Submodular Functions, Optimization, and Applications to Machine Learning
— Fall Quarter, Lecture 3 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/

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Oct 7th, 2020

\[ f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \]

- \( f(A) + 2f(C) + f(B) \)
- \( f(A) + f(C) + f(B) \)
- \( f(A \cap B) \)
Read chapter 1 from Fujishige’s book.
Class Road Map - EE563

L1(9/30): Motivation, Applications, Definitions, Properties
L2(10/5): Sums concave(modular), uses (diversity/costs, feature selection), information theory
L3(10/7): Monge, More Definitions, Graph and Combinatorial Examples, Matrix Rank, Properties
L4(10/12):
L5(10/14):
L6(10/19):
L7(10/21):
L8(10/26):
L9(10/28):
L10(11/2):
L11(11/4):
L12(11/9):
L–(11/11): Veterans Day, Holiday
L13(11/16):
L14(11/18):
L15(11/23):
L16(11/25):
L17(11/30):
L18(12/2):
L19(12/7):
L20(12/9): maximization.

Last day of instruction, Fri. Dec 11th. Finals Week: Dec 12-18, 2020
Two Equivalent **Submodular** Definitions

**Definition 3.2.1 (submodular concave)**

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

(3.7)

An alternate and (as we will soon see) equivalent definition is:

**Definition 3.2.2 (diminishing returns)**

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A \subseteq B \subseteq V$, and $v \in V \setminus B$, we have that:

$$f(A \cup \{v\}) - f(A) \geq f(B \cup \{v\}) - f(B)$$

(3.8)

- The incremental “value”, “gain”, or “cost” of $v$ decreases (diminishes) as the context in which $v$ is considered grows from $A$ to $B$.
- Gain notation: Define $f(v|A) \triangleq f(A + v) - f(A)$. Then function $f$ is submodular if $f(v|A) \geq f(v|B)$ for all $A \subseteq B \subseteq V \setminus \{v\}, v \in V$. 
Two Equivalent **Supermodular** Definitions

**Definition 3.2.1 (supermodular)**

A function \( f : 2^V \rightarrow \mathbb{R} \) is **supermodular** if for any \( A, B \subseteq V \), we have that:

\[
f(A) + f(B) \leq f(A \cup B) + f(A \cap B)
\] (3.7)

**Definition 3.2.2 (supermodular (improving returns))**

A function \( f : 2^V \rightarrow \mathbb{R} \) is **supermodular** if for any \( A \subseteq B \subseteq V \), and \( v \in V \setminus B \), we have that:

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- Incremental “value”, “gain”, or “cost” of \( v \) increases (improves) as the context in which \( v \) is considered grows from \( A \) to \( B \).
- A function \( f \) is submodular iff \( -f \) is supermodular.
- If \( f \) both submodular and supermodular, then \( f \) is said to be **modular**, and \( f(A) = c + \sum_{a \in A} \bar{f}(a) \) for some \( \bar{f} \) (often \( c = 0 \)).
**Monge Matrices**

- $m \times n$ matrices $C = [c_{ij}]_{ij}$ are called Monge matrices if they satisfy the Monge property, namely:

  $c_{ij} + c_{rs} \leq c_{is} + c_{rj}$ \hspace{1cm} (3.1)

  for all $1 \leq i < r \leq m$ and $1 \leq j < s \leq n$. 
Monge Matrices

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- Lined up indices

$$i < r \quad (3.2)$$
$$j < s \quad (3.3)$$
Monge Matrices

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- Lined up indices

$$i < r$$

(3.2)

$$j < s$$

(3.3)

- Equivalently, for all $1 \leq i, r \leq m, 1 \leq s, j \leq n$,

$$c_{\min(i,r),\min(s,j)} + c_{\max(i,r),\max(s,j)} \leq c_{is} + c_{rj}$$

(3.4)
Consider a non-negative matrix $D = (d_{i,j})$ of order $m \times n$ and form matrix $C = (c_{i,j})$ with $c_{i,j}$th entry, $1 \leq i \leq m$, $1 \leq j \leq n$:

$$c_{ij} = \sum_{k=i}^{m} \sum_{\ell=1}^{j} d_{k,\ell} \tag{3.5}$$
Monge Matrices Visuals

- Consider a non-negative matrix \( D = (d_{i,j}) \) of order \( m \times n \) and form matrix \( C = (c_{i,j}) \) with \( c_{i,j} \)th entry, \( 1 \leq i \leq m, 1 \leq j \leq n \):

\[
c_{ij} = \sum_{k=i}^{m} \sum_{\ell=1}^{j} d_{k,\ell}
\]  

(3.5)

- Consider four elements of the \( m \times n \) matrix:

\[c_{ij}, c_{rs}, c_{is}, c_{rj}\] and \[c_{ij}, c_{rs}, c_{is}, c_{rs}\]

\[c_{ij} + c_{r\ell} < c_{is} + c_{rj}\]
Consider a non-negative matrix $D = (d_{i,j})$ of order $m \times n$ and form matrix $C = (c_{i,j})$ with $c_{i,j}$th entry, $1 \leq i \leq m$, $1 \leq j \leq n$:

$$c_{ij} = \sum_{k=i}^{m} \sum_{\ell=1}^{j} d_{k,\ell}$$  \hspace{1cm} (3.5)

Consider four elements of the $m \times n$ matrix:

$$c_{ij} = A + B, \quad c_{rs} = B + D, \quad c_{rj} = B, \quad c_{is} = A + B + C + D.$$
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\[
c_{i,j} = \sum_{k=i}^{m} \sum_{\ell=1}^{j} d_{k,\ell}
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$$c_{ij} = \sum_{k=i}^{m} \sum_{\ell=1}^{j} d_{k,\ell} \quad (3.5)$$

Consider four elements of the $m \times n$ matrix:

\[
\begin{array}{cccc}
A & & & C \\
B & & & D \\
\end{array}
\]

\[
\begin{array}{cccc}
& & & \\
& & \\
& & & \\
\end{array}
\]

\[
\begin{array}{cccc}
& & c_{ij} & \\
& c_{ij} & & c_{is} \\
& & & c_{is} \\
\end{array}
\]

\[
\begin{array}{cccc}
& & c_{rs} & \\
& c_{rs} & & \\
& & & \\
\end{array}
\]

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Consider four elements of the $m \times n$ matrix:

$$c_{ij} = A + B, \quad c_{rs} = B + D, \quad c_{rj} = B, \quad c_{is} = A + B + C + D.$$

Then, $c_{ij} + c_{rs} < c_{is} + c_{rj}$, if $C > 0$. 
Monge Matrices, where useful

- Useful for speeding up transportation, dynamic programming, flow, search, lot-sizing and many other problems.

<table>
<thead>
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</table>

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**Example:** Hitchcock transportation problem

Given an $m \times n$ cost matrix $C = [c_{ij}]$, an non-negative supply vector $a \in \mathbb{R}_m^+$, an non-negative demand vector $b \in \mathbb{R}_n^+$ with $\sum_{i=1}^{m} a(i) = \sum_{j=1}^{n} b(j)$, we wish to optimally solve the following linear program:

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
\text{subject to} & \quad \sum_{i=1}^{m} x_{ij} = b(j) \quad \forall j = 1, \ldots, n \quad (3.7) \\
& \quad \sum_{j=1}^{n} x_{ij} = a(i) \quad \forall i = 1, \ldots, m \quad (3.8) \\
& \quad x_{ij} \geq 0 \quad \forall i, j \quad (3.9)
\end{align*}
\]
Monge Matrices, where useful

- Useful for speeding up transportation, dynamic programming, flow, search, lot-sizing and many other problems.

- **Example, Hitchcock transportation problem:** Given $m \times n$ cost matrix $C = [c_{ij}]_{ij}$, a non-negative supply vector $a \in \mathbb{R}^m_+$, a non-negative demand vector $b \in \mathbb{R}^n_+$ with $\sum_{i=1}^m a(i) = \sum_{j=1}^n b_j$, we wish to optimally solve the following linear program:

$$\text{minimize} \quad \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

subject to:

$$\sum_{i=1}^m x_{ij} = b_j \quad \forall j = 1, \ldots, n \quad (3.7)$$

$$\sum_{j=1}^n x_{ij} = a_i \quad \forall i = 1, \ldots, m \quad (3.8)$$

$$x_{i,j} \geq 0 \quad \forall i, j \quad (3.9)$$
## Monge Matrices, Hitchcock transportation

### Matrix $C$

<table>
<thead>
<tr>
<th></th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_1$</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>$b_2$</td>
<td>1</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>$b_3$</td>
<td>0</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>$b_4$</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

### Producers, Sources, or Supply
- $a_1$, $a_2$, $a_3$

### Consumers, Sinks, or Demand
- $b_1$, $b_2$, $b_3$, $b_4$

#### Solving the linear program
- The “North-West Corner Rule” can be used to solve the linear program optimally.
- This rule involves starting at the top-left cell and moving down or right to assign values.
- The time complexity is $O(m + n)$ if the matrix $C$ is Monge.

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Monge Matrices and Convex Polygons

- Can generate a Monge matrix from a convex polygon - delete two segments, then separately number vertices on each chain. Distances $c_{ij}$ satisfy Monge property (or quadrangle inequality).
Monge Matrices and Convex Polygons

Can generate a Monge matrix from a convex polygon - delete two segments, then separately number vertices on each chain. Distances $c_{ij}$ satisfy Monge property (or quadrangle inequality).

\[ d(q_3, p_2) + d(q_4, p_3) \leq d(q_4, p_2) + d(q_3, p_3) \]
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\[
d(q_3, p_2) + d(q_4, p_3) \leq d(q_4, p_2) + d(q_3, p_3)
\]  

(3.10)

Transport unit quantities from locations $q_3$ and $q_4$ to locations $p_2$ and $p_3$; to minimize total distance traveled, routes from $q_3$ and $q_4$ must not intersect.
Monge Matrices and Submodularity

- A submodular function has the form: \( f : 2^V \to \mathbb{R} \) which can be seen as \( f : \{0, 1\}^V \to \mathbb{R} \)
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We can generalize this to \( f : \{0, 1, \ldots, K\}^V \rightarrow \mathbb{R} \) for some constant \( K \in \mathbb{Z}_+ \).
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We may define submodularity as: for all $x, y \in \{0, 1, \ldots, K\}^V$, we have

$$f(x) + f(y) \geq f(x \lor y) + f(x \land y)$$  \hspace{1cm} (3.11)
Monge Matrices and Submodularity

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  \[
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- \( x \lor y \) is the (join) element-wise max of each element, that is \( (x \lor y)(v) = \max(x(v), y(v)) \) for \( v \in V \).
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- We may define submodularity as: for all \( x, y \in \{0, 1, \ldots, K\}^V \), we have
  \[
  f(x) + f(y) \geq f(x \vee y) + f(x \wedge y) \tag{3.11}
  \]
  where \( x \vee y \) is the (join) element-wise max of each element, that is \( (x \vee y)(v) = \max(x(v), y(v)) \) for \( v \in V \).
- \( x \wedge y \) is the (meet) element-wise min of each element, that is, \( (x \wedge y)(v) = \min(x(v), y(v)) \) for \( v \in V \).
- With \( K = 1 \), then this is the standard definition of submodularity.
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With $|V| = 2$, and $K + 1$ the side-dimension of the matrix, we get a Monge property (on square matrices).
A submodular function has the form: \( f : 2^V \rightarrow \mathbb{R} \) which can be seen as 
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With \( |V| = 2 \), and \( K + 1 \) the side-dimension of the matrix, we get a Monge property (on square matrices).

Non square: \( f : \{0, 1, \ldots, K_1\} \times \{0, 1, \ldots, K_2\} \rightarrow \mathbb{R} \).
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**Definition 3.4.1 (submodular concave)**

A function \( f : 2^V \rightarrow \mathbb{R} \) is **submodular** if for any \( A, B \subseteq V \), we have that:

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An alternate and (as we will soon see) equivalent definition is:

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A function \( f : 2^V \rightarrow \mathbb{R} \) is **submodular** if for any \( A \subseteq B \subseteq V \), and \( v \in V \setminus B \), we have that:

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    f(A \cup \{v\}) - f(A) \geq f(B \cup \{v\}) - f(B)
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The Submodular Square, and Hypercube Vertices

We can test submodularity via values on vertices of hypercube.

Example: with $|V| = n = 2$, this is easy:

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The Submodular Square, and Hypercube Vertices

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Example: with $|V| = n = 2$, this is easy:

$$f(100) + f(010) \geq f((100) \lor (010)) + f((100) \land (010)) = f(110) + f(000)$$

With $|V| = n = 3$, a bit harder.
The Submodular Square, and Hypercube Vertices

We can test submodularity via values on vertices of hypercube.

Example: with $|V| = n = 2$, this is easy:

\[
\begin{array}{c}
00 \\
01 \\
10 \\
11
\end{array}
\]

With $|V| = n = 3$, a bit harder.

\[
\begin{array}{c}
000 \\
001 \\
010 \\
011 \\
100 \\
101 \\
110 \\
111
\end{array}
\]

How many inequalities of form $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$?
The Submodular Square, and Hypercube Vertices

We can test submodularity via values on vertices of hypercube.

Example: with $|V| = n = 2$, this is easy:

$$f(x_i) + f(x_j) \geq f(x_i + x_j) + f(x)$$

Any $A \cup B$.

To verify submodularity when $|V| = 3$, need to check at most 9 unique squares.

Claim: need only check 6 inequalities. The 6 are:

1. $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$

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Definition 3.4.1 (subadditive)

A function $f : 2^V \to \mathbb{R}$ is subadditive if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \geq f(A \cup B) \quad (3.12)$$

This means that the “whole” is less than the sum of the parts.
Two Equivalent Supermodular Definitions

Definition 3.4.1 (supermodular)

A function \( f : 2^V \rightarrow \mathbb{R} \) is supermodular if for any \( A, B \subseteq V \), we have that:

\[
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- A function \( f \) is submodular iff \(-f\) is supermodular.
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Superadditive Definitions

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A function $f : 2^V \to \mathbb{R}$ is superadditive if for any $A, B \subseteq V$, we have that:

$$f(A) + f(B) \leq f(A \cup B) \tag{3.13}$$

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\[
f(A) + f(B) \leq f(A \cup B)
\]  

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- In general, submodular and subadditive (and supermodular and superadditive) are different properties.
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(3.13)

- This means that the “whole” is greater than the sum of the parts.
- In general, submodular and subadditive (and supermodular and superadditive) are different properties.
- Ex: Let $0 < k < |V|$, and consider $f : 2^V \to \mathbb{R}_+$ where:

$$f(A) = \begin{cases} 
1 & \text{if } |A| \leq k \\
0 & \text{else}
\end{cases}$$

(3.14)
Superadditive Definitions

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A function \( f : 2^V \rightarrow \mathbb{R} \) is superadditive if for any \( A, B \subseteq V \), we have that:

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\]  

(3.13)

- This means that the “whole” is greater than the sum of the parts.
- In general, submodular and subadditive (and supermodular and superadditive) are different properties.
- Ex: Let \( 0 < k < |V| \), and consider \( f : 2^V \rightarrow \mathbb{R}_+ \) where:

\[
f(A) = \begin{cases} 
1 & \text{if } |A| \leq k \\
0 & \text{else}
\end{cases}
\]  

(3.14)

- This function is subadditive but not submodular.

\[
f(A) + f(B) > f(A \cup B) + f(A \cap B)
\]
Definition 3.4.3 (modular)

A function that is both submodular and supermodular is called \textbf{modular}.

If \( f \) is a modular function, than for any \( A, B \subseteq V \), we have

\[
f(A) + f(B) = f(A \cap B) + f(A \cup B)
\]

(3.15)

In modular functions, elements do not interact (or cooperate, or compete, or influence each other), and have value based only on singleton values.

Proposition 3.4.4

\textbf{If} \( f \) is modular, \textit{it may be written as}

\[
f(A) = f(\emptyset) + \sum_{a \in A} \left( f(\{a\}) - f(\emptyset) \right) = c + \sum_{a \in A} f'(a)
\]

(3.16)

which has only \(|V| + 1\) parameters.

\textbf{Normalized} \Rightarrow \quad c = 0

\therefore \textbf{modular function} \quad \textbf{vector} \in \mathbb{IR}^V
Proof.

We inductively construct the value for \( A = \{a_1, a_2, \ldots, a_k\} \).

For \( k = 2 \),

\[
\begin{align*}
 f(a_1) + f(a_2) &= f(a_1, a_2) + f(\emptyset) \tag{3.17} \\
 \text{implies } f(a_1, a_2) &= f(a_1) - f(\emptyset) + f(a_2) - f(\emptyset) + f(\emptyset) \tag{3.18}
\end{align*}
\]

then for \( k = 3 \),

\[
\begin{align*}
 f(a_1, a_2) + f(a_3) &= f(a_1, a_2, a_3) + f(\emptyset) \tag{3.19} \\
 \text{implies } f(a_1, a_2, a_3) &= f(a_1, a_2) - f(\emptyset) + f(a_3) - f(\emptyset) + f(\emptyset) \tag{3.20} \\
 &= f(\emptyset) + \sum_{i=1}^{3} (f(a_i) - f(\emptyset)) \tag{3.21}
\end{align*}
\]

and so on . . .
Complement function

Given a function $f : 2^V \to \mathbb{R}$, we can find a complement function $\bar{f} : 2^V \to \mathbb{R}$ as $\bar{f}(A) = f(V \setminus A)$ for any $A$.

**Proposition 3.4.5**

$\bar{f}$ is submodular iff $f$ is submodular.

**Proof.**

$$\bar{f}(A) + \bar{f}(B) \geq \bar{f}(A \cup B) + \bar{f}(A \cap B) \quad (3.22)$$

follows from

$$f(V \setminus A) + f(V \setminus B) \geq f(V \setminus (A \cup B)) + f(V \setminus (A \cap B)) \quad (3.23)$$

which is true because $V \setminus (A \cup B) = (V \setminus A) \cap (V \setminus B)$ and $V \setminus (A \cap B) = (V \setminus A) \cup (V \setminus B)$ (De Morgan’s laws for sets).
Undirected Graphs

Let $G = (V, E)$ be a graph with vertices $V = V(G)$ and edges $E = E(G) \subseteq V \times V$. 
Undirected Graphs

- Let $G = (V, E)$ be a graph with vertices $V = V(G)$ and edges $E = E(G) \subseteq V \times V$.
- If $G$ is undirected, define

\[
E(X, Y) \triangleq \{\{x, y\} \in E(G) : x \in X \setminus Y, y \in Y \setminus X\} \tag{3.24}
\]

as the edges strictly between $X$ and $Y$. 

![Graph example](image-url)
Undirected Graphs

- Let $G = (V, E)$ be a graph with vertices $V = V(G)$ and edges $E = E(G) \subseteq V \times V$.
- If $G$ is undirected, define
  \[
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  \]
  (3.24)
  as the edges strictly between $X$ and $Y$.
- Nodes define cuts. Define the cut function $\delta(X) = E(X, V \setminus X)$, set of edges with exactly one vertex in $X$. 
Undirected Graphs

- Let $G = (V, E)$ be a graph with vertices $V = V(G)$ and edges $E = E(G) \subseteq V \times V$.
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  $$E(X, Y) = \{\{x, y\} \in E(G) : x \in X \setminus Y, y \in Y \setminus X\}$$

  (3.24)

  as the edges strictly between $X$ and $Y$.
- Nodes define cuts. Define the cut function $\delta(X) = E(X, V \setminus X)$, set of edges with exactly one vertex in $X$.

$$G = (V, E)$$

$S = \{a, b, c\}$

$$\delta_G(S) = \{\{u, v\} \in E : u \in S, v \in V \setminus S\} = \{\{a, d\}, \{b, d\}, \{b, e\}, \{c, e\}, \{c, f\}\}$$
Directed graphs, and cuts and flows

- If $G$ is directed, define

$$E^+(X, Y) \triangleq \{(x, y) \in E(G) : x \in X \setminus Y, y \in Y \setminus X\} \quad (3.25)$$

as the edges directed strictly from $X$ towards $Y$. 

Directed graphs, and cuts and flows

If $G$ is directed, define

$$E^+(X, Y) \triangleq \{(x, y) \in E(G) : x \in X \setminus Y, y \in Y \setminus X\}$$  (3.25)

as the edges directed strictly from $X$ towards $Y$.

- Nodes define cuts and flows. Define edges leaving $X$ (out-flow) as
  $$\delta^+(X) \triangleq E^+(X, V \setminus X)$$  (3.26)
  and edges entering $X$ (in-flow) as
  $$\delta^-(X) \triangleq E^+(V \setminus X, X)$$  (3.27)
Directed graphs, and cuts and flows

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  and edges entering $X$ (in-flow) as

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- Diagram:

  $\delta_G(S) = \{(v, u) \in E : u \in S, v \in V \setminus S\}.$

  $= \{(d,a), (d,b), (e,c)\}$

  $\delta^+_G(S) = \{(u, v) \in E : u \in S, v \in V \setminus S\}.$

  $= \{(b,e), (c,f)\}$
The Neighbor function in undirected graphs

- Given a set $X \subseteq V$, the neighbor function of $X$ is defined as

$$\Gamma(X) \triangleq \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$$  \hspace{1cm} (3.28)
The Neighbor function in undirected graphs

- Given a set $X \subseteq V$, the neighbor function of $X$ is defined as

$$\Gamma(X) \triangleq \{v \in V(G) \setminus X : E(X, \{v\}) \neq \emptyset\}$$  \hspace{1cm} (3.28)

- Example:

$G = (V, E)$

$S = \{a, b, c\}$

$\Gamma(S) = \{d, e, f\}$
Directed Cut function: property

Lemma 3.5.1

For a digraph $G = (V, E)$ and any $X, Y \subseteq V$: we have

$$|\delta^+(X)| + |\delta^+(Y)| = |\delta^+(X \cap Y)| + |\delta^+(X \cup Y)| + |E^+(X, Y)| + |E^+(Y, X)|$$  \hspace{1cm} (3.29)$$

and

$$|\delta^-(X)| + |\delta^-(Y)| = |\delta^-(X \cap Y)| + |\delta^-(X \cup Y)| + |E^-(X, Y)| + |E^-(Y, X)|$$  \hspace{1cm} (3.30)$$

\[ f^+(x) = |\delta^+(x)| \text{ is submodular} \]

\[ f^-(x) = |\delta^-(x)| \text{ is submodular} \]
Directed Cut function: proof of property

Proof.

We can prove Eq. (3.29) using a geometric counting argument (proof for $|\delta^-(X)|$ case is similar)

\[ w(\delta^+ (x)) = f(x) \]

\[ h(x) = f(\delta^+(x)) \]

\[ |A| = v(a) \]

Q: Why is \( (c) = |E^+(X, Y)| \)?
Directed cut/flow functions: submodular

Lemma 3.5.2

For a digraph $G = (V, E)$ and any $X, Y \subseteq V$: both functions $|\delta^+(X)|$ and $|\delta^-(X)|$ are submodular.

Proof.

$|E^+(X, Y)| \geq 0$ and $|E^-(X, Y)| \geq 0$.

More generally, in the non-negative weighted edge case, both in-flow and out-flow are submodular on subsets of the vertices.
Lemma 3.5.3

For an undirected graph \( G = (V, E) \) and any \( X, Y \subseteq V \): we have that both the undirected cut (or flow) function \( |\delta(X)| \) and the neighbor function \( |\Gamma(X)| \) are submodular. I.e.,

\[
|\delta(X)| + |\delta(Y)| = |\delta(X \cap Y)| + |\delta(X \cup Y)| + 2|E(X, Y)| \tag{3.31}
\]

and

\[
|\Gamma(X)| + |\Gamma(Y)| \geq |\Gamma(X \cap Y)| + |\Gamma(X \cup Y)| \tag{3.32}
\]

Proof.

- Eq. (3.31) follows from Eq. (3.29): we replace each undirected edge \( \{u, v\} \) with two oppositely-directed directed edges \((u, v)\) and \((v, u)\). Then we use same counting argument.

\[
|\delta(x)| = \frac{1|\delta^+(x)| + 1|\delta^-(x)|}{2}
\]

...
Lemma 3.5.3

For an undirected graph $G = (V, E)$ and any $X, Y \subseteq V$: we have that both the undirected cut (or flow) function $|\delta(X)|$ and the neighbor function $|\Gamma(X)|$ are submodular. I.e.,

$$|\delta(X)| + |\delta(Y)| = |\delta(X \cap Y)| + |\delta(X \cup Y)| + 2|E(X, Y)|$$ (3.31)

and

$$|\Gamma(X)| + |\Gamma(Y)| \geq |\Gamma(X \cap Y)| + |\Gamma(X \cup Y)|$$ (3.32)

Proof.

- Eq. (3.31) follows from Eq. (3.29): we replace each undirected edge $\{u, v\}$ with two oppositely-directed directed edges $(u, v)$ and $(v, u)$. Then we use same counting argument.

- Eq. (3.32) follows as shown in the following page.
Graphically, we can count and see that

\[
\Gamma(X) = (a) + (c) + (f) + (g) + (d)
\]
\[
\Gamma(Y) = (b) + (c) + (e) + (h) + (d)
\]
\[
\Gamma(X \cup Y) = (a) + (b) + (c) + (d)
\]
\[
\Gamma(X \cap Y) = (c) + (g) + (h)
\]

SO

\[
|\Gamma(X)| + |\Gamma(Y)| = (a) + (b) + 2(c) + 2(d) + (e) + (f) + (g) + (h)
\]
\[
\geq (a) + (b) + 2(c) + (d) + (g) + (h) = |\Gamma(X \cup Y)| + |\Gamma(X \cap Y)|
\]
Therefore, the undirected cut function $|\delta(A)|$ and the neighbor function $|\Gamma(A)|$ of a graph $G$ are both submodular.
Undirected cut/flow is submodular: alternate proof

- Another simple proof shows that $|\delta(X)|$ is submodular.
Undirected cut/flow is submodular: alternate proof

- Another simple proof shows that $|\delta(X)|$ is submodular.
- Define a graph $G_{uv} = (\{u, v\}, \{e\}, w)$ with two nodes $u, v$ and one edge $e = \{u, v\}$ with non-negative weight $w(e) \in \mathbb{R}_+$. 
Undirected cut/flow is submodular: alternate proof

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- Define a graph $G_{uv} = (\{u, v\}, \{e\}, w)$ with two nodes $u, v$ and one edge $e = \{u, v\}$ with non-negative weight $w(e) \in \mathbb{R}_+$. 
- Weighted cut function over those two nodes: $w(\delta_{u,v}(\cdot))$ has valuation:

\[
w(\delta_{u,v}(\emptyset)) = w(\delta_{u,v}(\{u,v\})) = 0 \tag{3.38}
\]

and

\[
w(\delta_{u,v}(\{u\})) = w(\delta_{u,v}(\{v\})) = w \geq 0 \tag{3.39}
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Undirected cut/flow is submodular: alternate proof

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and

$$w(\delta_{u,v}(\{u\})) = w(\delta_{u,v}(\{v\})) = w \geq 0 \quad (3.39)$$

- Thus, $w(\delta_{u,v}(\cdot))$ is submodular since $w(e) \geq 0$ and

$$w(\delta_{u,v}(\{u\})) + w(\delta_{u,v}(\{v\})) \geq w(\delta_{u,v}(\{u, v\})) + w(\delta_{u,v}(\emptyset)) \quad (3.40)$$
Undirected cut/flow is submodular: alternate proof

- Another simple proof shows that $|\delta(X)|$ is submodular.
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and

\[
w(\delta_{u,v}(\{u\})) = w(\delta_{u,v}(\{v\})) = w \geq 0 \quad (3.39)\]

- Thus, $w(\delta_{u,v}(\cdot))$ is submodular since $w(e) \geq 0$ and

\[
w(\delta_{u,v}(\{u\})) + w(\delta_{u,v}(\{v\})) \geq w(\delta_{u,v}(\{u, v\})) + w(\delta_{u,v}(\emptyset)) \quad (3.40)\]

- General non-negative weighted graph $G = (V, E, w)$, define $w(\delta(\cdot))$:

\[
f(X) = w(\delta(X)) = \sum_{(u,v) \in E(G)} w(\delta_{u,v}(X \cap \{u, v\})) \quad (3.41)\]
Undirected cut/flow is submodular: alternate proof

- Another simple proof shows that $|\delta(X)|$ is submodular.
- Define a graph $G_{uv} = (\{u, v\}, \{e\}, w)$ with two nodes $u, v$ and one edge $e = \{u, v\}$ with non-negative weight $w(e) \in \mathbb{R}_+$.
- Weighted cut function over those two nodes: $w(\delta_{u,v}(\cdot))$ has valuation:
  \[ w(\delta_{u,v}(\emptyset)) = w(\delta_{u,v}(\{u, v\})) = 0 \quad (3.38) \]

and
\[ w(\delta_{u,v}(\{u\})) = w(\delta_{u,v}(\{v\})) = w \geq 0 \quad (3.39) \]

Thus, $w(\delta_{u,v}(\cdot))$ is submodular since $w(e) \geq 0$ and
\[ w(\delta_{u,v}(\{u\})) + w(\delta_{u,v}(\{v\})) \geq w(\delta_{u,v}(\{u, v\})) + w(\delta_{u,v}(\emptyset)) \quad (3.40) \]

- General non-negative weighted graph $G = (V, E, w)$, define $w(\delta(\cdot))$:
  \[ f(X) = w(\delta(X)) = \sum_{(u,v) \in E(G)} w(\delta_{u,v}(X \cap \{u, v\})) \quad (3.41) \]

- This is easily shown to be submodular using properties we will soon see (namely, submodularity closed under summation and restriction).
These come from Narayanan’s book 1997. Let $G$ be an undirected graph.

- Let $V(X)$ be the vertices adjacent to some edge in $X \subseteq E(G)$, then $|V(X)|$ (the vertex function) is submodular.
Other graph functions that are submodular/supermodular

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- Let $E(S)$ be the edges with both vertices in $S \subseteq V(G)$. Then $|E(S)|$ (the interior edge function) is supermodular.

$$g(S) = \sum_{i,j \in S} w_{i,j}$$
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- Let $E(S)$ be the edges with both vertices in $S \subseteq V(G)$. Then $|E(S)|$ (the interior edge function) is supermodular.
- Let $I(S)$ be the edges with at least one vertex in $S \subseteq V(G)$. Then $|I(S)|$ (the incidence function) is submodular.

\[
g(A) = \sum_{i,j \in A} w_{i,j}
\]

\[
h(A) = \sum_{i,j \in V} w_{i,j} - g(A)
\]

\[
|I(S)| = h(V \Delta A)
\]
Other graph functions that are submodular/supermodular

These come from Narayanan’s book 1997. Let $G$ be an undirected graph.

- Let $V(X)$ be the vertices adjacent to some edge in $X \subseteq E(G)$, then $|V(X)|$ (the vertex function) is submodular.
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- Let $I(S)$ be the edges with at least one vertex in $S \subseteq V(G)$. Then $|I(S)|$ (the incidence function) is submodular.
- Recall $|\delta(S)|$, is the number of edges with exactly one vertex in $S \subseteq V(G)$ is submodular (cut function). Thus, we have $I(S) = E(S) \cup \delta(S)$ and $E(S) \cap \delta(S) = \emptyset$, and thus that $|I(S)| = |E(S)| + |\delta(S)|$. 

![Diagram of graph with vertices and edges labeled S and \delta(S)]
Other graph functions that are submodular/supermodular

These come from Narayanan’s book 1997. Let $G$ be an undirected graph.

- Let $V(X)$ be the vertices adjacent to some edge in $X \subseteq E(G)$, then $|V(X)|$ (the vertex function) is submodular.
- Let $E(S)$ be the edges with both vertices in $S \subseteq V(G)$. Then $|E(S)|$ (the interior edge function) is supermodular.
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Other graph functions that are submodular/supermodular

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- Let $V(X)$ be the vertices adjacent to some edge in $X \subseteq E(G)$, then $|V(X)|$ (the vertex function) is submodular.

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Other graph functions that are submodular/supermodular

These come from Narayanan’s book 1997. Let $G$ be an undirected graph.

- Let $V(X)$ be the vertices adjacent to some edge in $X \subseteq E(G)$, then $|V(X)|$ (the vertex function) is **submodular**.
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- Consider $f(A) = |\delta^+(A)| - |\delta^+(V \setminus A)|$. Guess, submodular, supermodular, modular, or neither? Exercise: determine which one and prove it.
Recall, $f : 2^V \to \mathbb{R}$ is submodular, then so is $\bar{f} : 2^V \to \mathbb{R}$ defined as $\bar{f}(S) = f(V \setminus S)$.
Number of connected components in a graph via edges

- Recall, $f : 2^V \rightarrow \mathbb{R}$ is submodular, then so is $\bar{f} : 2^V \rightarrow \mathbb{R}$ defined as $\bar{f}(S) = f(V \setminus S)$.
- Hence, if $g : 2^V \rightarrow \mathbb{R}$ is supermodular, then so is $\bar{g} : 2^V \rightarrow \mathbb{R}$ defined as $\bar{g}(S) = g(V \setminus S)$. 
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- Hence, if $g : 2^V \to \mathbb{R}$ is supermodular, then so is $\overline{g} : 2^V \to \mathbb{R}$ defined as $\overline{g}(S) = g(V \setminus S)$.
- Given a graph $G = (V, E)$, for each $A \subseteq E(G)$, let $c(A)$ denote the number of connected components of the (spanning) subgraph $(V(G), A)$, with $c : 2^E \to \mathbb{R}_+$. Thus, $c(\emptyset) = |V|$ and $c(E) \geq 1$.
Number of connected components in a graph via edges

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- Hence, if \( g : 2^V \to \mathbb{R} \) is supermodular, then so is \( \overline{g} : 2^V \to \mathbb{R} \) defined as \( \overline{g}(S) = g(V \setminus S) \).
- Given a graph \( G = (V,E) \), for each \( A \subseteq E(G) \), let \( c(A) \) denote the number of connected components of the (spanning) subgraph \((V(G), A)\), with \( c : 2^E \to \mathbb{R}_+ \). Thus, \( c(\emptyset) = |V| \), and \( c(E) \geq 1 \).
- \( c(A) \) is monotone non-increasing, \( c(A + a) - c(A) \leq 0 \).
Number of connected components in a graph via edges

- Recall, $f : 2^V \to \mathbb{R}$ is submodular, then so is $\overline{f} : 2^V \to \mathbb{R}$ defined as $\overline{f}(S) = f(V \setminus S)$.
- Hence, if $g : 2^V \to \mathbb{R}$ is supermodular, then so is $\overline{g} : 2^V \to \mathbb{R}$ defined as $\overline{g}(S) = g(V \setminus S)$.
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- $c(A)$ is monotone non-increasing, $c(A + a) - c(A) \leq 0$.
- Then $c(A)$ is supermodular, i.e.,

$$c(A + a) - c(A) \leq c(B + a) - c(B) \leq \emptyset$$

with $A \subseteq B \subseteq E \setminus \{a\}$. 

Intuition: an edge is "more" (no less) able to bridge separate components (and reduce the number of connected components) when edge is added in a smaller context than when added in a larger context.
Number of connected components in a graph via edges

- Recall, \( f : 2^V \to \mathbb{R} \) is submodular, then so is \( \bar{f} : 2^V \to \mathbb{R} \) defined as \( \bar{f}(S) = f(V \setminus S) \).
- Hence, if \( g : 2^V \to \mathbb{R} \) is supermodular, then so is \( \bar{g} : 2^V \to \mathbb{R} \) defined as \( \bar{g}(S) = g(V \setminus S) \).
- Given a graph \( G = (V, E) \), for each \( A \subseteq E(G) \), let \( c(A) \) denote the number of connected components of the (spanning) subgraph \((V(G), A)\), with \( c : 2^E \to \mathbb{R}_+ \). Thus, \( c(\emptyset) = |V| \), and \( c(E) \geq 1 \).
- \( c(A) \) is monotone non-increasing, \( c(A + a) - c(A) \leq 0 \).
- Then \( c(A) \) is supermodular, i.e.,
  \[
  c(A + a) - c(A) \leq c(B + a) - c(B)
  \]
  with \( A \subseteq B \subseteq E \setminus \{a\} \).
- Intuition: an edge is “more” (no less) able to bridge separate components (and reduce the number of connected components) when edge is added in a smaller context than when added in a larger context.
Number of connected components in a graph via edges

- Recall, \( f : 2^V \rightarrow \mathbb{R} \) is submodular, then so is \( \bar{f} : 2^V \rightarrow \mathbb{R} \) defined as \( \bar{f}(S) = f(V \setminus S) \).
- Hence, if \( g : 2^V \rightarrow \mathbb{R} \) is supermodular, then so is \( \bar{g} : 2^V \rightarrow \mathbb{R} \) defined as \( \bar{g}(S) = g(V \setminus S) \).
- Given a graph \( G = (V, E) \), for each \( A \subseteq E(G) \), let \( c(A) \) denote the number of connected components of the (spanning) subgraph \( (V(G), A) \), with \( c : 2^E \rightarrow \mathbb{R}_+ \). Thus, \( c(\emptyset) = |V| \), and \( c(E) \geq 1 \).
- \( c(A) \) is monotone non-increasing, \( c(A + a) - c(A) \leq 0 \).
- Then \( c(A) \) is supermodular, i.e.,
  \[
  c(A + a) - c(A) \leq c(B + a) - c(B) \tag{3.42}
  \]
  with \( A \subseteq B \subseteq E \setminus \{a\} \).
- Intuition: an edge is “more” (no less) able to bridge separate components (and reduce the number of connected components) when edge is added in a smaller context than when added in a larger context.
- \( \bar{c}(A) = c(E \setminus A) \) is number of connected components in \( G \) when we remove \( A \); supermodular monotone non-decreasing but not normalized.
Graph Strength

- So $\bar{c}(A) = c(E \setminus A)$, the number of connected components in $G$ when we remove $A$, is supermodular.
Graph Strength

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- Maximizing $\bar{c}(A)$ would be a goal for a network attacker — many connected components means that many points in the network have lost connectivity to many other points (unprotected network).
Graph Strength

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- If we can remove a small set $A$ and shatter the graph into many connected components, then the graph is weak.
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- Let $G = (V, E, w)$ with $w : E \rightarrow \mathbb{R}^+$ be a weighted graph with non-negative weights.
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- If we can remove a small set $A$ and shatter the graph into many connected components, then the graph is weak.
- An attacker wishes to choose a small number of edges (since it is cheap) to shatter the graph into as many components as possible.
- Let $G = (V, E, w)$ with $w : E \rightarrow \mathbb{R}^+$ be a weighted graph with non-negative weights.
- For $(u, v) = e \in E$, let $w(e)$ be a measure of the strength of the connection between vertices $u$ and $v$ (strength meaning the difficulty of cutting the edge $e$).
Then \( w(A) \) for \( A \subseteq E \) is a modular function

\[
w(A) = \sum_{e \in A} w_e
\]

so that \( w(E(G[S])) \) is the “internal strength” of the vertex set \( S \).

**Notation:** \( S \) is a set of nodes, \( G[S] \) is the vertex-induced subgraph of \( G \) induced by vertices \( S \), \( E(G[S]) \) are the edges contained within this induced subgraph, and \( w(E(G[S])) \) is the weight of these edges. \( w(E(G[S])) = \sum_{i,j \in S} w(i,j) \).
Then \( w(A) \) for \( A \subseteq E \) is a modular function

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Suppose removing \( A \) shatters \( G \) into a graph with \( \bar{c}(A) > 1 \) components —
Then \( w(A) \) for \( A \subseteq E \) is a modular function

\[
  w(A) = \sum_{e \in A} w_e
\]  \hspace{1cm} (3.43)

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Suppose removing \( A \) shatters \( G \) into a graph with \( \bar{c}(A) > 1 \) components — then \( w(A)/(\bar{c}(A) - 1) \) is like the “effort per achieved/additional component” for a network attacker.
Graph Strength

- Then $w(A)$ for $A \subseteq E$ is a modular function

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(3.43)

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- Suppose removing $A$ shatters $G$ into a graph with $\bar{c}(A) > 1$ components — then $w(A)/(\bar{c}(A) - 1)$ is like the “effort per achieved/additional component” for a network attacker.

- A form of graph strength can then be defined as the following:

$$\text{strength}(G, w) \overset{\Delta}{=} \min_{A \subseteq E(G) : \bar{c}(A) > 1} \frac{w(A)}{\bar{c}(A) - 1}$$

(3.44)
Graph Strength

- Then \( w(A) \) for \( A \subseteq E \) is a modular function
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- Graph strength is like the minimum effort per component. An attacker would use the argument of the min to choose which edges to attack. A network designer would maximize, over \( G \) and/or \( w \), the graph strength, \( \text{strength}(G, w) \).
Graph Strength

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Graph strength is like the minimum effort per component. An attacker would use the argument of the min to choose which edges to attack. A network designer would maximize, over \( G \) and/or \( w \), the graph strength, \( \text{strength}(G, w) \).

- Since submodularity, problems have strongly-poly-time solutions.
Lemma 3.5.4

Let $M \in \mathbb{R}^{n \times n}$ be a symmetric matrix and $m \in \mathbb{R}^n$ be a vector. Then $f : 2^V \rightarrow \mathbb{R}$ defined as

$$f(X) = m^T 1_X + \frac{1}{2} 1_X^T M 1_X$$

(3.45)

is submodular iff the off-diagonal elements of $M$ are non-positive.

Proof.
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Proof.

- Given a complete graph $G = (V, E)$, recall that $E(X)$ is the edge set with both vertices in $X \subseteq V(G)$, and that $|E(X)|$ is supermodular.
Lemma 3.5.4

Let $\mathbf{M} \in \mathbb{R}^{n \times n}$ be a symmetric matrix and $\mathbf{m} \in \mathbb{R}^n$ be a vector. Then $f : 2^V \to \mathbb{R}$ defined as

$$f(X) = \mathbf{m}^\top \mathbf{1}_X + \frac{1}{2} \mathbf{1}_X^\top \mathbf{M} \mathbf{1}_X$$

(3.45)

is submodular iff the off-diagonal elements of $\mathbf{M}$ are non-positive.

Proof.

- Given a complete graph $G = (V, E)$, recall that $E(X)$ is the edge set with both vertices in $X \subseteq V(G)$, and that $|E(X)|$ is supermodular.
- Non-negative modular weights $w^+ : E \to \mathbb{R}_+$, $w(E(X))$ is also supermodular, so $-w(E(X))$ is submodular.
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- \( f \) is a modular function \( m^T 1_A = m(A) \) added to a weighted submodular function, hence \( f \) is submodular.
Proof of Lemma 3.5.4 cont.

- Conversely, suppose \( f \) is submodular.
Proof of Lemma 3.5.4 cont.

- Conversely, suppose $f$ is submodular.
- Then $\forall u, v \in V, f(\{u\}) + f(\{v\}) \geq f(\{u, v\}) + f(\emptyset)$ and $f(\emptyset) = 0$. 

This requires:

$$0 \leq f(\{u\}) + f(\{v\}) - f(\{u, v\}) - f(\emptyset) = m(u) + \frac{1}{2}M_{u,u} + m(v) + \frac{1}{2}M_{v,v} - f(\{u, v\})$$

$$\leq m(u) + \frac{1}{2}M_{u,v} + m(v)$$

Therefore:

$$m(u) + m(v) + \frac{1}{2}M_{u,v} \geq f(\{u, v\})$$

So that $\forall u, v \in V, M_{u,v} \leq 0$. 

Proof of Lemma 3.5.4 cont.
Conversely, suppose \( f \) is submodular. Then \( \forall u, v \in V, f(\{u\}) + f(\{v\}) \geq f(\{u, v\}) + f(\emptyset) \) and \( f(\emptyset) = 0 \).

This requires:

\[
0 \leq f(\{u\}) + f(\{v\}) - f(\{u, v\}) = m(u) + \frac{1}{2} M_{u,u} + m(v) + \frac{1}{2} M_{v,v}
\]

\[
- \left( m(u) + m(v) + \frac{1}{2} M_{u,u} + M_{u,v} + \frac{1}{2} M_{v,v} \right)
\]

\[
= -M_{u,v}
\]

So that \( \forall u, v \in V, M_{u,v} \leq 0 \).
Set Cover and Maximum Coverage
just Special cases of Submodular Optimization

- We are given a finite set $U$ of $m$ elements and a set of subsets $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ of $n$ subsets of $U$, so that $U_i \subseteq U$ and $\bigcup_i U_i = U$. 
Set Cover and Maximum Coverage
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- The goal of minimum set cover is to choose the smallest subset $A \subseteq [n] \triangleq \{1, \ldots, n\}$ such that $\bigcup_{a \in A} U_a = U$. 
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- The goal of minimum set cover is to choose the smallest subset $A \subseteq [n] \triangleq \{1, \ldots, n\}$ such that $\bigcup_{a \in A} U_a = U$.

- Maximum $k$ cover: The goal in maximum coverage is, given an integer $k \leq n$, select $k$ subsets, say $\{a_1, a_2, \ldots, a_k\}$ with $a_i \in [n]$ such that $|\bigcup_{i=1}^k U_{a_i}|$ is maximized.
We are given a finite set \( U \) of \( m \) elements and a set of subsets \( \mathcal{U} = \{U_1, U_2, \ldots, U_n\} \) of \( n \) subsets of \( U \), so that \( U_i \subseteq U \) and \( \bigcup_i U_i = U \).

The goal of **minimum set cover** is to choose the smallest subset \( A \subseteq [n] \triangleq \{1, \ldots, n\} \) such that \( \bigcup_{a \in A} U_a = U \).

**Maximum \( k \)-cover**: The goal in **maximum coverage** is, given an integer \( k \leq n \), select \( k \) subsets, say \( \{a_1, a_2, \ldots, a_k\} \) with \( a_i \in [n] \) such that \( \left| \bigcup_{i=1}^k U_{a_i} \right| \) is maximized.

\( f : 2^{[n]} \to \mathbb{Z}_+ \) where for \( A \subseteq [n] \), \( f(A) = \left| \bigcup_{a \in A} U_a \right| \) is the set cover function and is submodular.
Set Cover and Maximum Coverage
just Special cases of Submodular Optimization

- We are given a finite set $U$ of $m$ elements and a set of subsets $U = \{U_1, U_2, \ldots, U_n\}$ of $n$ subsets of $U$, so that $U_i \subseteq U$ and $\bigcup_i U_i = U$.

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- $f : 2^{[n]} \rightarrow \mathbb{Z}_+$ where for $A \subseteq [n]$, $f(A) = |\bigcup_{a \in A} U_a|$ is the set cover function and is submodular.

- Weighted set cover: $f(A) = w(\bigcup_{a \in A} U_a)$ where $w : U \rightarrow \mathbb{R}_+$. 
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- We are given a finite set $U$ of $m$ elements and a set of subsets $\mathcal{U} = \{U_1, U_2, \ldots, U_n\}$ of $n$ subsets of $U$, so that $U_i \subseteq U$ and $\bigcup_i U_i = U$.

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- $f : 2^{[n]} \rightarrow \mathbb{Z}_+$ where for $A \subseteq [n]$, $f(A) = |\bigcup_{a \in A} U_a|$ is the set cover function and is submodular.

- Weighted set cover: $f(A) = w(\bigcup_{a \in A} U_a)$ where $w : U \rightarrow \mathbb{R}_+$.

- Both Set cover and maximum coverage are well known to be NP-hard, but have a fast greedy approximation algorithm, and hence are instances of submodular optimization.
Vertex and Edge Covers
Also instances of submodular optimization

Definition 3.5.5 (vertex cover)

A vertex cover (a “vertex-based cover of edges”) in graph $G = (V, E)$ is a set $S \subseteq V(G)$ of vertices such that every edge in $G$ is incident to at least one vertex in $S$.

Let $I(S)$ be the number of edges incident to vertex set $S$. Then we wish to find the smallest set $S \subseteq V$ subject to $I(S) = |E|$.

Definition 3.5.6 (edge cover)

A edge cover (an “edge-based cover of vertices”) in graph $G = (V, E)$ is a set $F \subseteq E(G)$ of edges such that every vertex in $G$ is incident to at least one edge in $F$.

Let $|V|(F)$ be the number of vertices incident to edge set $F$. Then we wish to find the smallest set $F \subseteq E$ subject to $|V|(F) = |V|$.
Graph Cut Problems
Also submodular optimization

- Minimum cut: Given a graph $G = (V, E)$, find a set of vertices $S \subseteq V$ that minimize the cut (set of edges) between $S$ and $V \setminus S$. 
Graph Cut Problems
Also submodular optimization

- **Minimum cut**: Given a graph \( G = (V, E) \), find a set of vertices \( S \subseteq V \) that minimize the cut (set of edges) between \( S \) and \( V \setminus S \).

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Graph Cut Problems
Also submodular optimization

- **Minimum cut**: Given a graph $G = (V, E)$, find a set of vertices $S \subseteq V$ that minimize the cut (set of edges) between $S$ and $V \setminus S$.

- **Maximum cut**: Given a graph $G = (V, E)$, find a set of vertices $S \subseteq V$ that maximize the cut (set of edges) between $S$ and $V \setminus S$.

- Let $\delta : 2^V \to \mathbb{R}_+$ be the cut function, namely for any given set of nodes $X \subseteq V$, $|\delta(X)|$ measures the number of edges between nodes $X$ and $V \setminus X$ — i.e., $\delta(x) = E(X, V \setminus X)$. 

Weighted versions, where rather than count, we sum the (non-negative) weights of the edges of a cut, $f(X) = \sum_{e \in E(X, V \setminus X)} w(e)$.

Hence, Minimum cut and Maximum cut are also special cases of submodular optimization.
Graph Cut Problems
Also submodular optimization

- Minimum cut: Given a graph $G = (V, E)$, find a set of vertices $S \subseteq V$ that minimize the cut (set of edges) between $S$ and $V \setminus S$.

- Maximum cut: Given a graph $G = (V, E)$, find a set of vertices $S \subseteq V$ that maximize the cut (set of edges) between $S$ and $V \setminus S$.

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Graph Cut Problems
Also submodular optimization

- **Minimum cut**: Given a graph $G = (V, E)$, find a set of vertices $S \subseteq V$ that minimize the cut (set of edges) between $S$ and $V \setminus S$.

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- Weighted versions, where rather than count, we sum the (non-negative) weights of the edges of a cut, $f(X) = w(\delta(X))$.

- Hence, Minimum cut and Maximum cut are also special cases of submodular optimization.
Matrix Rank functions

- Let $V$, with $|V| = m$ be an index set of a set of vectors in $\mathbb{R}^n$ for some $n$ (unrelated to $m$). Thus, $\forall v \in V$, $\exists x_v \in \mathbb{R}^n$. 
Matrix Rank functions

- Let $V$, with $|V| = m$ be an index set of a set of vectors in $\mathbb{R}^n$ for some $n$ (unrelated to $m$). Thus, $\forall v \in V, \exists x_v \in \mathbb{R}^n$.

- For a given set $\{v, v_1, v_2, \ldots, v_k\}$, it might or might not be possible to find $(\alpha_i)_i$ such that:

$$x_v = \sum_{i=1}^{k} \alpha_i x_{v_i}$$  \hspace{1cm} (3.50)

If not, then $x_v$ is **linearly independent** of $x_{v_1}, \ldots, x_{v_k}$.
Matrix Rank functions

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(3.50)

If not, then $x_v$ is **linearly independent** of $x_{v_1}, \ldots, x_{v_k}$.

- Let $r(S)$ for $S \subseteq V$ be the rank of the set of vectors $S$. Then $r(\cdot)$ is a submodular function, and in fact is called a **matric matroid rank function**.
Example: Rank function of a matrix

Given $n \times m$ matrix $X = (x_1, x_2, \ldots, x_m)$ with $x_i \in \mathbb{R}^n$ for all $i$. There are $m$ length-$n$ column vectors $\{x_i\}_i$. 

Skip matrix rank example
Example: Rank function of a matrix

- Given \( n \times m \) matrix \( X = (x_1, x_2, \ldots, x_m) \) with \( x_i \in \mathbb{R}^n \) for all \( i \). There are \( m \) length-\( n \) column vectors \( \{x_i\}_i \).
- Let \( V = \{1, 2, \ldots, m\} \) be the set of column vector indices.

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Example: Rank function of a matrix

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- Let $V = \{1, 2, \ldots, m\}$ be the set of column vector indices.
- For any $A \subseteq V$, let $r(A)$ be the rank of the column vectors indexed by $A$. 
Example: Rank function of a matrix

- Given \( n \times m \) matrix \( X = (x_1, x_2, \ldots, x_m) \) with \( x_i \in \mathbb{R}^n \) for all \( i \). There are \( m \) length-\( n \) column vectors \( \{x_i\}_i \).
- Let \( V = \{1, 2, \ldots, m\} \) be the set of column vector indices.
- For any \( A \subseteq V \), let \( r(A) \) be the rank of the column vectors indexed by \( A \).
- \( r(A) \) is the dimensionality of the vector space spanned by the set of vectors \( \{x_a\}_{a \in A} \).

▶ Skip matrix rank example
Example: Rank function of a matrix

Given $n \times m$ matrix $X = (x_1, x_2, \ldots, x_m)$ with $x_i \in \mathbb{R}^n$ for all $i$. There are $m$ length-$n$ column vectors $\{x_i\}_i$.

Let $V = \{1, 2, \ldots, m\}$ be the set of column vector indices.

For any $A \subseteq V$, let $r(A)$ be the rank of the column vectors indexed by $A$.

$r(A)$ is the dimensionality of the vector space spanned by the set of vectors $\{x_a\}_{a \in A}$.

Thus, $r(V)$ is the rank of the matrix $X$. 

Skip matrix rank example
Example: Rank function of a matrix

Consider the following $4 \times 8$ matrix, so $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 2 & 2 & 3 & 0 & 1 & 3 & 1 \\
2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5
\end{pmatrix}
= \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
| & | & | & | & | & | & | & | \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8
\end{pmatrix}
\]

- Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{6, 7\}$, $A_r = \{1\}$, $B_r = \{5\}$.
- Then $r(A) = 3$, $r(B) = 3$, $r(C) = 2$.
- $r(A \cup C) = 3$, $r(B \cup C) = 3$.
- $r(A \cup A_r) = 3$, $r(B \cup B_r) = 3$, $r(A \cup B_r) = 4$, $r(B \cup A_r) = 4$.
- $r(A \cup B) = 4$, $r(A \cap B) = 1 < r(C) = 2$. 


Example: Rank function of a matrix

Consider the following $4 \times 8$ matrix, so $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 2 & 2 & 3 & 0 & 1 & 3 & 1 \\
2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5
\end{pmatrix}
= \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\{x_1\} & \{x_2\} & \{x_3\} & \{x_4\} & \{x_5\} & \{x_6\} & \{x_7\} & \{x_8\}
\end{pmatrix}
\]

- Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{6, 7\}$, $A_r = \{1\}$, $B_r = \{5\}$.
- Then $r(A) = 3$, $r(B) = 3$, $r(C) = 2$.
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\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
0 & 2 & 2 & 3 & 0 & 1 & 3 & 1 \\
0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 5
\end{pmatrix}
= \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
& & & & & & & \\
& x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8
\end{pmatrix}
\]

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2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\x_1 & \x_2 & \x_3 & \x_4 & \x_5 & \x_6 & \x_7 & \x_8 \\
\end{pmatrix}
\]

Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{6, 7\}$, $A_r = \{1\}$, $B_r = \{5\}$.

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2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\vert & \vert & \vert & \vert & \vert & \vert & \vert & \vert & \vert \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
\vert & \vert & \vert & \vert & \vert & \vert & \vert & \vert & \vert \\
\end{pmatrix}
$$

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2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
| & | & | & | & | & | & | \\
\end{pmatrix}
\begin{pmatrix}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
| & | & | & | & | & | & | \\
\end{pmatrix}
\]

- Let $A = \{1, 2, 3\}, B = \{3, 4, 5\}, C = \{6, 7\}, A_r = \{1\}, \quad B_r = \{5\}$.
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\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 2 & 2 & 3 & 0 & 1 & 3 & 1 \\
2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\vert & \vert & \vert & \vert & \vert & \vert & \vert & \vert & \vert \\
\times_1 & \times_2 & \times_3 & \times_4 & \times_5 & \times_6 & \times_7 & \times_8 \\
\vert & \vert & \vert & \vert & \vert & \vert & \vert & \vert & \vert \\
\end{bmatrix}
$$

- Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{6, 7\}$, $A_r = \{1\}$, $B_r = \{5\}$.
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\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 2 & 2 & 3 & 0 & 1 & 3 & 1 \\
2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\| & \| & \| & \| & \| & \| & \| & \| & \| \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
\| & \| & \| & \| & \| & \| & \| & \| & \| \\
\end{pmatrix}
\]

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Prof. Jeff Bilmes

EE563/Spring 2020/Submodularity - Lecture 3 - Oct 7th, 2020

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Example: Rank function of a matrix

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$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 2 & 2 & 3 & 0 & 1 & 3 & 1 \\
2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
\end{pmatrix}
$$

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2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & | & | & | & | & | & | & | \\
\times_1 & \times_2 & \times_3 & \times_4 & \times_5 & \times_6 & \times_7 & \times_8 \\
1 & | & | & | & | & | & | & | \\
\end{pmatrix}
$$

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2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 5 \\
\end{pmatrix} =
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
| & | & | & | & | & | & | \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
| & | & | & | & | & | & | \\
\end{pmatrix}
$$

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Prof. Jeff Bilmes
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2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5
\end{pmatrix}
= \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\mid & \mid & \mid & \mid & \mid & \mid & \mid & \mid & \mid \\
X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \\
\mid & \mid & \mid & \mid & \mid & \mid & \mid & \mid & \mid
\end{pmatrix}
\]

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\begin{pmatrix}
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1 & 0 & 2 & 2 & 3 & 0 & 1 & 3 & 1 \\
2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5
\end{pmatrix}
= 
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\| & \| & \| & \| & \| & \| & \| & \| & \| & \| & \| & \| & \|
\end{pmatrix}
\begin{pmatrix}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8
\end{pmatrix}
$$

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2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
\end{pmatrix}
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2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\text{\textcolor{red}{x}}_1 & \text{\textcolor{red}{x}}_2 & \text{\textcolor{red}{x}}_3 & \text{\textcolor{red}{x}}_4 & \text{\textcolor{red}{x}}_5 & \text{\textcolor{red}{x}}_6 & \text{\textcolor{red}{x}}_7 & \text{\textcolor{red}{x}}_8 \\
\end{pmatrix}
$$

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2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
\end{pmatrix}
= \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & \vert & \vert & \vert & \vert & \vert & \vert & \vert & \vert \\
2 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
3 & \vert & \vert & \vert & \vert & \vert & \vert & \vert & \vert \\
4 & \vert & \vert & \vert & \vert & \vert & \vert & \vert & \vert \\
\end{pmatrix}
\]

- Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{6, 7\}$, $A_r = \{1\}$, $B_r = \{5\}$.
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- $r(A \cup C') = 3$, $r(B \cup C') = 3$.
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\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 0 & 2 & 3 & 0 & 1 & 3 & 1 \\
2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5
\end{pmatrix}
= 
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8
\end{pmatrix}
\]

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2 & 0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
3 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 5
\end{pmatrix}
= \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\left| \begin{array}{ccccccc}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8
\end{array} \right|
\end{pmatrix}
\]

- Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{6, 7\}$, $A_r = \{1\}$, $B_r = \{5\}$.
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\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
0 & 2 & 2 & 3 & 0 & 1 & 3 & 1 \\
0 & 3 & 0 & 4 & 0 & 0 & 2 & 4 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 5 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 5
\end{pmatrix}
= 
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x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8
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- Then $r(A) = 3$, $r(B) = 3$, $r(C) = 2$.
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- $r(A \cup A_r) = 3$, $r(B \cup B_r) = 3$, $r(A \cup B_r) = 4$, $r(B \cup A_r) = 4$.
- $r(A \cup B) = 4$, $r(A \cap B) = 1 < r(C) = 2$.
- $6 = r(A) + r(B) = r(A \cup B) + r(C) > r(A \cup B) + r(A \cap B) = 5$.
Rank function of a matrix

- Let $A, B \subseteq V$ be two subsets of column indices.
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- If some of the dimensions spanned by $A$ overlap some of the dimensions spanned by $B$ (i.e., if $\exists$ common span), then that area is counted twice in $r(A) + r(B)$, so the inequality will be strict.
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- If some of the dimensions spanned by $A$ overlap some of the dimensions spanned by $B$ (i.e., if $\exists$ common span), then that area is counted twice in $r(A) + r(B)$, so the inequality will be strict.
- Any function where the above inequality is true for all $A, B \subseteq V$ is called subadditive.
Vector sets $A$ and $B$ have a (possibly empty) common span and two (possibly empty) non-common residual spans.
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- Let $A_r$ index vectors spanning dimensions spanned by $A$ but not $B$. 

\[ r(A) = r(A_r) + r(C) + r(B_r) \]
\[ r(B) = r(C) + r(A_r) + r(B_r) \] (3.51)

But $r(A \cup B)$ counts the dimensions spanned by $C$ only once.

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Then, $r(A) = r(C) + r(A_r)$

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- Similarly, $r(B) = r(C) + r(B_r)$. 

\[ r(A) + r(B) \text{ counts the dimensions spanned by } C \text{ twice, i.e., } r(A) + r(B) = r(A_r) + 2r(C) + r(B_r). \] 

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Rank functions of a matrix

- Then $r(A) + r(B)$ counts the dimensions spanned by $C$ twice, i.e.,
  $$r(A) + r(B) = r(AR) + 2r(C) + r(BR)$$

- But $r(A \cup B)$ counts the dimensions spanned by $C$ only once.
  $$r(A \cup B) = r(AR) + r(C) + r(BR)$$
Rank functions of a matrix

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  $$r(A) + r(B) = r(A_r) + 2r(C) + r(B_r)$$

- But $r(A \cup B)$ counts the dimensions spanned by $C$ only once.
  
  $$r(A \cup B) = r(A_r) + r(C) + r(B_r)$$

- Thus, we have **subadditivity**: $r(A) + r(B) \geq r(A \cup B)$. Can we add more to the r.h.s. and still have an inequality? Yes.
Rank function of a matrix

Note, \( r(A \cap B) \leq r(C) \). Why? Vectors indexed by \( A \cap B \) (i.e., the common index set) span no more than the dimensions commonly spanned by \( A \) and \( B \) (namely, those spanned by the professed \( C \)).

\[
r(C) \geq r(A \cap B)
\]

In short:
Rank function of a matrix

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In short:
- Common span (blue) is “more” (no less) than span of common index (magenta).
Note, $r(A \cap B) \leq r(C)$. Why? Vectors indexed by $A \cap B$ (i.e., the common index set) span no more than the dimensions commonly spanned by $A$ and $B$ (namely, those spanned by the professed $C$).

$$r(C) \geq r(A \cap B)$$

In short:

- Common span (blue) is “more” (no less) than span of common index (magenta).
- More generally, common information (blue) is “more” (no less) than information within common index (magenta).
The Venn and Art of Submodularity

\[ r(A) + r(B) \geq r(A \cup B) + r(A \cap B) \]

\[ = r(A_r) + 2r(C) + r(B_r) \]

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\[ = r(A \cap B) \]

\[ r(A) + r(B) \]

\[ r(A[B) = r(Ar) + r(C) + r(Br) \]

Prof. Jeff Bilmes
EE563/Spring 2020/Submodularity - Lecture 3 - Oct 7th, 2020
F47/62 (pg.158/211)
Polymatroid rank function

- Let $S$ be a set of subspaces of a linear space (i.e., each $s \in S$ is a subspace of dimension $\geq 1$).
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- For each $X \subseteq S$, let $f(X)$ denote the dimensionality of the linear subspace spanned by the subspaces in $X$.
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- We can think of $S$ as a set of sets of vectors from the matrix rank example, and for each $s \in S$, let $X_s$ being a set of vector indices.
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- We can think of $S$ as a set of sets of vectors from the matrix rank example, and for each $s \in S$, let $X_s$ being a set of vector indices.
- Then, defining $f : 2^S \to \mathbb{R}_+$ as follows,

$$f(X) = r(\bigcup_{s \in X} X_s) \quad (3.53)$$

we have that $f$ is submodular, and is known to be a polymatroid rank function.
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- In general (as we will see) polymatroid rank functions are submodular, normalized $f(\emptyset) = 0$, and monotone non-decreasing ($f(A) \leq f(B)$ whenever $A \subseteq B$).
- We use the term non-decreasing rather than increasing, the latter of which is strict (also so that a constant function isn’t “increasing”).
Spanning trees

- Let $E$ be a set of edges of some graph $G = (V, E)$, and let $r(S)$ for $S \subseteq E$ be the maximum size (in terms of number of edges) spanning forest in the graph induced by edges $S$. 
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- Example: Given $G = (V, E)$, $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $E = \{1, 2, \ldots, 12\}$. $S = \{1, 2, 3, 4, 5, 8, 9\} \subseteq E$. Two spanning trees have the same edge count (the rank of $S$).
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- Then $r(S)$ is submodular, and is another matrix rank function corresponding to the incidence matrix of the graph.
Given $E$, let $f_1, f_2 : 2^E \rightarrow \mathbb{R}$ be two submodular functions. Then

$$f : 2^E \rightarrow \mathbb{R} \text{ with } f(A) = f_1(A) + f_2(A)$$

(3.58)

is submodular.
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is submodular. This follows easily since

$$f(A) + f(B) = f_1(A) + f_2(A) + f_1(B) + f_2(B)$$

(3.59)

$$\geq f_1(A \cup B) + f_2(A \cup B) + f_1(A \cap B) + f_2(A \cap B)$$

(3.60)

$$= f(A \cup B) + f(A \cap B).$$

(3.61)

I.e., it holds for each component of $f$ in each term in the inequality.
Summing Submodular Functions

Given $E$, let $f_1, f_2 : 2^E \to \mathbb{R}$ be two submodular functions. Then

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$$= f(A \cup B) + f(A \cap B).$$

I.e., it holds for each component of $f$ in each term in the inequality. In fact, any conic combination (i.e., non-negative linear combination) of submodular functions is submodular, as in $f(A) = \alpha_1 f_1(A) + \alpha_2 f_2(A)$ for $\alpha_1, \alpha_2 \geq 0$. 
Given $E$, let $f_1, m : 2^E \to \mathbb{R}$ be a submodular and a modular function.
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$$f : 2^E \to \mathbb{R} \text{ with } f(A) = f_1(A) - m(A)$$

is submodular (as is $f(A) = f_1(A) + m(A)$).
Summing Submodular and Modular Functions

Given $E$, let $f_1, m : 2^E \to \mathbb{R}$ be a submodular and a modular function. Then

$$f : 2^E \to \mathbb{R} \text{ with } f(A) = f_1(A) - m(A) \quad (3.62)$$

is submodular (as is $f(A) = f_1(A) + m(A)$). This follows easily since

$$f(A) + f(B) = f_1(A) - m(A) + f_1(B) - m(B) \geq f_1(A \cup B) - m(A \cup B) + f_1(A \cap B) - m(A \cap B) \quad (3.64)$$

$$= f(A \cup B) + f(A \cap B). \quad (3.65)$$
Given $E$, let $f_1, m : 2^E \to \mathbb{R}$ be a submodular and a modular function. Then

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$$= f(A \cup B) + f(A \cap B). \quad (3.65)$$

That is, the modular component with

$$m(A) + m(B) = m(A \cup B) + m(A \cap B)$$

never destroys the inequality. Note of course that if $m$ is modular then so is $-m$. 


Restricting Submodular functions

Given $E$, let $f : 2^E \to \mathbb{R}$ be a submodular functions. And let $S \subseteq E$ be an arbitrary fixed set. Then

$$f' : 2^E \to \mathbb{R} \text{ with } f'(A) \triangleq f(A \cap S)$$

is submodular.
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Proof.
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**Proof.**

Given $A \subseteq B \subseteq E \setminus v$, consider

$$f((A + v) \cap S') - f(A \cap S') \geq f((B + v) \cap S') - f(B \cap S')$$

(3.67)
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$$f((A + v) \cap S) - f(A \cap S) \geq f((B + v) \cap S) - f(B \cap S) \quad (3.67)$$

If $v \notin S$, then both differences on each size are zero. If $v \in S$, then we can consider this

$$f(A' + v) - f(A') \geq f(B' + v) - f(B') \quad (3.68)$$

with $A' = A \cap S$ and $B' = B \cap S$. Since $A' \subseteq B'$, this holds due to submodularity of $f$. 

□
Given $V$, let $f_1, f_2 : 2^V \rightarrow \mathbb{R}$ be two submodular functions and let $S_1, S_2 \subseteq V$ be two arbitrary fixed sets. Then

$$f : 2^V \rightarrow \mathbb{R} \text{ with } f(A) = f_1(A \cap S_1) + f_2(A \cap S_2)$$

is submodular. This follows easily from the preceding two results.
Given $V$, let $f_1, f_2 : 2^V \to \mathbb{R}$ be two submodular functions and let $S_1, S_2 \subseteq V$ be two arbitrary fixed sets. Then

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is submodular. This follows easily from the preceding two results.

Given $V$, let $C = \{C_1, C_2, \ldots, C_k\}$ be a set of subsets of $V$, and for each $C \in C$, let $f_C : 2^V \to \mathbb{R}$ be a submodular function. Then

$$f : 2^V \to \mathbb{R} \text{ with } f(A) = \sum_{C \in C} f_C(A \cap C') \quad (3.70)$$

is submodular.
Given $V$, let $f_1, f_2 : 2^V \to \mathbb{R}$ be two submodular functions and let $S_1, S_2 \subseteq V$ be two arbitrary fixed sets. Then

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Given $V$, let $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ be a set of subsets of $V$, and for each $C \in \mathcal{C}$, let $f_C : 2^V \to \mathbb{R}$ be a submodular function. Then

$$f : 2^V \to \mathbb{R} \text{ with } f(A) = \sum_{C \in \mathcal{C}} f_C(A \cap C) \quad (3.70)$$

is submodular. This property is critical for image processing and graphical models. For example, let $\mathcal{C}$ be all pairs of the form $\{\{u, v\} : u, v \in V\}$, or let it be all pairs corresponding to the edges of some undirected graphical model.
Max - normalized

Given $V$, let $c \in \mathbb{R}^V_+$ be a given fixed vector. Then $f : 2^V \rightarrow \mathbb{R}_+$, where

$$f(A) = \max_{j \in A} c_j$$

is submodular and normalized (we take $f(\emptyset) = 0$).

**Proof.**

Consider

$$\max_{j \in A} c_j + \max_{j \in B} c_j \geq \max_{j \in A \cup B} c_j + \max_{j \in A \cap B} c_j$$

which follows since we have that

$$\max(\max_{j \in A} c_j, \max_{j \in B} c_j) = \max_{j \in A \cup B} c_j$$

and

$$\min(\max_{j \in A} c_j, \max_{j \in B} c_j) \geq \max_{j \in A \cap B} c_j$$
Given $V$, let $c \in \mathbb{R}^V$ be a given fixed vector (not necessarily non-negative). Then $f : 2^V \to \mathbb{R}$, where

$$f(A) = \max_{j \in A} c_j$$

(3.75)

is submodular, where we take $f(\emptyset) \leq \min_j c_j$ (so the function need not be normalized).

**Proof.**

The proof is identical to the normalized case.
Facility/Plant Location (uncapacitated)

- Core problem in operations research, early motivation for submodularity.
- Goal: as efficiently as possible, place “facilities” (factories) at certain locations to satisfy sites (at all locations) having various demands.
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We can model this with a weighted bipartite graph \( G = (F, S, E, c) \) where \( F \) is set of possible factory/plant locations, \( S \) is set of sites needing service, \( E \) are edges indicating (factory, site) service possibility pairs, and \( c : E \to \mathbb{R}_+ \) is the benefit of a given pair.

Facility location function has form:

\[
f(A) = \sum_{i \in S} \max_{j \in A} c_{ij}. \tag{3.76}
\]
Let $F = \{1, \ldots, f\}$ be a set of possible factory/plant locations for facilities to be built.
Facility/Plant Location (uncapacitated) w. plant benefits

- Let $F = \{1, \ldots, f\}$ be a set of possible factory/plant locations for facilities to be built.
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We can define the (uncapacitated) facility location function

$$f(A) = \sum_{j \in A} m_j + \sum_{i \in S} \max_{j \in A} c_{ij}.$$  \hspace{1cm} (3.77)
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- Goal is to find a set $A$ that maximizes $f(A)$ (the benefit) placing a bound on the number of plants $A$ (e.g., $|A| \leq k$).
Facility Location

Given $V, E$, let $c \in \mathbb{R}^{V \times E}$ be a given $|V| \times |E|$ matrix. Then

$$f : 2^E \to \mathbb{R}, \text{ where } f(A) = \sum_{i \in V} \max_{j \in A} c_{ij}$$

(3.78)

is submodular.

**Proof.**

We can write $f(A)$ as $f(A) = \sum_{i \in V} f_i(A)$ where $f_i(A) = \max_{j \in A} c_{ij}$ is submodular (max of a $i^{th}$ row vector), so $f$ can be written as a sum of submodular functions.

Thus, the facility location function (which only adds a modular function to the above) is submodular.
Let $\Sigma$ be an $n \times n$ positive definite matrix. Let $V = \{1, 2, \ldots, n\} \equiv [n]$ be an index set, and for $A \subseteq V$, let $\Sigma_A$ be the (square) submatrix of $\Sigma$ obtained by including only entries in the rows/columns given by $A$. 

We have that:

$$f(A) = \log \det(\Sigma_A)$$

is submodular. (3.79)

The submodularity of the log determinant is crucial for determinantal point processes (DPPs) (defined later in the class).

Proof of submodularity of the logdet function.

Suppose $X \sim \mathcal{N}(\mu, \Sigma)$ is a multivariate Gaussian random variable, that is

$$p(x) = \frac{1}{\sqrt{|2\pi \Sigma|}} \exp\left(-\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu)\right).$$

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Then the (differential) entropy of the r.v. $X$ is given by

$$h(X) = \log \sqrt{|2\pi e\Sigma|} = \log \sqrt{(2\pi e)^n |\Sigma|} \quad (3.81)$$

and in particular, for a variable subset $A$,

$$f(A) = h(X_A) = \log \sqrt{(2\pi e)^{|A|} |\Sigma_A|} \quad (3.82)$$

Entropy is submodular (further conditioning reduces entropy), and moreover

$$f(A) = h(X_A) = m(A) + \frac{1}{2} \log |\Sigma_A| \quad (3.83)$$

where $m(A)$ is a modular function.

Note: still submodular in the semi-definite case as well.
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- SCCM is submodular $f(A) = \sum_{u \in U} \phi_u(m_u(A))$ where $m_u$ is a non-negative modular and $\phi_u$ is concave.
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- Restrictions preserve submodularity: $f'(A) = f(A \cap S)$.