Submodular Functions, Optimization, and Applications to Machine Learning
— Fall Quarter, Lecture 16 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/

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Nov 25th, 2020

\[ f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \]

Clockwise from top left:
László Lovász
Jack Edmonds
Satoru Fujishige
George Nemhauser
Laurence Wolsey
András Frank
Lloyd Shapley
H. Narayanan
Robert Bixby
William Cunningham
William Tutte
Richard Rado
Alexander Schrijver
Garrett Birkhoff
Hassler Whitney
Richard Dedekind
Announcements, Assignments, and Reminders

- Homework 3, due Wednesday (tonight), Nov 25th, 2020, 11:59pm.
- Office hours this week, Monday (11/23), Tues (11/24), & Wed (11/25), 10:00pm at our class zoom link. I can meet Monday night at 10:00pm as well on request.
- Happy Thanksgiving!
Class Road Map - EE563

- L1(9/30): Motivation, Applications, Definitions, Properties
- L2(10/5): Sums concave(modular), uses (diversity/costs, feature selection), information theory
- L3(10/7): Monge, More Definitions, Graph and Combinatorial Examples,
- L4(10/12): Graph & Combinatorial Examples, Matrix Rank, Properties, Other Defs, Independence
- L5(10/14): Properties,Defs of Submodularity, Independence
- L6(10/19): Matroids, Matroid Examples, Matroid Rank,
- L8(10/26): Transversal Matroid, Matroid and representation, Dual Matroid
- L9(10/28): Other Matroid Properties, Combinatorial Geometries, Matroid and Greedy, Polyhedra, Matroid Polytopes
- L10(11/2): Matroid Polytopes, Matroids → Polymatroids
- L11(11/4): Matroids → Polymatroids, Polymatroids
- L12(11/9): Polymatroids, Polymatroids and Greedy
- L13(11/11): Veterans Day, Holiday
- L14(11/16): Polymatroids and Greedy, Possible Polytopes, Extreme Points, Cardinality Constrained Maximization
- L15(11/18): Cardinality Constrained Maximization, Curvature
- L16(11/23): Curvature, Submodular Max w. Other Constraints, Start Cont. Extensions
- L17(11/25): Submodular Max w. Other Constraints, Cont. Extensions, Lovász extension
- L18(12/2):
- L19(12/7):
- L20(12/9):
- L21(12/14): final meeting (presentations) maximization.

Last day of instruction, Fri. Dec 11th. Finals Week: Dec 12-18, 2020
Rest of class

- Homework 4 will come out later this week, will be due about 1.5-2 weeks after that.

- Final project: Read and present a recent (past 5 years) paper on submodular/supermodular optimization. Paper should have both a theoretical and practical component. What is due: (1) 4-page paper summary, and (2) 10 minute presentation about the paper, will be giving presentations on Monday 12/14/2020. You must choose your paper before the 14th (this will be HW5), and you must turn in your slides and 4-page paper (this will be HW6).
We’ve been discussing algorithms and approximations for $\max_{A \in C} f(A)$ where $f$ is a polymatroid function and $C$ is a constraint set (e.g., cardinality, matroid, multiple matroids, knapsacks) as well as other problems (e.g., submodular welfare problem).
What About Non-monotone

- We sometimes wish to maximize non-monotone submodular functions. This includes of course graph cuts, and this problem is APX-hard, so maximizing non-monotone functions, even unconstrainedly, is hard.
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- Therefore, submodular function max in such case is inapproximable unless P=NP (since any such procedure would give us the sign of the max).
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- Therefore, submodular function max in such case is inapproximable unless $P=NP$ (since any such procedure would give us the sign of the max).

- Thus, any approximation algorithm must be for unipolar submodular functions. E.g., non-negative but otherwise arbitrary submodular functions.
Submodularity and local optima

Given any submodular function $f$, a set $S \subseteq V$ is a local maximum of $f$ if $f(S - v) \leq f(S)$ for all $v \in S$ and $f(S + v) \leq f(S)$ for all $v \in V \setminus S$ (i.e., local in a Hamming ball of radius 1).
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Idea of proof: Given \( v_1, v_2 \in S \), suppose \( f(S - v_1) \leq f(S) \) and \( f(S - v_2) \leq f(S) \). Submodularity requires
\[
f(S - v_1) + f(S - v_2) \geq f(S) + f(S - v_1 - v_2)
\]
which would be impossible unless \( f(S - v_1 - v_2) \leq f(S) \). I.e., if also
\[
f(S - v_1 - v_2) > f(S)
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were true, submodularity can’t hold.
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- Similarly, given $v_1, v_2 \notin S$, and $f(S + v_1) \leq f(S)$ and $f(S + v_2) \leq f(S)$. Submodularity requires $f(S + v_1) + f(S + v_2) \geq f(S) + f(S + v_1 + v_2)$ which requires $f(S + v_1 + v_2) \leq f(S)$. 
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- This is the approach can yield a $(\frac{1}{3} - \frac{\epsilon}{n})$ approximation algorithm for maximizing non-monotone non-negative submodular functions, with most $O(\frac{1}{\epsilon} n^3 \log n)$ function calls using approximate local maxima search.
Linear time algorithm unconstrained non-monotone max

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Algorithm 3: Randomized Linear-time non-monotone submodular max

1. Set $L \leftarrow \emptyset$; $U \leftarrow \mathbb{V}$ /*Lower $L$, upper $U$. Invariant: $L \subseteq U$*/
2. Order elements of $\mathbb{V} = (v_1, v_2, ..., v_n)$ arbitrarily;
3. for $i \leftarrow 0 ... |\mathbb{V}|$ do
4. $a \leftarrow [f(v_i \mid L)] +$;
5. $b \leftarrow [-f(U \mid U \{v_i\})] +$;
6. if $a = b = 0$ then $p \leftarrow 1/2$;
7. else $p \leftarrow a / (a + b)$;
8. if Flip of coin with Pr(heads) = $p$ draws heads then $L \leftarrow L \cup \{v_i\}$;
9. Otherwise /* if the coin drew tails, an event with prob. 1−$p$*/ $U \leftarrow U \{v_i\}$;
10. return $L$ (which is the same as $U$ at this point)
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Algorithm 4: Randomized Linear-time non-monotone submodular max

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3. for \(i \leftarrow 0 \ldots |V|\) do
   4. \(a \leftarrow [f(v_i|L)]_+;\) \(b \leftarrow [-f(U|U \setminus \{v_i\})]_+\);
   5. if \(a = b = 0\) then \(p \leftarrow 1/2\);
   6. else \(p \leftarrow a/(a + b)\);
   7. if Flip of coin with \(\Pr(\text{heads}) = p\) draws heads then
      8. \(L \leftarrow L \cup \{v_i\}\);
   9. Otherwise /* if the coin drew tails, an event with prob. \(1 - p\) */
      10. \(U \leftarrow U \setminus \{v_i\}\)
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- In practice, run it multiple times, each with a different random permutation of the elements, and then take the cumulative best.
- It may be possible to choose the random order smartly to get better results in practice.
- But note, even a random subset is a 1/4 approximation to the optimal solution for unconstrained non-monotone submodular maximization, in expectation (Feige, Mirrokni, Vondrak, Maximizing non-monotone submodular functions. SIAM Journal on Computing, 40(4):1133-1153, 2011.)
More general still: multiple constraints different types

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- Often the computational costs of the algorithms are prohibitive (e.g., exponential in $k$) with large constants, so these algorithms might not scale.

- On the other hand, these algorithms offer deep and interesting intuition into submodular functions, beyond what we have covered here.
As we’ve seen, we can get $1 - 1/e$ for non-negative monotone submodular (polymatroid) functions with greedy algorithm under cardinality constraints, and this is tight.
Some results on submodular maximization

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- More general constraints are possible too, as we see on the next table (for references, see Jan Vondrak’s publications http://theory.stanford.edu/~jvondrak/).
Venn Family of Subclusive Constraints

Figure idea from Amin Karbasi
## Submodular Max Summary - From J. Vondrak

### Monotone Maximization

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Approximation</th>
<th>Hardness</th>
<th>Technique</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>S</td>
<td>\leq k$</td>
<td>$1 - 1/e$</td>
</tr>
<tr>
<td>matroid</td>
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<td>multilinear ext.</td>
</tr>
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<td>$O(1)$ knapsacks</td>
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</tr>
<tr>
<td>$k$ matroids</td>
<td>$k + \epsilon$</td>
<td>$k / \log k$</td>
<td>local search</td>
</tr>
<tr>
<td>$k$ matroids and $O(1)$ knapsacks</td>
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<tr>
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<tbody>
<tr>
<td>Unconstrained</td>
<td>$1/2$</td>
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<td>combinatorial</td>
</tr>
<tr>
<td>matroid</td>
<td>$1/e$</td>
<td>$0.48$</td>
<td>multilinear ext.</td>
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Continuous Extensions of Discrete Set Functions

- Any function $f : 2^V \rightarrow \mathbb{R}$ (equivalently $f : \{0, 1\}^V \rightarrow \mathbb{R}$) can be extended to a continuous function in the sense $\tilde{f} : [0, 1]^V \rightarrow \mathbb{R}$. 

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- In fact, any such discrete function defined on the vertices of the \( n \)-D hypercube \( \{0, 1\}^n \) has a variety of both convex and concave extensions tight at the vertices (Crama & Hammer'11).

Example \( n = 1 \),

\[
\begin{align*}
\text{Discrete Function} & : \{0, 1\}^V \to \mathbb{R} \\
\text{Convex Extensions} & : [0, 1] \to \mathbb{R} \\
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  ![Diagram](image)

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\[ \tilde{f} : [0, 1] \rightarrow \mathbb{R} \quad \text{Convex Extensions} \]
\[ f : \{0, 1\}^V \rightarrow \mathbb{R} \quad \text{Discrete Function} \]
\[ \tilde{f} : [0, 1] \rightarrow \mathbb{R} \quad \text{Concave Extensions} \]

- Since there are an exponential number of vertices $\{0, 1\}^n$, important questions regarding such extensions is:
  1. When are they computationally feasible to obtain or estimate?
  2. When do they have nice mathematical properties?
Continuous Extensions of Discrete Set Functions

- Any function $f : 2^V \to \mathbb{R}$ (equivalently $f : \{0, 1\}^V \to \mathbb{R}$) can be extended to a continuous function in the sense $\tilde{f} : [0, 1]^V \to \mathbb{R}$.
- This may be tight (i.e., $\tilde{f}(1_A) = f(A)$ for all $A$). I.e., the extension $\tilde{f}$ coincides with $f$ at the hypercube vertices.
- In fact, any such discrete function defined on the vertices of the $n$-D hypercube $\{0, 1\}^n$ has a variety of both convex and concave extensions tight at the vertices (Crama & Hammer’11). Example $n = 1$,

Since there are an exponential number of vertices $\{0, 1\}^n$, important questions regarding such extensions is:

1. When are they computationally feasible to obtain or estimate?
2. When do they have nice mathematical properties?
3. When are they useful for something practical?
Def: Convex Envelope of a function

Given any function \( h : \mathcal{D}_h \to \mathbb{R} \), where \( \mathcal{D}_h \subseteq \mathbb{R}^n \), define the new function \( \tilde{h} : \mathbb{R}^n \to \mathbb{R} \) via:

\[
\tilde{h}(x) = \sup \{ g(x) : g \text{ is convex} & g(y) \leq h(y), \forall y \in \mathcal{D}_h \} \quad (16.1)
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- I.e., (1) \( \tilde{h}(x) \) is convex, (2) \( \tilde{h}(x) \leq h(x), \forall x \), and (3) if \( g(x) \) is any convex function having the property that \( g(x) \leq h(x), \forall x \), then \( g(x) \leq \tilde{h}(x) \).
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- Alternatively,

\[
\tilde{h}(x) = \inf \{ t : (x, t) \in \text{convexhull}(\text{epigraph}(h)) \} \quad (16.2)
\]
Given set function $f : 2^V \rightarrow \mathbb{R}$, an arbitrary (i.e., not necessarily submodular nor supermodular) set function, define a function $\tilde{f} : [0, 1]^V \rightarrow \mathbb{R}$, as

$$\tilde{f}(x) = \min_{p \in \Delta^n(x)} \sum_{S \subseteq V} p_S f(S)$$

(16.3)

where $\Delta^n(x) = \{ p \in \mathbb{R}^{2^n} : \sum_{S \subseteq V} p_S = 1, p_S \geq 0 \forall S \subseteq V, \text{ and } \sum_{S \subseteq V} p_S 1_S = x \}$. 

Hence, $\tilde{f}$ is a convex extension. Does it have any special properties?
Convex Closure of Discrete Set Functions

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where $\Delta^n(x) =$

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- Hence, $\Delta^n(x)$ is the set of all probability distributions over the $2^n$ vertices of the hypercube, and where the expected value of the characteristic vectors of those points is equal to $x$, i.e., for any $p \in \Delta^n(x)$, $E_{S \sim p}(1_S) = \sum_{S \subseteq V} p_S 1_S = x$. 


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where \( \Delta^n(x) = \{ p \in \mathbb{R}^{2^n} : \sum_{S \subseteq V} p_S = 1, \ p_S \geq 0 \forall S \subseteq V, \ & \sum_{S \subseteq V} p_S 1_S = x \} \)

- Hence, \( \Delta^n(x) \) is the set of all probability distributions over the \( 2^n \) vertices of the hypercube, and where the expected value of the characteristic vectors of those points is equal to \( x \), i.e., for any \( p \in \Delta^n(x) \), \( E_{S \sim p}(1_S) = \sum_{S \subseteq V} p_S 1_S = x \).

- Hence, \( \tilde{f}(x) = \min_{p \in \Delta^n(x)} E_{S \sim p}[f(S)] \)
Convex Closure of Discrete Set Functions

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$$\tilde{f}(x) = \min_{p \in \Delta^n(x)} \sum_{S \subseteq V} p_S f(S)$$ (16.3)

where $\Delta^n(x) = \left\{ p \in \mathbb{R}^{2^n} : \sum_{S \subseteq V} p_S = 1, \ p_S \geq 0 \forall S \subseteq V, \ & \sum_{S \subseteq V} p_S 1_S = x \right\}$

- Hence, $\Delta^n(x)$ is the set of all probability distributions over the $2^n$ vertices of the hypercube, and where the expected value of the characteristic vectors of those points is equal to $x$, i.e., for any $p \in \Delta^n(x)$, $E_{S \sim p}(1_S) = \sum_{S \subseteq V} p_S 1_S = x$.

- Hence, $\tilde{f}(x) = \min_{p \in \Delta^n(x)} E_{S \sim p}[f(S)]$

- So far, as far as we know, this is just a convex extension. Does it have any special properties?
Convex Closure of Discrete Set Functions

Given, \( \tilde{f}(x) = \min_{p \in \Delta^n(x)} E_{S \sim p}[f(S)] \), we can show:

1. \( \tilde{f} \) is tight (i.e., \( \forall S \subseteq V \), we have \( \tilde{f}(\mathbf{1}_S) = f(S) \)).
2. \( \tilde{f} \) is convex (and consequently, that any arbitrary set function has a tight convex extension).
3. that the convex closure \( \tilde{f} \) is the convex envelope of the function defined only on the hypercube vertices, and that takes value \( f(S) \) at \( \mathbf{1}_S \).
4. the definition of the Lovász extension of a set function, and that \( \tilde{f} \) is the Lovász extension iff \( f \) is submodular.
Convex Closure of Discrete Set Functions

Given, \( \tilde{f}(x) = \min_{p \in \Delta^n(x)} ES \sim_p [f(S)] \), we can show:

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Convex Closure of Discrete Set Functions

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Given, \( \tilde{f}(x) = \min_{p \in \Delta^n} E_{S \sim p}[f(S)] \), we can show:

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Lemma 16.4.1

∀A ⊆ V, we have ŵ(1_A) = f(A).

Proof.

- Define p^x to be an achieving argmin in ŵ(x) = \min_{p \in \triangle^n(x)} E_{S \sim p}[f(S)].
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∀A ⊆ V, we have \( \tilde{f}(1_A) = f(A) \).

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- Define \( p^x \) to be an achieving argmin in \( f(x) = \min_{p \in \Delta^n(x)} E_{S \sim p}[f(S)] \).
- Take an arbitrary \( A \), so that \( 1_A = \sum_{S \subseteq V} p^A_S 1_S \).
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- Define \( p^x \) to be an achieving argmin in \( \tilde{f}(x) = \min_{p \in \Delta^n(x)} E_{S \sim p}[f(S)] \).
- Take an arbitrary \( A \), so that \( 1_A = \sum_{S \subseteq V} p^A_S 1_S \).
- Suppose \( \exists S' \) with \( S' \setminus A \neq 0 \) having \( p^A_{S'} > 0 \). This would mean, for any \( v \in S' \setminus A \), that \( \left( \sum_S p^A_S 1_S \right)(v) > 0 \), a contradiction.
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- Suppose \( \exists S' \) s.t. \( A \setminus S' \neq \emptyset \) with \( p^1_A > 0 \).
- Then, for any \( v \in A \setminus S' \), consider below leading to a contradiction

\[
\underbrace{p_{S'} 1_{S'}}_{> 0} + \sum_{S \subseteq A, S \neq S'} p_S 1_S \Rightarrow \left( \sum_{S \subseteq A, S \neq S'} p_S 1_S \right)(v) < 1 \tag{16.4}
\]

i.e., \( v \in A \) so it must get value 1, but since \( v \notin S' \), \( v \) is deficient.
Lemma 16.4.2

\[ \tilde{f}(x) = \min_{p \in \Delta^n(x)} E_{S \sim p} [f(S)] \text{ is convex in } [0, 1]^V. \]

Proof.

- Let \( x, y \in [0, 1]^V \), \( 0 \leq \lambda \leq 1 \), and \( z = \lambda x + (1 - \lambda)y \), then

\[ \lambda \tilde{f}(x) + (1 - \lambda) \tilde{f}(y) = \lambda \sum_S p_S^x f(S) + (1 - \lambda) \sum_S p_S^y f(S) \tag{16.5} \]

\[ = \sum_S (\lambda p_S^x + (1 - \lambda)p_S^y) f(S) \tag{16.6} \]

\[ = \sum_S p_S^{z'} f(S) \geq \min_{p \in \Delta^n(z)} E_{S \sim p} [f(S)] \tag{16.7} \]

\[ = \tilde{f}(z) = \tilde{f}(\lambda x + (1 - \lambda)y) \tag{16.8} \]
Convexity of the Convex Closure

Lemma 16.4.2

\[ \hat{f}(x) = \min_{p \in \Delta^n(x)} E_{S \sim p}[f(S)] \] is convex in \([0, 1]^V\).

Proof.

- Let \(x, y \in [0, 1]^V\), \(0 \leq \lambda \leq 1\), and \(z = \lambda x + (1 - \lambda)y\), then

\[
\lambda \hat{f}(x) + (1 - \lambda) \hat{f}(y) = \lambda \sum_S p^x_S f(S) + (1 - \lambda) \sum_S p^y_S f(S) \quad (16.5)
\]

\[
= \sum_S (\lambda p^x_S + (1 - \lambda)p^y_S) f(S) \quad (16.6)
\]

\[
= \sum_S p^{z'}_S f(S) \geq \min_{p \in \Delta^n(z)} E_{S \sim p}[f(S)] \quad (16.7)
\]

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= \hat{f}(z) = \hat{f}(\lambda x + (1 - \lambda)y) \quad (16.8)
\]

- Note that \(p^{z'}_S = \lambda p^x_S + (1 - \lambda)p^y_S\) and is feasible in the min since \(\sum_S p^{z'}_S = 1\), \(p^{z'}_S \geq 0\) and \(\sum_S p^{z'}_S 1_S = z\).
Def: Convex Envelope of a function

- Given any function $h : \mathcal{D}_h \to \mathbb{R}$, where $\mathcal{D}_h \subseteq \mathbb{R}^n$, define the new function $\tilde{h} : \mathbb{R}^n \to \mathbb{R}$ via:

$$\tilde{h}(x) = \sup \{ g(x) : g \text{ is convex } & g(y) \leq h(y), \forall y \in \mathcal{D}_h \} \quad (16.1)$$

- I.e., (1) $\tilde{h}(x)$ is convex, (2) $\tilde{h}(x) \leq h(x), \forall x$, and (3) if $g(x)$ is any convex function having the property that $g(x) \leq h(x), \forall x$, then $g(x) \leq \tilde{h}(x)$.

- Alternatively,

$$\tilde{h}(x) = \inf \{ t : (x, t) \in \text{convexhull}(\text{epigraph}(h)) \} \quad (16.2)$$
Lemma 16.4.3

\( \tilde{f}(x) = \min_{p \in \triangle^n(x)} E_{S \sim p}[f(S)] \) is the convex envelope.

Proof.

- Suppose \( \exists \) a convex \( \bar{f} \) with \( \bar{f}(1_A) = f(A) = \tilde{f}(1_A), \forall A \subseteq V \) and \( \exists x \in [0, 1]^V \) s.t. \( \bar{f}(x) > \tilde{f}(x) \).

- Define \( p^x \) to be an achieving argmin in \( \tilde{f}(x) = \min_{p \in \triangle^n(x)} E_{S \sim p}[f(S)] \). Hence, we have \( x = \sum_S p^x_S 1_S \). Thus

\[
\tilde{f}(x) = \sum_S p^x_S f(S) = \sum_S p^x_S \bar{f}(1_S) \tag{16.9}
\]

\[
< \bar{f}(x) = \bar{f}(\sum_S p^x_S 1_S) \tag{16.10}
\]

but the inequality contradicts the convexity of \( \bar{f} \).
Polymatroid with labeled edge lengths

- Recall
  \[ f(e|A) = f(A+e) - f(A) \]

- Notice how submodularity,
  \[ f(e|B) \leq f(e|A) \]
  for \( A \subseteq B \), defines the shape of the polytope.

- In fact, we have strictness here
  \[ f(e|B) < f(e|A) \]
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- Also, consider how the greedy algorithm proceeds along the edges of the polytope.
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  \( A \subset B \).
- Also, consider how the greedy algorithm proceeds along the edges of the polytope.
**Optimization over \( P_f \)**

- Consider the following optimization. Given \( w \in \mathbb{R}^E \),
  \[
  \begin{align*}
  \text{maximize} & \quad w^\top x \\
  \text{subject to} & \quad x \in P_f
  \end{align*}
  \]  \hspace{1cm} (16.11a)

- Since \( P_f \) is down closed, if \( \exists e \in E \) with \( w(e) < 0 \) then the solution above is unboundedly large. Hence, assume \( w \in \mathbb{R}^E_+ \).

- In a future lecture, we will see that any \( x \in P_f \) with \( x \notin B_f \) is dominated by \( x \leq y \in B_f \) which can only increase \( w^\top x \leq w^\top y \) when \( w \in \mathbb{R}^E_+ \).

- Hence, the problem is equivalent to: given \( w \in \mathbb{R}^E_+ \),
  \[
  \begin{align*}
  \text{maximize} & \quad w^\top x \\
  \text{subject to} & \quad x \in B_f
  \end{align*}
  \]  \hspace{1cm} (16.12a)

- Moreover, we can have \( w \in \mathbb{R}^E \) if we insist on \( x \in B_f \).
Consider again optimization problem. Given $w \in \mathbb{R}^E$, 

$$\text{maximize } w^T x$$

$$\text{subject to } x \in B_f$$

(16.13a)  
(16.13b)
Consider again optimization problem. Given $w \in \mathbb{R}^E$, 

$$
\begin{align*}
\text{maximize} & \quad w^\top x \\
\text{subject to} & \quad x \in B_f
\end{align*}
$$

We may consider this optimization problem a function $\tilde{f} : \mathbb{R}^E \to \mathbb{R}$ of $w \in \mathbb{R}^E$, defined as:

$$
\tilde{f}(w) = \max(wx : x \in B_f)
$$
A continuous extension of $f$

- Consider again optimization problem. Given $w \in \mathbb{R}^E$,

$$\begin{align*}
\text{maximize} & \quad w^T x \\
\text{subject to} & \quad x \in B_f
\end{align*}$$

(16.13a) (16.13b)

- We may consider this optimization problem a function $\tilde{f} : \mathbb{R}^E \rightarrow \mathbb{R}$ of $w \in \mathbb{R}^E$, defined as:

$$\tilde{f}(w) = \max(wx : x \in B_f)$$

(16.14)

- Note that this is $\tilde{f}(w)$ which is a distinct notation from the convex closure $\tilde{f}(x)$ we defined before.
A continuous extension of \( f \)

- Consider again optimization problem. Given \( w \in \mathbb{R}^E \),

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\text{maximize} & \quad w^T x \\
\text{subject to} & \quad x \in B_f
\end{align*}
\]

(16.13a) (16.13b)

- We may consider this optimization problem a function \( \tilde{f} : \mathbb{R}^E \rightarrow \mathbb{R} \) of \( w \in \mathbb{R}^E \), defined as:

\[
\tilde{f}(w) = \max(wx : x \in B_f)
\]

(16.14)

- Note that this is \( \tilde{f}(w) \) which is a distinct notation from the convex closure \( \tilde{f}(x) \) we defined before.

- Hence, for any \( w \), from the solution to the above theorem (as we have seen), we can compute the value of this function using Edmond’s greedy algorithm.
Recall Edmond’s Theorem: The Greedy Algorithm

- Recall, Edmonds proved that the solution to $\tilde{f}(w) = \max(wx : x \in B_f)$ is solved by the greedy algorithm iff $f$ is submodular.
- In particular, sort choose element order $(e_1, e_2, \ldots, e_m)$ based on decreasing $w$, so that $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$.
- Define the chain with $i^{th}$ element $E_i = \{e_1, e_2, \ldots, e_i\}$.
- Define a vector $x^* \in \mathbb{R}^V$ where element $e_i$ has value $x(e_i) = f(e_i|E_{i-1})$ for all $i \in V$.
- Then $\langle w, x^* \rangle = \max(wx : x \in B_f)$

**Theorem 16.5.1 (Edmonds)**

If $f : 2^E \to \mathbb{R}_+$ is given, and $B_f$ is a polytope in $\mathbb{R}^E$ of the form $B_f = \{x \in \mathbb{R}^E : x(A) \leq f(A), \forall A \subseteq E, x(E) = f(E)\}$, then the greedy solution to the problem $\max(w^\top x : x \in B_f)$ is $\forall w$ optimum iff $f$ is monotone non-decreasing submodular (i.e., iff the corresponding $P_f$ is a polymatroid).
Greedy-based continuous extension of submodular $f$

That is, given a submodular function $f$, a $w \in \mathbb{R}^E$, choose element order $(e_1, e_2, \ldots, e_m)$ based on decreasing $w$, so that $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$. 
Greedy-based continuous extension of submodular $f$

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Define chain $\emptyset \triangleq E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m = E$ based on $w$, so the $i^{th}$ element of this change has $E_i = \{e_1, e_2, \ldots, e_i\}$. 
Greedy-based continuous extension of submodular $f$

- That is, given a submodular function $f$, a $w \in \mathbb{R}^E$, choose element order $(e_1, e_2, \ldots, e_m)$ based on decreasing $w$, so that $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$.

- Define chain $\emptyset \triangleq E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m = E$ based on $w$, so the $i^{th}$ element of this change has $E_i = \{e_1, e_2, \ldots, e_i\}$.

We have

\[ \tilde{f}(w) \]
Greedy-based continuous extension of submodular $f$

That is, given a submodular function $f$, a $w \in \mathbb{R}^E$, choose element order $(e_1, e_2, \ldots, e_m)$ based on decreasing $w$, so that $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$.

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We have

$$\tilde{f}(w) = \max(wx : x \in B_f)$$

(16.15)
Greedy-based continuous extension of submodular $f$

- That is, given a submodular function $f$, a $w \in \mathbb{R}^E$, choose element order $(e_1, e_2, \ldots, e_m)$ based on decreasing $w$, so that $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$.
- Define chain $\emptyset \triangleq E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m = E$ based on $w$, so the $i^{th}$ element of this change has $E_i = \{e_1, e_2, \ldots, e_i\}$.

We have

$$\tilde{f}(w) = \max(wx : x \in B_f)$$

$$= \sum_{i=1}^{m} w(e_i) f(e_i | E_{i-1}) = \sum_{i=1}^{m} w(e_i) x(e_i)$$

(16.15)

(16.16)
That is, given a submodular function \( f \), a \( w \in \mathbb{R}^E \), choose element order \((e_1, e_2, \ldots, e_m)\) based on decreasing \( w \), so that \( w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m) \).

Define chain \( \emptyset \triangleq E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m = E \) based on \( w \), so the \( i^{\text{th}} \) element of this change has \( E_i = \{e_1, e_2, \ldots, e_i\} \).

We have

\[
\tilde{f}(w) = \max(\sum w(e)x(e) : x \in B_f)
\]

\[
= \sum_{i=1}^{m} w(e_i)f(e_i|E_{i-1}) = \sum_{i=1}^{m} w(e_i)x(e_i)
\]

\[
= \sum_{i=1}^{m} w(e_i)(f(E_i) - f(E_{i-1}))
\]
Greedy-based continuous extension of submodular $f$

- That is, given a submodular function $f$, a $w \in \mathbb{R}^E$, choose element order $(e_1, e_2, \ldots, e_m)$ based on decreasing $w$, so that $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$.

- Define chain $\emptyset \triangleq E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m = E$ based on $w$, so the $i^{th}$ element of this change has $E_i = \{e_1, e_2, \ldots, e_i\}$.

We have

$$\tilde{f}(w) = \max(wx : x \in B_f)$$

$$= \sum_{i=1}^{m} w(e_i) f(e_i | E_{i-1}) = \sum_{i=1}^{m} w(e_i) x(e_i)$$

$$= \sum_{i=1}^{m} w(e_i)(f(E_i) - f(E_{i-1}))$$

$$= -w(e_1)f(E_0) + w(e_m)f(E_m) + \sum_{i=1}^{m-1} (w(e_i) - w(e_{i+1}))f(E_i)$$
Greedy-based continuous extension of submodular $f$

- Definition of the continuous extension, once again, for reference:

$$\bar{f}(w) = \max(wx : x \in B_f)$$  \hspace{1cm} (16.19)
Definition of the continuous extension, once again, for reference:

\[ \tilde{f}(w) = \max(wx : x \in B_f) \]  

Therefore, if \( f \) is a submodular function, we can write

\[ \tilde{f}(w) \]
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- Definition of the continuous extension, once again, for reference:

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- Therefore, if $f$ is a submodular function, we can write

$$\tilde{f}(w) = w(e_m)f(E_m) + \sum_{i=1}^{m-1} (w(e_i) - w(e_{i+1}))f(E_i)$$  \hspace{1cm} (16.20)
Greedy-based continuous extension of submodular $f$

- Definition of the continuous extension, once again, for reference:
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  \tilde{f}(w) = \max(wx : x \in B_f)
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  \[
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  \]  
  (16.20)
  \[
  = \sum_{i=1}^{m} \lambda_i f(E_i)
  \]  
  (16.21)
Greedy-based continuous extension of submodular $f$

- Definition of the continuous extension, once again, for reference:

$$\tilde{f}(w) = \max(wx : x \in B_f) \quad (16.19)$$

- Therefore, if $f$ is a submodular function, we can write

$$\tilde{f}(w) = w(e_m)f(E_m) + \sum_{i=1}^{m-1} (w(e_i) - w(e_{i+1}))f(E_i) \quad (16.20)$$

$$= \sum_{i=1}^{m} \lambda_i f(E_i) \quad (16.21)$$

where $\lambda_m = w(e_m)$ and otherwise $\lambda_i = w(e_i) - w(e_{i+1})$, where the elements are sorted descending according to $w$ as before.
Greedy-based continuous extension: properties?

So we go from $f$ to $B_f$ to $\tilde{f}(w) = \max(wx : x \in B_f) = \sum_{i=1}^{m} \lambda_i f(E_i)$ when $f$ is submodular.
So we go from $f$ to $B_f$ to $\tilde{f}(w) = \max(wx : x \in B_f) = \sum_{i=1}^{m} \lambda_i f(E_i)$ when $f$ is submodular.

Convex analysis $\Rightarrow \tilde{f}(w) = \max(wx : x \in P)$ is always convex in $w$ for any set $P \subseteq R^E$, since a maximum of a set of linear functions (true even when $f$ is not submodular or $P$ is not itself a convex set).
So we go from $f$ to $B_f$ to $\tilde{f}(w) = \max(wx : x \in B_f) = \sum_{i=1}^{m} \lambda_i f(E_i)$ when $f$ is submodular.

Convex analysis $\Rightarrow \tilde{f}(w) = \max(wx : x \in P)$ is always convex in $w$ for any set $P \subseteq R^E$, since a maximum of a set of linear functions (true even when $f$ is not submodular or $P$ is not itself a convex set).

What this says, though is that when $f$ is submodular, then $\sum_{i=1}^{m} \lambda_i f(E_i)$ is convex.
Greedy-based continuous extension: properties?

- So we go from $f$ to $B_f$ to $	ilde{f}(w) = \max(wx : x \in B_f) = \sum_{i=1}^{m} \lambda_i f(E_i)$ when $f$ is submodular.

- Convex analysis $\Rightarrow \tilde{f}(w) = \max(wx : x \in P)$ is always convex in $w$ for any set $P \subseteq \mathbb{R}^E$, since a maximum of a set of linear functions (true even when $f$ is not submodular or $P$ is not itself a convex set).

- What this says, though is that when $f$ is submodular, then $\sum_{i=1}^{m} \lambda_i f(E_i)$ is convex.

- What can we say about the form $\sum_{i=1}^{m} \lambda_i f(E_i)$ in general? What is this particular form if $f$ isn’t even submodular?
Reminder on notation

- Recall, for any such $w \in \mathbb{R}^E$, we have

$$
\begin{pmatrix}
  w_1 \\
  w_2 \\
  \vdots \\
  w_n
\end{pmatrix} = \underbrace{(w_1 - w_2)}_{\lambda_1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \underbrace{(w_2 - w_3)}_{\lambda_2} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \underbrace{(w_{n-1} - w_n)}_{\lambda_{m-1}} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 1 \end{pmatrix} + (w_m) \begin{pmatrix} 1 \\ 1 \end{pmatrix}
$$

(16.22)

- If we take $w$ in decreasing order, then each coefficient of the vectors is non-negative (except possibly the last one, $\lambda_m = w_m$).
- We often take $w \in \mathbb{R}_+^V$ or even $w \in [0, 1]^V$, making $\lambda_m \geq 0$. 
More reminder on notation

- Again, order the elements based on \( w \) and define \( E_i \) based on decreasing order of \( w \), giving \( E_i = \{e_1, e_2, \ldots, e_i\} \) for \( i = 0, \ldots, n \).

- Recall that

\[
1_{E_0} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad 1_{E_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad 1_{E_\ell} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\begin{pmatrix} \ell \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \text{etc.}
\]

- Hence, from the previous and current slide, we have \( w = \sum_{i=1}^{m} \lambda_i 1_{E_i} \)
From $\tilde{f}$ back to $f$, even when $f$ is not submodular

- From the continuous $\tilde{f}$, we can recover $f(A)$ for any $A \subseteq V$. 

\[
\text{From } \tilde{f} \text{ back to } f, \text{ even when } f \text{ is not submodular}
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\[
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From $f$ back to $f$, even when $f$ is not submodular

- From the continuous $\tilde{f}$, we can recover $f(A)$ for any $A \subseteq V$.
- Take $w = 1_A$ for some $A \subseteq E$, so $w$ is vertex of the hypercube.
From \( \tilde{f} \) back to \( f \), even when \( f \) is not submodular

- From the continuous \( \tilde{f} \), we can recover \( f(A) \) for any \( A \subseteq V \).
- Take \( w = 1_A \) for some \( A \subseteq E \), so \( w \) is vertex of the hypercube.
- Order and rename the elements of \( E \) in decreasing order of \( w \) so that \( w(e_1) \geq w(e_2) \geq w(e_3) \geq \cdots \geq w(e_m) \).
From \( \tilde{f} \) back to \( f \), even when \( f \) is not submodular

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  \[ w(e_1) \geq w(e_2) \geq w(e_3) \geq \cdots \geq w(e_m). \]
- This means
  \[ w = (w(e_1), w(e_2), \ldots, w(e_m)) = (1, 1, 1, \ldots, 1, 0, 0, \ldots, 0) \]
  \[ \text{for } |A| \text{ times and } m-|A| \text{ times} \]

so that \( 1_A(i) = 1 \) if \( i \leq |A| \), and \( 1_A(i) = 0 \) otherwise.
From \( \tilde{f} \) back to \( f \), even when \( f \) is not submodular

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  so that \( 1_A(i) = 1 \) if \( i \leq |A| \), and \( 1_A(i) = 0 \) otherwise.
- For any \( f : 2^E \rightarrow \mathbb{R} \), \( w = 1_A \), since \( E_{|A|} = \{ e_1, e_2, \ldots, e_{|A|} \} = A \):
  \[ \tilde{f}(w) \]
From $\tilde{f}$ back to $f$, even when $f$ is not submodular

- From the continuous $\tilde{f}$, we can recover $f(A)$ for any $A \subseteq V$.
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- For any $f : 2^E \to \mathbb{R}$, $w = 1_A$, since $E_{|A|} = \{e_1, e_2, \ldots, e_{|A|}\} = A$:

$$\tilde{f}(w) = \sum_{i=1}^{m} \lambda_i f(E_i)$$
From $\tilde{f}$ back to $f$, even when $f$ is not submodular

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- This means
  
  $$w = (w(e_1), w(e_2), \ldots, w(e_m)) = (1, 1, 1, \ldots, 1, 0, 0, \ldots, 0)$$
  \hspace{1cm} (16.23)

  \hspace{1cm} \text{\text{\textstyle{|A|}\times}} \text{\textstyle{m-|A|}\times}}

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- For any $f : 2^E \rightarrow \mathbb{R}$, $w = 1_A$, since $E_{|A|} = \{e_1, e_2, \ldots, e_{|A|}\} = A$:

  $$\tilde{f}(w) = \sum_{i=1}^m \lambda_i f(E_i) = w(e_m) f(E_m) + \sum_{i=1}^{m-1} (w(e_i) - w(e_{i+1})) f(E_i)$$

  \hspace{1cm} (16.24)
From $\bar{f}$ back to $f$, even when $f$ is not submodular

- From the continuous $\bar{f}$, we can recover $f(A)$ for any $A \subseteq V$.
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$$= 1_A(m) f(E_m) + \sum_{i=1}^{m-1} (1_A(i) - 1_A(i+1)) f(E_i)$$

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\]

\[
= 1_A(m) f(E_m) + \sum_{i=1}^{m-1} (1_A(i) - 1_A(i + 1)) f(E_i)
\]

(16.24)

\[
= (1_A(|A|) - 1_A(|A| + 1)) f(E_{|A|}) = f(E_{|A|})
\]
From \( \tilde{f} \) back to \( f \), even when \( f \) is not submodular

- From the continuous \( \tilde{f} \), we can recover \( f(A) \) for any \( A \subseteq V \).
- Take \( w = 1_A \) for some \( A \subseteq E \), so \( w \) is vertex of the hypercube.
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\[
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so that \( 1_A(i) = 1 \) if \( i \leq |A| \), and \( 1_A(i) = 0 \) otherwise.

- For any \( f : 2^E \rightarrow \mathbb{R} \), \( w = 1_A \), since \( E_{|A|} = \{e_1, e_2, \ldots, e_{|A|}\} = A \):

\[
\tilde{f}(w) = \sum_{i=1}^{m} \lambda_i f(E_i) = w(e_m) f(E_m) + \sum_{i=1}^{m-1} (w(e_i) - w(e_{i+1})) f(E_i)
\]

\[
= 1_A(m) f(E_m) + \sum_{i=1}^{m-1} (1_A(i) - 1_A(i+1)) f(E_i)
\] (16.24)

\[
= (1_A(|A|) - 1_A(|A| + 1)) f(E_{|A|}) = f(E_{|A|}) = f(A)
\] (16.25)
Summary so far

- To summarize, with \( \tilde{f}(1_A) = \sum_{i=1}^{m} \lambda_i f(E_i) \), we have

\[
\tilde{f}(1_A) = f(A),
\]  
(16.26)
Summary so far

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  \[
  \tilde{f}(1_A) = f(A),
  \]
  \[(16.26)\]

- Thus, we can view $\tilde{f} : [0, 1]^E \to \mathbb{R}$ defined on the hypercube, with $f$ defined as $\tilde{f}$ evaluated on the hypercube extreme points (vertices).
Summary so far

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- Thus, we can view $\tilde{f} : [0, 1]^E \rightarrow \mathbb{R}$ defined on the hypercube, with $f$ defined as $\tilde{f}$ evaluated on the hypercube extreme points (vertices).

  ... and when $f$ is submodular, we also have have

  $\tilde{f}(1_A) = \max \{1_A^T x : x \in B_f\}$ \hspace{1cm} (16.27)

  and

  $\tilde{f}(w) = \max \{w^T x : x \in B_f\} = \sum_{i=1}^{m} \lambda_i f(E_i)$ \hspace{1cm} (16.28)
Summary so far

- To summarize, with $\tilde{f}(1_A) = \sum_{i=1}^{m} \lambda_i f(E_i)$, we have
  \[ \tilde{f}(1_A) = f(A), \]  
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- Thus, we can view $\tilde{f} : [0, 1]^E \to \mathbb{R}$ defined on the hypercube, with $f$ defined as $\tilde{f}$ evaluated on the hypercube extreme points (vertices).

- ...and when $f$ is submodular, we also have
  \[ \tilde{f}(1_A) = \max \{ 1_A^T x : x \in B_f \} \]  
  (16.27)

  and
  \[ \tilde{f}(w) = \max \{ w^T x : x \in B_f \} = \sum_{i=1}^{m} \lambda_i f(E_i) \]  
  (16.28)

- When considering $\tilde{f} : [0, 1]^E \to \mathbb{R}$, then any $w \in [0, 1]^E$ is in positive orthant, and we have
  \[ \tilde{f}(w) = \max \{ w^T x : x \in P_f \} \]  
  (16.29)
The Lovász extension of an arbitrary $f : 2^V \rightarrow \mathbb{R}$

Thus, for any $f : 2^E \rightarrow \mathbb{R}$, even non-submodular $f$, we can define an extension, having $\tilde{f}(1_A) = f(A)$, $\forall A$, in this way where

$$\tilde{f}(w) = \sum_{i=1}^{m} \lambda_i f(E_i) \quad (16.30)$$

with the $E_i = \{e_1, \ldots, e_i\}$’s defined based on sorted descending order of $w$ as in $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$, and where

$$\lambda_i = \begin{cases} w(e_i) - w(e_{i+1}) & \text{if } i < m \\ w(e_m) & \text{if } i = m \end{cases} \quad (16.31)$$

so that $w = \sum_{i=1}^{m} \lambda_i 1_{E_i}$. 
The Lovász extension of an arbitrary $f : 2^V \rightarrow \mathbb{R}$

Thus, for any $f : 2^E \rightarrow \mathbb{R}$, even non-submodular $f$, we can define an extension, having $\tilde{f}(1_A) = f(A)$, $\forall A$, in this way where

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$$\text{for } i \in \{1, \ldots, m\}, \quad \lambda_i = \begin{cases} 
    w(e_i) - w(e_{i+1}) & \text{if } i < m \\
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\end{cases} \quad (16.31)$$

so that $w = \sum_{i=1}^{m} \lambda_i 1_{E_i}$.

$w = \sum_{i=1}^{m} \lambda_i 1_{E_i}$ is an interpolation of certain hypercube vertices.
The **Lovász extension** of an arbitrary \( f : 2^V \rightarrow \mathbb{R} \)

- Thus, for any \( f : 2^E \rightarrow \mathbb{R} \), even non-submodular \( f \), we can define an extension, having \( \tilde{f}(1_A) = f(A) \), \( \forall A \), in this way where

\[
\tilde{f}(w) = \sum_{i=1}^{m} \lambda_i f(E_i) \tag{16.30}
\]

with the \( E_i = \{e_1, \ldots, e_i\} \)'s defined based on sorted descending order of \( w \) as in \( w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m) \), and where

\[
\text{for } i \in \{1, \ldots, m\}, \quad \lambda_i = \begin{cases} 
  w(e_i) - w(e_{i+1}) & \text{if } i < m \\
  w(e_m) & \text{if } i = m 
\end{cases} \tag{16.31}
\]

so that

\[
w = \sum_{i=1}^{m} \lambda_i 1_{E_i}.
\]

- \( w = \sum_{i=1}^{m} \lambda_i 1_{E_i} \) is an interpolation of certain hypercube vertices.

- \( \tilde{f}(w) = \sum_{i=1}^{m} \lambda_i f(E_i) \) is the associated interpolation of the values of \( f \) at sets corresponding to each hypercube vertex.
The Lovász extension of an arbitrary \( f : 2^V \rightarrow \mathbb{R} \)

- Thus, for any \( f : 2^E \rightarrow \mathbb{R} \), even non-submodular \( f \), we can define an extension, having \( \tilde{f}(1_A) = f(A) \), \( \forall A \), in this way where

\[
\tilde{f}(w) = \sum_{i=1}^{m} \lambda_i f(E_i) \quad (16.30)
\]

with the \( E_i = \{e_1, \ldots, e_i\} \)'s defined based on sorted descending order of \( w \) as in \( w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m) \), and where

\[
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  w(e_i) - w(e_{i+1}) & \text{if } i < m \\
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\end{cases} \quad (16.31)
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so that \( w = \sum_{i=1}^{m} \lambda_i 1_{E_i} \).

- \( w = \sum_{i=1}^{m} \lambda_i 1_{E_i} \) is an interpolation of certain hypercube vertices.
- \( \tilde{f}(w) = \sum_{i=1}^{m} \lambda_i f(E_i) \) is the associated interpolation of the values of \( f \) at sets corresponding to each hypercube vertex.
- **This extension is called the Lovász extension!**
L.E.: Weighted gains or weighted functions, take your pick

- Again sorting $E$ descending in $w$, the extension summarized:

\[
\tilde{f}(w) = \sum_{i=1}^{m} w(e_i) f(e_i | E_{i-1})
\]

(16.32)

\[
= \sum_{i=1}^{m} w(e_i) (f(E_i) - f(E_{i-1}))
\]

(16.33)

\[
= w(e_m) f(E_m) + \sum_{i=1}^{m-1} (w(e_i) - w(e_{i+1})) f(E_i)
\]

(16.34)

\[
= \sum_{i=1}^{m} \lambda_i f(E_i)
\]

(16.35)
Again sorting $E$ descending in $w$, the extension summarized:

\[ \tilde{f}(w) = \sum_{i=1}^{m} w(e_i) f(e_i | E_{i-1}) \]  

(16.32)

\[ = \sum_{i=1}^{m} w(e_i)(f(E_i) - f(E_{i-1})) \]  

(16.33)

\[ = w(e_m)f(E_m) + \sum_{i=1}^{m-1} (w(e_i) - w(e_{i+1}))f(E_i) \]  

(16.34)

\[ = \sum_{i=1}^{m} \lambda_i f(E_i) \]  

(16.35)

So $\tilde{f}(w)$ seen either as sum of weighted gain evaluations (Eqn. (16.32)), or as sum of weighted function evaluations (Eqn. (16.35)).
Summary: comparison of the two extension forms

- So if $f$ is submodular, then we can write $f(w) = \max(wx : x \in B_f)$ (which is clearly convex) in the form:

$$\tilde{f}(w) = \max(wx : x \in B_f) = \sum_{i=1}^{m} \lambda_i f(E_i)$$  (16.36)

where $w = \sum_{i=1}^{m} \lambda_i \mathbf{1}_{E_i}$ and $E_i = \{e_1, \ldots, e_i\}$ defined based on sorted descending order $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$.
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- On the other hand, for any \( f \) (even non-submodular), we can produce an extension \( \tilde{f} \) having the form

  \[
  \tilde{f}(w) = \sum_{i=1}^{m} \lambda_i f(E_i) \tag{16.37}
  \]

  where \( w = \sum_{i=1}^{m} \lambda_i 1_{E_i} \) and \( E_i = \{e_1, \ldots, e_i\} \) defined based on sorted descending order \( w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m) \).
Submodular Max w. Other Constraints

Cont. Extensions

Lovász extension

Summary: comparison of the two extension forms

- So if $f$ is submodular, then we can write $\tilde{f}(w) = \max(wx : x \in B_f)$ (which is clearly convex) in the form:

$$\tilde{f}(w) = \max(wx : x \in B_f) = \sum_{i=1}^{m} \lambda_i f(E_i)$$  \hspace{1cm} (16.36)

where $w = \sum_{i=1}^{m} \lambda_i 1_{E_i}$ and $E_i = \{e_1, \ldots, e_i\}$ defined based on sorted descending order $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$.

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- In both Eq. (16.36) and Eq. (16.37), we have $\tilde{f}(1_A) = f(A)$, $\forall A$, but Eq. (16.37), might not be convex for non-submodular $f$. 
Summary: comparison of the two extension forms

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\tilde{f}(w) = \max(wx : x \in B_f) = \sum_{i=1}^{m} \lambda_i f(E_i) \tag{16.36}
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- In both Eq. (16.36) and Eq. (16.37), we have \( \tilde{f}(1_A) = f(A), \ \forall A \), but Eq. (16.37), might not be convex for non-submodular \( f \).

- Submodularity is sufficient for convexity, but is it also necessary?
The Lovász extension of $f : 2^E \to \mathbb{R}$

- This **continuous extension** $\tilde{f}$ of $f$, in any case ($f$ being submodular or not), is typically called the **Lovász extension** of $f$ (but also sometimes called the Choquet integral, or the Lovász-Edmonds extension since Edmonds showed many of the critical results).
The Lovász extension of $f : 2^E \rightarrow \mathbb{R}$

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- Lovász showed that if a function $\tilde{f}(w)$ defined as in Eqn. (16.30) is convex, then $f$ must be submodular.
Theorem 16.5.2

A function \( f : 2^E \to \mathbb{R} \) is submodular iff its Lovász extension \( \tilde{f} \) of \( f \) is convex.

Proof.

- We’ve already seen that if \( f \) is submodular, its extension can be written via Eqn.(16.30) due to the greedy algorithm, and therefore is also equivalent to \( \tilde{f}(w) = \max \{wx : x \in P_f\} \), and thus is convex.
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- Conversely, suppose the Lovász extension \( \tilde{f}(w) = \sum_i \lambda_i f(E_i) \) of some function \( f : 2^E \rightarrow \mathbb{R} \) is a convex function.
Theorem 16.5.2

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- Conversely, suppose the Lovász extension $\tilde{f}(w) = \sum_i \lambda_i f(E_i)$ of some function $f : 2^E \to \mathbb{R}$ is a convex function.

- We note that, based on the extension definition, in particular the definition of the $\{\lambda_i\}_i$, we have that $\tilde{f}(\alpha w) = \alpha \tilde{f}(w)$ for any $\alpha \in \mathbb{R}_+$. I.e., $f$ is a positively homogeneous convex function.
Earlier, we saw that \( \tilde{f}(1_A) = f(A) \) for all \( A \subseteq E \).
Earlier, we saw that $\tilde{f}(1_A) = f(A)$ for all $A \subseteq E$.

Now, given $A, B \subseteq E$, we will show that

$$\tilde{f}(1_A + 1_B) = \tilde{f}(1_{A \cup B} + 1_{A \cap B})$$

$$= f(A \cup B) + f(A \cap B).$$

(16.38)

(16.39)
... proof of Thm. 16.5.2 cont.

- Earlier, we saw that \( \tilde{f}(1_A) = f(A) \) for all \( A \subseteq E \).
- Now, given \( A, B \subseteq E \), we will show that
  \[
  \tilde{f}(1_A + 1_B) = \tilde{f}(1_{A\cup B} + 1_{A\cap B}) \\
  = f(A \cup B) + f(A \cap B). 
  \]  
  (16.38)  
  (16.39)

- Let \( C = A \cap B \), order \( E \) based on decreasing \( w = 1_A + 1_B \) so that
  
  \[
  w = (w(e_1), w(e_2), \ldots, w(e_m)) \\
  = (2, 2, \ldots, 2, 1, 1, \ldots, 1, 0, 0, \ldots, 0) \\
  \begin{align*}
  &i \in C \\
  &i \in A \Delta B \\
  &i \in E \setminus (A \cup B)
  \end{align*}
  \]  
  (16.40)  
  (16.41)
... proof of Thm. 16.5.2 cont.

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  = (2, 2, \ldots, 2, 1, 1, \ldots, 1, 0, 0, \ldots, 0) \tag{16.41}
  \]
    \begin{array}{ccc}
    i \in C & i \in A \Delta B & i \in E \setminus (A \cup B)
    \end{array}
  \]
- Then, considering \( \tilde{f}(w) = \sum_i \lambda_i f(E_i) \), we have \( \lambda_{|C|} = 1 \), \( \lambda_{|A \cup B|} = 1 \), and \( \lambda_i = 0 \) for \( i \notin \{|C|, |A \cup B|\} \).
Earlier, we saw that $\tilde{f}(1_A) = f(A)$ for all $A \subseteq E$.

Now, given $A, B \subseteq E$, we will show that
\[
\tilde{f}(1_A + 1_B) = \tilde{f}(1_{A \cup B} + 1_{A \cap B}) \\
= f(A \cup B) + f(A \cap B).
\] (16.38) (16.39)

Let $C = A \cap B$, order $E$ based on decreasing $w = 1_A + 1_B$ so that
\[
w = (w(e_1), w(e_2), \ldots, w(e_m)) \\
= (2, 2, \ldots, 2, 1, 1, \ldots, 1, 0, 0, \ldots, 0)
\] (16.40) (16.41)

Then, considering $\tilde{f}(w) = \sum_i \lambda_i f(E_i)$, we have $\lambda_{|C|} = 1$, $\lambda_{|A \cup B|} = 1$, and $\lambda_i = 0$ for $i \notin \{|C|, |A \cup B|\}$.

But then $E_{|C|} = A \cap B$ and $E_{|A \cup B|} = A \cup B$. Therefore,
\[
\tilde{f}(w) = \tilde{f}(1_A + 1_B) = f(A \cap B) + f(A \cup B).
\]
Also, since $\tilde{f}$ is convex (by assumption) and positively homogeneous, we have for any $A, B \subseteq E$,

$$0.5[f(A \cap B) + f(A \cup B)]$$

(16.45)
... proof of Thm. 16.5.2 cont.

Also, since $\tilde{f}$ is convex (by assumption) and positively homogeneous, we have for any $A, B \subseteq E$,

$$0.5[f(A \cap B) + f(A \cup B)] = 0.5[\tilde{f}(1_A + 1_B)]$$

(16.42)
... proof of Thm. 16.5.2 cont.

Also, since $\tilde{f}$ is convex (by assumption) and positively homogeneous, we have for any $A, B \subseteq E$,

$$0.5[f(A \cap B) + f(A \cup B)] = 0.5[\tilde{f}(1_A + 1_B)]$$

$$= \tilde{f}(0.51_A + 0.51_B)$$

(16.42)

(16.43)

(16.44)
Also, since $\tilde{f}$ is convex (by assumption) and positively homogeneous, we have for any $A, B \subseteq E$,

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$$= \tilde{f}(0.51_A + 0.51_B)$$

$$\leq 0.5\tilde{f}(1_A) + 0.5\tilde{f}(1_B)$$
... proof of Thm. 16.5.2 cont.

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0.5[f(A \cap B) + f(A \cup B)] = 0.5[\tilde{f}(1_A + 1_B)]
\]

\[
= \tilde{f}(0.51_A + 0.51_B)
\]

\[
\leq 0.5\tilde{f}(1_A) + 0.5\tilde{f}(1_B)
\]

\[
= 0.5(f(A) + f(B))
\]
... proof of Thm. 16.5.2 cont.

- Also, since \( \tilde{f} \) is convex (by assumption) and positively homogeneous, we have for any \( A, B \subseteq E \),

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0.5[f(A \cap B) + f(A \cup B)] = 0.5[\tilde{f}(1_A + 1_B)] = \tilde{f}(0.51_A + 0.51_B) \\
\leq 0.5\tilde{f}(1_A) + 0.5\tilde{f}(1_B) = 0.5(f(A) + f(B))
\]

- Thus, we have shown that for any \( A, B \subseteq E \),

\[
f(A \cup B) + f(A \cap B) \leq f(A) + f(B)
\]

so \( f \) must be submodular.
The above theorem showed that the Lovász extension is convex iff $f$ is submodular.
Lovász ext. vs. the concave closure of submodular function

- The above theorem showed that the Lovász extension is convex iff $f$ is submodular.

- Our next theorem shows that the Lovász extension coincides precisely with the convex closure iff $f$ is submodular.
Lovász ext. vs. the concave closure of submodular function

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- Our next theorem shows that the Lovász extension coincides precisely with the convex closure iff $f$ is submodular.
- I.e., not only is the Lovász extension convex for $f$ submodular and tight at each $1_A$, it is the actual convex closure when $f$. 
The above theorem showed that the Lovász extension is convex iff $f$ is submodular.

Our next theorem shows that the Lovász extension coincides precisely with the convex closure iff $f$ is submodular.

I.e., not only is the Lovász extension convex for $f$ submodular and tight at each $1_A$, it is the actual convex closure when $f$.

Hence, convex closure (which normally looks daunting) is easy to evaluate (the greedy algorithm) when $f$ is submodular and takes this particular form iff $f$ is submodular.
**Theorem 16.5.3**

Let \( \tilde{f}(w) = \max(wy : y \in B_f) = \sum_{i=1}^{m} \lambda_i f(E_i) \) be the Lovász extension and \( \tilde{f}(x) = \min_{p \in \Delta^n(x)} ES_{\sim p}[f(S)] \) be the convex closure. Then \( \tilde{f} \) and \( \tilde{f} \) coincide iff \( f \) is submodular, i.e., \( \tilde{f}(w) = \tilde{f}(w), \forall w \in [0, 1] \).

**Proof.**

- Assume \( f \) is submodular.
Lovász ext. vs. the concave closure of submodular function

**Theorem 16.5.3**

Let \( \tilde{f}(w) = \max(wy : y \in B_f) = \sum_{i=1}^{m} \lambda_i f(E_i) \) be the Lovász extension and \( \check{f}(x) = \min_{p \in \Delta^n(x)} E_{S \sim p}[f(S)] \) be the convex closure. Then \( \tilde{f} \) and \( \check{f} \) coincide iff \( f \) is submodular, i.e., \( \tilde{f}(w) = \check{f}(w), \forall w \in [0, 1] \).

**Proof.**

- Assume \( f \) is submodular.
- Given \( x \), let \( p^x \) be an achieving argmin in \( \tilde{f}(x) \) that also maximizes \( \sum_{S} p^x_S |S|^2 \).
Lovász ext. vs. the concave closure of submodular function

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**Proof.**

- Assume \( f \) is submodular.
- Given \( x \), let \( p^x \) be an achieving argmin in \( \tilde{f}(x) \) that also maximizes \( \sum_S p^x_S |S|^2 \).
- Suppose \( \exists A, B \subseteq V \) that are crossing (i.e., \( A \not\subseteq B, B \not\subseteq A \)) with positive \( p^x_A, p^x_B \). W.l.o.g., \( p^x_A \geq p^x_B > 0 \).
Lovász ext. vs. the concave closure of submodular function

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**Proof.**

- Assume \( f \) is submodular.
- Given \( x \), let \( p^x \) be an achieving argmin in \( \tilde{f}(x) \) that also maximizes \( \sum_S p^x_S |S|^2 \).
- Suppose \( \exists A, B \subseteq V \) that are crossing (i.e., \( A \nsubseteq B, B \nsubseteq A \)) with positive \( p^x_A, p^x_B \). W.l.o.g., \( p^x_A \geq p^x_B > 0 \).
- Then we may update \( p^x \), keeping it a distribution, as follows:

\[
\begin{align*}
\tilde{p}_A^x &\leftarrow p_A^x - p_B^x \\
\tilde{p}_B^x &\leftarrow p_B^x - p_B^x \\
\tilde{p}_{A \cup B}^x &\leftarrow p_{A \cup B}^x + p_B^x \\
\tilde{p}_{A \cap B}^x &\leftarrow p_{A \cap B}^x + p_B^x
\end{align*}
\]  \hspace{1cm} (16.47)

and by submodularity, this does not increase \( \sum_S p^x_S f(S) \). ...
This does increase $\sum_S p^x_S |S|^2$ however since $|S|^2$ is supermodular, i.e.:

$$|A \cup B|^2 + |A \cap B|^2 = (|A| + |B \setminus A|)^2 + (|B| - |B \setminus A|)^2$$

$$= |A|^2 + |B|^2 + 2|B \setminus A||A| - |B| + |B \setminus A|$$

$$= |A|^2 + |B|^2 + 2|B \setminus A||A \setminus B|$$

$$> |A|^2 + |B|^2$$
This does increase \( \sum_S p^x_S |S|^2 \) however since \( |S|^2 \) is supermodular, i.e.:

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|A \cup B|^2 + |A \cap B|^2 = (|A| + |B \setminus A|)^2 + (|B| - |B \setminus A|)^2
\]

\[
= |A|^2 + |B|^2 + 2|B \setminus A|(|A| - |B| + |B \setminus A|)
\]

\[
= |A|^2 + |B|^2 + 2|B \setminus A||A \setminus B|
\]

\[
> |A|^2 + |B|^2
\]

Contradiction! Hence, there can be no crossing sets \( A, B \) and we must have, for any \( A, B \) with \( p^x_A > 0 \) and \( p^x_B > 0 \) either \( A \subset B \) or \( B \subset A \).
...proof cont.

- This does increase \( \sum_S p^x_S |S|^2 \) however since \( |S|^2 \) is supermodular, i.e.:
  \[
  |A \cup B|^2 + |A \cap B|^2 = (|A| + |B \setminus A|)^2 + (|B| - |B \setminus A|)^2
  \]
  \[
  = |A|^2 + |B|^2 + 2|B \setminus A||A| - |B| + 2|B \setminus A||B \setminus A|
  \]
  \[
  = |A|^2 + |B|^2 + 2|B \setminus A||A \setminus B|
  \]
  \[
  > |A|^2 + |B|^2
  \]  (16.49)  (16.50)  (16.51)  (16.52)

- Contradiction! Hence, there can be no crossing sets \( A, B \) and we must have, for any \( A, B \) with \( p^x_A > 0 \) and \( p^x_B > 0 \) either \( A \subset B \) or \( B \subset A \).

- Hence, the sets \( \{ A \subseteq V : p^x_A > 0 \} \) form a chain and can be at most as large as size \( n = |V| \).
This does increase $\sum_S p^x_S |S|^2$ however since $|S|^2$ is supermodular, i.e.:

$$|A \cup B|^2 + |A \cap B|^2 = (|A| + |B \setminus A|)^2 + (|B| - |B \setminus A|)^2$$

$$= |A|^2 + |B|^2 + 2|B \setminus A||A| - |B| + |B \setminus A||A \setminus B|$$

$$= |A|^2 + |B|^2 + 2|B \setminus A| |A \setminus B|$$

$$> |A|^2 + |B|^2$$

Contradiction! Hence, there can be no crossing sets $A, B$ and we must have, for any $A, B$ with $p^x_A > 0$ and $p^x_B > 0$ either $A \subset B$ or $B \subset A$.

Hence, the sets $\{A \subseteq V : p^x_A > 0\}$ form a chain and can be at most as large as size $n = |V|$.

The only feasible chain is the same chain that defines the Lovász extension $\tilde{f}(x)$, namely $\emptyset = E_0 \subseteq E_1 \subseteq E_2 \subset \ldots$ where $E_i = \{e_1, e_2, \ldots, e_i\}$ and $e_i$ is ordered as $x(e_1) \geq x(e_2) \geq \cdots \geq x(e_n)$. 
Next, assume $f$ is not submodular. We must show that the Lovász extension $\tilde{f}(x)$ and the concave closure $\tilde{\tilde{f}}(x)$ need not coincide.
Next, assume $f$ is not submodular. We must show that the Lovász extension $\tilde{f}(x)$ and the concave closure $\tilde{\tilde{f}}(x)$ need not coincide.

Since $f$ is not submodular, $\exists S$ and $i, j \notin S$ such that $f(S) + f(S + i + j) > f(S + i) + f(S + j)$, a strict violation of submodularity.
Next, assume $f$ is not submodular. We must show that the Lovász extension $\tilde{f}(x)$ and the concave closure $\tilde{\tilde{f}}(x)$ need not coincide.

Since $f$ is not submodular, $\exists S$ and $i, j \notin S$ such that $f(S) + f(S + i + j) > f(S + i) + f(S + j)$, a strict violation of submodularity.

Consider $x = 1_S + \frac{1}{2}1_{\{i,j\}}$. 

\[
\tilde{f}(x) = 1_S + \frac{1}{2}1_{\{i,j\}} \quad \text{and} \quad \tilde{\tilde{f}}(x) = 1_S + \frac{1}{2}1_{\{i,j\}}.
\]

This gives $\tilde{\tilde{f}}(x) \leq \frac{1}{2}[f(S + i) + f(S + j)] < \tilde{f}(x)$, meaning $\tilde{\tilde{f}}(x) \neq \tilde{f}(x)$.
Next, assume \( f \) is not submodular. We must show that the Lovász extension \( \tilde{f}(x) \) and the concave closure \( \tilde{\tilde{f}}(x) \) need not coincide.

Since \( f \) is not submodular, \( \exists S \) and \( i, j \notin S \) such that
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\]
a strict violation of submodularity.

Consider \( x = 1_S + \frac{1}{2} 1_{\{i,j\}} \).

Then L.E. has \( \tilde{f}(x) = \frac{1}{2} f(S) + \frac{1}{2} f(S + i + j) \) and this \( p \) is feasible for \( \tilde{f}(x) \) with \( p_S = 1/2 \) and \( p_{S+i+j} = 1/2 \).
Next, assume $f$ is not submodular. We must show that the Lovász extension $\tilde{f}(x)$ and the concave closure $\tilde{\tilde{f}}(x)$ need not coincide.

Since $f$ is not submodular, $\exists S$ and $i, j \notin S$ such that $f(S) + f(S + i + j) > f(S + i) + f(S + j)$, a strict violation of submodularity.

Consider $x = 1_S + \frac{1}{2} 1_{\{i,j\}}$.

Then L.E. has $\tilde{f}(x) = \frac{1}{2} f(S) + \frac{1}{2} f(S + i + j)$ and this $p$ is feasible for $\tilde{\tilde{f}}(x)$ with $p_S = 1/2$ and $p_{S+i+j} = 1/2$.

An alternate feasible distribution for $\tilde{f}(x)$ in the convex closure is $\bar{p}_{S+i} = \bar{p}_{S+j} = 1/2$. 
Next, assume $f$ is not submodular. We must show that the Lovász extension $\tilde{f}(x)$ and the concave closure $\tilde{\tilde{f}}(x)$ need not coincide.

Since $f$ is not submodular, $\exists S$ and $i, j \notin S$ such that $f(S) + f(S + i + j) > f(S + i) + f(S + j)$, a strict violation of submodularity.

Consider $x = 1_S + \frac{1}{2} 1_{\{i, j\}}$.

Then L.E. has $\tilde{f}(x) = \frac{1}{2} f(S) + \frac{1}{2} f(S + i + j)$ and this $p$ is feasible for $\tilde{f}(x)$ with $p_S = 1/2$ and $p_{S+i+j} = 1/2$.

An alternate feasible distribution for $\tilde{f}(x)$ in the convex closure is $\bar{p}_{S+i} = \bar{p}_{S+j} = 1/2$.

This gives

$$\tilde{f}(x) \leq \frac{1}{2} [f(S + i) + f(S + j)] < \tilde{\tilde{f}}(x)$$  \hspace{1cm} (16.53)

meaning $\tilde{f}(x) \neq \tilde{\tilde{f}}(x)$. 