Submodular Functions, Optimization, and Applications to Machine Learning

— Fall Quarter, Lecture 14 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/

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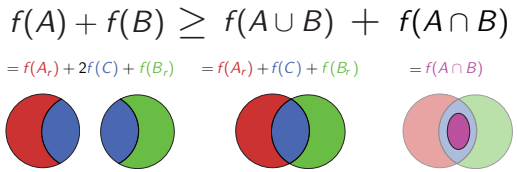
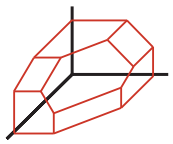
Nov 18th, 2020





Announcements, Assignments, and Reminders

- Homework 3, out, due next Wednesday, Nov 25th, 2020, 11:59pm.
- Office hours this week, Wed & Thur, 10:00pm at our class zoom link.
- Next week office hours, Tues (11/24) & Wed (11/25), 10:00pm at our class zoom link. I can meet Monday night at 10:00pm as well (let me know if you want to meet then).
- Reminder, all lectures are being recorded and posted to YouTube. To get the links, see our announcements (`https://canvas.uw.edu/courses/1397085/announcements`).

Class Road Map - EE563

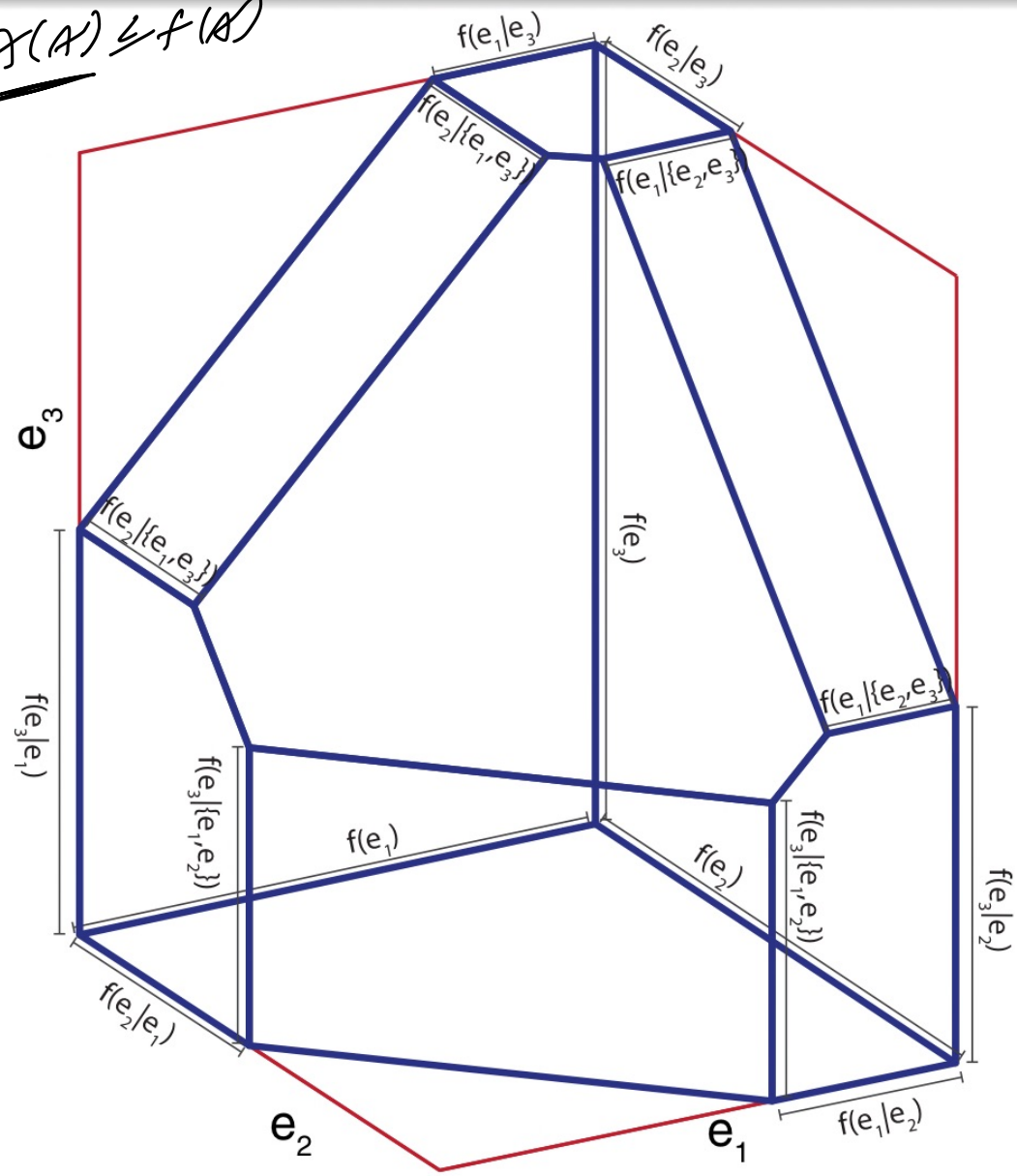
- L1(9/30): Motivation, Applications, Definitions, Properties
- L2(10/5): Sums concave(modular), uses (diversity/costs, feature selection), information theory
- L3(10/7): Monge, More Definitions, Graph and Combinatorial Examples,
- L4(10/12): Graph & Combinatorial Examples, Matrix Rank, Properties, Other Defs, Independence
- L5(10/14): Properties, Defs of Submodularity, Independence
- L6(10/19): Matroids, Matroid Examples, Matroid Rank,
- L7(10/21): Matroid Rank, More on Partition Matroid, Laminar Matroids, System of Distinct Reps, Transversals
- L8(10/26): Transversal Matroid, Matroid and representation, Dual Matroid
- L9(10/28): Other Matroid Properties, Combinatorial Geometries, Matroid and Greedy, Polyhedra, Matroid Polytopes
- L10(11/2): Matroid Polytopes, Matroids $\rightarrow$ Polymatroids
- L11(11/4): Matroids $\rightarrow$ Polymatroids, Polymatroids
- L12(11/9): Polymatroids, Polymatroids and Greedy
- L–(11/11): Veterans Day, Holiday
- L13(11/16): Polymatroids and Greedy, Possible Polytopes, Extreme Points, Cardinality Constrained Maximization
- L14(11/18): Cardinality Constrained Maximization, Curvature
- L15(11/23):
- L16(11/25):
- L17(11/30):
- L18(12/2):
- L19(12/7):
- L20(12/9): maximization.

Last day of instruction, Fri. Dec 11th. Finals Week: Dec 12-18, 2020

Polymatroid with labeled edge lengths

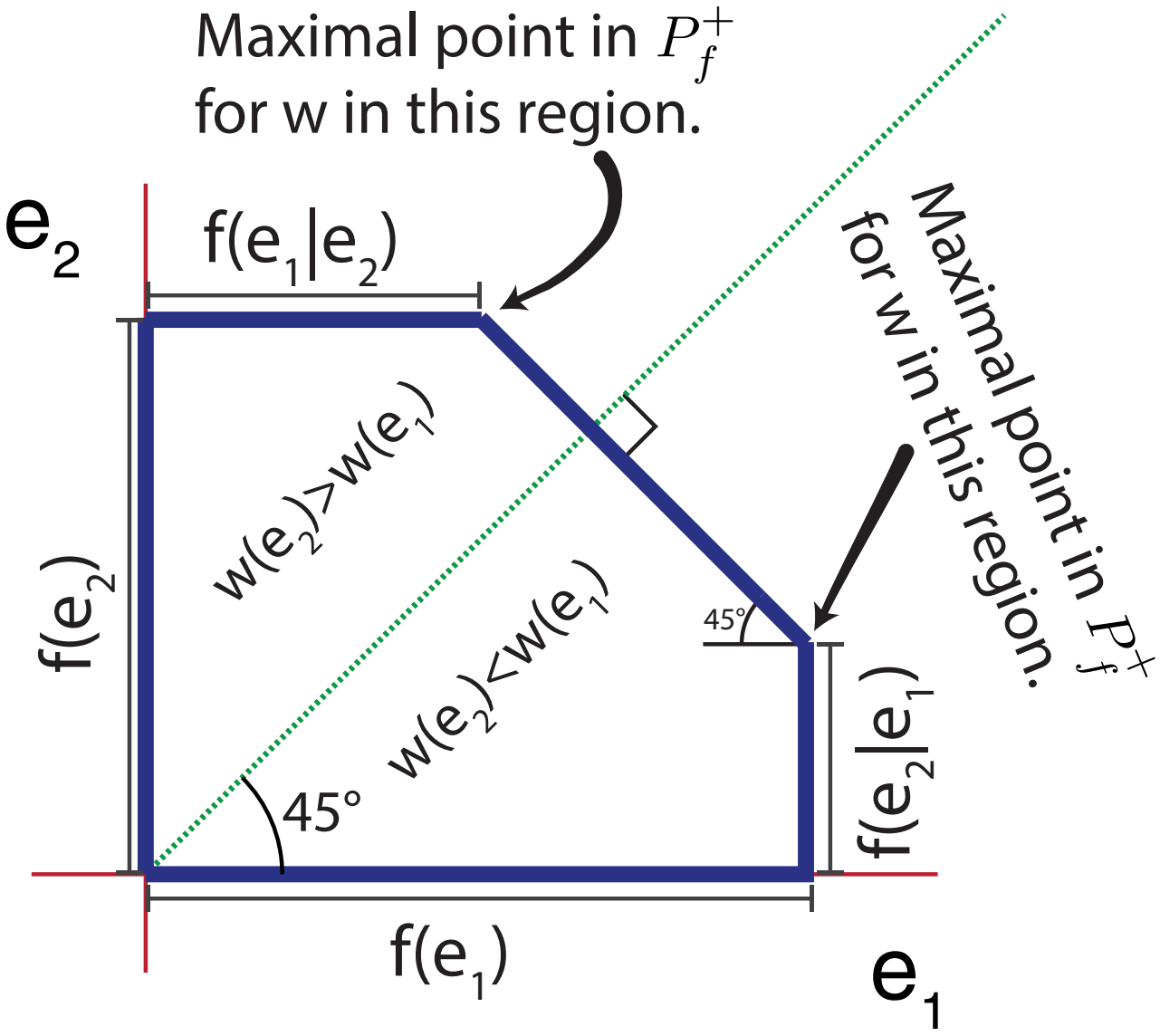
- Recall $f(e|A) = f(A+e) - f(A)$
- Notice how submodularity, $f(e|B) \leq f(e|A)$ for $A \subseteq B$, defines the shape of the polytope.
- In fact, we have strictness here $f(e|B) < f(e|A)$ for $A \subset B$.
- Also, consider how the greedy algorithm proceeds along the edges of the polytope.

$f(A) \leq f(B)$



Intuition: why greedy works with polymatroids

- Given w, the goal is to find $x = (x(e_1), x(e_2))$ that maximizes $x^\intercal w = x(e_1)w(e_1) + x(e_2)w(e_2)$.
- If $w(e_2) > w(e_1)$ the upper extreme point indicated maximizes $x^\intercal w$ over $x \in P_f^+$.
- If $w(e_2) < w(e_1)$ the lower extreme point indicated maximizes $x^\intercal w$ over $x \in P_f^+$.



Maximization of Submodular Functions

- Submodular maximization is quite useful.
- Applications: sensor placement, facility location, document summarization, or any kind of covering problem (choose a small set of elements that cover some domain as much as possible).
- f in this case is a model of dispersion, diversity, representativeness, or information. $\max f(A)$ s.t. $A \subseteq V$, $|A| \leq k$
- For polymatroid function (or any monotone non-decreasing function), unconstrained maximization is trivial (take ground set).
- Thus, when we do monotone submodular maximization we find the maximum under some constraint.
- There is also a sort of dual problem that is often considered together with max, and those are minimum cover problems (to be defined).

The Set Cover Problem

- Let E be a set and let $\{E_i\}_{i=1}^m$ be a set of m subsets $E_i \subseteq E$ with $\bigcup_i E_i = E$.
- Let $V = \{1, 2, \ldots, m\}$ be the set of integers.
- Define $f : 2^V \to \mathbb{Z}_+$ as $f(X) = |\bigcup_{v \in X} E_v|$
- Then f is the set cover function. As we say, f is monotone submodular (a polymatroid).
- The set cover problem asks for the smallest subset X of V such that $f(X) = |E|$ (smallest subset of the subsets of E) where E is still covered. I.e.,

$\bigcup_{i \in X} E_i = E$

$$\text{minimize} |X| \text{ subject to } f(X) \geq |E| \tag{14.1}$$

- We might wish to use a more general modular function $m(X)$ rather than cardinality $|X|$.
- This problem is NP-hard, and Feige in 1998 showed that it cannot be approximated with ratio better than $(1 - \epsilon) \log n$ for any $\epsilon > 0$ unless NP is slightly superpolynomial ($n^{O(\log \log n)}$).

The Max k-Cover Problem

- Let E be a set and let $E_1, E_2, \ldots, E_m$ be a set of subsets.
- Let $V = \{1, 2, \ldots, m\}$ be the set of integers.
- Define $f : 2^V \to \mathbb{Z}_+$ as $f(X) = |\bigcup_{v \in V} E_v|$
- Then f is the set cover function. As we saw, f is monotone submodular (a polymatroid).
- The max k cover problem asks, given a k, what sized k set of sets X can we choose that covers the most? I.e., that maximizes $f(X)$ as in:

$$\max f(X) \text{ subject to } |X| \leq k \tag{14.1}$$

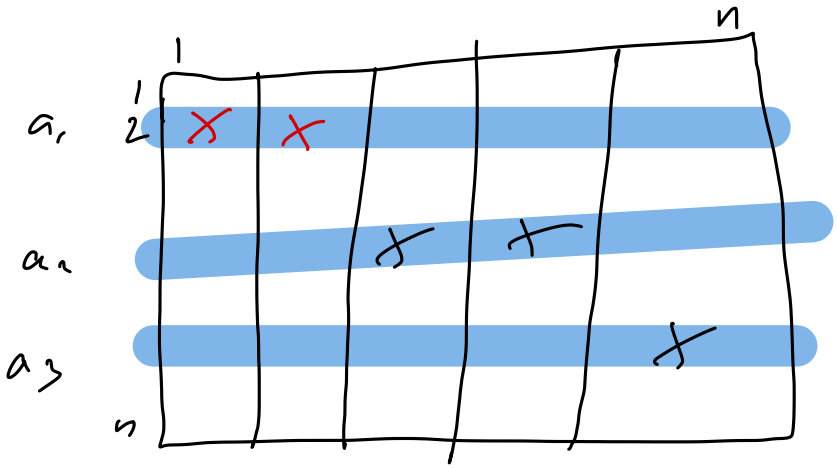
- This problem is NP-hard, and Feige in 1998 showed that it cannot be approximated with a ratio better than $(1 - 1/e)$.

Cardinality Constrained Max. of Facility Location

- Recall the (uncapacitated) facility location function $f(A) = \sum_{v \in V} \max_{a \in A} s_{a,v}$ where $s_{a,v}$ is the similarity (affinity, value) between a and v. Alternatively, we can think of this as a representativeness matrix, where $s_{a,v}$ is seen as how how good a is as acting as a representative for v (which might not be the same as $s_{v,a}$).

clients

possible facilities



$A = \{a_1, a_2, a_3\}$

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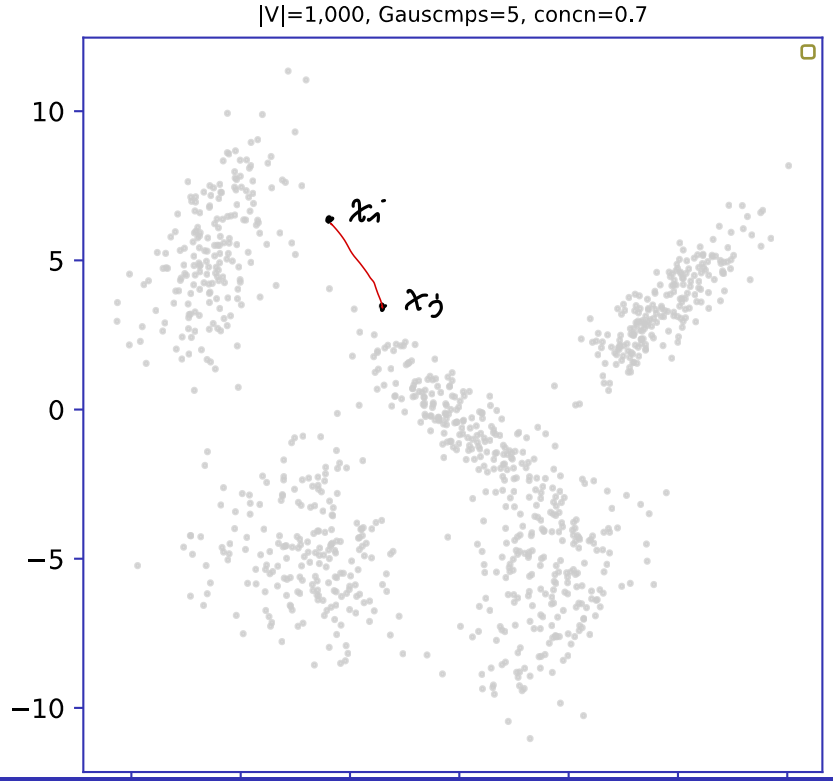
  Example:

$s_{ij} = \exp(-d(x_i, x_j))$

$d(x_i, x_j) = \|x_i - x_j\|_2$



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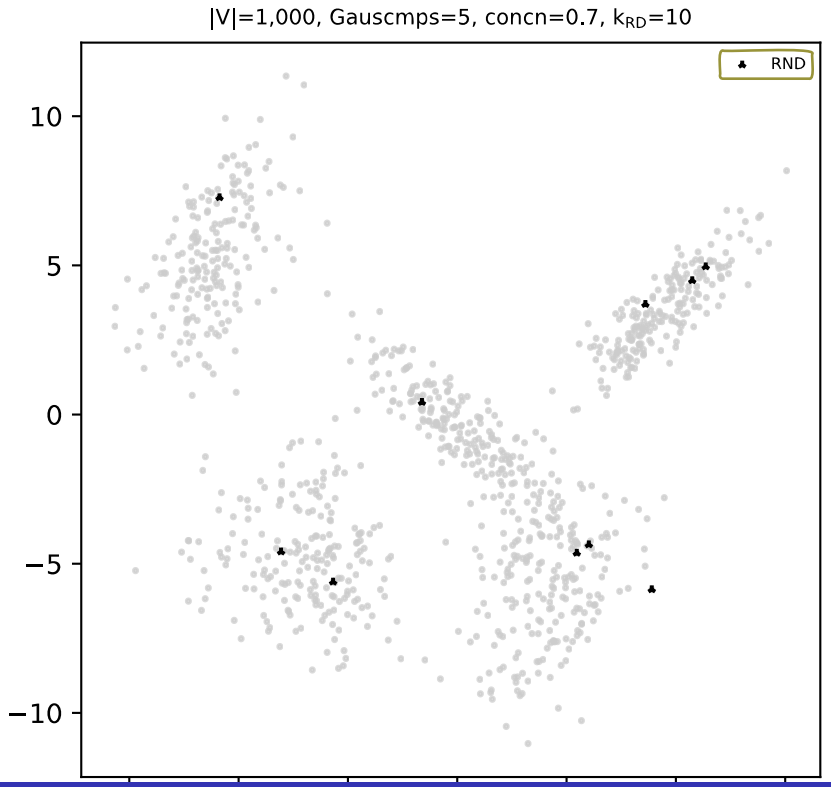

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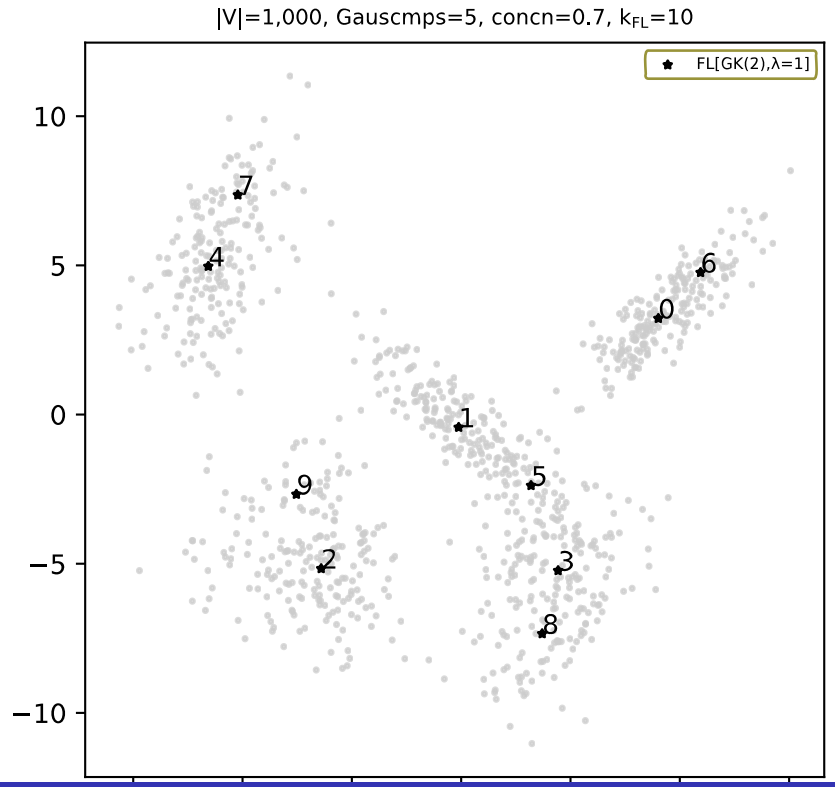

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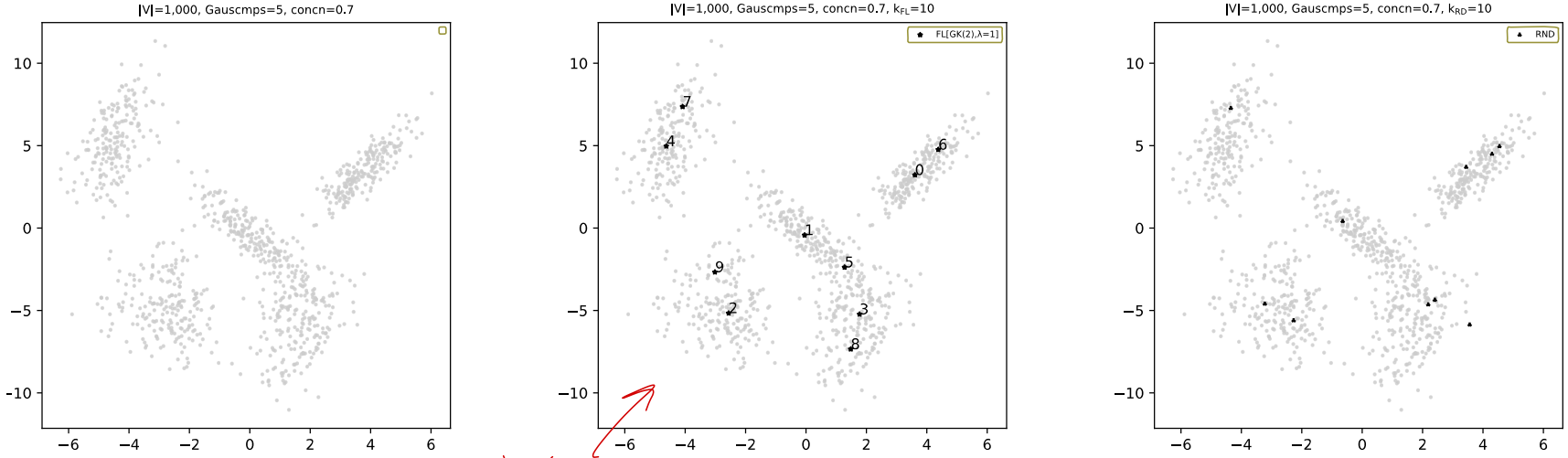
- Middle example is estimate of $\max_{A \subseteq V : |A| \leq k} f(A)$, right is uniformly-at-random randomly chosen set of size k, for $k = 10$.



$1/\binom{1000}{10}$

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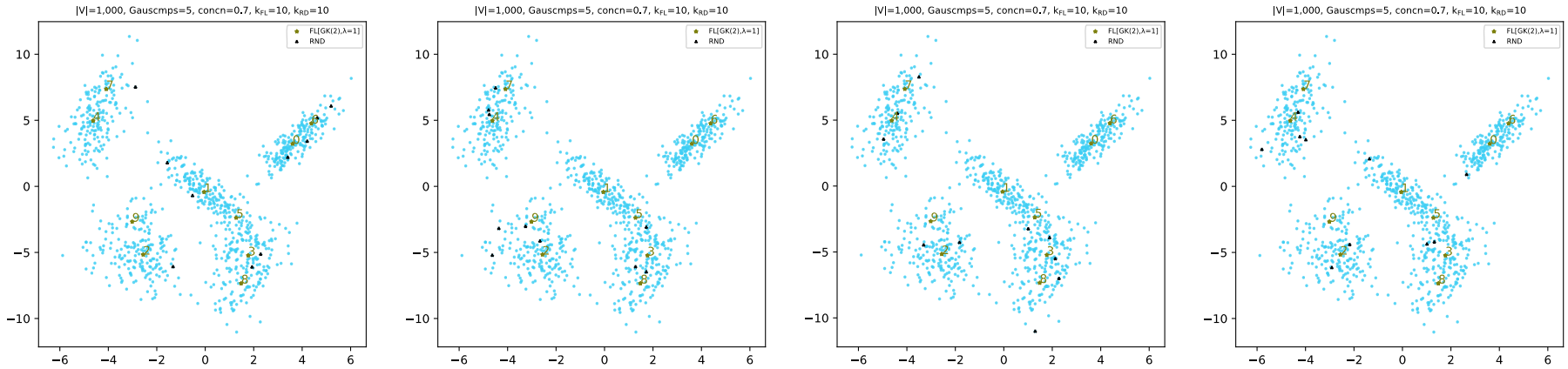
- Middle example is estimate of $\max_{A \subseteq V : |A| \leq k} f(A)$, right is uniformly-at-random randomly chosen set of size k, for $k = 10$.
- Multiple uniformly-at-random random summaries.



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- An important result by Nemhauser et. al. (1978) states that for normalized ($f(\emptyset) = 0$) monotone submodular functions (i.e., polymatroids) can be approximately maximized using a simple greedy algorithm.

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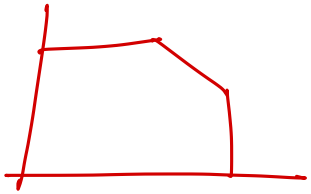
- An important result by Nemhauser et. al. (1978) states that for normalized ($f(\emptyset) = 0$) monotone submodular functions (i.e., polymatroids) can be approximately maximized using a simple greedy algorithm.
- Starting with $S_0 = \emptyset$, we repeat the following greedy step for $i = 0 \ldots (k-1)$:

$$S_{i+1} = S_i \cup \left\{ \operatorname*{argmax}_{v \in V \setminus S_i} f(S_i \cup \{v\}) \right\} \tag{14.1}$$

The Greedy Algorithm for Submodular Max

$\max(w \cdot x : x \in P_f)$



A bit more precisely:

Algorithm 1: The Greedy Algorithm

Set $S_0 \leftarrow \emptyset$;
for $i \leftarrow 0 \ldots |E| - 1$ **do**
  Choose v_i as follows:
  $v_i \in \text{argmax}_{v \in V \setminus S_i} f(\{v\}|S_i) = \text{argmax}_{v \in V \setminus S_i} f(S_i \cup \{v\})$;
  Set $S_{i+1} \leftarrow S_i \cup \{v_i\}$;

$O(n^2 \cdot OE)$

$f(v|S_i) = f(S_i + v) - f(S_i)$

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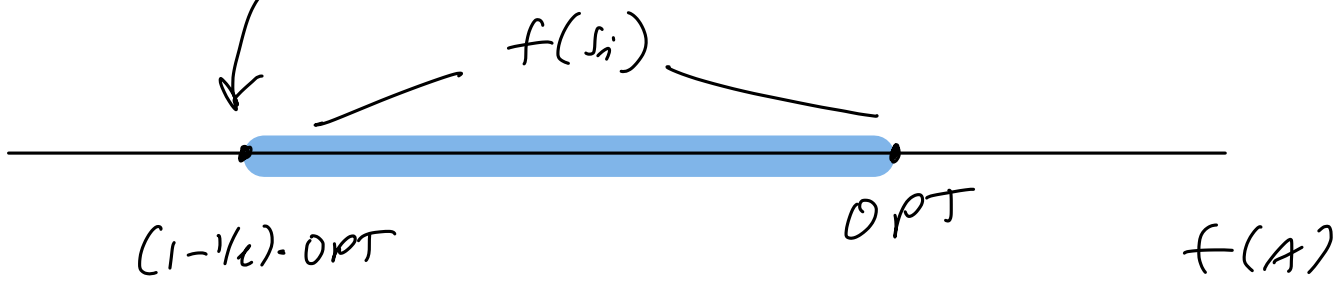
Theorem 14.3.1

Given a polymatroid function f , the above greedy algorithm returns sets S_i such that for each i we have $f(S_i) \geq (1 - 1/e) \max_{|S| \leq i} f(S)$.

OPT_i



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- We can think of a "greedy operator" $\tilde{A} \in \operatorname{gargmax} \{f(A) : |A| \leq k\}$
- Again, since this generalizes max k-cover, Feige (1998) showed that this can't be improved. Unless $P = NP$, no polynomial time algorithm can do better than $(1 - 1/e + \epsilon)$ for any $\epsilon > 0$.

The Greedy Algorithm: $1 - 1/e$ intuition.

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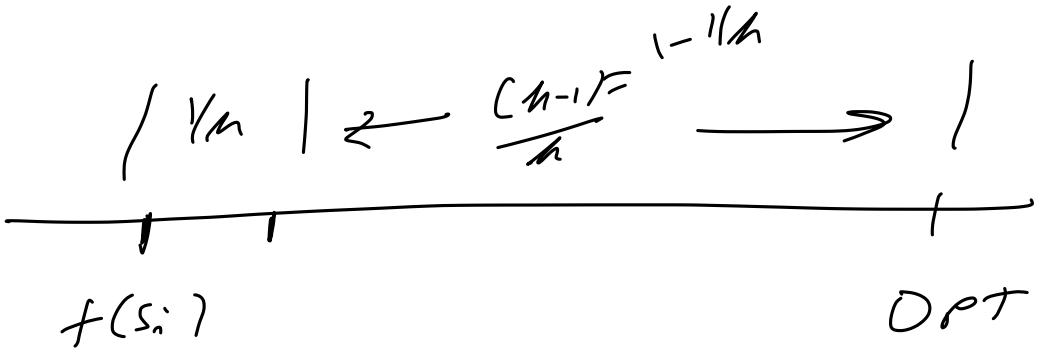
$$\exists v \in V \setminus S_i : f(v|S_i) = f(S_i + v|S_i) \geq \frac{1}{k}(\text{OPT} - f(S_i)) \tag{14.2}$$



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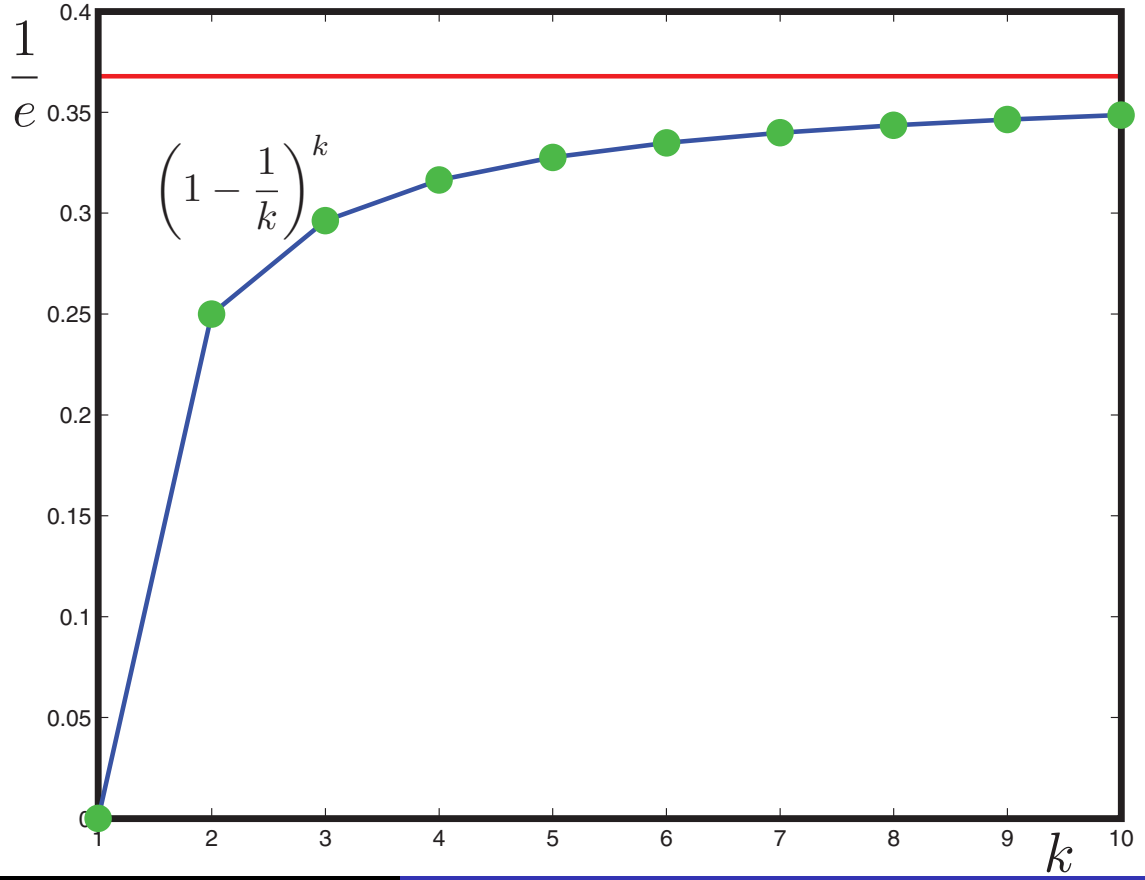

Equation (14.11) will show that Equation (14.2) $\Rightarrow$:

$$\begin{aligned}
&\text{OPT} - f(S_{i+1}) \\
&\qquad \leq (1 - 1/k)(\text{OPT} - f(S_i)) \\
\Rightarrow\ &\text{OPT} - f(S_k) \\
&\qquad \leq (1 - 1/k)^k \text{OPT} \\
&\qquad \leq 1/e\text{OPT} \\
\Rightarrow\ &\text{OPT}(1 - 1/e) \leq f(S_k)
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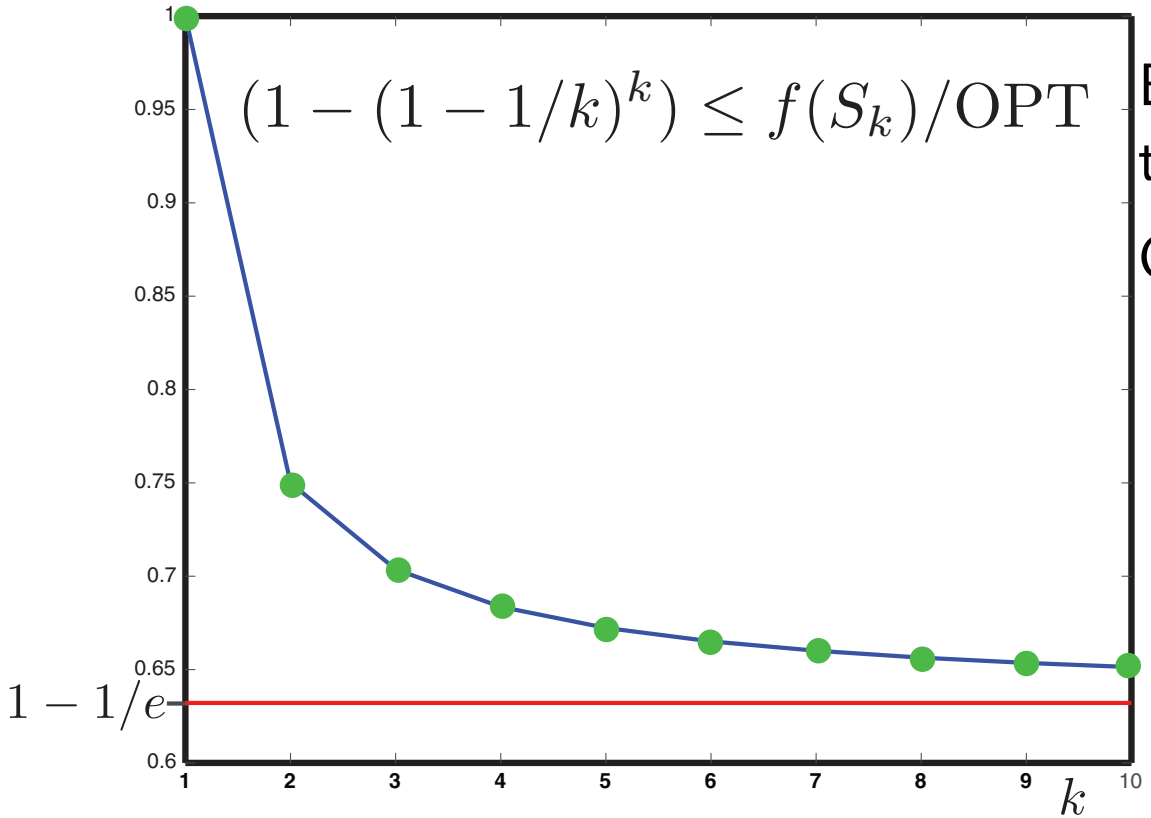

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Cardinality Constrained Polymatroid Max Theorem

Theorem 14.3.2 (Nemhauser et al. 1978)

Given non-negative monotone submodular function $f : 2^V \to \mathbb{R}_+$, define $\{S_i\}_{i \geq 0}$ to be the chain formed by the greedy algorithm (Eqn. (14.1)). Then for all $k, \ell \in \mathbb{Z}_{++}$, we have:

$$f(S_\ell) \geq (1 - e^{-\ell/k}) \overbrace{\max_{S:|S| \leq k} f(S)}^{OPT_k} \tag{14.3}$$

and in particular, for $\ell = k$, we have $f(S_k) \geq (1 - 1/e) \max_{S:|S| \leq k} f(S)$.

$OPT = f(S_k^*)$

S_ℓ

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- Intuitively, bound should get worse when $\ell < k$ and get better when $\ell > k$.

Cardinality Constrained Polymatroid Max Theorem

Proof of Theorem 14.3.2.

. . .

Cardinality Constrained Polymatroid Max Theorem

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Cardinality Constrained Polymatroid Max Theorem

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- Order $S^* = (v_1^*, v_2^*, \ldots, v_k^*)$ arbitrarily.
- Let $S_i = (v_1, v_2, \ldots, v_i)$ be the greedy order chain chosen by the algorithm, for $i \in \{1, 2, \ldots, \ell\}$.

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- Let $S_i = (v_1, v_2, \ldots, v_i)$ be the greedy order chain chosen by the algorithm, for $i \in \{1, 2, \ldots, \ell\}$.
- Then the following inequalities (on the next slide) follow:

...

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 14.3.2 cont.

. . .

Cardinality Constrained Polymatroid Max Theorem

...proof of Theorem 14.3.2 cont.

- For all $i < \ell$, we have

$$f(S^*)$$

...

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 14.3.2 cont.

- For all $i < \ell$, we have

$$f(S^*) \leq f(S^* \cup S_i)$$

. . .

Cardinality Constrained Polymatroid Max Theorem

...proof of Theorem 14.3.2 cont.

- For all $i < \ell$, we have

$$f(S^*) \leq f(S^* \cup S_i) = f(S_i) + f(S^*|S_i) \tag{14.4}$$

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Cardinality Constrained Polymatroid Max Theorem

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Cardinality Constrained Polymatroid Max Theorem

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$$\leq f(S_i) + \sum_{v \in S^*} f(v_{i+1}|S_i) = f(S_i) + \sum_{v \in S^*} f(S_{i+1}|S_i) \tag{14.7}$$

$$f(v_{i+1}|S_i) = f(S_i + v_{i+1}) - f(S_i)$$
$$= f(S_{i+1}) - f(S_i)$$
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$$= f(S_i) + k f(S_{i+1}|S_i) \tag{14.8}$$

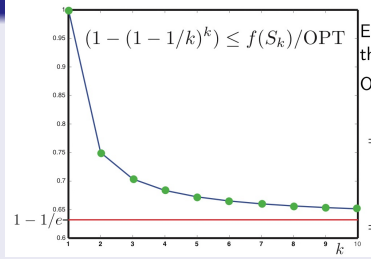
...

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 14.3.2 cont.

- Let S^* be optimal solution (of size k) and OPT $= f(S^*)$. By submodularity, we will show:

$$\exists v \in V \setminus S_i : f(v|S_i) = f(S_i + v|S_i) \geq \frac{1}{k}(\text{OPT} - f(S_i)) \tag{14.2}$$



Equation (14.11) will show that Equation (14.2) $\Rightarrow$:

$$\begin{aligned} &\text{OPT} - f(S_{i+1}) \\ &\quad \leq (1-1/k)(\text{OPT} - f(S_i)) \\ \Rightarrow\ &\text{OPT} - f(S_k) \\ &\quad \leq (1-1/k)^k \text{OPT} \\ &\quad \leq 1/e \text{OPT} \\ \Rightarrow\ &\text{OPT}(1-1/e) \leq f(S_k) \end{aligned}$$

- For all $i < \ell$, we have

$$\begin{aligned} f(S^*) &\leq f(S^* \cup S_i) = f(S_i) + f(S^*|S_i) \\ &= f(S_i) + \sum_{j=1}^{k} f(v_j^*|S_i \cup \{v_1^*, v_2^*, \ldots, v_{j-1}^*\}) && (14.5) \\ &\leq f(S_i) + \sum_{v \in S^*} f(v|S_i) && (14.6) \\ &\leq f(S_i) + \sum_{v \in S^*} f(v_{i+1}|S_i) = f(S_i) + \sum_{v \in S^*} f(S_{i+1}|S_i) && (14.7) \\ &= f(S_i) + k f(S_{i+1}|S_i) && (14.8) \end{aligned}$$

- Therefore, we have shown Equation 14.2 above, i.e.,:

$$f(S^*) - f(S_i) \leq k f(S_{i+1}|S_i) = k(f(S_{i+1}) - f(S_i)) \tag{14.9}$$

. . .

Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 14.3.2 cont.

Cardinality Constrained Polymatroid Max Theorem

...proof of Theorem 14.3.2 cont.

- Define the gap $\delta_i \triangleq f(S^*) - f(S_i)$. Thus $\delta_i - \delta_{i+1} = f(S_{i+1}) - f(S_i)$,

Cardinality Constrained Polymatroid Max Theorem

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or rearranging slightly

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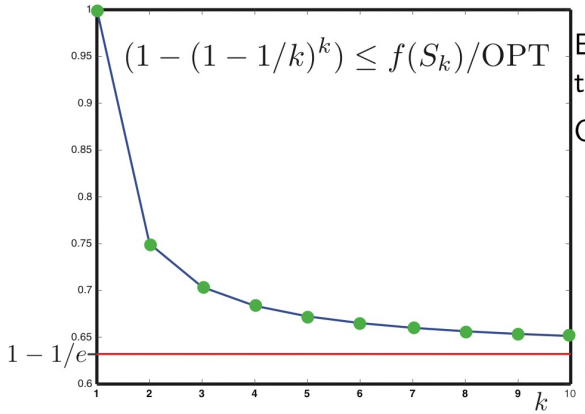

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$$\delta_\ell \leq (1 - \frac{1}{k})^\ell \delta_0 \tag{14.12}$$

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- Now, $\delta_0 = f(S^*) - f(\emptyset) \leq f(S^*)$ since $f \geq 0$.

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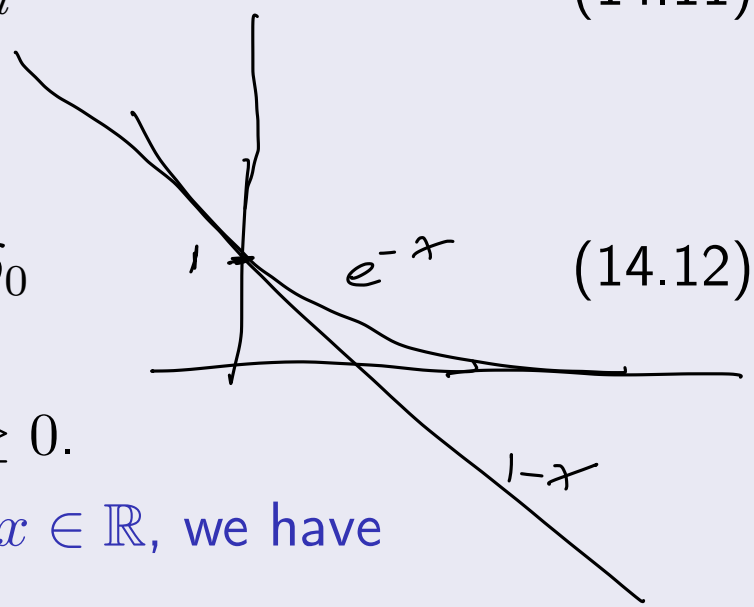

- Now, $\delta_0 = f(S^*) - f(\emptyset) \leq f(S^*)$ since $f \geq 0$.
- Also, by variational bound $1 - x \leq e^{-x}$ for $x \in \mathbb{R}$, we have

$$\delta_\ell \leq (1 - \frac{1}{k})^\ell \delta_0 \leq e^{-\ell/k} f(S^*) \tag{14.13}$$

Cardinality Constrained Polymatroid Max Theorem

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- When we identify $\delta_\ell = f(S^*) - f(S_\ell)$, a bit of rearranging then gives:

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- With $\ell = k$, when picking k items, greedy gets $(1 - 1/e) \approx 0.6321$ bound. This means that if S_k is greedy solution of size k, and S^* is an optimal solution of size k, $f(S_k) \geq (1 - 1/e) f(S^*) \approx 0.6321 f(S^*)$.

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Cardinality Constrained Polymatroid Max Theorem

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if you get $f(v_{i+1} \mid S_i) = 0$ at some point $i < k$, then you have OPT.

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- So solution, in the worst case, quickly gets very good. Typical/practical case is much better.

Greedy running time

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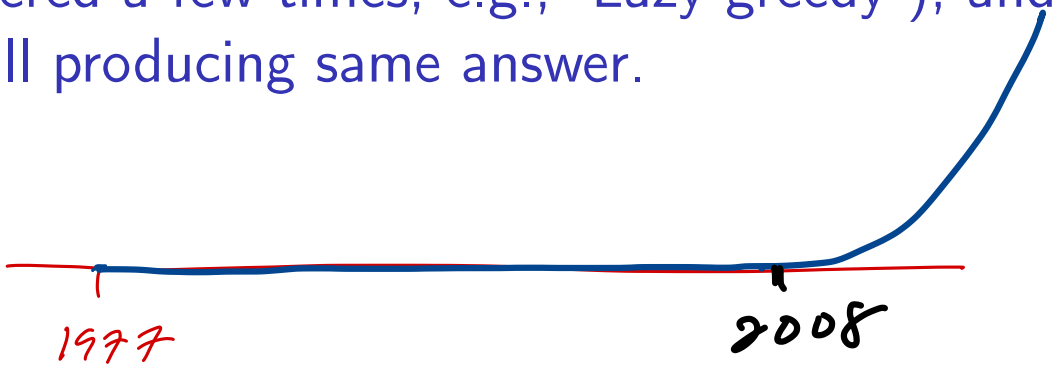
- This is called Minoux's 1977 Accelerated Greedy strategy (and has been rediscovered a few times, e.g., "Lazy greedy"), and runs much faster while still producing same answer.



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- This is called Minoux's 1977 Accelerated Greedy strategy (and has been rediscovered a few times, e.g., "Lazy greedy"), and runs much faster while still producing same answer.
- We describe it next:

Minoux's Accelerated Greedy for Submodular Functions

- At stage i in the algorithm, we have a set of gains $f(v|S_i)$ for all $v \notin S_i$. Store these values $\alpha_v \leftarrow f(v|S_i)$ in sorted priority queue.

$$\max_{v \in V \setminus S_i} f(v|S_i)$$

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- For $v \notin S_{i+1}$ we have $f(v|S_{i+1}) \leq f(v|S_i)$ by submodularity.

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- Once we choose a max v, then set $S_{i+1} \leftarrow S_i + v$.
- For $v \notin S_{i+1}$ we have $f(v|S_{i+1}) \leq f(v|S_i)$ by submodularity.
- Therefore, if we find a v' such that $f(v'|S_{i+1}) \geq \alpha_v$ for all $v \neq v'$ in the queue, then since

$$f(v'|S_{i+1}) \geq \alpha_v \overbrace{= f(v|S_i)}^{\text{previous gains but it could be even earlier.}} \underbrace{\geq f(v|S_{i+1})}_{\text{by submodularity}} \tag{14.15}$$

we have the true max, and we need not re-evaluate gains of other elements again.

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  we have the true max, and we need not re-evaluate gains of other elements again.
- Strategy is: find $v' \in \operatorname{argmax}_{v \in V \setminus S_{i+1}} \alpha_v$, and then compute the real "fresh" $f(v'|S_{i+1})$. If it is greater than all other α_v's in the queue, then that's the next greedy step. Otherwise, replace $\alpha_{v'}$ with its fresh value, resort ($O(\log n)$), and repeat.

Minoux's Accelerated Greedy for Submodular Functions

- Minoux's algorithm is exact, in that it has the same guarantees as does the standard $O(n^2)$ greedy algorithm (will return the same answers, i.e., those having the $1 - 1/e$ guarantee).

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- In practice: Minoux's trick has enormous speedups ($\approx 700\times$) over the standard greedy procedure due to reduced function evaluations and use of good data structures (priority queue).
- When choosing a set of size k, naïve greedy algorithm is $O(nk)$ but accelerated variant at the very best does $O(n \log n + k)$, so this limits the speedup.

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- In practice: Minoux's trick has enormous speedups ($\approx 700\times$) over the standard greedy procedure due to reduced function evaluations and use of good data structures (priority queue).
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- Can be used used for "big data" sets (e.g., social networks, selecting blogs of greatest influence, document summarization, etc.).
- Very good if there are many elements v with $f(v|S_{i+1}) \geq f(u|S_i)$ for many other elements u, for many i.

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- On next slide, we call a popped item "fresh" if the value (v, α) popped has the correct value $\alpha = f(v|S_i)$. Use extra "bit" to store this info

Minoux's Accelerated Greedy Algorithm Submodular Max

Algorithm 2: Minoux's Accelerated Greedy Algorithm

1 Set $S_0 \leftarrow \emptyset$; $i \leftarrow 0$; Initialize priority queue Q ;
2 **for** $v \in E$ **do**
3 INSERT$(Q, f(v))$ $O(n)$
4 **repeat**
5 $(v, \alpha) \leftarrow$ pop(Q) ;
6 **if** α *not "fresh"* **then**
7 recompute $\alpha \leftarrow f(v|S_i)$
8 **if** *(popped α in line 5 was "fresh") OR ($\alpha \geq$ max(Q))* **then**
9 Set $S_{i+1} \leftarrow S_i \cup \{v\}$; and mark other items in Q as stale ;
10 $i \leftarrow i + 1$;
11 **else**
12 insert$(Q, (v, \alpha))$ as a fresh item
13 **until** $i = |E|$;

On the accelerated greedy algorithm

- If a popped item is fresh, it must be the maximum — this can happen if, at given iteration, v was first popped and neither fresh nor maximum so placed back in the queue, and it then percolates back to the top at next iteration at which point it is fresh — thereby avoid extra queue check.

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- Note, we can avoid keeping track of fresh/stale bit and always do an unconditional evaluation at the top of each loop.

(Minimum) Submodular Set Cover

- Given polymatroid f, goal is to find a covering set of minimum cost:

$$S^* \in \operatorname*{argmin}_{S \subseteq V} |S| \text{ such that } f(S) \geq \alpha \tag{14.19}$$

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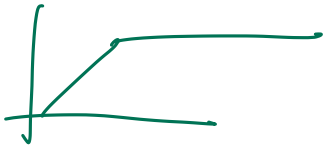

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- Normally take $\alpha = f(V)$ but defining $f'(A) = \min\{f(A), \alpha\}$ we can take any α. Hence, we have equivalent formulation:

$f'(V) \leq \alpha$

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- Greedy Algorithm: Pick the first chain item S_i chosen by aforementioned greedy algorithm such that $f(S_i) \geq \alpha$ and output that as solution.

(Minimum) Submodular Set Cover: Approximation Analysis

- For integer valued f, this greedy algorithm has an $O(\log(\max_{s \in V} f(\{s\})))$ approximation. Let S^* be optimal, and S^{G} be greedy solution, then

$$|S^{\mathsf{G}}| \leq |S^*| H(\max_{s \in V} f(\{s\})) = |S^*| O(\log_e(\max_{s \in V} f(\{s\}))) \tag{14.21}$$

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$$|S^{\mathsf{G}}| \leq |S^*| \Big(1 + \log_e \frac{f(V)}{f(V) - f(S_{T-1})}\Big) \tag{14.22}$$

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- As we mentioned earlier, even set cover (a special case of submodular set cover) is hard to approximate with a factor better than $(1 - \epsilon) \log \alpha$, where α is the desired cover constraint.

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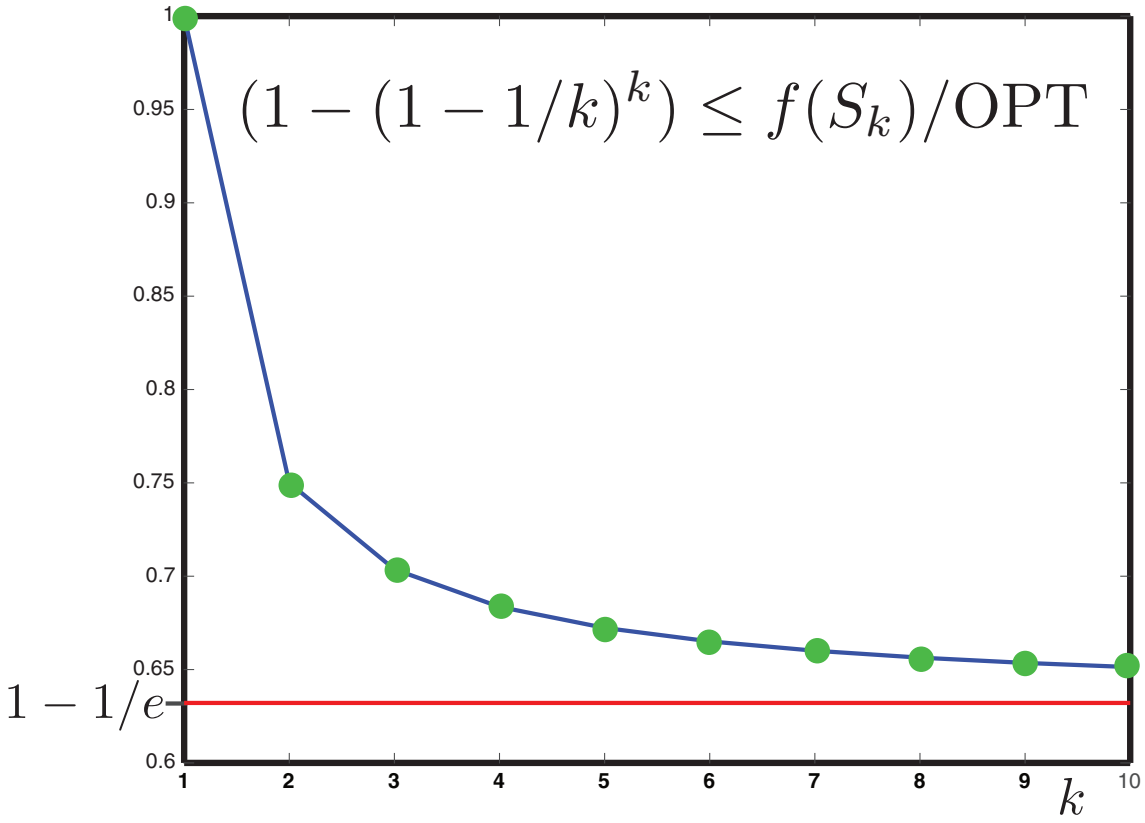
- Submodular cover: min. $|S|$ s.t. $f(S) \geq \alpha$.
- Minoux's accelerated greedy trick.

The Greedy Algorithm: $1 - 1/e$ intuition.

- At step $i < k$, greedy chooses v_i to maximize $f(v|S_i)$.
- Let S^* be optimal solution (of size k) and OPT $= f(S^*)$. By submodularity, we will show:

$$\exists v \in V \setminus S_i : f(v|S_i) = f(S_i + v|S_i) \geq \frac{1}{k}(\text{OPT} - f(S_i)) \tag{14.2}$$



Equation (14.11) will show that Equation (14.2) $\Rightarrow$:

$$\begin{aligned}
&\text{OPT} - f(S_{i+1}) \\
&\qquad \leq (1 - 1/k)(\text{OPT} - f(S_i)) \\
\Rightarrow\ &\text{OPT} - f(S_k) \\
&\qquad \leq (1 - 1/k)^k \text{OPT} \\
&\qquad \leq 1/e\text{OPT} \\
\Rightarrow\ &\text{OPT}(1 - 1/e) \leq f(S_k)
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- Will be on Homework 4.

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- For any submodular function, we have $f(j|S) \leq f(j|\emptyset)$ so that $f(j|S)/f(j|\emptyset) \leq 1$ whenever $f(j|\emptyset) \neq 0$.

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$0 = f(j) \geq \underbrace{f(j|S)}_{=0} \geq 0$

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$$c_f \triangleq 1 - \min_{S, j \notin S : f(j|\emptyset) \neq 0} \frac{f(j|S)}{f(j|\emptyset)} = 1 - \min_{j : f(j) \neq 0} \frac{f(j|V \setminus j)}{f(j)} \tag{14.24}$$

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- For a matroid rank function, we might have $c_r = 1$. E.g., $r(A) = \min(|A|, n-1)$ has $r(j|V \setminus j)/r(j) = 0$. Theorem 9.5.1 said that greedy is optimal for max weight indep. set of a matroid.

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- Thus far, curvature extremes doesn't tell us much. However ...

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- For f with curvature c_f, then $\forall A \subseteq V$, $\forall v \notin a$, $\forall c' \geq c_f$:

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- Hence both upper and lower bound: $f(v) \geq f(v|A) \geq (1 - c_f)f(v)$.
- When $c_f = 1$ then the submodular function said to be "maximally" (or fully, or totally) curved, i.e., there exists is a subset that fully spans some other element.

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- Curvature definition again (by submodularity, both forms are the same):

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- Recall the notion of "partial dependence" within polymatroid functions.

Curvature and approximation: key theorem

- Curvature-based approximation bound for max k cardinality.

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Theorem 14.4.1

Given a polymatroid function $f : 2^V \to \mathbb{R}_+$ with curvature $c_f \in [0, 1]$ defined above. Then the greedy algorithm's solution to the problem $\max_{A \subseteq V : |A| \leq k} f(A)$ has the following approximation bound:

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$c_f = 1 \implies 1 - e^{-1}$

$\lim_{c_f \to 0} \frac{1}{c_f}(1 - e^{-c_f}) = 1$

Curvature and approximation: key theorem

- Curvature-based approximation bound for max k cardinality.

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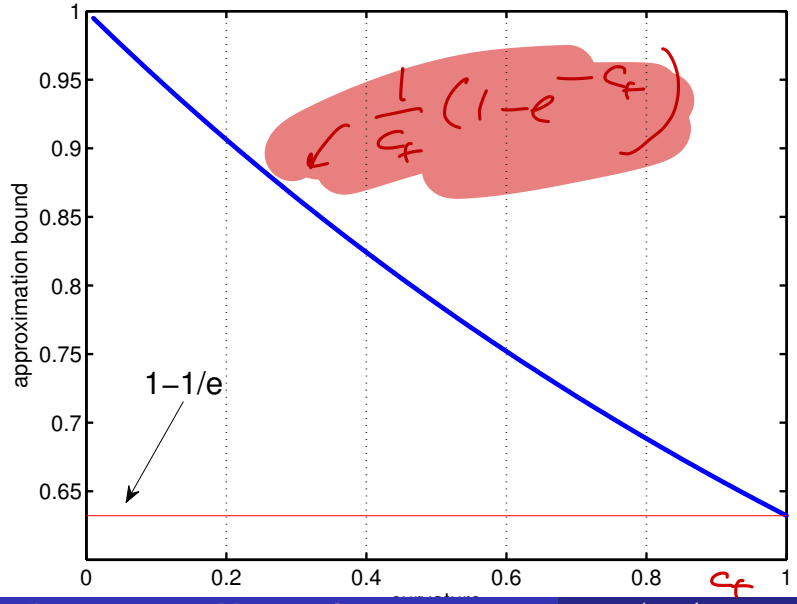

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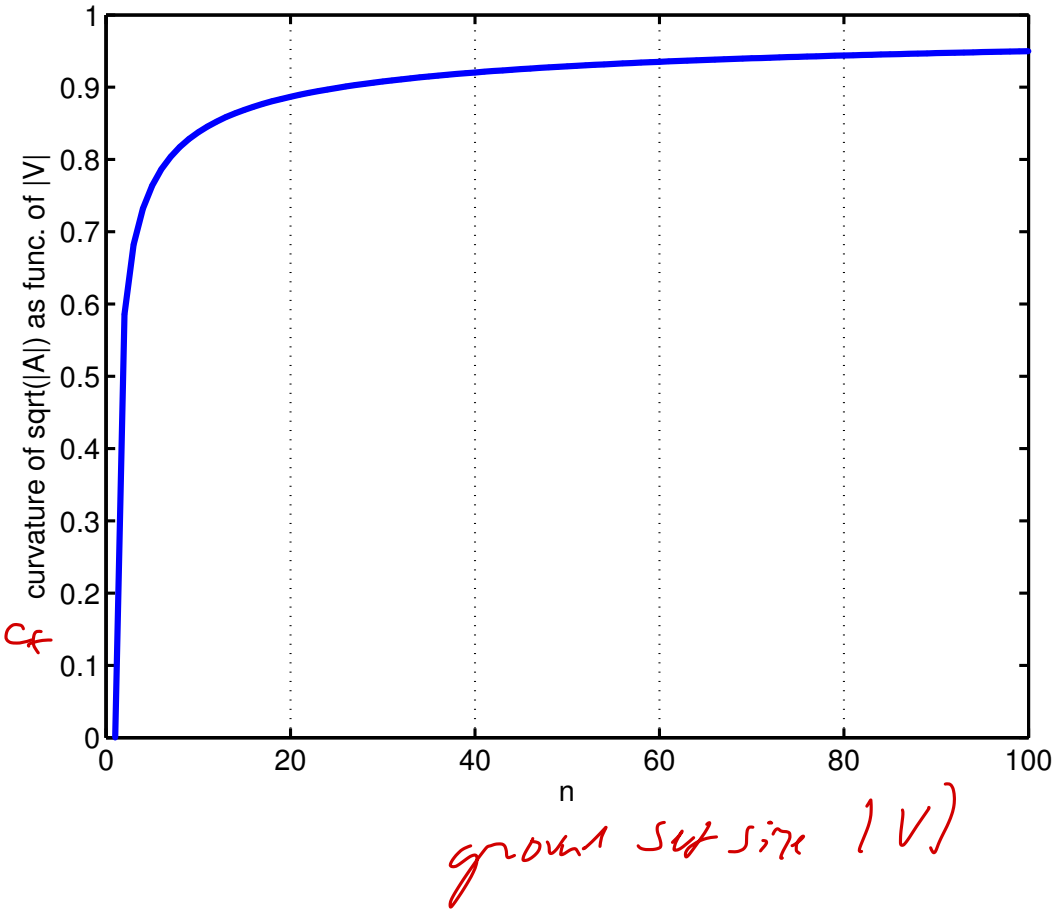
Curvature of $f(S) = \sqrt{|S|}$ as function of $|V| = n$



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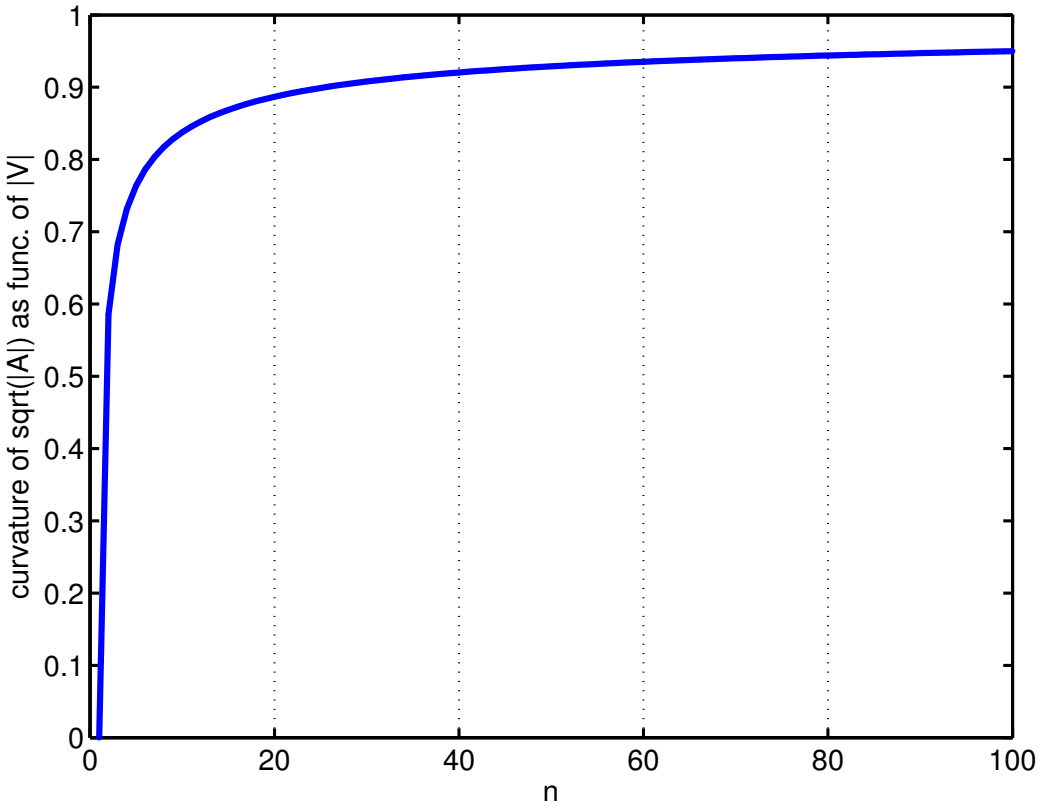

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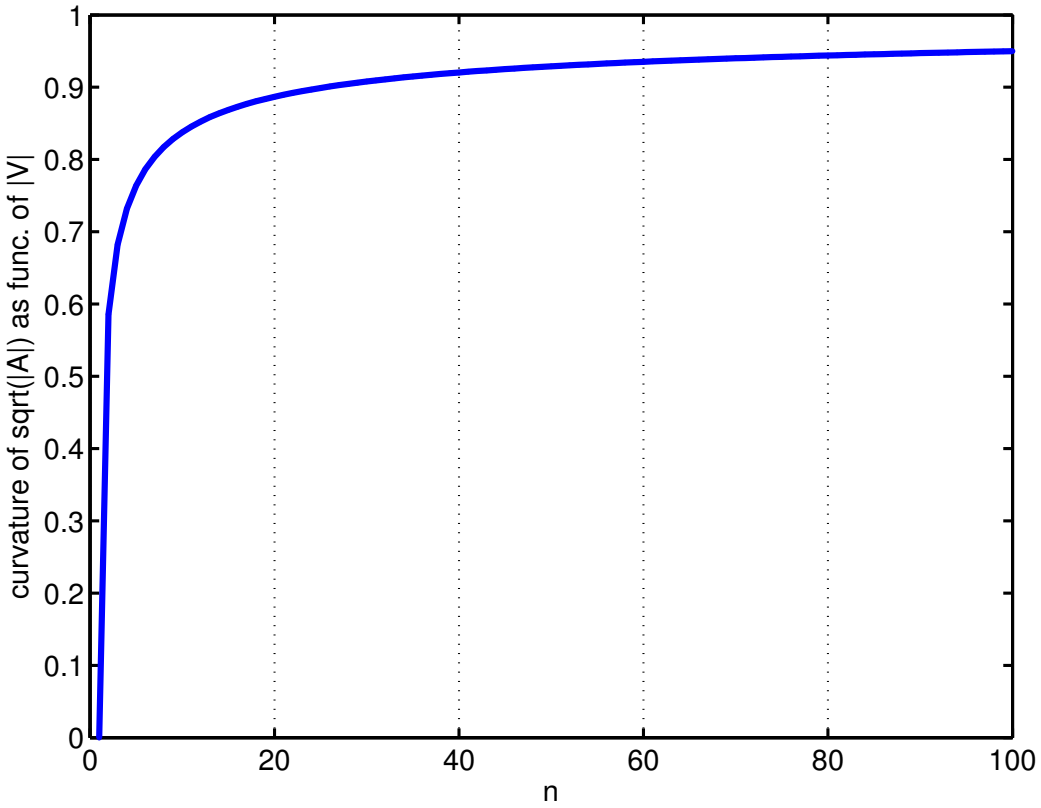

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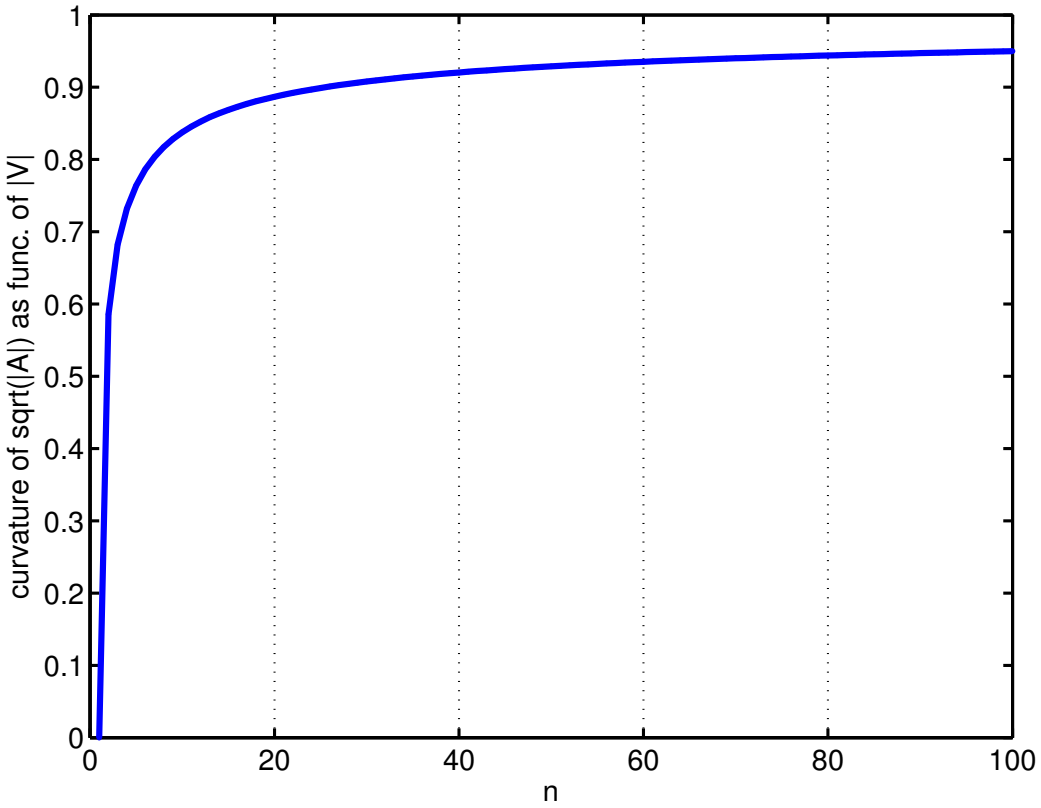

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- Quite good news for greedy and submodular functions.

Submodular and Supermodular Curvature Approximation

- Let f be a polymatroid function and let g be a non-negative monotone non-decreasing supermodular function (e.g., $g(A) = \phi(m(A))$ where $\phi()$ is non-decreasing convex and $m : V \to \mathbb{R}_+$).

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- Form function $h(A) = f(A) + g(A)$, then h is neither suBmodular nor suPermodular, but is known as a BP-function.

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- Both curvatures are very easy to compute given BP decomposition.

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- Another analogous concepts, submodular degree.