v = \overline{I} - v \) is still spanning in \( M \). That is, removing \( v \) from \( V \setminus I \) doesn’t make \( (V \setminus I) \setminus v \) not spanning in \( M \).
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**Theorem 8.5.5**

*Given matroid* \( M = (V, \mathcal{I}) \), *let* \( M^* = (V, \mathcal{I}^*) \) *be as previously defined. Then* \( M^* \) *is a matroid.*

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- Consider \( I, J \in \mathcal{I}^* \) with \( |I| < |J| \). We need to show that there is some member \( v \in J \setminus I \) such that \( I + v \) is independent in \( M^* \), which means that \( V \setminus (I + v) = (V \setminus I) \setminus v = \bar{I} - v \) is still spanning in \( M \). That is, removing \( v \) from \( V \setminus I \) doesn’t make \( (V \setminus I) \setminus v \) not spanning in \( M \).

- Since \( V \setminus J \) is spanning in \( M \), \( V \setminus J \) contains some base (say \( B_J \subseteq V \setminus J \)) of \( M \). Also, \( V \setminus I \) contains a base of \( M \), say \( B_{\bar{I}} \subseteq V \setminus I \).
The dual of a matroid is (indeed) a matroid

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*Given matroid* $M = (V, \mathcal{I})$, let $M^* = (V, \mathcal{I}^*)$ be as previously defined. Then $M^*$ is a matroid.*

**Proof.**

- Consider $I, J \in \mathcal{I}^*$ with $|I| < |J|$. We need to show that there is some member $v \in J \setminus I$ such that $I + v$ is independent in $M^*$, which means that $V \setminus (I + v) = (V \setminus I) \setminus v = \bar{I} - v$ is still spanning in $M$. That is, removing $v$ from $V \setminus I$ doesn’t make $(V \setminus I) \setminus v$ not spanning in $M$.

- Since $V \setminus J$ is spanning in $M$, $V \setminus J$ contains some base (say $B_{\bar{J}} \subseteq V \setminus J$) of $M$. Also, $V \setminus I$ contains a base of $M$, say $B_{\bar{I}} \subseteq V \setminus I$.

- Since $B_{\bar{J}} \setminus I \subseteq V \setminus I$, and $B_{\bar{J}} \setminus I$ is independent in $M$, we can choose the base $B_{\bar{I}}$ of $M$ s.t. $B_{\bar{J}} \setminus I \subseteq B_{\bar{I}} \subseteq V \setminus I$. 

...
The dual of a matroid is (indeed) a matroid

**Theorem 8.5.5**

*Given matroid* \( M = (V, \mathcal{I}) \), *let* \( M^* = (V, \mathcal{I}^*) \) *be as previously defined. Then* \( M^* \) *is a matroid."

**Proof.**

- Consider \( I, J \in \mathcal{I}^* \) with \( |I| < |J| \). We need to show that there is some member \( v \in J \setminus I \) such that \( I + v \) is independent in \( M^* \), which means that \( V \setminus (I + v) = (V \setminus I) \setminus v = \bar{I} - v \) is still spanning in \( M \). That is, removing \( v \) from \( V \setminus I \) doesn’t make \( (V \setminus I) \setminus v \) not spanning in \( M \).

- Since \( V \setminus J \) is spanning in \( M \), \( V \setminus J \) contains some base (say \( B_{\bar{J}} \subseteq V \setminus J \)) of \( M \). Also, \( V \setminus I \) contains a base of \( M \), say \( B_{\bar{I}} \subseteq V \setminus I \).

- Since \( B_{\bar{J}} \setminus I \subseteq V \setminus I \), and \( B_{\bar{J}} \setminus I \) is independent in \( M \), we can choose the base \( B_{\bar{I}} \) of \( M \) s.t. \( B_{\bar{J}} \setminus I \subseteq B_{\bar{I}} \subseteq V \setminus I \).

- Since \( B_{\bar{J}} \) and \( J \) are disjoint, we have both: 1) \( B_{\bar{J}} \setminus I \) and \( J \setminus I \) are disjoint; and 2) \( B_{\bar{J}} \cap I \subseteq I \setminus J \). Also note, \( B_{\bar{I}} \) and \( I \) are disjoint. ...
The dual of a matroid is (indeed) a matroid

**Theorem 8.5.5**

Given matroid $M = (V, \mathcal{I})$, let $M^* = (V, \mathcal{I}^*)$ be as previously defined. Then $M^*$ is a matroid.

**Proof.**

Now $J \setminus I \not\subseteq B_I$, since otherwise (i.e., assuming $J \setminus I \subseteq B_I$):

$$|B_J| = |B_J \cap I| + |B_J \setminus I|$$

$$\leq |I \setminus J| + |B_J \setminus I|$$

$$< |J \setminus I| + |B_J \setminus I| \leq |B_I|$$

which is a contradiction. The last inequality on the right follows since $J \setminus I \subseteq B_I$ (by assumption) and $B_J \setminus I \subseteq B_I$ implies that $(J \setminus I) \cup (B_J \setminus I) \subseteq B_I$, but since $J$ and $B_J$ are disjoint, we have that $|J \setminus I| + |B_J \setminus I| \leq |B_I|$. 
The dual of a matroid is (indeed) a matroid

**Theorem 8.5.5**

*Given matroid* $M = (V, \mathcal{I})$, *let* $M^* = (V, \mathcal{I}^*)$ *be as previously defined. Then* $M^*$ *is a matroid.*

**Proof.**

- Now $J \setminus I \not\subseteq B_{\overline{I}}$, since otherwise (i.e., assuming $J \setminus I \subseteq B_{\overline{I}}$):

  $$|B_J| = |B_J \cap I| + |B_J \setminus I|$$

  $$\leq |I \setminus J| + |B_J \setminus I|$$

  $$< |J \setminus I| + |B_J \setminus I| \leq |B_{\overline{I}}|$$

  which is a contradiction.

- Therefore, $J \setminus I \not\subseteq B_{\overline{I}}$, and there is a $v \in J \setminus I$ s.t. $v \notin B_{\overline{I}}$. 

  ...
The dual of a matroid is (indeed) a matroid

**Theorem 8.5.5**

Given matroid $M = (V, \mathcal{I})$, let $M^* = (V, \mathcal{I}^*)$ be as previously defined. Then $M^*$ is a matroid.

**Proof.**

- Now $J \setminus I \not\subseteq B_{\bar{I}}$, since otherwise (i.e., assuming $J \setminus I \subseteq B_{\bar{I}}$):

  \begin{align*}
  |B_J| &= |B_J \cap I| + |B_J \setminus I| \\
  &\leq |I \setminus J| + |B_J \setminus I| \\
  &< |J \setminus I| + |B_J \setminus I| \leq |B_{\bar{I}}|
  \end{align*}  

  which is a contradiction.

- Therefore, $J \setminus I \not\subseteq B_{\bar{I}}$, and there is a $v \in J \setminus I$ s.t. $v \notin B_{\bar{I}}$.

- So $B_{\bar{I}}$ is disjoint with $I \cup \{v\}$, means $B_{\bar{I}} \subseteq V \setminus (I \cup \{v\})$, or $V \setminus (I \cup \{v\})$ is spanning in $M$, and therefore $I \cup \{v\} \in \mathcal{I}^*$. 

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Theorem 8.5.6

Let \( M \) be an \( \mathbb{F} \)-representable matroid (i.e., one that can be represented by a finite sized matrix over field \( \mathbb{F} \)). Then \( M^* \) is also \( \mathbb{F} \)-representable.

Hence, for matroids as general as matric matroids, duality does not extend the space of matroids that can be used.
Theorem 8.5.6

Let $M$ be an $\mathbb{F}$-representable matroid (i.e., one that can be represented by a finite sized matrix over field $\mathbb{F}$). Then $M^*$ is also $\mathbb{F}$-representable.

Hence, for matroids as general as matric matroids, duality does not extend the space of matroids that can be used.

Theorem 8.5.7

Let $M$ be a graphic matroid (i.e., one that can be represented by a graph $G = (V, E)$). Then $M^*$ is not necessarily also graphic.

Hence, for graphic matroids, duality can increase the space and power of matroids, and since they are based on a graph, they are relatively easy to use: 1) all cuts are dependent sets; 2) minimal cuts are cycles; 3) bases of a cut are any one edge removed from minimal cuts; 4) independent sets are edges that are not cuts (minimal or otherwise); 5) bases of matroid are maximal non-cuts (non-cut containing edge sets).
Theorem 8.5.8

The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified in terms of the rank $r_M$ in matroid $M$ as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \quad (8.19)$$

Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2. I.e., $|X|$ is modular, complement $f(V \setminus X)$ is submodular if $f$ is submodular, $r_M(V)$ is a constant, and summing submodular functions and a constant preserves submodularity.
Theorem 8.5.8

The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified in terms of the rank $r_M$ in matroid $M$ as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V)$$

(8.19)

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2.
- Non-negativity integral follows since $|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V)$. The right inequality follows since $r_M$ is submodular.
Theorem 8.5.8

The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified in terms of the rank $r_M$ in matroid $M$ as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \quad (8.19)$$

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2.
- Non-negativity integral follows since

  $$|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V).$$

- Monotone non-decreasing follows since, as $X$ increases by one, $|X|$ always increases by 1, while $r_M(V \setminus X)$ decreases by one or zero.
The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified in terms of the rank $r_M$ in matroid $M$ as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \tag{8.19}$$

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2.
- Non-negativity integral follows since $|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V)$.
- Monotone non-decreasing follows since, as $X$ increases by one, $|X|$ always increases by 1, while $r_M(V \setminus X)$ decreases by one or zero.
- Therefore, $r_{M^*}$ is the rank function of a matroid. That it is the dual matroid rank function is shown in the next proof.
The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified in terms of the rank $r_M$ in matroid $M$ as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \quad (8.19)$$

**Proof.**

A set $X$ is independent in $(V, r_{M^*})$ if and only if

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) = |X| \quad (8.20)$$
Theorem 8.5.8

The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified in terms of the rank $r_M$ in matroid $M$ as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \quad (8.19)$$

Proof.

A set $X$ is independent in $(V, r_{M^*})$ if and only if

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) = |X| \quad (8.20)$$

or

$$r_M(V \setminus X) = r_M(V) \quad (8.21)$$
The rank function $r_{M^*}$ of the dual matroid $M^*$ may be specified in terms of
the rank $r_M$ in matroid $M$ as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V)$$  \hfill (8.19)

Proof.

A set $X$ is independent in $(V, r_{M^*})$ if and only if

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) = |X|$$  \hfill (8.20)

or

$$r_M(V \setminus X) = r_M(V)$$  \hfill (8.21)

But a subset $X$ is independent in $M^*$ only if $V \setminus X$ is spanning in $M$ (by
the definition of the dual matroid).
Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then

$$\mathcal{I}_Y = \{Z : Z \subseteq Y, Z \in \mathcal{I}\}$$

is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid with rank $r(M_Y) = r(Y)$. 
Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then

$$\mathcal{I}_Y = \{Z : Z \subseteq Y, Z \in \mathcal{I}\}$$

(8.22)

is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid with rank $r(M_Y) = r(Y)$.

This is called the restriction of $M$ to $Y$, and is often written $M|_Y$. 

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Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then

$$\mathcal{I}_Y = \{Z : Z \subseteq Y, Z \in \mathcal{I}\}$$

is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid with rank $r(M_Y) = r(Y)$. This is called the restriction of $M$ to $Y$, and is often written $M|Y$.

If $Y = V \setminus X$, then we have that $M|Y$ has the form:

$$\mathcal{I}_Y = \{Z : Z \cap X = \emptyset, Z \in \mathcal{I}\}$$

is considered a deletion of $X$ from $M$, and is often written $M \setminus X$. 
Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then

$$\mathcal{I}_Y = \{ Z : Z \subseteq Y, Z \in \mathcal{I} \}$$

(8.22)

is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid with rank $r(M_Y) = r(Y)$.

This is called the restriction of $M$ to $Y$, and is often written $M|Y$.

If $Y = V \setminus X$, then we have that $M|Y$ has the form:

$$\mathcal{I}_Y = \{ Z : Z \cap X = \emptyset, Z \in \mathcal{I} \}$$

(8.23)

is considered a deletion of $X$ from $M$, and is often written $M \setminus X$.

Hence, $M|Y = M \setminus (V \setminus Y)$, and $M|(V \setminus X) = M \setminus X$. 
Matroid restriction/deletion

- Let $M = (V, \mathcal{I})$ be a matroid and let $Y \subseteq V$, then

$$\mathcal{I}_Y = \{Z : Z \subseteq Y, Z \in \mathcal{I}\}$$

is such that $M_Y = (Y, \mathcal{I}_Y)$ is a matroid with rank $r(M_Y) = r(Y)$.

- This is called the restriction of $M$ to $Y$, and is often written $M|Y$.

- If $Y = V \setminus X$, then we have that $M|Y$ has the form:

$$\mathcal{I}_Y = \{Z : Z \cap X = \emptyset, Z \in \mathcal{I}\}$$

is considered a deletion of $X$ from $M$, and is often written $M \setminus X$.

- Hence, $M|Y = M \setminus (V \setminus Y)$, and $M|(V \setminus X) = M \setminus X$.

- The rank function is of the same form. I.e., $r_Y : 2^Y \to \mathbb{Z}_+$, where $r_Y(Z) = r(Z)$ for $Z \subseteq Y$, $Y = V \setminus X$. 
Matroid contraction $M/Z$

- Contraction by $Z$ is dual to deletion, and is like a forced inclusion of a contained base $B_Z$ of $Z$, but with a similar ground set removal by $Z$.
- Contracting $Z$ is written $M/Z$. Updated ground set in $M/Z$ is $V \setminus Z$. 

The rank function takes the form:

$$r_{M/Z}(Y) = r(Y \cup Z) - r(Z) = r(Y \mid Z)$$  \hspace{1cm} (8.24)$$

$$= r(Y \cup B_Z) - r(B_Z) = r(Y \mid B_Z)$$  \hspace{1cm} (8.25)$$

So given $I \subseteq V \setminus Z$ and $B_Z$ is a base of $Z$, $r_{M/Z}(I) = |I|$ is identical to $r(I \cup Z) = |I| + r(Z) = |I| + |B_Z|$. Since $r(I \cup Z) = r(I \cup B_Z)$, this implies $r(I \cup B_Z) = |I| + |B_Z|$, or $I \cup B_Z$ is independent in $M$.  

A minor of a matroid is any matroid obtained via a series of deletions and contractions of some matroid. In fact, it is the case $M/Z = (M^* \setminus Z)^*$ (Exercise: show why).
Contraction by $Z$ is dual to deletion, and is like a forced inclusion of a contained base $B_Z$ of $Z$, but with a similar ground set removal by $Z$. Contracting $Z$ is written $M/Z$. Updated ground set in $M/Z$ is $V \setminus Z$.

Let $Z \subseteq V$ and let $B_Z$ be a base of $Z$. Then a subset $I \subseteq V \setminus Z$ is independent in $M/Z$ iff $I \cup B_Z$ is independent in $M$. 

The rank function takes the form:

$$r_{M/Z}(Y) = r(Y \cup Z) - r(Z) = r(Y | Z)$$

(8.24)

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Matroid Intersection

Let $M_1 = (V, \mathcal{I}_1)$ and $M_2 = (V, \mathcal{I}_2)$ be two matroids. Consider their common independent sets $\mathcal{I}_1 \cap \mathcal{I}_2$. 

Theorem 8.6.1

Let $M_1$ and $M_2$ be given as above, with rank functions $r_1$ and $r_2$. Then the size of the maximum size set in $\mathcal{I}_1 \cap \mathcal{I}_2$ is given by

$$\left( r_1 \ast r_2 \right)(V) \equiv \min_{X \subseteq V} \left( r_1(X) + r_2(V \setminus X) \right)$$

(8.26)

This is an instance of the convolution of two submodular functions, $f_1$ and $f_2$ that, evaluated at $Y \subseteq V$, is written as:

$$(f_1 \ast f_2)(Y) = \min_{X \subseteq Y} \left( f_1(X) + f_2(Y \setminus X) \right)$$

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Recall Hall’s theorem, that a transversal exists iff for all $X \subseteq V$, we have $|\Gamma(X)| \geq |X|$.

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$\Leftrightarrow |\Gamma(X)| - |X| \geq 0, \forall X$
Recall Hall’s theorem, that a transversal exists if and only if for all $X \subseteq V$, we have $|\Gamma(X)| \geq |X|$. 

$\iff |\Gamma(X)| - |X| \geq 0, \forall X$

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So Hall’s theorem can be expressed as convolution. Exercise: define $g(A) = [\Gamma(\cdot) \ast |\cdot|](A)$, prove that $g$ is submodular.
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So Hall’s theorem can be expressed as convolution. Exercise: define $g(A) = [\Gamma(\cdot) \ast |\cdot|](A)$, prove that $g$ is submodular.

Note, in general, convolution of two submodular functions does not preserve submodularity (but in certain special cases it does).
Matroid Union

Definition 8.6.2

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, ..., $M_k = (V_k, \mathcal{I}_k)$ be matroids. We define the union of matroids as

$$M_1 \vee M_2 \vee \cdots \vee M_k = (V_1 \cup V_2 \cup \cdots \cup V_k, \mathcal{I}_1 \vee \mathcal{I}_2 \vee \cdots \vee \mathcal{I}_k),$$

where

$$\mathcal{I}_1 \vee \mathcal{I}_2 \vee \cdots \vee \mathcal{I}_k = \{I_1 \cup I_2 \cup \cdots \cup I_k | I_1 \in \mathcal{I}_1, \ldots, I_k \in \mathcal{I}_k\} \quad (8.28)$$

Note $A \cup B$ designates the disjoint union of $A$ and $B$. 
**Matroid Union**

**Definition 8.6.2**

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, \ldots, $M_k = (V_k, \mathcal{I}_k)$ be matroids. We define the **union** of matroids as

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Note $A \cup B$ designates the disjoint union of $A$ and $B$.

**Theorem 8.6.3**

Let $M_1 = (V_1, \mathcal{I}_1)$, $M_2 = (V_2, \mathcal{I}_2)$, \ldots, $M_k = (V_k, \mathcal{I}_k)$ be matroids, with rank functions $r_1, \ldots, r_k$. Then the union of these matroids is still a matroid, having rank function

$$r(Y) = \min_{X \subseteq Y} \left( |Y \setminus X| + r_1(X \cap V_1) + \cdots + r_k(X \cap V_k) \right) \quad (8.29)$$

for any $Y \subseteq V_1 \cup \ldots \cup V_k$. 

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Exercise: Matroid Union, and Matroid duality

Exercise: Fully characterize $M \vee M^*$. 
Matroids of three or fewer elements are graphic

- All matroids up to and including three elements (edges) are graphic.
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(a) The only matroid with zero elements.
(b) The two one-element matroids.
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This is a nice way to visualize matroids with very low ground set sizes.

What about matroids that are low rank but with many elements?
**Affine Matroids**

- Given an $n \times m$ matrix with entries over some field $\mathbb{F}$, we say that a subset $S \subseteq \{1, \ldots, m\}$ of indices (with corresponding column vectors $\{v_i : i \in S\}$, with $|S| = k \leq m$) is **affinely dependent** if $m \geq 1$ and there exists elements $\{a_1, \ldots, a_k\} \in \mathbb{F}$, not all zero with $\sum_{i=1}^{k} a_i = 0$, such that $\sum_{i=1}^{k} a_i v_i = 0$. 

Concisely: points $\{v_1, v_2, \ldots, v_k\}$ are affinely independent if $v_2 - v_1$, $v_3 - v_1$, ..., $v_k - v_1$ are linearly independent.

Example: in 2D, three collinear points are affinely dependent, three non-collinear points are affinely independent, and $\geq 4$ collinear or non-collinear points are affinely dependent.

**Proposition 8.7.1 (affine matroid)**

Let ground set $E = \{1, \ldots, m\}$ index column vectors of a matrix, and let $I$ be the set of subsets $X$ of $E$ such that $X$ indices affinely independent vectors. Then $(E, I)$ is a matroid.

Exercise: prove this.
Affine Matroids

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- Otherwise, the set is called **affinely independent**.
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**Exercise:** prove this.
Consider the affine matroid with \( n \times m = 2 \times 6 \) matrix on the field \( \mathbb{F} = \mathbb{R} \), and let the elements be \( \{(0, 0), (1, 0), (2, 0), (0, 1), (0, 2), (1, 1)\} \).
Euclidean Representation of Low-rank Matroids

- Consider the affine matroid with $n \times m = 2 \times 6$ matrix on the field $\mathbb{F} = \mathbb{R}$, and let the elements be $\{(0, 0), (1, 0), (2, 0), (0, 1), (0, 2), (1, 1)\}$.
- We can plot the points in $\mathbb{R}^2$ as on the right:
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Lines indicate collinear sets with $\geq 3$ points, while any two points have rank 2.

Dependent sets consist of all subsets with $\geq 4$ elements (rank 3), or 3 collinear elements (rank 2). Any two points have rank 2.
As another example on the right, a rank 4 matroid
As another example on the right, a rank 4 matroid

All sets of 5 points are dependent. The only other sets of dependent points are coplanar ones of size 4. Namely:

\{(0, 0, 0), (0, 1, 0), (1, 1, 0), (1, 0, 0)\},
\{(0, 0, 0), (0, 0, 1), (0, 1, 1), (0, 1, 0)\}, and
\{(0, 0, 1), (0, 1, 1), (1, 1, 0), (1, 0, 0)\}. 
Euclidean Representation of Low-rank Matroids

- In general, for a matroid $\mathcal{M}$ of rank $m + 1$ with $m \leq 3$, then a subset $X$ in a geometric representation in $\mathbb{R}^m$ is dependent if:

  1. $|X| \geq 2$ and the points are identical;
  2. $|X| \geq 3$ and the points are collinear;
  3. $|X| \geq 4$ and the points are coplanar; or
  4. $|X| \geq 5$ and the points are anywhere in space.

When they exist, loops are represented in a geometry by a separate box indicating how many loops there are. Parallel elements, when they exist in a matroid, are indicated by a multiplicity next to a point.

Theorem 8.7.2

Any matroid of rank $m \leq 4$ can be represented by an affine matroid in $\mathbb{R}^{m-1}$. True regardless of how big $|V|$ is.
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(see Oxley 2011 for more details)
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- If diagram has at most one plane, then any two distinct lines meet in at most one point.
- If diagram has more than one plane, then: 1) any two distinct planes meeting in more than two points do so in a line; 2) any two distinct lines meeting in a point do so in at most one point and lie in on a common plane; 3) any line not lying on a plane intersects it in at most one point.
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- Example: Is there a matroid that is not representable (i.e., not linear for some field)?

Consider the matroid

\[
\begin{pmatrix}
1 & 7 & 8 & 9 \\
2 & 3 & 6 & 5 & 4
\end{pmatrix}
\]

Called the non-Pappus matroid. Has rank three, but any matric matroid with the above dependencies would require that \{7, 8, 9\} is dependent, hence requiring an additional line in the above.
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Euclidean Representation of Low-rank Matroids: A test

Is this a matroid?

1 2 3
4
7
5
6

Check rank's submodularity: Let $X = \{1, 2, 3, 6, 7\}$, $Y = \{1, 4, 5, 6, 7\}$. So $r(X) = 3$, and $r(Y) = 3$, and $r(X \cup Y) = 4$, so we must have, by submodularity, that $r(\{1, 6, 7\}) = r(X \cap Y) \leq r(X) + r(Y) - r(X \cup Y) = 2$.

However, from the diagram, we have that since 1, 6, 7 are distinct non-collinear points, we have that $r(X \cap Y) = 3$. If we extend the line from 6-7 to 1, then is it a matroid? Hence, not all 2D or 3D graphs of points and lines are matroids.

Prof. Jeff Bilmes
EE563/Spring 2020/Submodularity - Lecture 8 - Oct 26th, 2020
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Is this a matroid?

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Hence, not all 2D or 3D graphs of points and lines are matroids.
Consider the following geometry on $|V| = 8$ points with $V = \{a, b, c, d, e, f, g, h\}$. 

Exercise: Is this a matroid? Exercise: If so, is it representable?
Consider the following geometry on $|V| = 8$ points with $V = \{a, b, c, d, e, f, g, h\}$.

Note, we are given that the points $\{b, d, h, f\}$ are not coplanar. However, the following sets of points are coplanar: $\{a, b, e, f\}$, $\{d, c, g, h\}$, $\{a, d, h, e\}$, $\{b, c, g, f\}$, $\{b, c, d, a\}$, $\{f, g, h, e\}$, and $\{a, c, g, e\}$.
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Other examples can be more complex, consider the following two matroids (from Oxley, 2011):

- Projective Geometries: Other Examples

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- Hence, lines (in 2D) which are rank 2 sets may be curved; planes (in 3D) can be twisted.
Matroids with $|V| \leq 3$ are graphic.
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Matroids, Representation and Equivalence: Summary

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- Matroids can be seen as related to projective geometries (and are sometimes called combinatorial geometries).
- Exists much research on different subclasses of matroids, and if/when they are contained in (or isomorphic to) each other.
Matroid Further Reading

- “The Coming of the Matroids”, William Cunningham, 2012 (a nice history)
- Crapo & Rota, “On the Foundations of Combinatorial Theory: Combinatorial Geometries”, 1970 (while this is old, it is very readable).
- Schrijver, “Combinatorial Optimization”, 2003
The greedy algorithm

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- Greedy is good since it can be made to run very fast $O(n \log n)$.
- Often, however, greedy is heuristic (it might work well in practice, but worst-case performance can be unboundedly poor).
- We will next see that the greedy algorithm working optimally is a defining property of a matroid, and is also a defining property of a polymatroid function.
Matroid and the greedy algorithm

Let \((E, \mathcal{I})\) be an independence system, and we are given a non-negative modular weight function \(w : E \rightarrow \mathbb{R}_+\).
Matroid and the greedy algorithm

Let \((E, I)\) be an independence system, and we are given a non-negative modular weight function \(w : E \to \mathbb{R}_+\).

**Algorithm 1: The Matroid Greedy Algorithm**

1. Set \(X \leftarrow \emptyset\);
2. while \(\exists v \in E \setminus X\) s.t. \(X \cup \{v\} \in I\) do
3. \hspace{0.5cm} \(v \in \text{argmax} \{w(v) : v \in E \setminus X, X \cup \{v\} \in I\}\);
4. \hspace{0.5cm} \(X \leftarrow X \cup \{v\}\);
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Same as sorting items by decreasing weight \(w\), and then choosing items in that order that retain independence.

Theorem 8.8.1

Let \((E, \mathcal{I})\) be an independence system. Then the pair \((E, \mathcal{I})\) is a matroid if and only if for each weight function \(w \in \mathbb{R}_+^E\), Algorithm 1 above leads to a set \(I \in \mathcal{I}\) of maximum weight \(w(I)\).
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The next slide is from Lecture 6.
In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

**Theorem 8.8.3 (Matroid (by bases))**

Let $E$ be a set and $\mathcal{B}$ be a nonempty collection of subsets of $E$. Then the following are equivalent.

1. $\mathcal{B}$ is the collection of bases of a matroid;
2. if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.
3. if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.

Properties 2 and 3 are called “exchange properties.”

Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.
proof of Theorem 8.8.1.

- Assume \((E, I)\) is a matroid and \(w : E \to \mathcal{R}_+\) is given.

\[\text{...}\]
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- Assume $(E, I)$ is a matroid and $w : E \to \mathcal{R}_+$ is given.
- Let $A = (a_1, a_2, \ldots, a_r)$ be the solution returned by greedy, where $r = r(M)$ the rank of the matroid, and we order the elements as they were chosen (so $w(a_1) \geq w(a_2) \geq \cdots \geq w(a_r)$).
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- $A$ is a base of $M$, and let $B = (b_1, \ldots, b_r)$ be any another base of $M$ with elements also ordered decreasing by weight, so $w(b_1) \geq w(b_2) \geq \cdots \geq w(b_r)$.
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- Assume \((E, \mathcal{I})\) is a matroid and \(w : E \to \mathbb{R}_+\) is given.

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- \(A\) is a base of \(M\), and let \(B = (b_1, \ldots, b_r)\) be any another base of \(M\) with elements also ordered decreasing by weight, so \(w(b_1) \geq w(b_2) \geq \cdots \geq w(b_r)\).

- We next show that not only is \(w(A) \geq w(B)\) but that \(w(a_i) \geq w(b_i)\) for all \(i\).
proof of Theorem 8.8.1.

Assume otherwise, and let $k$ be the first (smallest) integer such that $w(a_k) < w(b_k)$. Hence $w(a_j) \geq w(b_j)$ for $j < k$. 

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- But $w(b_i) \geq w(b_k) > w(a_k)$, and so the greedy algorithm would have chosen $b_i$ rather than $a_k$, contradicting what greedy does.
Given an independence system \((E, I)\), suppose the greedy algorithm leads to an independent set of max weight for every non-negative weight function. We’ll show \((E, I)\) is a matroid.
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- Define the following modular weight function \(w\) on \(E\), and define \(k = |I|\).

\[
w(v) = \begin{cases} 
  k + 2 & \text{if } v \in I, \\
  k + 1 & \text{if } v \in J \setminus I, \\
  0 & \text{if } v \in E \setminus (I \cup J)
\end{cases} \quad (8.30)
\]
converse proof of Theorem 8.8.1.

- Now greedy will, after $k$ iterations, recover $I$, but it cannot choose any element in $J \setminus I$ by assumption. Thus, greedy chooses a set of weight $k(k + 2) = w(I)$. 

On the other hand, $J$ has weight $w(J) \geq |J|(k + 1) \geq (k + 1)(k + 1) > k(k + 2) = w(I)$ (8.31), so $J$ has strictly larger weight but is still independent, contradicting greedy's optimality.

Therefore, there must be a $z \in J \setminus I$ such that $I \cup \{z\} \in I$, and since $I$ and $J$ are arbitrary, $(E, I)$ must be a matroid.
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Therefore, there must be a $z \in J \setminus I$ such that $I \cup \{z\} \in \mathcal{I}$, and since $I$ and $J$ are arbitrary, $(E, \mathcal{I})$ must be a matroid.
As given, the theorem asked for a modular function $w \in \mathbb{R}^E_+$. 

- This will not only return an independent set, but it will return a base if we keep going even if the weights are 0.
- If we don't want elements with weight 0, we can stop once (and if) the weight hits zero, thus giving us a maximum weight independent set.
- We don't need non-negativity, we can use any $w \in \mathbb{R}^E_+$ and keep going until we have a base.
- If we stop at a negative value, we'll once again get a maximum weight independent set.
- Exercise: what if we keep going until a base even if we encounter negative values?
- We can instead do as small as possible thus giving us a minimum weight independent set/base.
Matroid and greedy

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Summary of Important (for us) Matroid Definitions

Given an independence system, matroids are defined equivalently by any of the following:

- All maximally independent sets have the same size.
- A monotone non-decreasing submodular integral rank function with unit increments.
- The greedy algorithm achieves the maximum weight independent set for all weight functions.