Submodular Functions, Optimization, and Applications to Machine Learning
— Fall Quarter, Lecture 7 —

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$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

$= f(A_r) + 2f(C) + f(B_r)$ $= f(A_r) + f(C) + f(B_r)$ $= f(A \cap B)$

Announcements, Assignments, and Reminders

- Homework 2.
- Reminder, all lectures are being recorded and posted to youtube. To get the links, see our announcements
  (https://canvas.uw.edu/courses/1397085/announcements).

Class Road Map - EE563

- L1(9/30): Motivation, Applications, Definitions, Properties
- L2(10/5): Sums concave(modular), uses (diversity/costs, feature selection), information theory
- L3(10/7): Monge, More Definitions, Graph and Combinatorial Examples,
- L4(10/12): Graph & Combinatorial Examples, Matrix Rank, Properties, Other Defs, Independence
- L5(10/14): Properties, Defs of Submodularity, Independence
- L6(10/19): Matroids, Matroid Examples, Matroid Rank,
- L7(10/21): Matroid Rank, More on Partition Matroid, Laminar Matroids, System of Distinct Reps, Transversals, Transversal Matroid, Matroid and representation, Dual Matroid
- L8(10/26):
- L9(10/28):
- L10(11/2):

- L11(11/4):
- L12(11/9):
- L–(11/11): Veterans Day, Holiday
- L13(11/16):
- L14(11/18):
- L15(11/23):
- L16(11/25):
- L17(11/30):
- L18(12/2):
- L19(12/7):
- L20(12/9): maximization.

Last day of instruction, Fri. Dec 11th. Finals Week: Dec 12-18, 2020

Many (Equivalent) Definitions of Submodularity

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B), \quad \forall A, B \subseteq V \tag{7.16}$$

$$f(j|S) \geq f(j|T), \forall S \subseteq T \subseteq V, \text{ with } j \in V \setminus T \tag{7.17}$$

$$f(C|S) \geq f(C|T), \forall S \subseteq T \subseteq V, \text{ with } C \subseteq V \setminus T \tag{7.18}$$

$$f(j|S) \geq f(j|S \cup \{k\}), \forall S \subseteq V \text{ with } j \in V \setminus (S \cup \{k\}) \tag{7.19}$$

$$f(A \cup B|A \cap B) \leq f(A|A \cap B) + f(B|A \cap B), \quad \forall A, B \subseteq V \tag{7.20}$$

$$f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S) - \sum_{j \in S \setminus T} f(j|S \cup T - \{j\}), \forall S, T \subseteq V \tag{7.21}$$

$$f(T) \leq f(S) + \sum_{j \in T \setminus S} f(j|S), \forall S \subseteq T \subseteq V \tag{7.22}$$

$$f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|S \setminus \{j\}) + \sum_{j \in T \setminus S} f(j|S \cap T) \forall S, T \subseteq V \tag{7.23}$$

$$f(T) \leq f(S) - \sum_{j \in S \setminus T} f(j|S \setminus \{j\}), \forall T \subseteq S \subseteq V \tag{7.24}$$

Matroid

Independent set definition of a matroid is perhaps most natural. Note, if $J \in \mathcal{I}$, then J is said to be an independent set.

Definition 7.2.3 (Matroid)

A set system $(E, \mathcal{I})$ is a Matroid if

(I1) $\emptyset \in \mathcal{I}$

(I2) $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ (down-closed or subclusive)

(I3) $\forall I, J \in \mathcal{I}$, with $|I| = |J| + 1$, then there exists $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$.

Why is (I1) is not redundant given (I2)? Because without (I1) could have a non-matroid where $\mathcal{I} = \{\}$.

Matroids - important property

Proposition 7.2.3

In a matroid $M = (E, \mathcal{I})$, for any $U \subseteq E(M)$, any two bases of U have the same size.

- In matrix terms, given a set of vectors U, all sets of independent vectors that span the space spanned by U have the same size.
- In fact, under (I1),(I2), this condition is equivalent to (I3). Exercise: show the following is equivalent to the above.

Definition 7.2.4 (Matroid)

A set system $(V, \mathcal{I})$ is a Matroid if

(I1') $\emptyset \in \mathcal{I}$ (emptyset containing)

(I2') $\forall I \in \mathcal{I}, J \subset I \Rightarrow J \in \mathcal{I}$ (down-closed or subclusive)

(I3') $\forall X \subseteq V$, and $I_1, I_2 \in \text{maxInd}(X)$, we have $|I_1| = |I_2|$ (all maximally independent subsets of X have the same size).

Partition Matroid

- Let V be our ground set.
- Let $V = V_1 \cup V_2 \cup \cdots \cup V_\ell$ be a partition of V into ℓ blocks (i.e., disjoint sets). Define a set of subsets of V as

$$\mathcal{I} = \{X \subseteq V : |X \cap V_i| \leq k_i \text{ for all } i = 1, \ldots, \ell\}. \tag{7.4}$$

  where $k_1, \ldots, k_\ell$ are fixed "limit" parameters, $k_i \geq 0$. Then $M = (V, \mathcal{I})$ is a matroid.
- Note that a k-uniform matroid is a trivial example of a partition matroid with $\ell = 1$, $V_1 = V$, and $k_1 = k$.
- Parameters associated with a partition matroid: ℓ and $k_1, k_2, \ldots, k_\ell$ although often the k_i's are all the same.
- We'll show that property (I3') in Def ?? holds. First note, for any $X \subseteq V$, $|X| = \sum_{i=1}^{\ell} |X \cap V_i|$ since $\{V_1, V_2, \ldots, V_\ell\}$ is a partition.
- If $X, Y \in \mathcal{I}$ with $|Y| > |X|$, then there must be at least one i with $|Y \cap V_i| > |X \cap V_i|$. Therefore, adding one element $e \in V_i \cap (Y \setminus X)$ to X won't break independence.

Matroids - rank function is submodular

Lemma 7.2.3

The rank function $r : 2^E \to \mathbb{Z}_+$ of a matroid is submodular, that is
$r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$

Proof.

1. Let $X \in \mathcal{I}$ be an inclusionwise maximal set with $X \subseteq A \cap B$
2. Let $Y \in \mathcal{I}$ be inclusionwise maximal set with $X \subseteq Y \subseteq A \cup B$.
3. Since M is a matroid, we know that $r(A \cap B) = r(X) = |X|$, and $r(A \cup B) = r(Y) = |Y|$. Also, for any $U \in \mathcal{I}$, $r(A) \geq |A \cap U|$.
4. Then we have (since $X \subseteq A \cap B$, $X \subseteq Y$, and $Y \subseteq A \cup B$),

$$r(A) + r(B) \geq |Y \cap A| + |Y \cap B| \tag{7.4}$$
$$= |Y \cap (A \cap B)| + |Y \cap (A \cup B)| \tag{7.5}$$
$$\geq |X| + |Y| = r(A \cap B) + r(A \cup B) \tag{7.6}$$

$\square$

A matroid is defined from its rank function

Theorem 7.3.1 (Matroid from rank)

Let E be a set and let $r : 2^E \to \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with r being its rank function if and only if for all $A, B \subseteq E$:

(R1) $\forall A \subseteq E \ \ 0 \leq r(A) \leq |A|$ *(non-negative cardinality bounded)*

(R2) $r(A) \leq r(B)$ *whenever $A \subseteq B \subseteq E$ (monotone non-decreasing)*

(R3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ *for all $A, B \subseteq E$ (submodular)*

- From above, $r(\emptyset) = 0$. Let $v \notin A$, then by monotonicity and submodularity, $r(A) \leq r(A \cup \{v\}) \leq r(A) + r(\{v\})$ which gives only two possible values to $r(A \cup \{v\})$, namely $r(A)$ or $r(A) + 1$.
- Hence, unit increment (if $r(A) = k$, then either $r(A \cup \{v\}) = k$ or $r(A \cup \{v\}) = k + 1$) holds.
- Thus, submodularity, normalized, monotone non-decreasing, & unit increment of rank is necessary & sufficient to define matroids.
- Can refer to matroid as (E, r), E is ground set, r is rank function.

Matroids from rank

Proof of Theorem 7.3.1 (matroid from rank).

- Given a matroid $M = (E, \mathcal{I})$, we see its rank function as defined in Eq. 6.7 satisfies (R1), (R2), and, as we saw in Lemma 6.5.1, (R3) too.
- Conversely, assume we have r satisfying (R1), (R2), and (R3). Define $\mathcal{I} = \{X \subseteq E : r(X) = |X|\}$. We will show that $(E, \mathcal{I})$ is a matroid.
- First, $\emptyset \in \mathcal{I}$.
- Also, if $Y \in \mathcal{I}$ and $X \subseteq Y$ then by submodularity,

$$r(X) \geq r(Y) - r(Y \setminus X) + r(\emptyset) \tag{7.1}$$
$$\geq |Y| - |Y \setminus X| \tag{7.2}$$
$$= |X| \tag{7.3}$$

implying $r(X) = |X|$, and thus $X \in \mathcal{I}$.

...

Matroids from rank

Proof of Theorem 7.3.1 (matroid from rank) cont.

- Let $A, B \in \mathcal{I}$, with $|A| < |B|$, so $r(A) = |A| < r(B) = |B|$. Let $B \setminus A = \{b_1, b_2, \ldots, b_k\}$ (note $1 \le k \le |B|$).
- Suppose, to the contrary, that $\forall b \in B \setminus A$, $A + b \notin \mathcal{I}$, which means for all such b, $r(A + b) = r(A) = |A| < |A| + 1$. Then

$$r(B) \le r(A \cup B) \qquad (7.4)$$
$$\le r(A \cup (B \setminus \{b_1\})) + r(A \cup \{b_1\}) - r(A) \qquad (7.5)$$
$$= r(A \cup (B \setminus \{b_1\})) \qquad (7.6)$$
$$\le r(A \cup (B \setminus \{b_1, b_2\})) + r(A \cup \{b_2\}) - r(A) \qquad (7.7)$$
$$= r(A \cup (B \setminus \{b_1, b_2\})) \qquad (7.8)$$
$$\le \ldots \le r(A) = |A| < |B| \qquad (7.9)$$

giving a contradiction since $B \in \mathcal{I}$. $\qquad \square$

Matroids from rank II

Another way of using function r to define a matroid.

Theorem 7.3.2 (Matroid from rank II)

Let E be a finite set and let $r : 2^E \to \mathbb{Z}_+$ be a function. Then $r(\cdot)$ defines a matroid with r being its rank function if and only if for all $X \subseteq E$, and $x, y \in E$:

(R1') $r(\emptyset) = 0$;

(R2') $r(X) \le r(X \cup \{y\}) \le r(X) + 1$;

(R3') *If $r(X \cup \{x\}) = r(X \cup \{y\}) = r(X)$, then $r(X \cup \{x, y\}) = r(X)$.*

Matroids by submodular functions

Theorem 7.3.3 (Matroid by submodular functions)

Let $f : 2^E \to \mathbb{Z}$ be a integer valued monotone non-decreasing submodular function. Define a set of sets as follows:

$$\mathcal{C}(f) = \Big\{ C \subseteq E : C \text{ is non-empty,}$$
$$\text{is inclusionwise-minimal,}$$
$$\text{and has } f(C) < |C| \Big\} \qquad (7.10)$$

Then $\mathcal{C}(f)$ is the collection of circuits of a matroid on E.

Inclusionwise-minimal in this case means that if $C \in \mathcal{C}(f)$, then there exists no $C' \subset C$ with $C' \in \mathcal{C}(f)$ (i.e., $C' \subset C$ would either be empty or have $f(C') \geq |C'|$). Also, recall inclusionwise-minimal in Definition 6.3.10, the definition of a circuit.

Summarizing: Many ways to define a Matroid

Summarizing what we've so far seen, we saw that it is possible to uniquely define a matroid based on any of:

- Independence (define the independent sets).
- Base axioms (exchangeability)
- Circuit axioms
- Closure axioms (we didn't see this, but it is possible)
- Rank axioms (normalized, monotone, cardinality bounded, non-negative integral, submodular)
- Matroids by integral submodular functions.

Maximization problems for matroids

- Given a matroid $M = (E, \mathcal{I})$ and a modular value function $c : E \to \mathbb{R}$, the task is to find an $X \in \mathcal{I}$ such that $c(X) = \sum_{x \in X} c(x)$ is maximum.
- This seems remarkably similar to the max spanning tree problem.

Minimization problems for matroids

- Given a matroid $M = (E, \mathcal{I})$ and a modular cost function $c : E \to \mathbb{R}$, the task is to find a <u>basis</u> $B \in \mathcal{B}$ such that $c(B)$ is minimized.
- This sounds like a set cover problem (find the minimum cost covering set of sets).

Partition Matroid

- What is the partition matroid's rank function?
- A partition matroids rank function:

$$r(A) = \sum_{i=1}^{\ell} \min(|A \cap V_i|, k_i) \tag{7.11}$$

which we also immediately see is submodular using properties we spoke about last week. That is:

1. $|A \cap V_i|$ is submodular (in fact modular) in A
2. $\min(\text{submodular}(A), k_i)$ is submodular in A since $|A \cap V_i|$ is monotone.
3. sums of submodular functions are submodular.

- $r(A)$ is also non-negative integral monotone non-decreasing, so it defines a matroid (the partition matroid).

From 2-partition matroid rank to truncated matroid rank

- Example: 2-partition matroid rank function: Given natural numbers $a, b \in \mathbb{Z}_+$ with $a < b$, and any set $R \subseteq V$ with $|R| = b$.
- Create two-block partition $V = (R, \bar{R})$, where $\bar{R} = V \setminus R$ so $|\bar{R}| = |V| - b$. Gives 2-partition matroid rank function as follows:

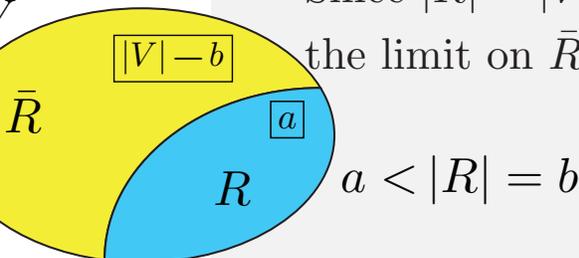$$r(A) = \min(|A \cap R|, a) + \min(|A \cap \bar{R}|, |\bar{R}|) \tag{7.12}$$
$$= \min(|A \cap R|, a) + |A \cap \bar{R}| \tag{7.13}$$
$$= \min(|A \cap \bar{R}| + |A \cap R|, |A \cap \bar{R}| + a) \tag{7.14}$$
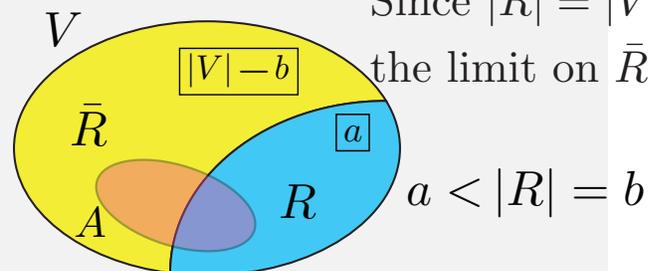$$= \min(|A|, |A \cap \bar{R}| + a) \tag{7.15}$$

- Figure showing partition blocks and partition matroid limits.

Since $|\bar{R}| = |V| - b$ the limit on $\bar{R}$ is vacuous.

$a < |R| = b$

Since $|\bar{R}| = |V| - b$ the limit on $\bar{R}$ is vacuo

$a < |R| = b$

Truncated Matroid Rank Function

- Define truncated matroid rank function. Start with 2-partition matroid rank $r(A) = \min(|A \cap R|, a) + \min(|A \cap \bar{R}|, |\bar{R}|)$, $a < b$. Define:

$$f_R(A) = \min \left\{ \boxed{r(A)}, b \right\} \tag{7.16}$$

$$= \min \left\{ \boxed{\min(|A|, |A \cap \bar{R}| + a)}, b \right\} \tag{7.17}$$

$$= \min \left\{ |A|, a + |A \cap \bar{R}|, b \right\} \tag{7.18}$$

- Defines a matroid $M = (V, f_R) = (V, \mathcal{I})$ (Goemans et. al.) with

$$\mathcal{I} = \{ I \subseteq V : |I| \leq b \text{ and } |I \cap R| \leq a \}, \tag{7.19}$$

Useful for showing hardness of constrained submodular minimization.
Consider sets $\mathcal{B} = \{ B \subseteq V : |B| = b \}$. Note $R \in \mathcal{B}$ fixed.

- For R, we have $f_R(R) = \min(b, a, b) = a < b$.
- For $B \in \mathcal{B}$ with $|B \cap R| \leq a$, $f_R(B) = b > a$.
- For $B \in \mathcal{B}$ with $|B \cap R| = \ell$, with $a < \ell < b$, $f_R(B) = a + b - \ell > a$.
- R, the set with minimum valuation amongst size-b sets $\mathcal{B}$, is hidden within an exponentially larger set of size-b sets with larger valuation.

Partition Matroid, rank as matching

- A partition matroid can be viewed using a bipartite graph.
- Letting V denote the ground set, and $V_1, V_2, \ldots$ the partition, the bipartite graph is $G = (V, I, E)$ where V is the ground set, I is a set of "indices", and E is the set of edges.
- $I = (I_1, I_2, \ldots, I_\ell)$ is a set of $k = \sum_{i=1}^\ell k_i$ nodes, grouped into ℓ clusters, where there are k_i nodes in the ith group I_i, and $|I_i| = k_i$.
- $(v, i) \in E(G)$ iff $v \in V_j$ and $i \in I_j$.

Partition Matroid, rank as matching
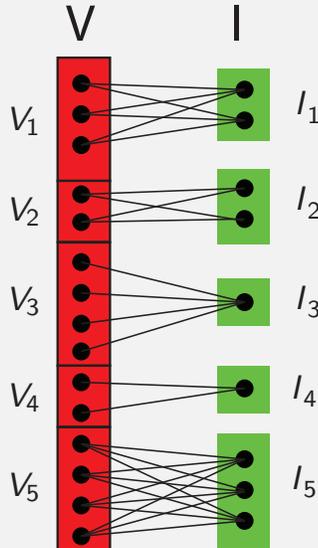
- Example where $\ell = 5$, $(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3)$.

- Recall, $\Gamma : 2^V \to \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of X is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.
- Here, for $X \subseteq V$, we have $\Gamma(X) = \{i \in I : (v, i) \in E(G) \text{ and } v \in X\}$.
- For such a constructed bipartite graph, the rank function of a partition matroid is $r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i) =$ the maximum matching involving X.

Laminar Family and Laminar Matroid

- We can define a matroid with structures richer than just partitions.
- A set system $(V, \mathcal{F})$ is called a laminar family if for any two sets $A, B \in \mathcal{F}$, at least one of the three sets $A \cap B$, $A \setminus B$, or $B \setminus A$ is empty.
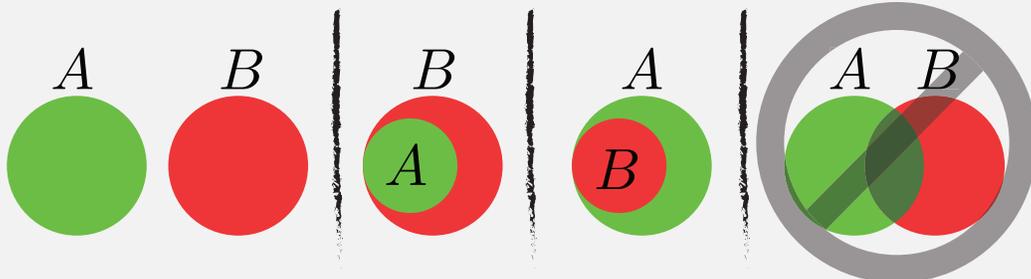
- Family is laminar $\exists$ no two <u>properly intersecting</u> members: $\forall A, B \in \mathcal{F}$, either A, B disjoint ($A \cap B = \emptyset$) or comparable ($A \subseteq B$ or $B \subseteq A$).
- Suppose we have a laminar family $\mathcal{F}$ of subsets of V and an integer k_A for every set $A \in \mathcal{F}$. Then $(V, \mathcal{I})$ defines a matroid where

$$\mathcal{I} = \{I \subseteq E : |I \cap A| \leq k_A \text{ for all } A \in \mathcal{F}\} \qquad (7.20)$$
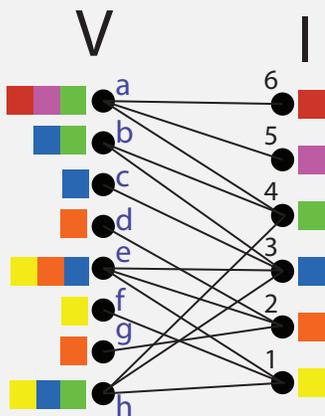
- Exercise: what is the rank function here?

System of Representatives

- Let $(V, \mathcal{V})$ be a set system (i.e., $\mathcal{V} = (V_i : i \in I)$ where $\emptyset \subset V_i \subseteq V$ for all i), and I is an index set. Hence, $|I| = |\mathcal{V}|$.
- Here, the sets $V_i \in \mathcal{V}$ are like "groups" and any $v \in V$ with $v \in V_i$ is a member of group i. Groups need not be disjoint (e.g., interest groups of individuals).
- A family $(v_i : i \in I)$ with $v_i \in V$ is said to be a system of representatives of $\mathcal{V}$ if $\exists$ a bijection $\pi : I \to I$ such that $v_i \in V_{\pi(i)}$.
- v_i is the representative of set (or group) $V_{\pi(i)}$, meaning the ith representative is meant to represent set $V_{\pi(i)}$.
- Example: Consider the house of representatives, $v_i =$ "Pramila Jayapal", while $i =$ "King County, WA-7".
- In a system of representatives, there is no requirement for the representatives to be distinct. I.e., we could have some $v_1 \in V_1 \cap V_2$, where v_1 represents both V_1 and V_2.
- We can view this as a bipartite graph.
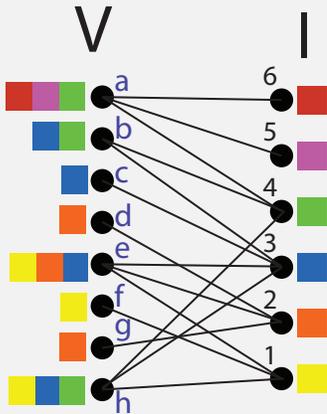

System of Representatives

- We can view this as a bipartite graph. The groups of V are marked by color tags on the left, and also via right neighbors in the graph.
- Here, $\ell = 6$ groups, with $\mathcal{V} = (V_1, V_2, \ldots, V_6)$

$$= \left(\boxed{\{e, f, h\}}, \boxed{\{d, e, g\}}, \boxed{\{b, c, e, h\}}, \boxed{\{a, b, h\}}, \boxed{\{a\}}, \boxed{\{a\}} \right).$$

- A system of representatives would make sure that there is a representative for each color group. For example,
- The representatives $(\{a, c, d, f, h\})$ are shown as colors on the left.
- Here, the set of representatives is not distinct. Why? In fact, due to the red and pink group, a distinct group of representatives is impossible (since there is only one common choice to represent both color groups).
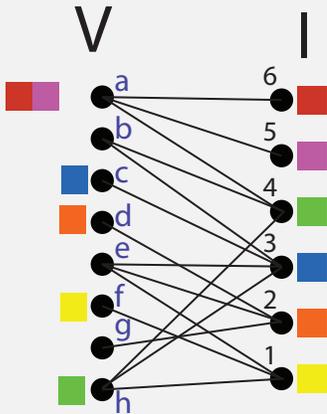
System of Representatives

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$$= \left(\{e, f, h\}, \{d, e, g\}, \{b, c, e, h\}, \{a, b, h\}, \{a\}, \{a\} \right).$$

- A system of representatives would make sure that there is a representative for each color group. For example,
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System of Distinct Representatives

- Let $(V, \mathcal{V})$ be a set system (i.e., $\mathcal{V} = (V_i : i \in I)$ where $V_i \subseteq V$ for all i), and I is an index set. Hence, $|I| = |\mathcal{V}|$.
- A family $(v_i : i \in I)$ with $v_i \in V$ is said to be a system of distinct representatives of $\mathcal{V}$ if $\exists$ a bijection $\pi : I \leftrightarrow I$ such that $v_i \in V_{\pi(i)}$ and $v_i \neq v_j$ for all $i \neq j$.
- In a system of distinct representatives, there is a requirement for the representatives to be distinct. We can re-state (and rename) this as a:

Definition 7.6.1 (transversal)

Given a set system $(V, \mathcal{V})$ and index set I for $\mathcal{V}$ as defined above, a set $T \subseteq V$ is a transversal of $\mathcal{V}$ if there is a bijection $\pi : T \leftrightarrow I$ such that

$$x \in V_{\pi(x)} \text{ for all } x \in T \tag{7.21}$$

- Note that due to $\pi : T \leftrightarrow I$ being a bijection, all of I and T are "covered" (so this makes things distinct automatically).

Transversals are Subclusive

- A set $T' \subseteq V$ is a partial transversal if T' is a transversal of some subfamily $\mathcal{V}' = (V_i : i \in I')$ where $I' \subseteq I$.
- Therefore, for any transversal T, any subset $T' \subseteq T$ is a partial transversal.
- Thus, transversals are down closed (subclusive).

When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
- Given a set system $(V, \mathcal{V})$ with $\mathcal{V} = (V_i : i \in I)$, and $V_i \subseteq V$ for all i. Then, for any $J \subseteq I$, let

$$V(J) = \cup_{j \in J} V_j \tag{7.22}$$

so $|V(J)| : 2^I \to \mathbb{Z}_+$ is the set cover func. (we know is submodular).
- We have

Theorem 7.7.1 (Hall's theorem)

Given a set system $(V, \mathcal{V})$, the family of subsets $\mathcal{V} = (V_i : i \in I)$ has a transversal $(v_i : i \in I)$ iff for all $J \subseteq I$
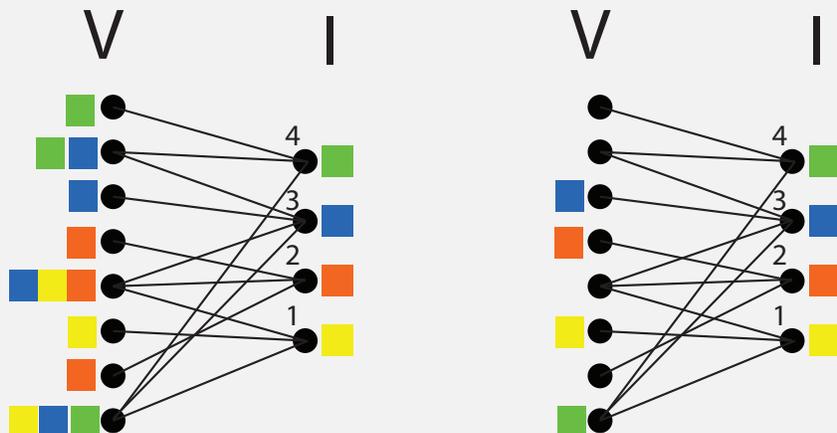
$$|V(J)| \geq |J| \tag{7.23}$$

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- As we saw, a transversal might not always exist. How to tell?
- Given a set system $(V, \mathcal{V})$ with $\mathcal{V} = (V_i : i \in I)$, and $V_i \subseteq V$ for all i. Then, for any $J \subseteq I$, let

$$V(J) = \cup_{j \in J} V_j \tag{7.22}$$

so $|V(J)| : 2^I \to \mathbb{Z}_+$ is the set cover func. (we know is submodular).
- Hall's theorem ($\forall J \subseteq I, |V(J)| \geq |J|$) as a bipartite graph.

When do transversals exist?

- As we saw, a transversal might not always exist. How to tell?
- Given a set system $(V, \mathcal{V})$ with $\mathcal{V} = (V_i : i \in I)$, and $V_i \subseteq V$ for all i. Then, for any $J \subseteq I$, let

$$V(J) = \cup_{j \in J} V_j \tag{7.22}$$

so $|V(J)| : 2^I \to \mathbb{Z}_+$ is the set cover func. (we know is submodular).
- Moreover, we have

Theorem 7.7.2 (Rado's theorem (1942))

If $M = (V, r)$ is a matroid on V with rank function r, then the family of subsets $(V_i : i \in I)$ of V has a transversal $(v_i : i \in I)$ that is <u>independent in M</u> iff for all $J \subseteq I$

$$r(V(J)) \geq |J| \tag{7.24}$$

- Note, a transversal T independent in M means that $r(T) = |T|$.

Application's of Hall's theorem

- Consider a set of jobs I and a set of applicants V to the jobs. If an applicant $v \in V$ is qualified for job $i \in I$, we add edge (v, i) to the bipartite graph $G = (V, I, E)$.
- We wish all jobs to be filled, and hence Hall's condition $(\forall J \subseteq I, |V(J)| \geq |J|)$ is a necessary and sufficient condition for this to be possible.
- Note if $|V| = |I|$, then Hall's theorem is the Marriage Theorem (Frobenious 1917), where an edge (v, i) in the graph indicate compatibility between two individuals $v \in V$ and $i \in I$ coming from two separate groups V and I.
- If $\forall J \subseteq I, |V(J)| \geq |J|$, then all individuals in each group can be matched with a compatible mate.

More general conditions for existence of transversals

Theorem 7.7.3 (Polymatroid transversal theorem)

If $\mathcal{V} = (V_i : i \in I)$ is a finite family of non-empty subsets of V, and $f : 2^V \to \mathbb{Z}_+$ is a non-negative, integral, monotone non-decreasing, and submodular function, then $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that

$$f(\cup_{i \in J}\{v_i\}) \geq |J| \text{ for all } J \subseteq I \tag{7.25}$$

if and only if

$$f(V(J)) \geq |J| \text{ for all } J \subseteq I \tag{7.26}$$

- Given Theorem 7.7.3, we immediately get Theorem 7.7.1 by taking $f(S) = |S|$ for $S \subseteq V$. *In which case, Eq. 7.25 requires the system of representatives to be distinct.*
- We get Theorem 7.7.2 by taking $f(S) = r(S)$ for $S \subseteq V$, the rank function of the matroid. *where, Eq. 7.25 insists the system of representatives is independent in M, and hence also distinct.*

On Distinctiveness

- In Hall's Theorem 7.7.1, we say a "transversal" exists iff the condition $|V(J)| \geq |J|$ is true for all $J \subseteq I$. The condition guarantees that it is possible for the system of representative to be distinct (i.e., a transversal).

- In Rado's variant (Theorem 7.7.2), we say it is a transversal that is independent if $r(V(J)) \geq |J|$. This guarantees it is a transversal since $r(V(J)) \leq |V(J)|$ so $|V(J)| \geq r(V(J)) \geq |J|$ and we again can have distinctiveness.

- In third polymatroid case, Theorem 7.7.3, the condition $f(\cup_{j \in J} v_j) \geq |J|$ ensures a transversal (distinctiveness) in the cardinality case (i.e., $f(A) = |A|$) since $|\cup_{j \in J}\{v_j\}| \geq |J|$ ensures things are distinct. Also, we get Rado's theorem simply by taking $f = r$.

- Theorem 7.7.3 is the most general, therefore, and we prove this one.

Submodular Composition with Set-to-Set functions

- Note the condition in Theorem 7.7.3 is $f(V(J)) \geq |J|$ for all $J \subseteq I$, where $f : 2^V \to \mathbb{Z}_+$ is non-negative, integral, monotone non-decreasing and submodular, and $V(J) = \cup_{j \in J} V_j$ with $V_i \subseteq V$.
- Note $V(\cdot) : 2^I \to 2^V$ is a set-to-set function, composable with a submodular function.
- Define $g : 2^I \to \mathbb{Z}$ with $g(J) = f(V(J)) - |J|$, then the condition for the existence of a system of representatives, with quality Equation 7.25, becomes:

$$\min_{J \subseteq I} g(J) \geq 0 \qquad (7.27)$$

- What kind of function is g?

Proposition 7.7.4

g as given above is submodular.

- Hence, the condition for existence can be solved by (a special case of) submodular function minimization, or vice verse!

More general conditions for existence of transversals

First part proof of Theorem 7.7.3.

- Suppose $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that Eq. 7.25 (i.e., $f(\cup_{i \in J}\{v_i\}) \geq |J|$ for all $J \subseteq I$) is true.
- Then since f is monotone, and since $V(J) \supseteq \cup_{i \in J}\{v_i\}$ when $(v_i : i \in I)$ is a system of representatives, then Eq. 7.26 (i.e., $f(V(J)) \geq |J|$ for all $J \subseteq I$) immediately follows.

$\dots$

To show the converse, we first prove a contraction lemma.

More general conditions for existence of transversals

Lemma 7.7.5 (contraction lemma)

Suppose Eq. 7.26 ($f(V(J)) \geq |J|, \forall J \subseteq I$) is true for $\mathcal{V} = (V_i : i \in I)$, and there exists an i such that $|V_i| \geq 2$ (w.l.o.g., say $i = 1$). Then there exists $\bar{v} \in V_1$ such that the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 7.26.

Proof.

- When Eq. 7.26 holds, this means that for any subsets $J_a, J_b \subseteq I \setminus \{1\}$, we have that, for $J \in \{J_a, J_b\}$,
$$f(V(J \cup \{1\})) \geq |J \cup \{1\}| \qquad (7.28)$$
and hence
$$f(V_1 \cup V(J_a)) \geq |J_a| + |\{1\}| = |J_a| + 1 \qquad (7.29)$$
$$f(V_1 \cup V(J_b)) \geq |J_b| + |\{1\}| = |J_b| + 1 \qquad (7.30)$$

. . .

More general conditions for existence of transversals

Lemma 7.7.5 (contraction lemma)

Suppose Eq. 7.26 ($f(V(J)) \geq |J|, \forall J \subseteq I$) is true for $\mathcal{V} = (V_i : i \in I)$, and there exists an i such that $|V_i| \geq 2$ (w.l.o.g., say $i = 1$). Then there exists $\bar{v} \in V_1$ such that the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 7.26.

Proof.

- Suppose, to the contrary, the consequent is false. Then we may take any $\bar{v}_1, \bar{v}_2 \in V_1$ as two distinct elements in V_1 ...
- ... and there must exist subsets J_a, J_b of $I \setminus \{1\}$ such that

$$f((V_1 \setminus \{\bar{v}_1\}) \cup V(J_a)) < |J_a| + |\{1\}| = |J_a| + 1, \quad (7.31)$$
$$f((V_1 \setminus \{\bar{v}_2\}) \cup V(J_b)) < |J_b| + |\{1\}| = |J_b| + 1, \quad (7.32)$$

(note that either one or both of J_a, J_b could be empty).

. . .

More general conditions for existence of transversals

Lemma 7.7.5 (contraction lemma)

Suppose Eq. 7.26 ($f(V(J)) \geq |J|, \forall J \subseteq I$) is true for $\mathcal{V} = (V_i : i \in I)$, and there exists an i such that $|V_i| \geq 2$ (w.l.o.g., say $i = 1$). Then there exists $\bar{v} \in V_1$ such that the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 7.26.

Proof.

- Taking $X = (V_1 \setminus \{\bar{v}_1\}) \cup V(J_a)$ and $Y = (V_1 \setminus \{\bar{v}_2\}) \cup V(J_b)$, we have $f(X) \leq |J_a|$, $f(Y) \leq |J_b|$, and that:

$$X \cup Y = V_1 \cup V(J_a \cup J_b), \qquad (7.33)$$
$$X \cap Y \supseteq V(J_a \cap J_b), \qquad (7.34)$$

  and

$$|J_a| + |J_b| \geq f(X) + f(Y)$$
$$\geq f(X \cup Y) + f(X \cap Y) \quad (7.35)$$

  $\ldots$

More general conditions for existence of transversals

Lemma 7.7.5 (contraction lemma)

Suppose Eq. 7.26 ($f(V(J)) \geq |J|, \forall J \subseteq I$) is true for $\mathcal{V} = (V_i : i \in I)$, and there exists an i such that $|V_i| \geq 2$ (w.l.o.g., say $i = 1$). Then there exists $\bar{v} \in V_1$ such that the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 7.26.

Proof.

- since f submodular monotone non-decreasing, & Eqs 7.33-7.35,

$$|J_a| + |J_b| \geq f(V_1 \cup V(J_a \cup J_b)) + f(V(J_a \cap J_b)) \qquad (7.36)$$

- Since $\mathcal{V}$ satisfies Eq. 7.26, $1 \notin J_a \cup J_b$, & Eqs 7.29-7.30, this gives

$$|J_a| + |J_b| \geq |J_a \cup J_b| + 1 + |J_a \cap J_b| \qquad (7.37)$$

  which is a contradiction since cardinality is modular.

  $\ldots$

More general conditions for existence of transversals

Theorem 7.7.3 (Polymatroid transversal theorem)

If $\mathcal{V} = (V_i : i \in I)$ is a finite family of non-empty subsets of V, and $f : 2^V \to \mathbb{Z}_+$ is a non-negative, integral, monotone non-decreasing, and submodular function, then $\mathcal{V}$ has a system of representatives $(v_i : i \in I)$ such that

$$f(\cup_{i \in J}\{v_i\}) \geq |J| \text{ for all } J \subseteq I \tag{7.25}$$

if and only if

$$f(V(J)) \geq |J| \text{ for all } J \subseteq I \tag{7.26}$$

- Given Theorem 7.7.3, we immediately get Theorem 7.7.1 by taking $f(S) = |S|$ for $S \subseteq V$. *In which case, Eq. 7.25 requires the system of representatives to be distinct.*
- We get Theorem 7.7.2 by taking $f(S) = r(S)$ for $S \subseteq V$, the rank function of the matroid. *where, Eq. 7.25 insists the system of representatives is independent in M, and hence also distinct.*

More general conditions for existence of transversals

converse proof of Theorem 7.7.3.

- Conversely, suppose Eq. 7.26 ($f(V(J)) \geq |J|, \forall J \subseteq I$) is true.
- If each V_i is a singleton set, then the result, Eq. 7.25 ($f(\cup_{i \in J}\{v_i\}) \geq |J|$ for all $J \subseteq I$) follows immediately.
- W.l.o.g., let $|V_1| \geq 2$, then by Lemma 7.7.5, the family of subsets $(V_1 \setminus \{\bar{v}\}, V_2, \ldots, V_{|I|})$ also satisfies Eq 7.26 for the right $\bar{v}$.
- We can continue to reduce the family, deleting elements from V_i for some i while $|V_i| \geq 2$, until we arrive at a family of singleton sets.
- This family will be the required system of representatives.

  □

This theorem can be used to produce a variety of other results quite easily, and shows how submodularity is the key ingredient in its truth.

Transversal Matroid

Transversals, themselves, define a matroid.

Theorem 7.8.1

If $\mathcal{V}$ is a family of finite subsets of a ground set V, then the collection of partial transversals of $\mathcal{V}$ is the set of independent sets of a matroid $M = (V, \mathcal{V})$ on V.

- This means that the transversals of $\mathcal{V}$ are the bases of matroid M.
- Therefore, all maximal partial transversals of $\mathcal{V}$ have the same cardinality!

Transversals and Bipartite Matchings

- Transversals correspond exactly to matchings in bipartite graphs.
- Given a set system $(V, \mathcal{V})$, with $\mathcal{V} = (V_i : i \in I)$, we can define a bipartite graph $G = (V, I, E)$ associated with $\mathcal{V}$ that has edge set $\{(v, i) : v \in V, i \in I, v \in V_i\}$.
- A matching in this graph is a set of edges no two of which that have a common endpoint. In fact, we easily have:

Lemma 7.8.2

A subset $T \subseteq V$ is a partial transversal of $\mathcal{V}$ iff there is a matching in (V, I, E) in which every edge has one endpoint in T (T matched into I).
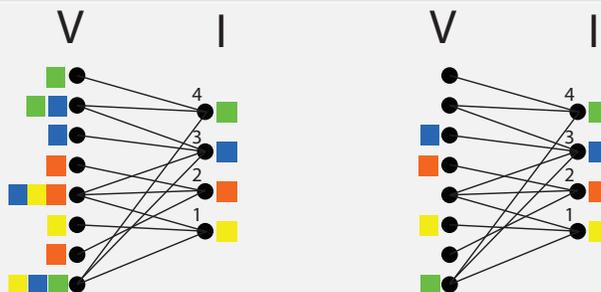
Arbitrary Matchings and Matroids?

- Are arbitrary matchings matroids?
- Consider the following graph (left), and two max-matchings (two right instances)
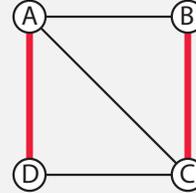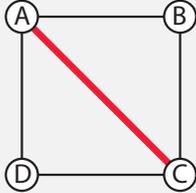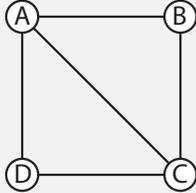
- $\{AC\}$ is a maximum matching, as is $\{AD, BC\}$, but they are not the same size.
- Let $\mathcal{M}$ be the set of matchings in an arbitrary graph $G = (V, E)$. Hence, $(E, \mathcal{M})$ is a set system. I1 holds since $\emptyset \in \mathcal{M}$. I2 also holds since if $M \in \mathcal{M}$ is a matching, then so is any $M' \subseteq M$. I3 doesn't hold (as seen above). Exercise: fully characterize the problem of finding the largest subset $\mathcal{M}' \subset \mathcal{M}$ of matchings so that $(E, \mathcal{M}')$ also satisfies I3?

Review

Next slide is from earlier in lecture.

Partition Matroid, rank as matching
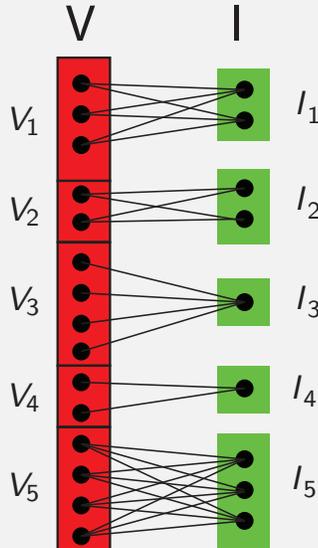
- Example where $\ell = 5$, $(k_1, k_2, k_3, k_4, k_5) = (2, 2, 1, 1, 3)$.

- Recall, $\Gamma : 2^V \to \mathbb{R}$ as the neighbor function in a bipartite graph, the neighbors of X is defined as $\Gamma(X) = \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$, and recall that $|\Gamma(X)|$ is submodular.
- Here, for $X \subseteq V$, we have $\Gamma(X) = \{i \in I : (v, i) \in E(G) \text{ and } v \in X\}$.
- For such a constructed bipartite graph, the rank function of a partition matroid is $r(X) = \sum_{i=1}^{\ell} \min(|X \cap V_i|, k_i) = $ the maximum matching involving X.

Morphing Partition Matroid Rank

- Recall the partition matroid rank function. Note, $k_i = |I_i|$ in the bipartite graph representation, and since a matroid, w.l.o.g., $|V_i| \geq k_i$ (also, recall, $V(J) = \cup_{j \in J} V_j$).
- Start with partition matroid rank function in the subsequent equations.

$$r(A) = \sum_{i \in \{1, \ldots, \ell\}} \min(|A \cap V_i|, k_i) \tag{7.38}$$

$$= \sum_{i=1}^{\ell} \min(|A \cap V(I_i)|, |I_i|) \tag{7.39}$$

$$= \sum_{i \in \{1, \ldots, \ell\}} \min_{J_i \in \{\emptyset, I_i\}} \left(\left\{ \begin{array}{ll} |A \cap V(I_i)| & \text{if } J_i \neq \emptyset \\ 0 & \text{if } J_i = \emptyset \end{array} \right\} + |I_i \setminus J_i| \right) \tag{7.40}$$

$$= \sum_{i \in \{1, \ldots, \ell\}} \min_{J_i \subseteq I_i} \left(\left\{ \begin{array}{ll} |A \cap V(I_i)| & \text{if } J_i \neq \emptyset \\ 0 & \text{if } J_i = \emptyset \end{array} \right\} + |I_i \setminus J_i| \right) \tag{7.41}$$

$$= \sum_{i \in \{1, \ldots, \ell\}} \min_{J_i \subseteq I_i} \left(|V(J_i) \cap A| + |I_i \setminus J_i| \right) \tag{7.42}$$

... Morphing Partition Matroid Rank

- Continuing,

$$r(A) = \sum_{i=1}^{\ell} \min_{J_i \subseteq I_i} \left(|V(J_i) \cap V(I_i) \cap A| - |I_i \cap J_i| + |I_i| \right) \qquad (7.43)$$

$$= \min_{J \subseteq I} \left(\sum_{i=1}^{\ell} |V(J) \cap V(I_i) \cap A| - |I_i \cap J| + |I_i| \right) \qquad (7.44)$$

$$= \min_{J \subseteq I} \left(|V(J) \cap V(I) \cap A| - |J| + |I| \right) \qquad (7.45)$$

$$= \min_{J \subseteq I} \left(|V(J) \cap A| - |J| + |I| \right) \qquad (7.46)$$

- In fact, this bottom (more general) expression is the expression for the rank of a transversal matroid.

Partial Transversals Are Independent Sets in a Matroid

In fact, we have

Theorem 7.8.3

Let $(V, \mathcal{V})$ where $\mathcal{V} = (V_1, V_2, \ldots, V_\ell)$ be a subset system. Let $I = \{1, \ldots, \ell\}$. Let $\mathcal{I}$ be the set of partial transversals of $\mathcal{V}$. Then $(V, \mathcal{I})$ is a matroid.

Proof.

- We note that $\emptyset \in \mathcal{I}$ since the empty set is a transversal of the empty subfamily of $\mathcal{V}$, thus (I1') holds.

- We already saw that if T is a partial transversal of $\mathcal{V}$, and if $T' \subseteq T$, then T' is also a partial transversal. So (I2') holds.

- Suppose that T_1 and T_2 are partial transversals of $\mathcal{V}$ such that $|T_1| < |T_2|$. Exercise: show that (I3') holds.

Transversal Matroid Rank

- Transversal matroid has rank

$$r(A) = \min_{J \subseteq I} \left(|V(J) \cap A| - |J| + |I| \right) \tag{7.47}$$

$$= \min_{J \subseteq I} m_J(I) \tag{7.48}$$

- Therefore, this function is submodular.

- Note that it is a minimum over a set of modular functions in I. Is this true in general? Exercise:

- Exercise: Can you identify a set of sufficient properties over a set of modular functions $m_i : V \to \mathbb{R}_+$ so that $f(A) = \min_i m_i(A)$ is submodular? Can you identify both necessary and sufficient conditions?

Matroid loops

- A circuit in a matroids is well defined, a subset $A \subseteq E$ is circuit if it is an inclusionwise minimally dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

- There is no reason in a matroid such an A could not consist of a single element.

- Such an $\{a\}$ is called a loop.

- In a matric (i.e., linear) matroid, the only such loop is the value $\mathbf{0}$, as all non-zero vectors have rank 1. The $\mathbf{0}$ can appear > 1 time with different indices, as can a self loop in a graph appear on different nodes.

- Note, we also say that two elements s, t are said to be parallel if $\{s, t\}$ is a circuit (e.g., in a matrix, two column vectors, one of which is a scalar multiple of the other).

Representable

Definition 7.9.1 (Matroid isomorphism)

Two matroids M_1 and M_2 respectively on ground sets V_1 and V_2 are isomorphic if there is a bijection $\pi : V_1 \to V_2$ which preserves independence (equivalently, rank, circuits, and so on).

- Let $\mathbb{F}$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $\mathbb{F}$, such as a Galois field GF(p) where p is prime (such as GF(2)), but not $\mathbb{Z}$. Succinctly: A field is a set with $+$, $*$, closure, associativity, commutativity, and additive and multiplicative identities and inverses.
- We can more generally define matroids on a field.

Definition 7.9.2 (linear matroids on a field)

Let $\mathbf{X}$ be an $n \times m$ matrix and $E = \{1, \ldots, m\}$, where $\mathbf{X}_{ij} \in \mathbb{F}$ for some field, and let $\mathcal{I}$ be the set of subsets of E such that the columns of $\mathbf{X}$ are linearly independent over $\mathbb{F}$.

Representable

Definition 7.9.1 (Matroid isomorphism)

Two matroids M_1 and M_2 respectively on ground sets V_1 and V_2 are isomorphic if there is a bijection $\pi : V_1 \to V_2$ which preserves independence (equivalently, rank, circuits, and so on).

- Let $\mathbb{F}$ be any field (such as $\mathbb{R}$, $\mathbb{Q}$, or some finite field $\mathbb{F}$, such as a Galois field GF(p) where p is prime (such as GF(2)), but not $\mathbb{Z}$. Succinctly: A field is a set with $+$, $*$, closure, associativity, commutativity, and additive and multiplicative identities and inverses.
- We can more generally define matroids on a field.

Definition 7.9.3 (representable (as a linear matroid))

Any matroid isomorphic to a linear matroid on a field is called representable over $\mathbb{F}$

Representability of Transversal Matroids

- Piff and Welsh in 1970, and Adkin in 1972 proved an important theorem about representability of transversal matroids.
- In particular:

Theorem 7.9.4

Transversal matroids are representable over all finite fields of sufficiently large cardinality, and are representable over any infinite field.

Converse: Representability of Transversal Matroids

The converse is not true, however.

Example 7.9.5

Let $V = \{1, 2, 3, 4, 5, 6\}$ be a ground set and let $M = (V, \mathcal{I})$ be a set system where $\mathcal{I}$ is all subsets of V of cardinality ≤ 2 except for the pairs $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$.

- It can be shown that this is a matroid and is representable.
- However, this matroid is not isomorphic to any transversal matroid.

Review from Lecture 6

The next frame comes from lecture 6.

Matroids, other definitions using matroid rank $r : 2^V \to \mathbb{Z}_+$

Definition 7.11.3 (closed/flat/subspace)

A subset $A \subseteq E$ is closed (equivalently, a flat or a subspace) of matroid M if for all $x \in E \setminus A$, $r(A \cup \{x\}) = r(A) + 1$.

Definition: A hyperplane is a flat of rank $r(M) - 1$.

Definition 7.11.4 (closure)

Given $A \subseteq E$, the closure (or span) of A, is defined by
$\mathrm{span}(A) = \{b \in E : r(A \cup \{b\}) = r(A)\}$.

Therefore, a closed set A has $\mathrm{span}(A) = A$, and the span of a set is closed.

Definition 7.11.5 (circuit)

A subset $A \subseteq E$ is circuit or a cycle if it is an inclusionwise-minimal dependent set (i.e., if $r(A) < |A|$ and for any $a \in A$, $r(A \setminus \{a\}) = |A| - 1$).

Spanning Sets

- We have the following definitions:

Definition 7.11.1 (spanning set of a set)

Given a matroid $\mathcal{M} = (V, \mathcal{I})$, and a set $Y \subseteq V$, then any set $X \subseteq Y$ such that $r(X) = r(Y)$ is called a spanning set of Y.

Definition 7.11.2 (spanning set of a matroid)

Given a matroid $\mathcal{M} = (V, \mathcal{I})$, any set $A \subseteq V$ such that $r(A) = r(V)$ is called a spanning set of the matroid.

- A base of a matroid is a minimal spanning set (and it is independent) but supersets of a base are also spanning.
- V is always trivially spanning.
- Consider the terminology: "spanning tree in a graph", comes from spanning in a matroid sense.

Dual of a Matroid

- Given a matroid $M = (V, \mathcal{I})$, a dual matroid $M^* = (V, \mathcal{I}^*)$ can be defined on the same ground set V, but using a very different set of independent sets $\mathcal{I}^*$.
- We define the set of sets $\mathcal{I}^*$ for M^* as follows:

$$\mathcal{I}^* = \{A \subseteq V : V \setminus A \text{ is a spanning set of } M\} \qquad (7.49)$$
$$= \{V \setminus S : S \subseteq V \text{ is a spanning set of } M\} \qquad (7.50)$$

  i.e., $\mathcal{I}^*$ are complements of spanning sets of M.
- That is, a set A is independent in the dual matroid M^* if removal of A from V does not decrease the rank in M:

$$\mathcal{I}^* = \{A \subseteq V : \text{rank}_M(V \setminus A) = \text{rank}_M(V)\} \qquad (7.51)$$

- In other words, a set $A \subseteq V$ is independent in the dual M^* (i.e., $A \in \mathcal{I}^*$) if A's complement is spanning in M (residual $V \setminus A$ must contain a base in M).
- Dual of the dual: Note, we have that $(M^*)^* = M$.

Dual of a Matroid: Bases

- The smallest spanning sets are bases. Hence, a base B of M (where $B = V \setminus B^*$ is as small as possible while still spanning) is the complement of a base B^* of M^* (where $B^* = V \setminus B$ is as large as possible while still being independent).
- In fact, we have that

Theorem 7.11.3 (Dual matroid bases)

Let $M = (V, \mathcal{I})$ be a matroid and $\mathcal{B}(M)$ be the set of bases of M. Then define

$$\mathcal{B}^*(M) = \{V \setminus B : B \in \mathcal{B}(M)\}. \tag{7.52}$$

Then $\mathcal{B}^*(M)$ is the set of basis of M^* (that is, $\mathcal{B}^*(M) = \mathcal{B}(M^*)$).

An exercise in duality Terminology

- $\mathcal{B}^*(M)$, the bases of M^*, are called cobases of M.
- The circuits of M^* are called cocircuits of M.
- The hyperplanes of M^* are called cohyperplanes of M.
- The independent sets of M^* are called coindependent sets of M.
- The spanning sets of M^* are called cospanning sets of M.

Proposition 7.11.4 (from Oxley 2011)

Let $M = (V, \mathcal{I})$ be a matroid, and let $X \subseteq V$. Then

1. X is independent in M iff $V \setminus X$ is cospanning in M (spanning in M^*).
2. X is spanning in M iff $V \setminus X$ is coindependent in M (independent in M^*).
3. X is a hyperplane in M iff $V \setminus X$ is a cocircuit in M (circuit in M^*).
4. X is a circuit in M iff $V \setminus X$ is a cohyperplane in M (hyperplane in M^*).

Example duality: graphic matroid

- Using a graphic/cycle matroid, we can already see how dual matroid concepts demonstrates the extraordinary flexibility and power that a matroid can have.
- Recall, in cycle matroid, a spanning set of G is any set of edges that are incident to all nodes (i.e., any superset of a spanning forest), a minimal spanning set is a spanning tree (or forest), and a circuit has a nice visual interpretation (a cycle in the graph).
- A cut in a graph G is a set of edges, the removal of which increases the number of connected components. I.e., $X \subseteq E(G)$ is a cut in G if $k(G) < k(G \setminus X)$.
- A minimal cut in G is a cut $X \subseteq E(G)$ such that $X \setminus \{x\}$ is not a cut for any $x \in X$.
- A cocycle (cocircuit) in a graphic matroid is a minimal graph cut.
- A mincut is a circuit in the dual "cocycle" (or "cut") matroid.
- All dependent sets in a cocycle matroid are cuts (i.e., a dependent set is a minimal cut or contains one).

Example: cocycle matroid (sometimes "cut matroid")

- The dual of the cycle matroid is called the cocycle matroid. Recall,
  $$\mathcal{I}^* = \{A \subseteq V : V \setminus A \text{ is a spanning set of } M\}$$
- $\mathcal{I}^*$ consists of all sets of edges the complement of which contains a spanning tree — i.e., an independent set can't consist of edges that, if removed, would render the graph non-spanning.
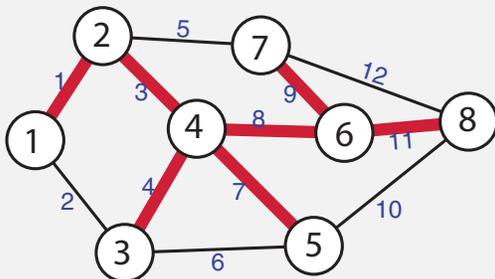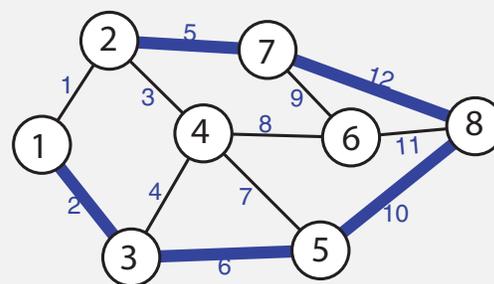
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A graph G

Example: cocycle matroid (sometimes "cut matroid")

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Minimally spanning in M (and thus a base (maximally independent) in M)
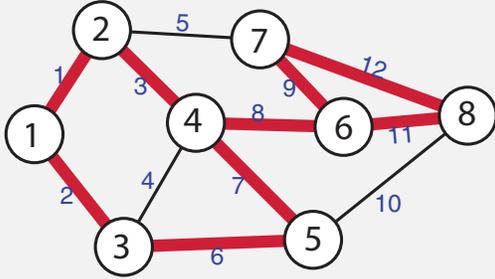
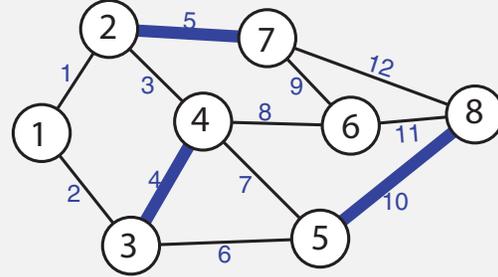Maximally independent in M* (thus a base, minimally spanning, in M*)

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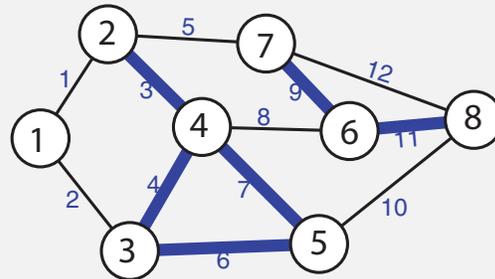Maximally independent in M* (thus a base, minimally spanning, in M*)

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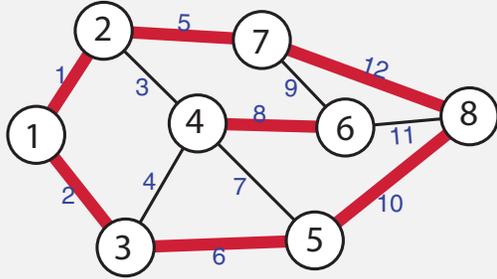
Independent but not spanning in M, and not closed in M.

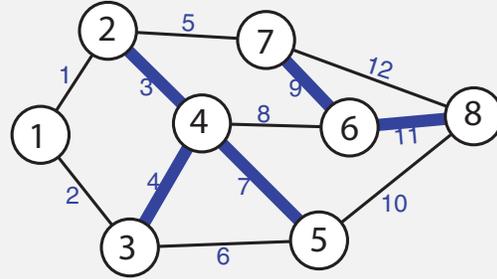Dependent in M* (contains a cocycle, is a nonminimal cut)

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Spanning in M, but not a base, and not independent (has cycles)

Independent in M* (does not contain a cut)

Example: cocycle matroid (sometimes "cut matroid")

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Independent but not spanning in M, and not closed in M.
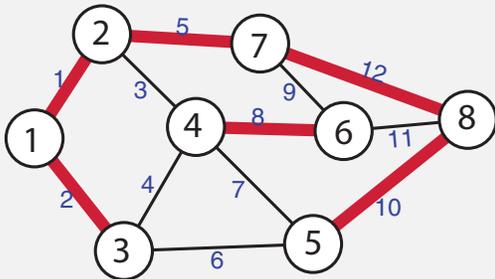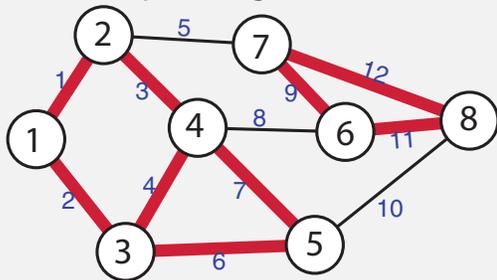
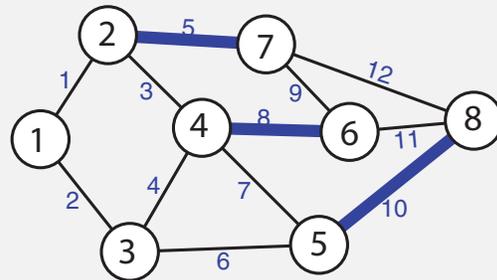Dependent in M* (contains a cocycle, is a nonminimal cut)

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A hyperplane in M, dependent but not spanning in M

A cycle in M* (minimally dependent in M*, a cocycle, or a minimal cut)

Example: cocycle matroid (sometimes "cut matroid")

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Cycle Matroid - independent sets have no cycles.
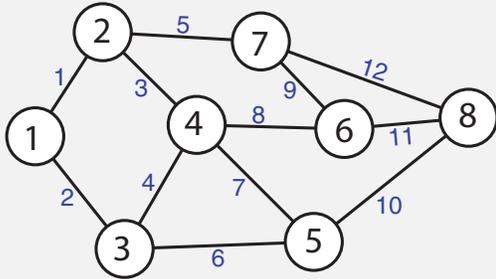
Cocycle matroid, independent sets contain no cuts.
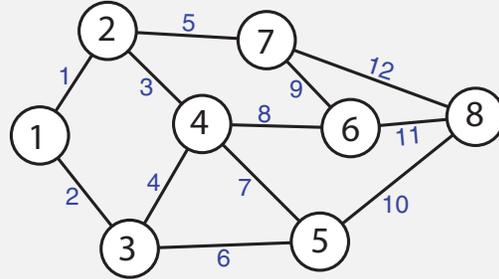
The dual of a matroid is (indeed) a matroid

Theorem 7.11.5

Given matroid $M = (V, \mathcal{I})$, let $M^* = (V, \mathcal{I}^*)$ be as previously defined. Then M^* is a matroid.

Proof.

- Since $V \setminus \emptyset$ is spanning in primal, clearly $\emptyset \in \mathcal{I}^*$, so (I1') holds.
- Also, if $I \subseteq J \in \mathcal{I}^*$, then clearly also $I \in \mathcal{I}^*$ since if $V \setminus J$ is spanning in M, so must $V \setminus I$. Therefore, (I2') holds.
- Next, given $I, J \in \mathcal{I}^*$ with $|I| < |J|$, it must be the case that $\bar{I} = V \setminus I$ and $\bar{J} = V \setminus J$ are both spanning in M with $|\bar{I}| > |\bar{J}|$.

The dual of a matroid is (indeed) a matroid

Theorem 7.11.5

Given matroid $M = (V, \mathcal{I})$, let $M^* = (V, \mathcal{I}^*)$ be as previously defined. Then M^* is a matroid.

Proof.

- Consider $I, J \in \mathcal{I}^*$ with $|I| < |J|$. We need to show that there is some member $v \in J \setminus I$ such that $I + v$ is independent in M^*, which means that $V \setminus (I + v) = (V \setminus I) \setminus v = \bar{I} - v$ is still spanning in M. That is, removing v from $V \setminus I$ doesn't make $(V \setminus I) \setminus v$ not spanning in M.
- Since $V \setminus J$ is spanning in M, $V \setminus J$ contains some base (say $B_{\bar{J}} \subseteq V \setminus J$) of M. Also, $V \setminus I$ contains a base of M, say $B_{\bar{I}} \subseteq V \setminus I$.
- Since $B_{\bar{J}} \setminus I \subseteq V \setminus I$, and $B_{\bar{J}} \setminus I$ is independent in M, we can choose the base $B_{\bar{I}}$ of M s.t. $B_{\bar{J}} \setminus I \subseteq B_{\bar{I}} \subseteq V \setminus I$.
- Since $B_{\bar{J}}$ and J are disjoint, we have both: 1) $B_{\bar{J}} \setminus I$ and $J \setminus I$ are disjoint; and 2) $B_{\bar{J}} \cap I \subseteq I \setminus J$. Also note, $B_{\bar{I}}$ and I are disjoint.

$\cdots$

Proof.

- Now $J \setminus I \not\subseteq B_{\bar{I}}$, since otherwise (i.e., assuming $J \setminus I \subseteq B_{\bar{I}}$):

$$\begin{align}
|B_{\bar{J}}| &= |B_{\bar{J}} \cap I| + |B_{\bar{J}} \setminus I| \tag{7.53} \\
&\leq |I \setminus J| + |B_{\bar{J}} \setminus I| \tag{7.54} \\
&< |J \setminus I| + |B_{\bar{J}} \setminus I| \leq |B_{\bar{I}}| \tag{7.55}
\end{align}$$

which is a contradiction. *The last inequality on the right follows since $J \setminus I \subseteq B_{\bar{I}}$ (by assumption) and $B_{\bar{J}} \setminus I \subseteq B_{\bar{I}}$ implies that $(J \setminus I) \cup (B_{\bar{J}} \setminus I) \subseteq B_{\bar{I}}$, but since J and $B_{\bar{J}}$ are disjoint, we have that $|J \setminus I| + |B_{\bar{J}} \setminus I| \leq |B_{\bar{I}}|$.*

- Therefore, $J \setminus I \not\subseteq B_{\bar{I}}$, and there is a $v \in J \setminus I$ s.t. $v \notin B_{\bar{I}}$.
- So $B_{\bar{I}}$ is disjoint with $I \cup \{v\}$, means $B_{\bar{I}} \subseteq V \setminus (I \cup \{v\})$, or

Matroid Duals and Representability

Theorem 7.11.6

Let M be an $\mathbb{F}$-representable matroid (i.e., one that can be represented by a finite sized matrix over field $\mathbb{F}$). Then M^* is also $\mathbb{F}$-representable.

Hence, for matroids as general as matric matroids, duality does not extend the space of matroids that can be used.

Theorem 7.11.7

Let M be a graphic matroid (i.e., one that can be represented by a graph $G = (V, E)$). Then M^* is not necessarily also graphic.

Hence, for graphic matroids, duality can increase the space and power of matroids, and since they are based on a graph, they are relatively easy to use: 1) all cuts are dependent sets; 2) minimal cuts are cycles; 3) bases of a cut are any one edge removed from minimal cuts; 4) independent sets are edges that are not cuts (minimal or otherwise); 5) bases of matroid are maximal non-cuts (non-cut containing edge sets).

Dual Matroid Rank

Theorem 7.11.8

The rank function r_{M^*} of the dual matroid M^* may be specified in terms of the rank r_M in matroid M as follows. For $X \subseteq V$:

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) \tag{7.56}$$

- Note, we again immediately see that this is submodular by the properties of submodular functions we saw in lectures 1 and 2. *I.e., $|X|$ is modular, complement $f(V \setminus X)$ is submodular if f is submodular, $r_M(V)$ is a constant, and summing submodular functions and a constant preserves submodularity.*
- Non-negativity integral follows since $|X| + r_M(V \setminus X) \geq r_M(X) + r_M(V \setminus X) \geq r_M(V)$. *The right inequality follows since r_M is submodular.*
- Monotone non-decreasing follows since, as X increases by one, $|X|$ always increases by 1, while $r_M(V \setminus X)$ decreases by one or zero.
- Therefore, r_{M^*} is the rank function of a matroid. That it is the dual

Dual Matroid Rank

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Proof.

A set X is independent in (V, r_{M^*}) if and only if

$$r_{M^*}(X) = |X| + r_M(V \setminus X) - r_M(V) = |X| \tag{7.57}$$

or

$$r_M(V \setminus X) = r_M(V) \tag{7.58}$$

But a subset X is independent in M^* only if $V \setminus X$ is spanning in M (by the definition of the dual matroid). □