Submodular Functions, Optimization, and Applications to Machine Learning

— Fall Quarter, Lecture 13 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/

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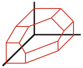
$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

$= f(A_r) + 2f(C) + f(B_r)$ $\quad = f(A_r) + f(C) + f(B_r)$ $\quad = f(A \cap B)$



Announcements, Assignments, and Reminders

- Homework 3, out, due next Wednesday, Nov 25th, 2020, 11:59pm.
- Reminder, all lectures are being recorded and posted to youtube. To get the links, see our announcements (`https://canvas.uw.edu/courses/1397085/announcements`).
- Office hours this week, Wed & Thur, 10:00pm at our class zoom link.
- Next week office hours, Tues (11/24) & Wed (11/25), 10:00pm at our class zoom link.

Class Road Map - EE563

- L1(9/30): Motivation, Applications, Definitions, Properties
- L2(10/5): Sums concave(modular), uses (diversity/costs, feature selection), information theory
- L3(10/7): Monge, More Definitions, Graph and Combinatorial Examples,
- L4(10/12): Graph & Combinatorial Examples, Matrix Rank, Properties, Other Defs, Independence
- L5(10/14): Properties, Defs of Submodularity, Independence
- L6(10/19): Matroids, Matroid Examples, Matroid Rank,
- L7(10/21): Matroid Rank, More on Partition Matroid, Laminar Matroids, System of Distinct Reps, Transversals
- L8(10/26): Transversal Matroid, Matroid and representation, Dual Matroid
- L9(10/28): Other Matroid Properties, Combinatorial Geometries, Matroid and Greedy, Polyhedra, Matroid Polytopes
- L10(11/2): Matroid Polytopes, Matroids $\rightarrow$ Polymatroids
- L11(11/4): Matroids $\rightarrow$ Polymatroids, Polymatroids
- L12(11/9): Polymatroids, Polymatroids and Greedy
- L–(11/11): Veterans Day, Holiday
- L13(11/16): Polymatroids and Greedy, Possible Polytopes, Extreme Points, Cardinality Constrained Maximization
- L14(11/18):
- L15(11/23):
- L16(11/25):
- L17(11/30):
- L18(12/2):
- L19(12/7):
- L20(12/9): maximization.

Last day of instruction, Fri. Dec 11th. Finals Week: Dec 12-18, 2020

A polymatroid is a polymatroid function's polytope

- So, when f is a polymatroid function, P_f^+ is a polymatroid.
- Is it the case that, conversely, for any polymatroid P, there is an associated polymatroidal function f such that $P = P_f^+$?

Theorem 13.2.1

For any polymatroid P (compact subset of $\mathbb{R}_+^E$, zero containing, down-monotone, and $\forall x \in \mathbb{R}_+^E$ any maximal independent subvector $y \leq x$ has same component sum $y(E) = \text{rank}(x)$), there is a polymatroid function $f : 2^E \to \mathbb{R}$ (normalized, monotone non-decreasing, submodular) such that $P = P_f^+$ where $P_f^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\}$.

Tight sets $\mathcal{D}(y)$ are closed, and max tight set $\text{sat}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{A : A \subseteq E,\ y(A) = f(A)\} \tag{13.1}$$

Theorem 13.2.1

For any $y \in P_f^+$, with f a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

Proof.

We have already proven this as part of Theorem ?? □

Also recall the definition of $\text{sat}(y)$, the maximal set of tight elements relative to $y \in \mathbb{R}_+^E$.

$$\text{sat}(y) \stackrel{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\} \tag{13.2}$$

Join $\vee$ and meet $\wedge$ for $x, y \in \mathbb{R}_+^E$

- For $x, y \in \mathbb{R}_+^E$, define vectors $x \wedge y \in \mathbb{R}_+^E$ and $x \vee y \in \mathbb{R}_+^E$ such that, for all $e \in E$

$$(x \vee y)(e) = \max(x(e), y(e)) \tag{13.1}$$

$$(x \wedge y)(e) = \min(x(e), y(e)) \tag{13.2}$$

Hence,

$$x \vee y \triangleq \Big(\max\big(x(e_1), y(e_1)\big), \max\big(x(e_2), y(e_2)\big), \dots, \max\big(x(e_n), y(e_n)\big)\Big)$$

and similarly

$$x \wedge y \triangleq \Big(\min\big(x(e_1), y(e_1)\big), \min\big(x(e_2), y(e_2)\big), \dots, \min\big(x(e_n), y(e_n)\big)\Big)$$

- From this, we can define things like an lattices, and other constructs.

Vector rank, rank(x), is submodular

- Recall that the matroid rank function $r(A) = \max(|I| : I \subseteq A : I \in \mathcal{I})$ is submodular.
- The vector rank function $\text{rank}(x) = \max\left(y(E) : y \leq x, y \in P\right)$ also satisfies a form of submodularity, namely one defined on the real lattice.

Theorem 13.2.1 (vector rank and submodularity)

Let P be a polymatroid polytope. The vector rank function rank : $\mathbb{R}_+^E \to \mathbb{R}$ *with rank*$(x) = \max\left(y(E) : y \leq x, y \in P\right)$ *satisfies, for all* $u, v \in \mathbb{R}_+^E$

$$\textit{rank}(u) + \textit{rank}(v) \geq \textit{rank}(u \vee v) + \textit{rank}(u \wedge v) \tag{13.1}$$

- Note what happens when $u, v \in \{0, 1\}^E \subseteq \mathbb{R}_+^E$.

Polymatroidal polyhedron and the greedy solution

- What is the greedy solution for $\max\left\{wx : x \in P_f^+\right\}$, when $w \in \mathbb{R}^E$?
- Sort elements of E w.r.t. w so that, w.l.o.g. $E = (e_1, e_2, \dots, e_m)$ with $w(e_1) \geq w(e_2) \geq \dots \geq w(e_m)$.
- Let $k+1$ be the first point (if any) at which we are non-positive, i.e., $w(e_k) > 0$ and $0 \geq w(e_{k+1})$.
- Next define partial accumulated sets E_i, for $i = 0 \dots m$, we have w.r.t. the above sorted order:

$$E_i \stackrel{\text{def}}{=} \{e_1, e_2, \dots e_i\} \tag{13.22}$$

(note $E_0 = \emptyset$, $f(E_0) = 0$, and E and E_i is always sorted w.r.t w).
- The greedy solution is the vector $x \in \mathbb{R}_+^E$ with elements defined as:

$$x(e_1) \stackrel{\text{def}}{=} f(E_1) = f(e_1) = f(e_1|E_0) = f(e_1|\emptyset) \tag{13.23}$$

$$x(e_i) \stackrel{\text{def}}{=} f(E_i) - f(E_{i-1}) = f(e_i|E_{i-1}) \text{ for } i = 2 \dots k \tag{13.24}$$

$$x(e_i) \stackrel{\text{def}}{=} 0 \text{ for } i = k+1 \dots m = |E| \tag{13.25}$$

Polymatroidal Polyhedron and Greedy: Optimality

Theorem 13.2.2

The vector $x \in \mathbb{R}_+^E$ as previously defined using the greedy algorithm maximizes wx over P_f^+, with $w \in \mathbb{R}_+^E$, if f is submodular.

Proof.

- Consider the LP strong duality equation:

$$\max(wx : x \in P_f^+) = \min\Big(\sum_{A \subseteq E} y_A f(A) : y \in \mathbb{R}_+^{2^E}, \sum_{A \subseteq E} y_A \mathbf{1}_A \geq w\Big) \tag{13.21}$$

- Sort E by w descending, and define the following vector $y \in \mathbb{R}_+^{2^E}$ as

$$y_{E_i} \leftarrow w(e_i) - w(e_{i+1}) \text{ for } i = 1 \ldots (m-1), \tag{13.22}$$

$$y_E \leftarrow w(e_m), \text{ and} \tag{13.23}$$

$$y_A \leftarrow 0 \text{ otherwise} \tag{13.24}$$

Polymatroidal polyhedron and greedy

Theorem 13.2.2

Conversely, suppose P_f^+ *is a polytope of form* $P_f^+ = \{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\}$*, then the greedy solution to* $\max(wx : x \in P_f^+)$ *is optimum only if* f *is submodular.*

Proof.

- Choose A and B arbitrarily, and then order elements of E as $(e_1, e_2, \ldots, e_m)$, with $E_i = (e_1, e_2, \ldots, e_i)$, so the following is true:
- For $1 \leq p \leq q \leq m$, $A = \{e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_p\} = E_p$ and $B = \{e_1, e_2, \ldots, e_k, e_{p+1}, \ldots, e_q\} = E_k \cup (E_q \setminus E_p) = (A \cap B) \cup (B \setminus A)$
- Note, then we have $A \cap B = \{e_1, \ldots, e_k\} = E_k$, and $A \cup B = E_q$.

Review from Lecture 9

- The next slide comes from lecture 9.

Matroid and the greedy algorithm

- Let $(E, \mathcal{I})$ be an independence system, and we are given a non-negative modular weight function $w : E \to \mathbb{R}_+$.

Algorithm 1: The Matroid Greedy Algorithm

```
1 Set X ← ∅ ;
2 while ∃v ∈ E \ X s.t. X ∪ {v} ∈ I do
3     v ∈ argmax {w(v) : v ∈ E \ X, X ∪ {v} ∈ I} ;
4     X ← X ∪ {v} ;
```

- Same as sorting items by decreasing weight w, and then choosing items in that order that retain independence.

Theorem 13.3.4

Let $(E, \mathcal{I})$ be an independence system. Then the pair $(E, \mathcal{I})$ is a matroid if and only if for each weight function $w \in \mathcal{R}_+^E$, Algorithm ?? above leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.

Polymatroidal polyhedron and greedy

- Thus, restating the above results into a single complete theorem, we have a result very similar to what we saw for matroids (i.e., Theorem 10.3.2)

Theorem 13.3.1

If $f : 2^E \to \mathbb{R}_+$ is given, and P is a polytope in $\mathbb{R}_+^E$ of the form $P = \left\{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\right\}$, then the greedy solution to the problem $\max(w^\intercal x : x \in P)$ is $\forall w$ optimum iff f is monotone non-decreasing submodular (i.e., iff P is a polymatroid).

Multiple Polytopes associated with arbitrary f

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- If $f(\emptyset) \neq 0$, can set $f'(A) = f(A) - f(\emptyset)$ without destroying submodularity. This does not change any minima, (i.e., $\text{argmin}_A f(A) = \text{argmin}_{A'} f'(A)$) so we often assume all functions are normalized $f(\emptyset) = 0$.
  Note that due to constraint $x(\emptyset) \leq f(\emptyset)$, we must have $f(\emptyset) \geq 0$ since if not (i.e., if $f(\emptyset) < 0$), then P_f^+ doesn't exist.
  Another form of normalization takes the form:

$$f'(A) = \begin{cases} f(A) & \text{if } A \neq \emptyset \\ 0 & \text{if } A = \emptyset \end{cases} \tag{13.1}$$

  This preserves submodularity due to $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$, and if $A \cap B = \emptyset$ then r.h.s. only gets smaller when $f(\emptyset) \geq 0$.

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- We can define several polytopes:

$$P_f = \left\{x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E\right\} \tag{13.1}$$

$$P_f^+ = P_f \cap \left\{x \in \mathbb{R}^E : x \geq 0\right\} \tag{13.2}$$

$$B_f = P_f \cap \left\{x \in \mathbb{R}^E : x(E) = f(E)\right\} \tag{13.3}$$

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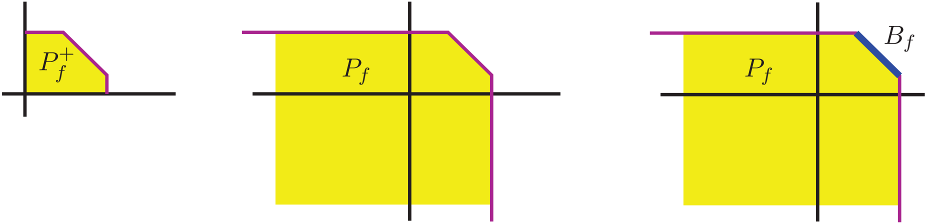
- P_f is what is sometimes called the extended polytope (sometimes notated as EP_f.
- P_f^+ is P_f restricted to the positive orthant.
- B_f is called the base polytope, analogous to the base in matroid.

Multiple Polytopes in 2D associated with f



$$P_f^+ = P_f \cap \left\{x \in \mathbb{R}^E : x \geq 0\right\} \tag{13.4}$$

$$P_f = \left\{x \in \mathbb{R}^E : x(S) \leq f(S), \forall S \subseteq E\right\} \tag{13.5}$$

$$B_f = P_f \cap \left\{x \in \mathbb{R}^E : x(E) = f(E)\right\} \tag{13.6}$$

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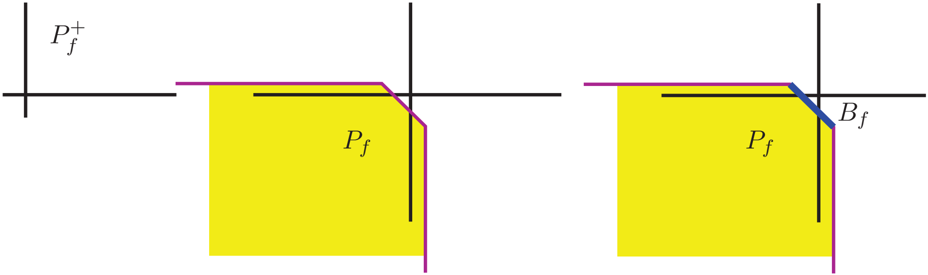

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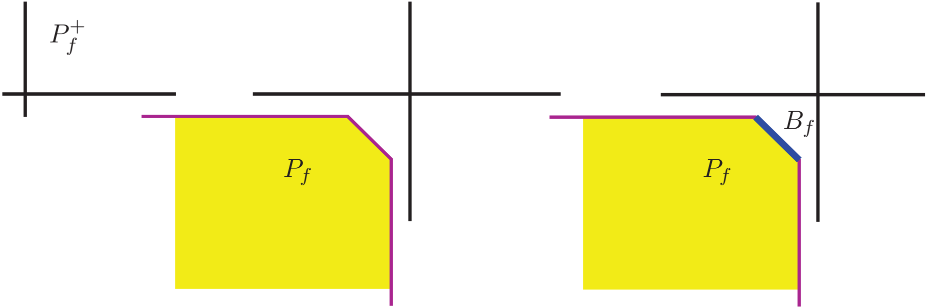

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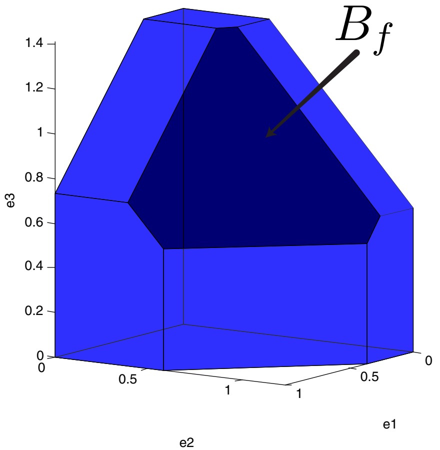
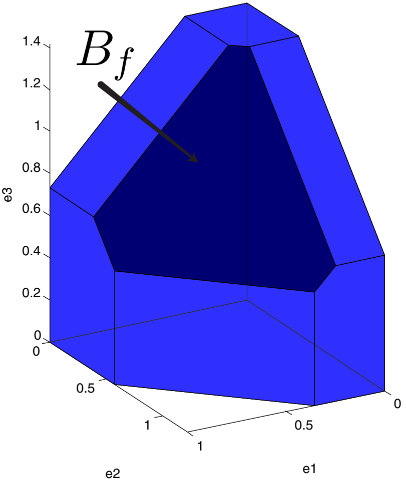
Base Polytope in 3D





$$P_f = \left\{x \in \mathbb{R}^E : x(S) \le f(S), \forall S \subseteq E\right\} \tag{13.7}$$

$$B_f = P_f \cap \left\{x \in \mathbb{R}^E : x(E) = f(E)\right\} \tag{13.8}$$

A polymatroid function's polyhedron is a polymatroid.

Theorem 13.4.1

Let f be a submodular function defined on subsets of E. For any $x \in \mathbb{R}^E$, we have:

$$\textit{rank}(x) = \max\left(y(E) : y \leq x, y \in P_f\right) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{13.9}$$

Essentially the same theorem as Theorem 11.4.1, but note P_f rather than P_f^+. Taking $x = 0$ we get:

Corollary 13.4.2

Let f be a submodular function defined on subsets of E. We have:

$$\textit{rank}(0) = \max\left(y(E) : y \leq 0, y \in P_f\right) = \min\left(f(A) : A \subseteq E\right) \tag{13.10}$$

Proof of Theorem 13.4.1

Proof Thm 13.4.1: $\max\left(y(E) : y \leq x, y \in P_f\right) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right)$.

- Let y^* be optimal solution of the l.h.s. and let $A \subseteq E$ be any subset.

$\square$

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- Let y^* be optimal solution of the l.h.s. and let $A \subseteq E$ be any subset.
- Then $y^*(E) = y^*(A) + y^*(E \setminus A) \leq f(A) + x(E \setminus A)$ (since if $y^* \in P_f$ then $y^*(A) \leq f(A)$, and since $y^* \leq x$ then $y^*(E \setminus A) \leq x(E \setminus A)$). This is a form of weak duality.

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- For any $e \in E$, if $y^*(e) < x(e)$, must be some reason other than constraint $y^* \leq x$, namely must be that $\exists T \in \mathcal{D}(y^*)$ with $e \in T$ (i.e., e is a member of at least one of the tight sets).

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- Hence, for all $e \notin \text{sat}(y^*)$ we have $y^*(e) = x(e)$, and moreover $y^*(\text{sat}(y^*)) = f(\text{sat}(y^*))$ by definition.

$\square$

Proof of Theorem 13.4.1

Proof Thm 13.4.1: $\max\left(y(E) : y \leq x, y \in P_f\right) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right)$.

- Let y^* be optimal solution of the l.h.s. and let $A \subseteq E$ be any subset.
- Then $y^*(E) = y^*(A) + y^*(E \setminus A) \leq f(A) + x(E \setminus A)$ (since if $y^* \in P_f$ then $y^*(A) \leq f(A)$, and since $y^* \leq x$ then $y^*(E \setminus A) \leq x(E \setminus A)$). This is a form of weak duality.
- For any $e \in E$, if $y^*(e) < x(e)$, must be some reason other than constraint $y^* \leq x$, namely must be that $\exists T \in \mathcal{D}(y^*)$ with $e \in T$ (i.e., e is a member of at least one of the tight sets). I.e., given $e \notin \text{sat}(y^*)$, then $y^*(A) < f(A) \forall A \ni e$ including $\{e\}$, hence $x(e) < f(e)$. Conversely, $e \in \text{sat}(y^*)$ means $\exists T \in \mathcal{D}(y^*)$, w. $e \in T$ & $y^*(T) = f(T)$.
- Hence, for all $e \notin \text{sat}(y^*)$ we have $y^*(e) = x(e)$, and moreover $y^*(\text{sat}(y^*)) = f(\text{sat}(y^*))$ by definition.
- Thus $y^*(\text{sat}(y^*)) + y^*(E \setminus \text{sat}(y^*)) = f(\text{sat}(y^*)) + x(E \setminus \text{sat}(y^*))$, strong duality, showing that the two sides are equal for y^*. □

Greedy and P_f

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- We might thus be more interested in $\max(wx : x \in B_f)$ when w is an arbitrary vector.
- In fact, we will see, in the next section, that the full run of the greedy algorithm producing x is in fact a vertex of B_f.

Greedy and P_f

- Recall that Theorem 11.4.1 states that
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- Theorem 12.4.1 states that greedy algorithm maximizes wx over P_f^+ for $w \in \mathbb{R}_+^E$ with f being submodular.
- Above implies that Theorem 12.4.1 can be generalized to over P_f and that greedy solution gives a point in B_f, even for arbitrary finite w.

Polymatroid extreme points

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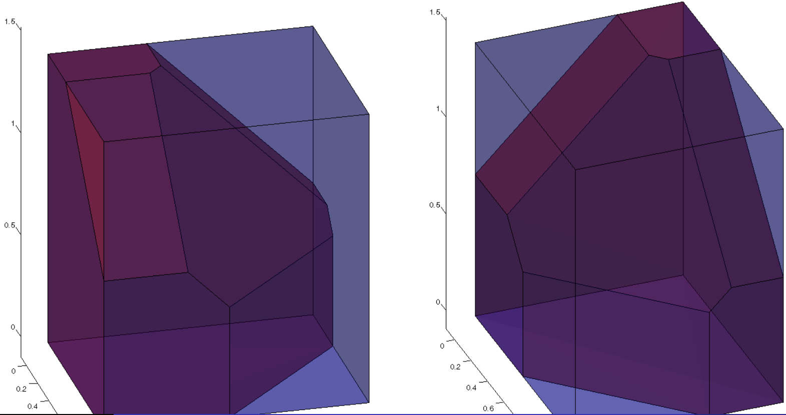

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$$B_f = P_f \cap \left\{x \in \mathbb{R}_+^E : x(E) = f(E)\right\} \tag{13.11}$$

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- We formalize this next:

Polymatroid extreme points

- Given any arbitrary order of $E = (e_1, e_2, \ldots, e_m)$, define $E_i = (e_1, e_2, \ldots, e_i)$.

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- An extreme point of P_f is a point that is not a convex combination of two other distinct points in P_f. Equivalently, an extreme point corresponds to setting certain inequalities ($|E|$ of them) in the specification of P_f to be equalities, so that there is a unique single point solution.

Polymatroid extreme points

Theorem 13.5.1

For a given ordering $E = (e_1, \ldots, e_m)$ *of* E *and a given* $E_i = (e_1, \ldots, e_i)$ *and* x *generated by* E_i *using the greedy procedure (*$x(e_i) = f(e_i|E_{i-1})$*), then* x *is an extreme point of* P_f *when* f *is submodular.*

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Proof.

- We already saw that $x \in P_f$ (Theorem 12.4.1).

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Proof.

- We already saw that $x \in P_f$ (Theorem 12.4.1).
- To show that x is an extreme point of P_f, note that it is the unique solution of the following system of equations

$$x(E_j) = f(E_j) \text{ for } 1 \le j \le i \le m \tag{13.14}$$

$$x(e) = 0 \text{ for } e \in E \setminus E_i \tag{13.15}$$

There are $i \le m$ equations and $i \le m$ unknowns, and simple Gaussian elimination gives us back the x constructed via the Greedy algorithm!!

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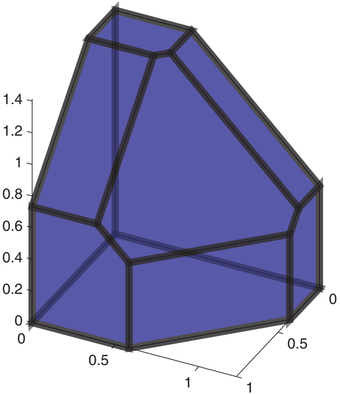
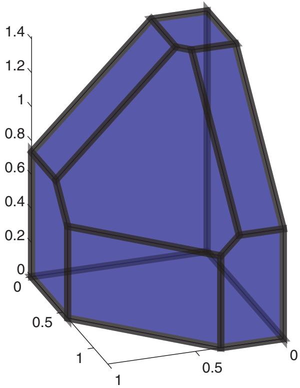
- Thus, the greedy procedure provides a modular function lower bound on f that is tight on all points E_i in the order. This can be useful in its own right, as it provides subgradients and subdifferential structure.

Polymatroid extreme points

some examples





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- Moreover, we can show that

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- For arbitrary x, $\mathrm{supp}(x)$ is not necessarily tight, but for an extreme point, $\mathrm{supp}(x)$ is.

Polymatroid with labeled edge lengths

- Recall $f(e|A) = f(A+e) - f(A)$
- Notice how submodularity, $f(e|B) \leq f(e|A)$ for $A \subseteq B$, defines the shape of the polytope.
- In fact, we have strictness here $f(e|B) < f(e|A)$ for $A \subset B$.
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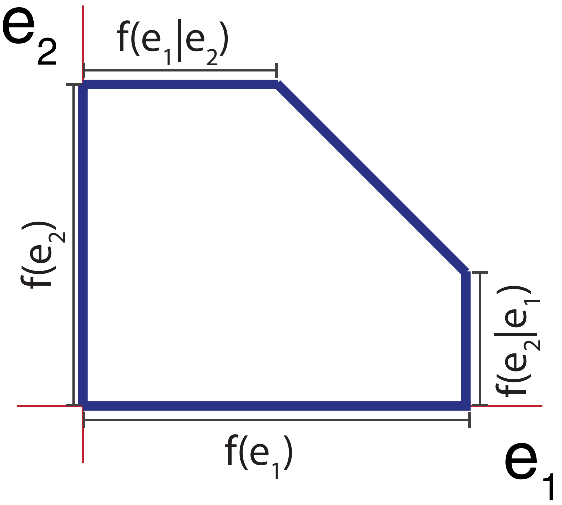

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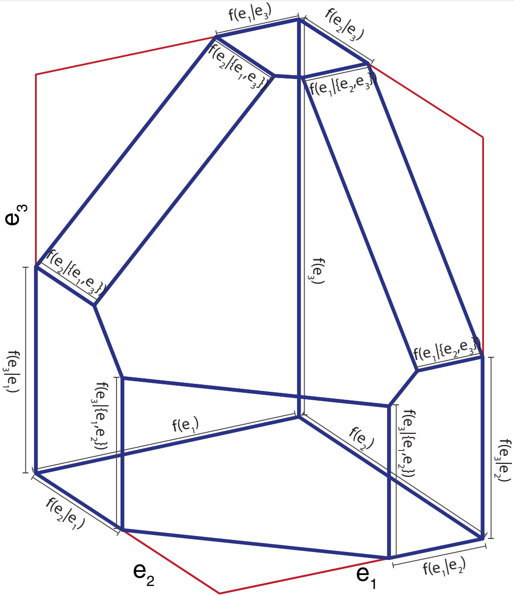
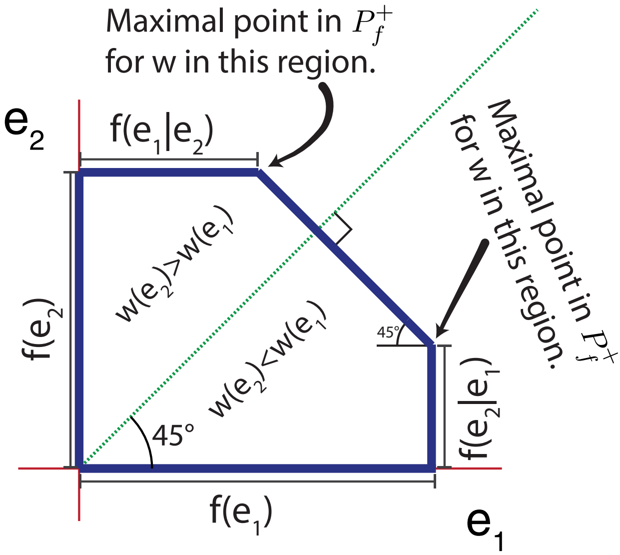


Intuition: why greedy works with polymatroids

- Given w, the goal is to find $x = (x(e_1), x(e_2))$ that maximizes $x^\intercal w = x(e_1)w(e_1) + x(e_2)w(e_2)$.
- If $w(e_2) > w(e_1)$ the upper extreme point indicated maximizes $x^\intercal w$ over $x \in P_f^+$.
- If $w(e_2) < w(e_1)$ the lower extreme point indicated maximizes $x^\intercal w$ over $x \in P_f^+$.



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- For polymatroid function (or any monotone non-decreasing function), unconstrained maximization is trivial (take ground set).

Maximization of Submodular Functions

- Submodular maximization is quite useful.
- Applications: sensor placement, facility location, document summarization, or any kind of covering problem (choose a small set of elements that cover some domain as much as possible).
- f in this case is a model of dispersion, diversity, representativeness, or information.
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- Thus, when we do monotone submodular maximization we find the maximum under some constraint.
- There is also a sort of dual problem that is often considered together with max, and those are minimum cover problems (to be defined).

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- Let E be a set and let $E_1, E_2, \ldots, E_m$ be a set of subsets.

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- This problem is NP-hard, and Feige in 1998 showed that it cannot be approximated with a ratio better than $(1 - \epsilon) \log n$ unless NP is slightly superpolynomial ($n^{O(\log \log n)}$).

What About Non-monotone

- So even simple case of cardinality constrained submodular function maximization is NP-hard.
- This will be true of most submodular max (and related) problems.
- Hence, the only hope is approximation algorithms. Question is, what is the tradeoff between running time and approximation quality, and is it possible to get tight bounds (i.e., an algorithm that achieves an approximation ratio, and a proof that one can't do better than that unless some extremely unlike event were to be true, such as P=NP).

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Cardinality Constrained Max. of Facility Location

- Recall facility location function $f(A) = \sum_{v \in V} \max_{a \in A} s_{a,v}$ where $s_{a,v}$ is the similarity between a and v. Alternatively, we can think of this as a representativeness matrix, where $s_{a,v}$ is seen as how how good a is as acting as a representative for v (which might not be the same as $s_{v,a}$).

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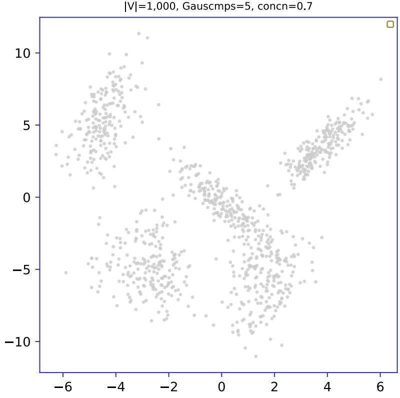

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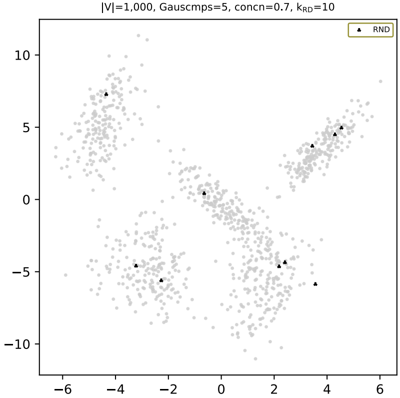

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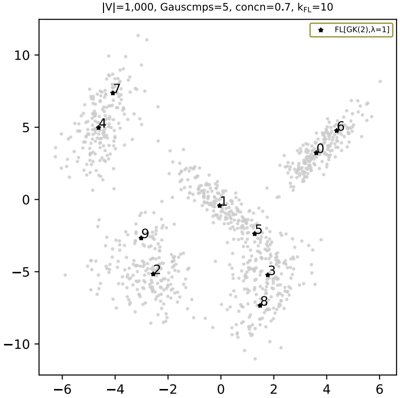

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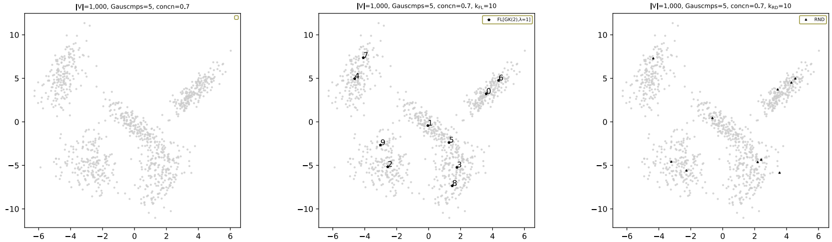
- Middle example is estimate of $\max_{A \subseteq V : |A| \leq k} f(A)$, right is uniformly-at-random randomly chosen set of size k, for $k = 10$.



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- An important result by Nemhauser et. al. (1978) states that for normalized ($f(\emptyset) = 0$) monotone submodular functions (i.e., polymatroids) can be approximately maximized using a simple greedy algorithm.

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- An important result by Nemhauser et. al. (1978) states that for normalized ($f(\emptyset) = 0$) monotone submodular functions (i.e., polymatroids) can be approximately maximized using a simple greedy algorithm.
- Starting with $S_0 = \emptyset$, we repeat the following greedy step for $i = 0 \ldots (k-1)$:

$$S_{i+1} = S_i \cup \left\{ \operatorname*{argmax}_{v \in V \setminus S_i} f(S_i \cup \{v\}) \right\} \tag{13.20}$$

The Greedy Algorithm for Submodular Max

A bit more precisely:

Algorithm 1: The Greedy Algorithm

Set $S_0 \leftarrow \emptyset$;
for $i \leftarrow 0 \ldots |E| - 1$ **do**
    Choose v_i as follows:
    $v_i \in \operatorname{argmax}_{v \in V \setminus S_i} f(\{v\}|S_i) = \operatorname{argmax}_{v \in V \setminus S_i} f(S_i \cup \{v\})$;
    Set $S_{i+1} \leftarrow S_i \cup \{v_i\}$;

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Theorem 13.6.1

Given a polymatroid function f , the above greedy algorithm returns sets S_i such that for each i we have $f(S_i) \geq (1 - 1/e) \max_{|S| \leq i} f(S)$.

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- To approximately find $A^* \in \operatorname{argmax} \{f(A) : |A| \leq k\}$, we repeat the greedy step until $k = i + 1$ in Algorithm 4:
- Again, since this generalizes max k-cover, Feige (1998) showed that this can't be improved. Unless $P = NP$, no polynomial time algorithm can do better than $(1 - 1/e + \epsilon)$ for any $\epsilon > 0$.

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$$\exists v \in V \setminus S_i : f(v|S_i) = f(S_i + v|S_i) \geq \frac{1}{k}(\text{OPT} - f(S_i)) \tag{13.21}$$

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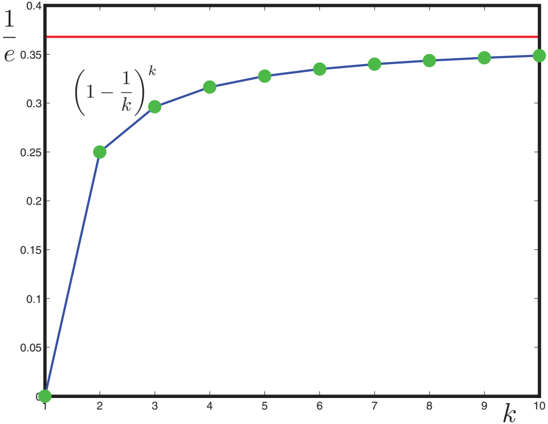

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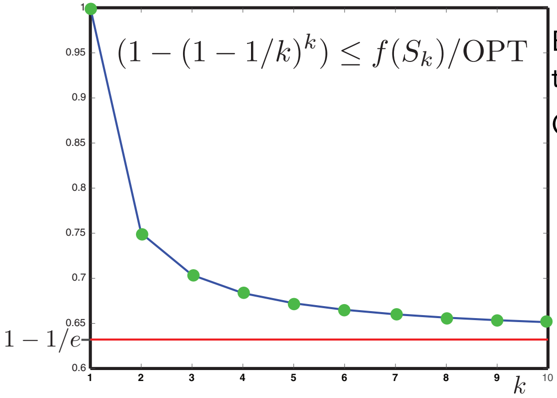

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Theorem 13.6.2 (Nemhauser et al. 1978)

Given non-negative monotone submodular function $f : 2^V \to \mathbb{R}_+$*, define* $\{S_i\}_{i \geq 0}$ *to be the chain formed by the greedy algorithm (Eqn.* (13.20)*). Then for all* $k, \ell \in \mathbb{Z}_{++}$*, we have:*

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- Intuitively, bound should get worse when $\ell < k$ and get better when $\ell > k$.

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Proof of Theorem 13.6.2.

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- Then the following inequalities (on the next slide) follow:

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Cardinality Constrained Polymatroid Max Theorem

. . . proof of Theorem 13.6.2 cont.

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Cardinality Constrained Polymatroid Max Theorem

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- For all $i < \ell$, we have

$$f(S^*)$$

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- For all $i < \ell$, we have

$$f(S^*) \leq f(S^* \cup S_i)$$

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$$= f(S_i) + \sum_{j=1}^{k} f(v_j^*|S_i \cup \{v_1^*, v_2^*, \dots, v_{j-1}^*\}) \tag{13.24}$$

$$\leq f(S_i) + \sum_{v \in S^*} f(v|S_i) \tag{13.25}$$

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Cardinality Constrained Polymatroid Max Theorem

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- Therefore, we have Equation 13.21, i.e.,:

$$f(S^*) - f(S_i) \leq kf(S_{i+1}|S_i) = k(f(S_{i+1}) - f(S_i)) \tag{13.28}$$

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- The relationship between δ_0 and δ_ℓ is then
$$\delta_l \leq (1 - \frac{1}{k})^\ell \delta_0 \tag{13.31}$$

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- Now, $\delta_0 = f(S^*) - f(\emptyset) \leq f(S^*)$ since $f \geq 0$.
- Also, by variational bound $1 - x \leq e^{-x}$ for $x \in \mathbb{R}$, we have

$$\delta_\ell \leq (1 - \frac{1}{k})^\ell \delta_0 \leq e^{-\ell/k} f(S^*) \tag{13.32}$$

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- So solution, in the worst case, quickly gets very good. Typical/practical case is much better.

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- We describe it next:

Minoux's Accelerated Greedy for Submodular Functions

- At stage i in the algorithm, we have a set of gains $f(v|S_i)$ for all $v \notin S_i$. Store these values $\alpha_v \leftarrow f(v|S_i)$ in sorted priority queue.

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- For $v \notin S_{i+1}$ we have $f(v|S_{i+1}) \leq f(v|S_i)$ by submodularity.
- Therefore, if we find a v' such that $f(v'|S_{i+1}) \geq \alpha_v$ for all $v \neq v'$, then since

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  we have the true max, and we need not re-evaluate gains of other elements again.
- Strategy is: find the $\operatorname{argmax}_{v' \in V \setminus S_{i+1}} \alpha_{v'}$, and then compute the real $f(v'|S_{i+1})$. If it is greater than all other α_v's then that's the next greedy step. Otherwise, replace $\alpha_{v'}$ with its real value, resort $(O(\log n))$, and repeat.

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- Can be used used for "big data" sets (e.g., social networks, selecting blogs of greatest influence, document summarization, etc.).
- Very good if there are many elements v with $f(v) < f(u|V \setminus \{u\})$ for enough u elements (gain of v is evaluated only once).

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- If a popped item is fresh, it must be the maximum — this can happen if, at given iteration, v was first popped and neither fresh nor maximum so placed back in the queue, and it then percolates back to the top at which point it is fresh — thereby avoid extra queue check.

Minoux's Accelerated Greedy Algorithm Submodular Max

Algorithm 2: Minoux's Accelerated Greedy Algorithm

```
1  Set S_0 ← ∅ ; i ← 0 ; Initialize priority queue Q ;
2  for v ∈ E do
3      INSERT(Q, f(v))
4  repeat
5      (v, α) ← pop(Q) ;
6      if α not "fresh" then
7          recompute α ← f(v|S_i)
8      if (popped α in line 5 was "fresh") OR (α ≥ max(Q)) then
9          Set S_{i+1} ← S_i ∪ {v} ;
10         i ← i + 1 ;
11     else
12         insert(Q, (v, α))
13 until i = |E|;
```

(Minimum) Submodular Set Cover

- Given polymatroid f, goal is to find a covering set of minimum cost:

$$S^* \in \operatorname*{argmin}_{S \subseteq V} |S| \text{ such that } f(S) \geq \alpha \tag{13.38}$$

where α is a "cover" requirement.

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- Normally take $\alpha = f(V)$ but defining $f'(A) = \min\{f(A), \alpha\}$ we can take any α. Hence, we have equivalent formulation:

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- Note that this immediately generalizes standard set cover, in which case $f(A)$ is the cardinality of the union of sets indexed by A.
- Greedy Algorithm: Pick the first chain item S_i chosen by aforementioned greedy algorithm such that $f(S_i) \geq \alpha$ and output that as solution.

(Minimum) Submodular Set Cover: Approximation Analysis

- For integer valued f, this greedy algorithm an $O(\log(\max_{s \in V} f(\{s\})))$ approximation. Let S^* be optimal, and S^{G} be greedy solution, then

$$|S^{\mathsf{G}}| \leq |S^*| H(\max_{s \in V} f(\{s\})) = |S^*| O(\log_e(\max_{s \in V} f(\{s\}))) \tag{13.40}$$

where H is the harmonic function, i.e., $H(d) = \sum_{i=1}^{d} (1/i)$.

(Minimum) Submodular Set Cover: Approximation Analysis

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$$|S^{\mathsf{G}}| \leq |S^*| H(\max_{s \in V} f(\{s\})) = |S^*| O(\log_e(\max_{s \in V} f(\{s\}))) \tag{13.40}$$

  where H is the harmonic function, i.e., $H(d) = \sum_{i=1}^{d} (1/i)$.
- If f is not integral value, then bounds we get are of the form:

$$|S^{\mathsf{G}}| \leq |S^*| \left(1 + \log_e \frac{f(V)}{f(V) - f(S_{T-1})}\right) \tag{13.41}$$

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- Set cover is hard to approximate with a factor better than $(1 - \epsilon) \log \alpha$, where α is the desired cover constraint.

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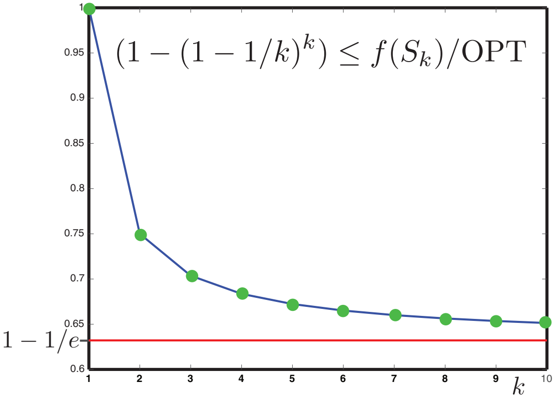
- Minoux's accelerated greedy trick.

The Greedy Algorithm: $1 - 1/e$ intuition.

- At step $i < k$, greedy chooses v_i to maximize $f(v|S_i)$.
- Let S^* be optimal solution (of size k) and OPT $= f(S^*)$. By submodularity, we will show:

$$\exists v \in V \setminus S_i : f(v|S_i) = f(S_i + v|S_i) \geq \frac{1}{k}(\text{OPT} - f(S_i)) \tag{13.21}$$



Equation (13.30) will show that Equation (13.21) $\Rightarrow$:

$$\text{OPT} - f(S_{i+1}) \leq (1 - 1/k)(\text{OPT} - f(S_i))$$

$$\Rightarrow \text{OPT} - f(S_k) \leq (1 - 1/k)^k \text{OPT} \leq 1/e\text{OPT}$$

$$\Rightarrow \text{OPT}(1 - 1/e) \leq f(S_k)$$

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- Suppose the following holds:

$$E[f(a_{i+1}|A_i)] \geq \frac{f(OPT) - f(A_i)}{k} \tag{13.42}$$

where $A_i = (a_1, a_2, \ldots, a_i)$ are the first i elements chosen by the strategy.