Submodular Functions, Optimization, and Applications to Machine Learning

— Fall Quarter, Lecture 12 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/

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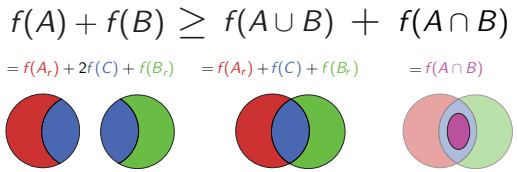
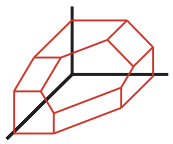
Nov 9th, 2020





Announcements, Assignments, and Reminders

- Homework 3, out soon.
- Reminder, all lectures are being recorded and posted to youtube. To get the links, see our announcements (`https://canvas.uw.edu/courses/1397085/announcements`).
- Office hours, Wed & Thur, 10:00pm at our class zoom link.

Class Road Map - EE563

- L1(9/30): Motivation, Applications, Definitions, Properties
- L2(10/5): Sums concave(modular), uses (diversity/costs, feature selection), information theory
- L3(10/7): Monge, More Definitions, Graph and Combinatorial Examples,
- L4(10/12): Graph & Combinatorial Examples, Matrix Rank, Properties, Other Defs, Independence
- L5(10/14): Properties, Defs of Submodularity, Independence
- L6(10/19): Matroids, Matroid Examples, Matroid Rank,
- L7(10/21): Matroid Rank, More on Partition Matroid, Laminar Matroids, System of Distinct Reps, Transversals
- L8(10/26): Transversal Matroid, Matroid and representation, Dual Matroid
- L9(10/28): Other Matroid Properties, Combinatorial Geometries, Matroid and Greedy, Polyhedra, Matroid Polytopes
- L10(11/2): Matroid Polytopes, Matroids $\rightarrow$ Polymatroids
- L11(11/4): Matroids $\rightarrow$ Polymatroids, Polymatroids
- L12(11/9): Polymatroids, Polymatroids and Greedy
- L–(11/11): Veterans Day, Holiday
- L13(11/16):
- L14(11/18):
- L15(11/23):
- L16(11/25):
- L17(11/30):
- L18(12/2):
- L19(12/7):
- L20(12/9): maximization.

Last day of instruction, Fri. Dec 11th. Finals Week: Dec 12-18, 2020

Polymatroidal polyhedron (or a "polymatroid")

Definition 12.2.1 (polymatroid)

A polymatroid is a compact set $P \subseteq \mathbb{R}_+^E$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For every $x \in \mathbb{R}_+^E$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any P-basis of x), has the same component sum $y(E)$

- Vectors within P (i.e., any $y \in P$) are called independent, and any vector outside of P is called dependent.
- Since all P-bases of x have the same component sum, if $\mathcal{B}_x$ is the set of P-bases of x, than $\text{rank}(x) = y(E)$ for any $y \in \mathcal{B}_x$.

Matroid and Polymatroid: side-by-side

A Matroid is:

1. a set system $(E, \mathcal{I})$
2. empty-set containing $\emptyset \in \mathcal{I}$
3. down closed, $\emptyset \subseteq I' \subseteq I \in \mathcal{I} \Rightarrow I' \in \mathcal{I}$.
4. any maximal set I in $\mathcal{I}$, bounded by another set A, has the same matroid rank (any maximal independent subset $I \subseteq A$ has same size $|I|$).

A Polymatroid is:

1. a compact set $P \subseteq \mathbb{R}_+^E$
2. zero containing, $\mathbf{0} \in P$
3. down monotone, $0 \leq y \leq x \in P \Rightarrow y \in P$
4. any maximal vector y in P, bounded by another vector x, has the same vector rank (any maximal independent subvector $y \leq x$ has same sum $y(E)$).

Polymatroid function and its polyhedron.

P,

Definition 12.2.1

A polymatroid function is a real-valued function f defined on subsets of E which is normalized, non-decreasing, and submodular. That is we have

1. $f(\emptyset) = 0$ (normalized)
2. $f(A) \leq f(B)$ for any $A \subseteq B \subseteq E$ (monotone non-decreasing)
3. $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$ for any $A, B \subseteq E$ (submodular)

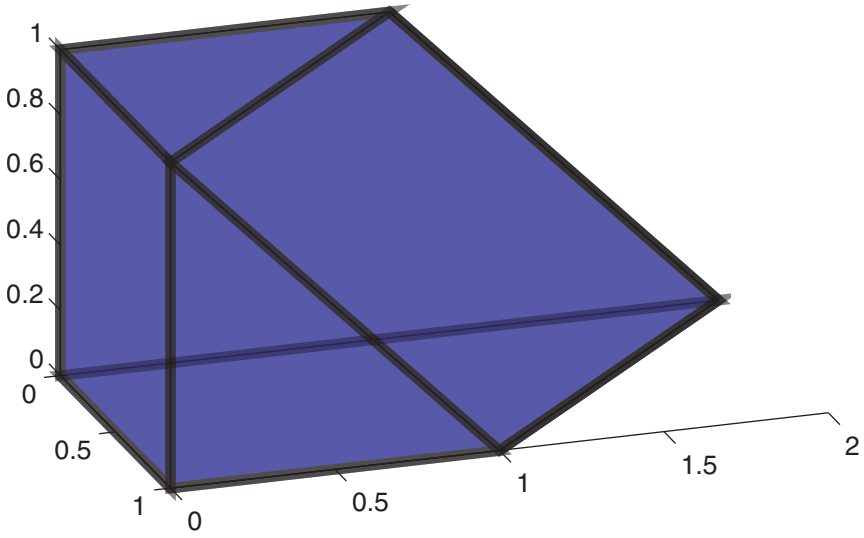
We can define the polyhedron P_f^+ associated with a polymatroid function as follows

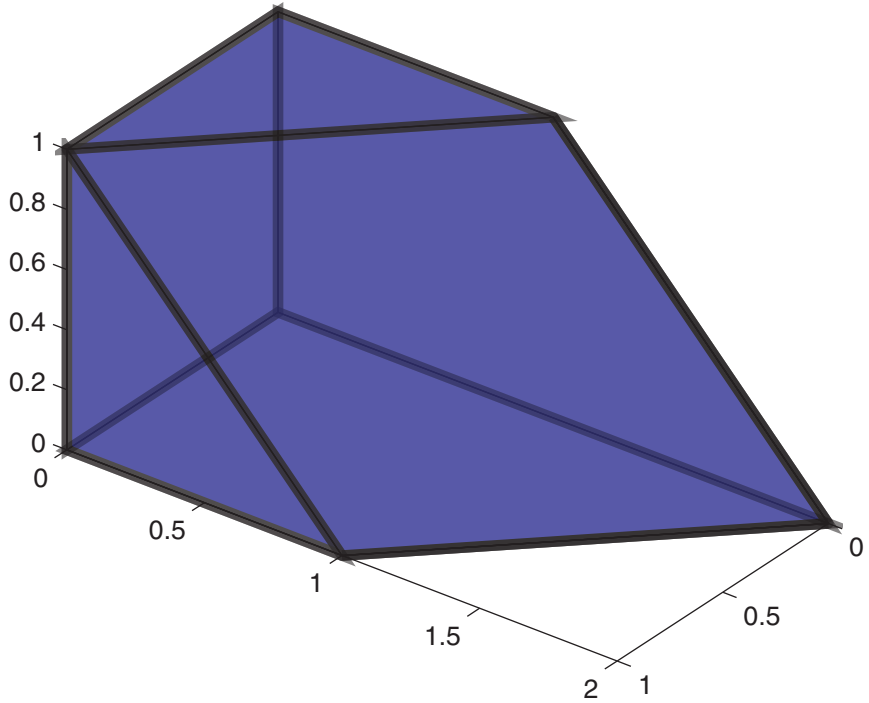
$$P_f^+ = \{y \in \mathbb{R}_+^E : y(A) \leq f(A) \text{ for all } A \subseteq E\} \tag{12.1}$$

$$= \{y \in \mathbb{R}^E : y \geq 0, y(A) \leq f(A) \text{ for all } A \subseteq E\} \tag{12.2}$$

Associated polyhedron with a polymatroid function

- Consider the asymmetric graph cut function on the simple chain graph $v_1 - v_2 - v_3$. That is, $f(S) = |\{(v, s) \in E(G) : v \in V, s \in S\}|$ is count of any edges within S or between S and $V \setminus S$, so that $\delta(S) = f(S) + f(V \setminus S) - f(V)$ is the standard graph cut.
- Observe: P_f^+ (at two views):

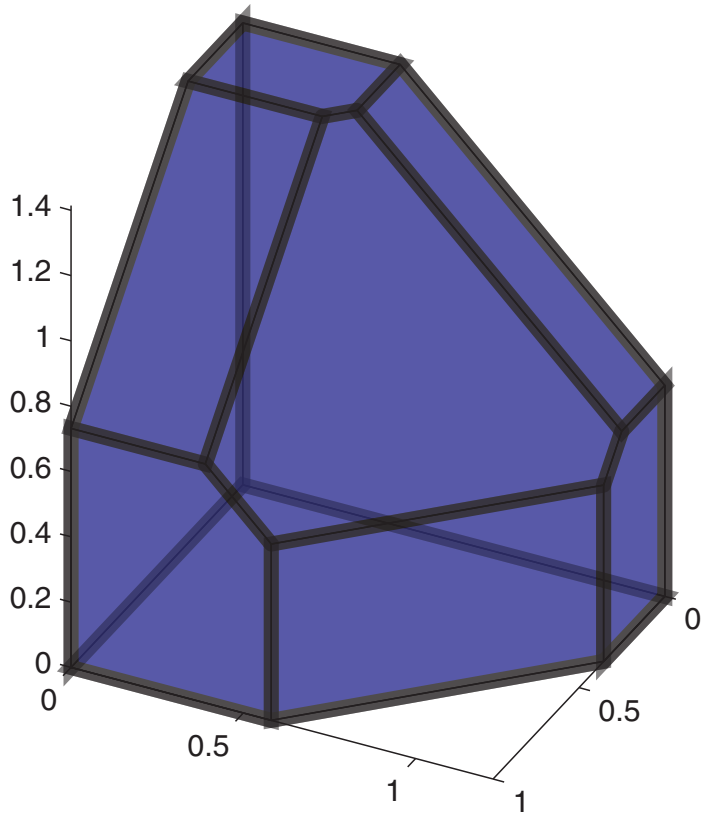


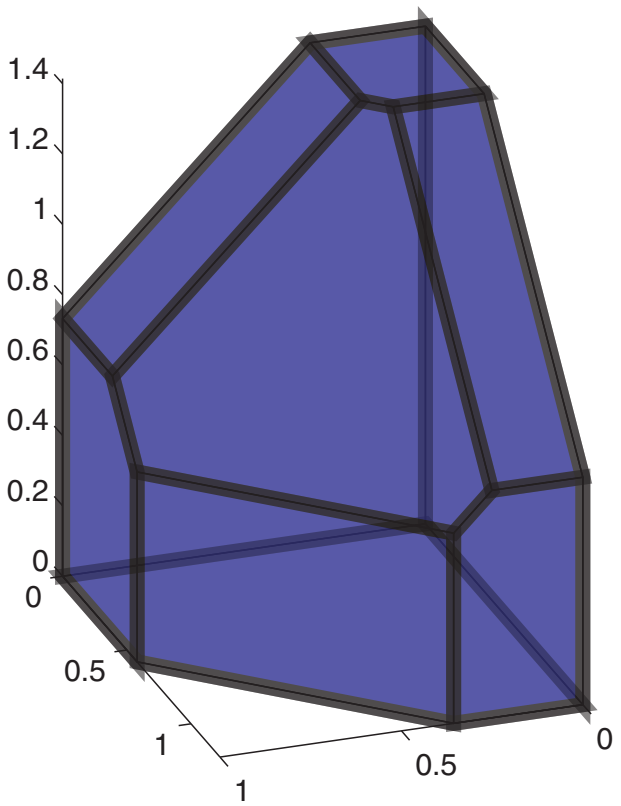


- which axis is which?

Associated polyhedron with a polymatroid function

- Consider modular function $w : V \to \mathbb{R}_+$ as $w = (1, 1.5, 2)^\intercal$, and then the submodular function $f(S) = \sqrt{w(S)}$.
- Observe: P_f^+ (at two views):





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A polymatroid vs. a polymatroid function's polyhedron

- Summarizing the above, we have:
  - Given a polymatroid function f, its associated polytope is given as

$$P_f^+ = \{y \in \mathbb{R}_+^E : y(A) \leq f(A) \text{ for all } A \subseteq E\} \tag{12.10}$$

  - We also have the definition of a polymatroidal polytope P (compact subset, zero containing, down-monotone, and $\forall x$ any maximal independent subvector $y \leq x$ has same component sum $y(E)$).
- Is there any relationship between these two polytopes?
- In the next theorem, that we proved last lecture. we show that any P_f^+-basis has the same component sum, when f is a polymatroid function, and P_f^+ satisfies the other properties so that P_f^+ is a polymatroid.
- After this, we will prove that for all polymatroid polytopes P, there exits a polymatroid function f such that $P = P_f^+$.
- Along the way, we will see that the vector rank function itself satisfies a form of submodularity.

A polymatroid function's polyhedron is a polymatroid.

Theorem 12.2.1

Let f be a polymatroid function defined on subsets of E. For any $x \in \mathbb{R}_+^E$, and <u>any</u> P_f^+-basis $y^x \in \mathbb{R}_+^E$ of x, the component sum of y^x is

$$\begin{aligned} y^x(E) = \textit{rank}(x) &\triangleq \max\left(y(E) : y \leq x, y \in P_f^+\right) \\ &= \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \qquad (12.10) \\ &= \min\left(x(E\setminus A) + f(A) : A \subseteq E\right) \end{aligned}$$

As a consequence, P_f^+ is a polymatroid, since r.h.s. is constant w.r.t. y^x.

Taking $E \setminus B = \text{supp}(x)$ (so elements B are all zeros in x), and for $b \notin B$ we make $x(b)$ is big enough, the r.h.s. min has solution $A^* = B$. We recover submodular function from the polymatroid polyhedron via the following:

$$\text{rank}\left(\frac{1}{\epsilon}\mathbf{1}_{\overset{A}{E\setminus B}}\right) = f(\overset{A}{E \setminus B}) = \max\left\{y(\overset{A}{E \setminus B}) : y \in P_f^+\right\} \qquad (12.11)$$

In fact, we will ultimately see a number of important consequences of this theorem (other than just that P_f^+ is a polymatroid)

A polymatroid is a polymatroid function's polytope

- So, when f is a polymatroid function, P_f^+ is a polymatroid.

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- Is it the case that, conversely, for any polymatroid P, there is an associated polymatroidal function f such that $P = P_f^+$?

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Theorem 12.3.1

For any polymatroid P (compact subset of $\mathbb{R}_+^E$, zero containing, down-monotone, and $\forall x \in \mathbb{R}_+^E$ any maximal independent subvector $y \leq x$ has same component sum $y(E) = \text{rank}(x)$), there is a polymatroid function $f : 2^E \to \mathbb{R}$ (normalized, monotone non-decreasing, submodular) such that $P = P_f^+$ where $P_f^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\}$.

Tight sets $\mathcal{D}(y)$ are closed, and max tight set $\text{sat}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{A : A \subseteq E,\ y(A) = f(A)\} \tag{12.1}$$

Theorem 12.3.2

For any $y \in P_f^+$, with f a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

i.e., if $A, B \in \mathcal{D}(y)$
then $A \cup B, A \cap B \in \mathcal{D}(y)$.

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Proof.

We have already proven this as part of Theorem ?? □

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Also recall the definition of $\text{sat}(y)$, the maximal set of tight elements relative to $y \in \mathbb{R}_+^E$.

$$\text{sat}(y) \stackrel{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\} \tag{12.2}$$

$\therefore \quad y(\text{sat}(y)) = f(\text{sat}(y))$

Join $\vee$ and meet $\wedge$ for $x, y \in \mathbb{R}_+^E$

- For $x, y \in \mathbb{R}_+^E$, define vectors $x \wedge y \in \mathbb{R}_+^E$ and $x \vee y \in \mathbb{R}_+^E$ such that, for all $e \in E$

$$(x \vee y)(e) = \max(x(e), y(e)) \tag{12.3}$$

$$(x \wedge y)(e) = \min(x(e), y(e)) \tag{12.4}$$

Hence,

$$x \vee y \triangleq \Big(\max\Big(x(e_1), y(e_1)\Big), \max\Big(x(e_2), y(e_2)\Big), \ldots, \max\Big(x(e_n), y(e_n)\Big)\Big)$$

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- From this, we can define things like an lattices, and other constructs.

Vector rank, rank(x), is submodular

- Recall that the matroid rank function $r(A) = \max(|I| : I \subseteq A : I \in \mathcal{I})$ is submodular.

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- The vector rank function rank$(x) = \max\left(y(E) : y \leq x, y \in P\right)$ also satisfies a form of submodularity, namely one defined on the real lattice.

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Theorem 12.3.3 (vector rank and submodularity)

Let P be a polymatroid polytope. The vector rank function rank $: \mathbb{R}_+^E \to \mathbb{R}$ with rank$(x) = \max\left(y(E) : y \leq x, y \in P\right)$ satisfies, for all $u, v \in \mathbb{R}_+^E$

$$\textit{rank}(u) + \textit{rank}(v) \geq \textit{rank}(u \vee v) + \textit{rank}(u \wedge v) \tag{12.5}$$

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$r(A) = \text{rank}(1_A)$

- Note what happens when $u, v \in \{0,1\}^E \subseteq \mathbb{R}_+^E$.

$u = 1_A, \quad v = 1_B \qquad u \vee v = 1_{A \cup B} \qquad u \wedge v = 1_{A \cap B}$

$\therefore \text{rank}(1_A) + \text{rank}(1_B) \geq \text{rank}(1_{A \cup B}) + \text{rank}(1_{A \cap B})$

Vector rank rank(x) is submodular, proof

Proof of Theorem 12.3.3.

- Let $a \in \mathbb{R}_+^E$ be a P-basis of $u \wedge v$, so rank$(u \wedge v) = a(E)$.

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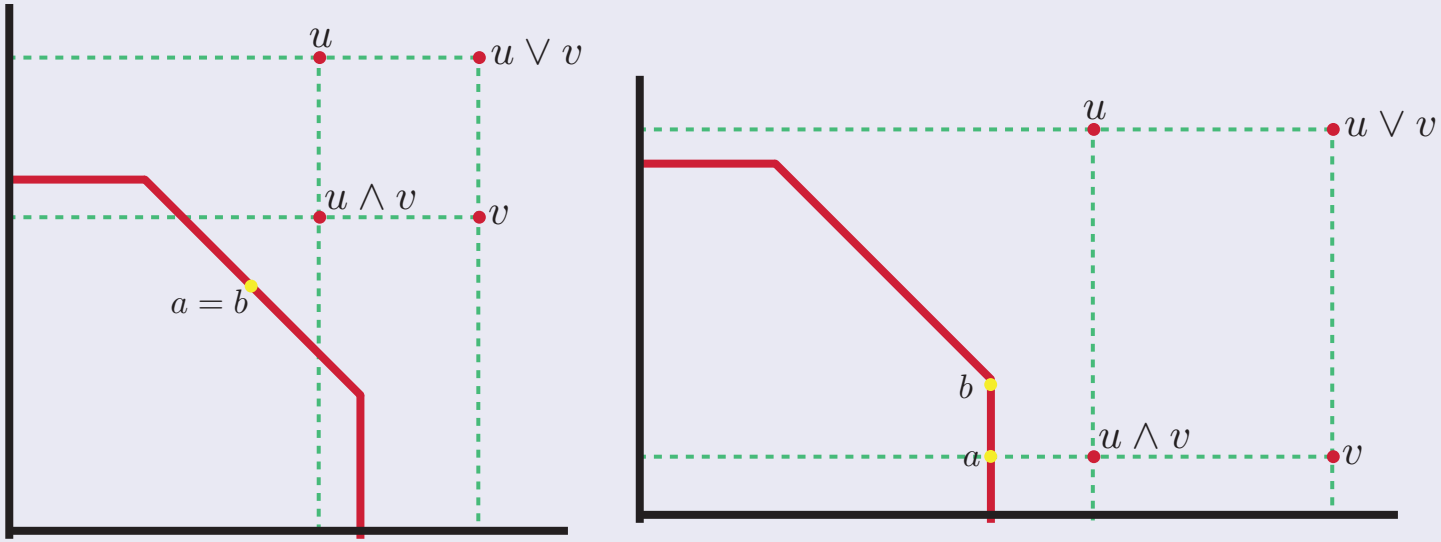
Ex
$|E|=2$



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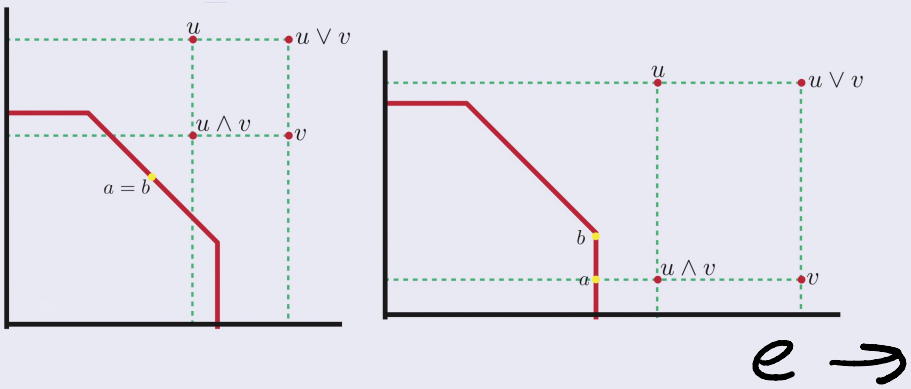

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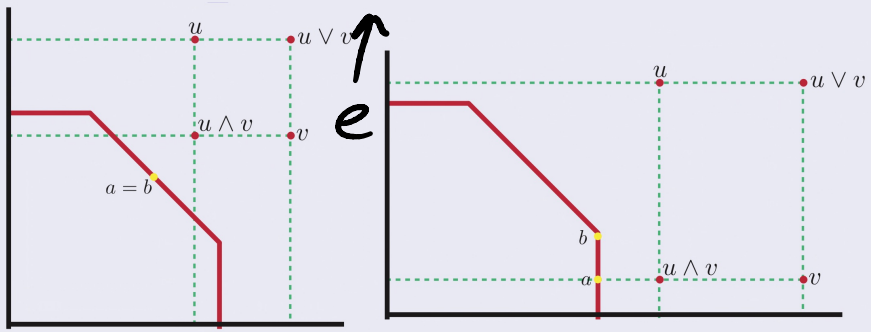

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- ... and since $b \leq u \vee v$, we get

$$a + b \tag{12.6}$$

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- Let $a \in \mathbb{R}_+^E$ be a P-basis of $u \wedge v$, so rank$(u \wedge v) = a(E)$.
- Claim (shown below): $\exists$ an independent $b \in P$ such that: $a \leq b \leq u \vee v$ and also such that rank$(b) = b(E) =$ rank$(u \vee v)$, so b is a P-basis of $u \vee v$, and thus $b \leq u \vee v$. Follows by polymatroid property since ...
- Given any $e \in E$: if $a(e)$ is maximal due to P, then $a(e) = b(e)$ $\leq \min(u(e), v(e))$ ...
- otherwise, if $a(e)$ is maximal due to $(u \wedge v)(e)$, then $a(e) = \min(u(e), v(e)) \leq b(e)$.
- Therefore, in either case, $a = b \wedge (u \wedge v)$ ...
- ... and since $b \leq u \vee v$, we get

$$a + b = b \wedge u \wedge v + b = b \wedge u + b \wedge v \tag{12.6}$$

How? Note $b(e) \leq \max(u(e), v(e))$. *Suppose w.l.o.g.* $u(e) \leq v(e)$. *then* $\min(b(e), u(v), v(e)) + b(e) = \min(b(e), u(e)) + b(e) =$ $\min(b(e), u(e)) + \min(b(e), v(e))$.

...

Vector rank rank(x) is submodular, proof

$a + b = b \wedge u + b \wedge v$

. . . proof of Theorem 12.3.3.

- b is independent, and $b \wedge u$ and $b \wedge v$ are independent subvectors of u and v respectively, so $(b \wedge u)(E) \leq \text{rank}(u)$ and $(b \wedge v)(E) \leq \text{rank}(v)$.

□

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- Hence,

$$\text{rank}(u \wedge v) + \text{rank}(u \vee v)$$

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$$\text{rank}(u \wedge v) + \text{rank}(u \vee v) = a(E) + b(E) \tag{12.7}$$

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$$= (b \wedge u)(E) + (b \wedge v)(E) \tag{12.8}$$

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$$\leq \text{rank}(u) + \text{rank}(v) \tag{12.9}$$

□

A polymatroid function's polyhedron vs. a polymatroid.

- Note the remarkable similarity between the proof of Theorem 12.3.3 and the proof of Theorem ?? that the standard matroid rank function is submodular.

the corresponding proof of the submodularity of matroid rank.

A polymatroid function's polyhedron vs. a polymatroid.

- Note the remarkable similarity between the proof of Theorem 12.3.3 and the proof of Theorem **??** that the standard matroid rank function is submodular.
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A polymatroid function's polyhedron vs. a polymatroid.

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- Next, we prove Theorem 12.3.1, that any polymatroid polytope P has a polymatroid function f such that $P = P_f^+$.
- Given this result, we can conclude that a polymatroid is really an extremely natural polyhedral generalization of a matroid. This was all realized by Jack Edmonds in the mid 1960s (and published in 1969 in his landmark paper "Submodular Functions, Matroids, and Certain Polyhedra").

A polymatroid is a polymatroid function's polytope

- So, when f is a polymatroid function, P_f^+ is a polymatroid.
- Is it the case that, conversely, for any polymatroid P, there is an associated polymatroidal function f such that $P = P_f^+$?

Theorem 12.3.1

For any polymatroid P (compact subset of $\mathbb{R}_+^E$, zero containing, down-monotone, and $\forall x \in \mathbb{R}_+^E$ any maximal independent subvector $y \leq x$ has same component sum $y(E) = \text{rank}(x)$), there is a polymatroid function $f : 2^E \to \mathbb{R}$ (normalized, monotone non-decreasing, submodular) such that $P = P_f^+$ where $P_f^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\}$.

Method to prove Theorem 12.3.1

start with polymatroid P and

- To show Theorem 12.3.1, we will first define a function f, show that that it is monotone non-decreasing submodular, which allows us to define P_f^+, and that we show that $P \subseteq P_f^+$.

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- This results in that $P_f^+ = P$ to complete the proof.

Proof of Theorem 12.3.1

Proof of Theorem 12.3.1 ($\forall P, \exists f$ s.t. $P = P_f^+$).

- We are given a polymatroid P.

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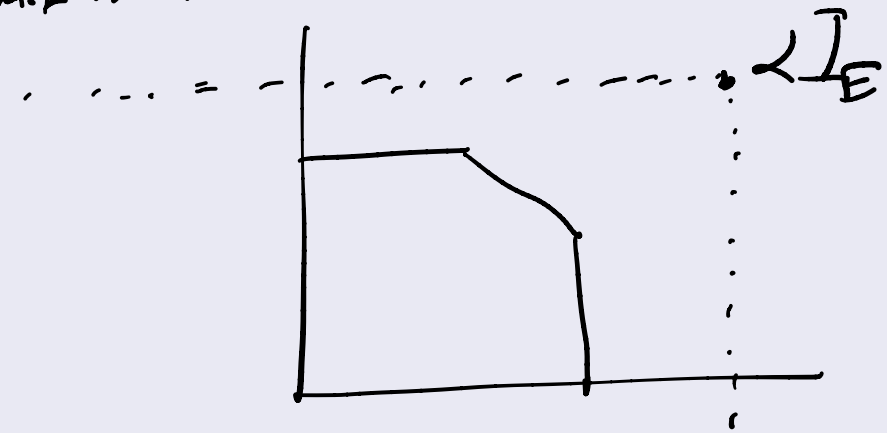
- We are given a polymatroid P.
- Define $\alpha_{\text{max}} \triangleq \max\{x(E) : x \in P\}$, and note that $\alpha_{\text{max}} > 0$ when P is non-empty, and $\alpha_{\text{max}} = \lim_{\alpha\to\infty} \text{rank}(\alpha \mathbf{1}_E) = \text{rank}(\alpha_{\text{max}} \mathbf{1}_E)$.

α_{max} is the component sum of the base of the polymatroid.



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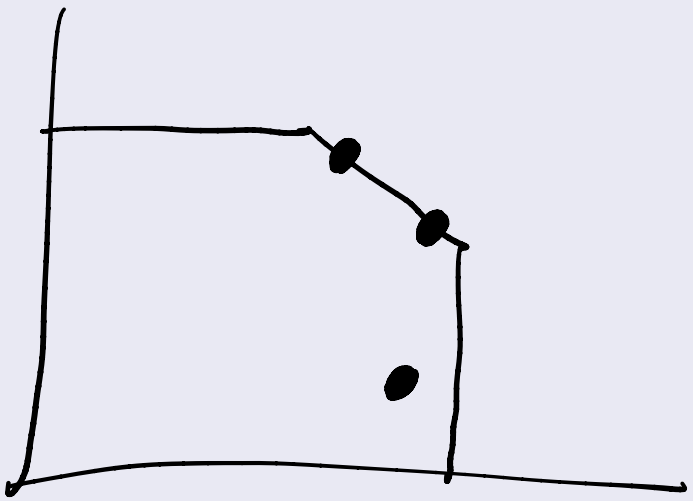
- Hence, for any $x \in P$, and $\forall e \in E$, we have $x(e) \leq x(E) \leq \alpha_{\text{max}}$.



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- Hence, for any $x \in P$, and $\forall e \in E$, we have $x(e) \leq x(E) \leq \alpha_{\text{max}}$.
- Define a function $f : 2^V \to \mathbb{R}$ as, for any $A \subseteq E$,

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$$\geq \text{rank}(\underbrace{\alpha_{\text{max}} \mathbf{1}_A \vee \alpha_{\text{max}} \mathbf{1}_B}_{\alpha_{\text{max}} 1_{A\cup B}}) + \text{rank}(\underbrace{\alpha_{\text{max}} \mathbf{1}_A \wedge \alpha_{\text{max}} \mathbf{1}_B}_{\alpha_{\text{max}} 1_{A\cap B}}) \tag{12.12}$$

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Proof of Theorem 12.3.1

Proof of Theorem 12.3.1 ($\forall P, \exists f$ s.t. $P = P_f^+$).

- Moreover, we have that f is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).

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- Hence, f is a polymatroid function.
- Definition: for any $A \subseteq E$, define $x_A \in \mathbb{R}_+^E$ as

$$x_A(e) = \begin{cases} x(e) & \text{if } e \in A \\ 0 & \text{else} \end{cases} \tag{12.15}$$

note this is an analogous definition to $\mathbf{1}_A$ but for a not necessarily unity vector x.

i.e., $x_A = x \wedge \left[\left(\max_{a \in A} x(a) \right) \cdot \mathbf{1}_A \right]$

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- Hence $x_A(A) = x(A)$ and $x_A(E \setminus A) = 0$.
- Using $f(A) \triangleq \text{rank}(\alpha_{\text{max}} \mathbf{1}_A)$, consider the polytope P_f^+ defined as:

$$P_f^+ = \left\{ x \in \mathbb{R}_+^E : x(A) \leq f(A), \ \forall A \subseteq E \right\} \tag{12.16}$$

Proof of Theorem 12.3.1

Proof of Theorem 12.3.1 ($\forall P, \exists f$ s.t. $P = P_f^+$).

- Given an $x \in P$, then for any $A \subseteq E$, $x_A \leq \alpha_{\mathsf{max}} \mathbf{1}_A$, and thus $x(A) \leq \alpha_{\mathsf{max}} |A|$.

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Proof of Theorem 12.3.1

Proof of Theorem 12.3.1 ($\forall P, \exists f$ s.t. $P = P_f^+$).

- Given an $x \in P$, then for any $A \subseteq E$, $x_A \leq \alpha_{\text{max}} \mathbf{1}_A$, and thus $x(A) \leq \alpha_{\text{max}}|A|$.
- Therefore,

$$
\begin{aligned}
x(A) &\leq \max\{z(A) : z \in P, z_A \leq \alpha_{\text{max}} \mathbf{1}_A\} && (12.17)\\
&= \max\{z(A) : z \in P, z \leq \alpha_{\text{max}} \mathbf{1}_A\} && (12.18)\\
&\leq \max\{z(E) : z \in P, z \leq \alpha_{\text{max}} \mathbf{1}_A\} && (12.19)\\
&= \text{rank}(\alpha_{\text{max}} \mathbf{1}_A) && (12.20)\\
&= f(A) && (12.21)
\end{aligned}
$$

P_f^+

Therefore $x \in P_f^+$.

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  Therefore $x \in P_f^+$.
- Hence, $P \subseteq P_f^+$.
- We will next show that $P_f^+ \subseteq P$ to complete the proof.

...

Proof of Theorem 12.3.1

Proof of Theorem 12.3.1 ($\forall P, \exists f$ s.t. $P = P_f^+$).

- Let $x \in P_f^+$ be chosen arbitrarily (goal is to show that $x \in P$).

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- Let $x \in P_f^+$ be chosen arbitrarily (goal is to show that $x \in P$).
- Suppose $x \notin P$.

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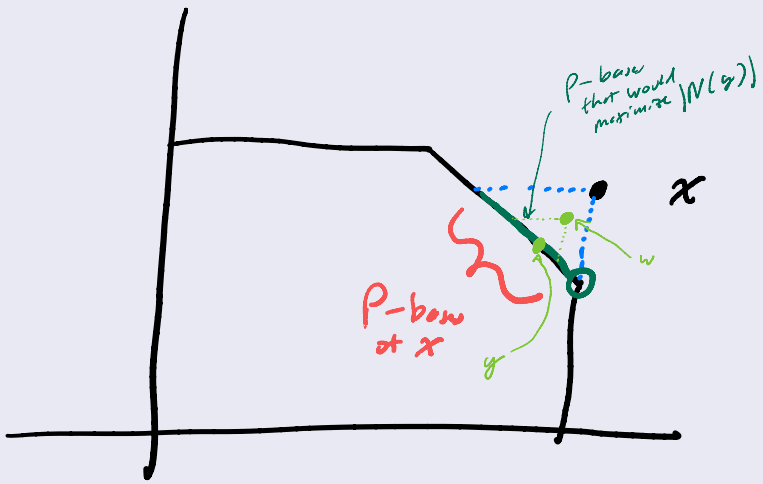
- Let $x \in P_f^+$ be chosen arbitrarily (goal is to show that $x \in P$).
- Suppose $x \notin P$. Then, choose y to be a P-basis of x that maximizes the number of y elements strictly less than the corresponding x element. I.e., that maximizes $|N(y)|$, where

$$N(y) = \{e \in E : y(e) < x(e)\} \tag{12.22}$$



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- Choose w between y and x, so that $y \leq x$

$$y \leq w \triangleq (y + x)/2 \leq x \tag{12.23}$$

so y is also a P-basis of w. Thus, $\forall e \in N(y)$, $y(e) < w(e) < x(e)$.

...

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- Choose w between y and x, so that

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so y is also a P-basis of w. Thus, $\forall e \in N(y)$, $y(e) < w(e) < x(e)$.
- Hence, rank(x) = rank$(w) = y(E)$, and the set of P-bases of w are also P-bases of x.

...

Proof of Theorem 12.3.1

Proof of Theorem 12.3.1 ($\forall P, \exists f$ s.t. $P = P_f^+$).

- Now, we have

$$y(N(y)) < w(N(y)) \leq f(N(y)) \overset{\triangle}{=} \text{rank}(\alpha_{\max} \mathbf{1}_{N(y)}) \tag{12.24}$$

the last inequality follows since $w \leq x$ and $x \in P_f^+$ (so $x(A) \leq f(A), \forall A$). aw $N(y) \subseteq E$.

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$$y(N(y)) < w(N(y)) \leq f(N(y)) = \text{rank}(\alpha_{\text{max}} \mathbf{1}_{N(y)}) \tag{12.24}$$

  the last inequality follows since $w \leq x$ and $x \in P_f^+$ (so $x(A) \leq f(A), \forall A$).
- Thus, $y \wedge x_{N(y)}$ is not a P-basis of $w \wedge x_{N(y)}$ since, over $N(y)$, it is neither tight at w nor tight at the rank (i.e., not a maximal independent subvector on $N(y)$).

...

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- We can extend $y \wedge x_{N(y)}$ to be a P-basis of $w \wedge x_{N(y)}$ since $y \wedge x_{N(y)} < w \wedge x_{N(y)}$.

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- and also that $\hat{y}(E) = y(E)$ (since both y and $\hat{y}$ are P-bases of w and x),

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- This contradiction means that we must have had $x \in P$.

$\therefore P_f^+ \subseteq P$

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- This contradiction means that we must have had $x \in P$.
- Therefore, $P_f^+ = P$. $\square$

More on polymatroids

Theorem 12.3.4

A polymatroid can equivalently be defined as a pair (E, P) where E is a finite ground set and $P \subseteq R_+^E$ is a compact non-empty set of independent vectors such that

1. *every subvector of an independent vector is independent (if $x \in P$ and $y \leq x$ then $y \in P$, i.e., down closed)*

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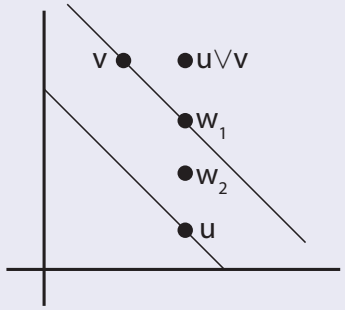
2. *If $u, v \in P$ (i.e., are independent) and $u(E) < v(E)$, then there exists a vector $w \in P$ such that*

$$u < w \leq u \vee v \qquad (12.25)$$



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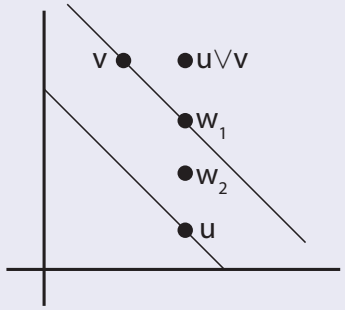

Corollary 12.3.5

The independent vectors of a polymatroid form a convex polyhedron in $\mathbb{R}_+^E$.

Review

- The next slide comes from lecture 6.

Matroids by bases

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

Theorem 12.3.3 (Matroid (by bases))

Let E be a set and $\mathcal{B}$ be a nonempty collection of subsets of E. Then the following are equivalent.

1. *$\mathcal{B}$ is the collection of bases of a matroid;*
2. *if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.*
3. *If $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.*

Properties 2 and 3 are called "exchange properties."
Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.

More on polymatroids

For any compact set P, b is a base of P if it is a maximal subvector within P. Recall the bases of matroids. In fact, we can define a polymatroid via vector bases (analogous to how a matroid can be defined via matroid bases).

Theorem 12.3.6

A polymatroid can equivalently be defined as a pair (E, P) where E is a finite ground set and $P \subseteq R_+^E$ is a compact non-empty set of independent vectors such that

1. *every subvector of an independent vector is independent (if $x \in P$ and $y \leq x$ then $y \in P$, i.e., down closed)*
2. *if b, c are bases of P and d is such that $b \wedge c < d < b$, then there exists an f, with $d \wedge c < f \leq c$ such that $d \vee f$ is a base of P*
3. *All of the bases of P have the same rank.*

Note, all three of the above are required for a polymatroid (a matroid analogy would require the equivalent of only the first two).

A word on terminology & notation

- Recall how a matroid is sometimes given as (E, r) where r is the rank function.

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A word on terminology & notation

- Recall how a matroid is sometimes given as (E, r) where r is the rank function.
- We mention also that the term "polymatroid" is sometimes not used for the polytope itself, but instead but for the pair (E, f),
- But now we see that (E, f) is equivalent to a polymatroid polytope, so this is sensible.

Where are we going with this?

- Consider the right hand side of Theorem ??:
  $\min\left(x(A) + f(E \setminus A) : A \subseteq E\right)$

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Where are we going with this?

- Consider the right hand side of Theorem ??:
  $\min\left(x(A) + f(E \setminus A) : A \subseteq E\right)$ $= \min\left(x(E \setminus A) + f(A) : A \subseteq E\right)$
- We are going to study this problem, and approaches that address it, as part of our ultimate goal which is to present strategies for submodular function minimization (that we will ultimately get to, in near future lectures).
- As a bit of a hint on what's to come, recall that we can write it as: $x(E) + \min\left(\underbrace{f(A) - x(A)}_{\text{any submodular function.}} : A \subseteq E\right)$ where f is a polymatroid function.

Another Interesting Fact: Matroids from polymatroid functions

Theorem 12.3.7

Given integral polymatroid function f, let $(E, \mathcal{F})$ be a set system with ground set E and set of subsets $\mathcal{F}$ such that

$$\forall F \in \mathcal{F}, \ \ \forall \emptyset \subset S \subseteq F, |S| \leq f(S) \tag{12.26}$$

Then $M = (E, \mathcal{F})$ is a matroid.

Proof.

Exercise □

And its rank function is Exercise.

Matroid instance of Theorem ??

- Considering Theorem ??, the matroid case is now a special case, where we have that:

Corollary 12.3.8

We have that:

$$\max\{y(E) : y \in P_{\textit{ind. set}}(M), y \leq x\} = \min\{r_M(A) + x(E \setminus A) : A \subseteq E\} \tag{12.27}$$

where r_M is the matroid rank function of some matroid.

Review

- The next two slides come respectively from Lecture 11 and Lecture 10.

Polymatroidal polyhedron (or a "polymatroid")

Definition 12.4.1 (polymatroid)

A polymatroid is a compact set $P \subseteq \mathbb{R}_+^E$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For every $x \in \mathbb{R}_+^E$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any P-basis of x), has the same component sum $y(E)$

- Vectors within P (i.e., any $y \in P$) are called independent, and any vector outside of P is called dependent.
- Since all P-bases of x have the same component sum, if $\mathcal{B}_x$ is the set of P-bases of x, than $\text{rank}(x) = y(E)$ for any $y \in \mathcal{B}_x$.

Maximum weight independent set via greedy weighted rank

Theorem 12.4.1

Let $M = (V, \mathcal{I})$ be a matroid, with rank function r, then for any weight function $w \in \mathbb{R}_+^V$, there exists a chain of sets $U_1 \subset U_2 \subset \cdots \subset U_n \subseteq V$ such that

$$\max \{w(I) | I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i) \tag{12.4}$$

where $\lambda_i \geq 0$ satisfy

$$w = \sum_{i=1}^{n} \lambda_i \mathbf{1}_{U_i} \tag{12.5}$$

Polymatroidal polyhedron and greedy

- Let $(E, \mathcal{I})$ be a set system and $w \in \mathbb{R}_+^E$ be a weight vector.

Polymatroidal polyhedron and greedy

- Let $(E, \mathcal{I})$ be a set system and $w \in \mathbb{R}_+^E$ be a weight vector.
- Recall greedy algorithm: Set $A = \emptyset$, and repeatedly choose $y \in E \setminus A$ such that $A \cup \{y\} \in \mathcal{I}$ with $w(y)$ as large as possible, stopping when no such y exists.

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- For a matroid, we saw that independence system $(E, \mathcal{I})$ is a matroid iff for each weight function $w \in \mathbb{R}_+^E$, the greedy algorithm leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.

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- Stated succinctly, considering $\max\{w(I) : I \in \mathcal{I}\}$, then $(E, \mathcal{I})$ is a matroid iff greedy works for this maximization.

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  $\langle w, 1_I \rangle$
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- That is, if we consider $\max \left\{ \overset{\langle w, x \rangle}{wx} : x \in P_f^+ \right\}$, where P_f^+ represents the "independent vectors", is it the case that P_f^+ is a polymatroid iff greedy works for this maximization?

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- Can we also characterize a polymatroid in this way?
- That is, if we consider $\max\left\{wx : x \in P_f^+\right\}$, where P_f^+ represents the "independent vectors", is it the case that P_f^+ is a polymatroid iff greedy works for this maximization?
- Can we, ultimately, even relax things so that $w \in \mathbb{R}^E$?

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- What is the greedy solution in this setting, when $w \in \mathbb{R}^E$?

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- Sort elements of E w.r.t. w so that, w.l.o.g. $E = (e_1, e_2, \ldots, e_m)$ with $w(e_1) \geq w(e_2) \geq \cdots \geq w(e_m)$.

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- Let $k+1$ be the first point (if any) at which we are non-positive, i.e., $w(e_k) > 0$ and $0 \geq w(e_{k+1})$.
  That is, we have

$$w(e_1) \geq w(e_2) \geq \cdots \geq w(e_k) > 0 \geq \underbrace{w(e_{k+1}) \geq \cdots \geq w(e_m)}_{\text{0 or negative.}} \tag{12.28}$$

Polymatroidal polyhedron and greedy

- What is the greedy solution in this setting, when $w \in \mathbb{R}^E$?
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- Let $k+1$ be the first point (if any) at which we are non-positive, i.e., $w(e_k) > 0$ and $0 \geq w(e_{k+1)}$.
- Next define partial accumulated sets E_i, for $i = 0 \dots m$, we have w.r.t. the above sorted order:

$$E_i \stackrel{\text{def}}{=} \{e_1, e_2, \dots e_i\} \tag{12.29}$$

(note $E_0 = \emptyset$, $f(E_0) = 0$, and <u>E and E_i is always sorted w.r.t w</u>).

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- What is the greedy solution in this setting, when $w \in \mathbb{R}^E$?
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- The greedy solution is the vector $x \in \mathbb{R}_+^E$ with elements defined as:

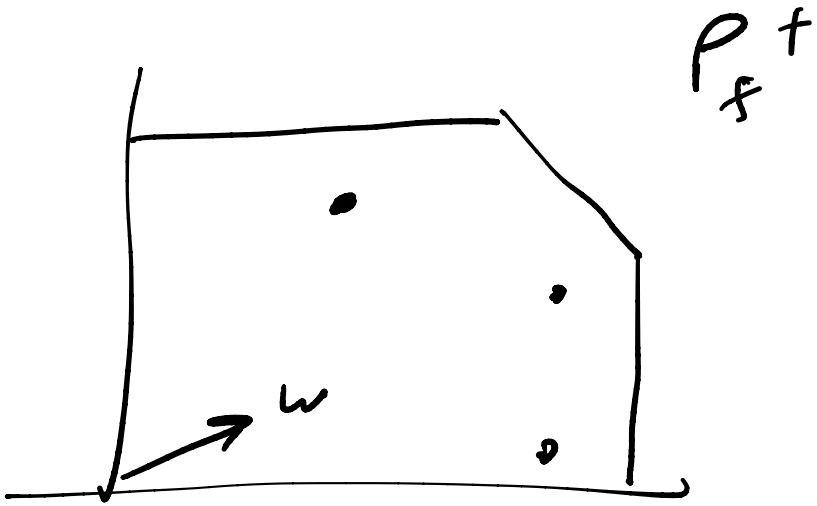
$$x(e_1) \stackrel{\text{def}}{=} f(E_1) = f(e_1) = f(e_1|E_0) = f(e_1|\emptyset) \tag{12.30}$$

$$x(e_i) \stackrel{\text{def}}{=} f(E_i) - f(E_{i-1}) = f(e_i|E_{i-1}) \text{ for } i = 2 \dots k \tag{12.31}$$

$$x(e_i) \stackrel{\text{def}}{=} 0 \text{ for } i = k+1 \dots m = |E| \tag{12.32}$$

Some Intuition: greedy and gain

- Note $x(e_i) = f(e_i|E_{i-1}) \leq f(e_i|E')$ for any $E' \subseteq E_{i-1}$



goal: find x s.t. $\langle w, x \rangle$ is max.

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- Note $x(e_i) = f(e_i|E_{i-1}) \leq f(e_i|E')$ for any $E' \subseteq E_{i-1}$
- So $x(e_1) = f(e_1)$ and this corresponds to $w(e_1) \geq w(e_i)$ for all $i \neq 1$.

$$\langle x, w \rangle = x(e_1) \cdot w(e_1) + \text{rest.}$$

$$f(e_1) \geq f(e_1|A) \quad \forall A.$$

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- Hence, for the largest value of w (namely $w(e_1)$), we use for $x(e_1)$ the largest possible gain value of e_1 (namely $f(e_1|\emptyset) \geq f(e_1|A)$ for any $A \subseteq E \setminus \{e_1\}$).

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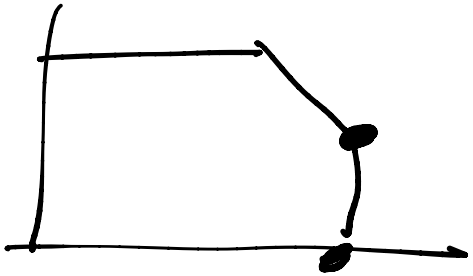
- For the next largest value of w (namely $w(e_2)$), we use for $x(e_2)$ the next largest gain value of e_2 (namely $f(e_2|e_1)$), while still ensuring (as we will soon see in Theorem 12.4.1) that the resulting $x \in P_f$.

if we need to condition on e_1



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- For the next largest value of w (namely $w(e_2)$), we use for $x(e_2)$ the next largest gain value of e_2 (namely $f(e_2|e_1)$), while still ensuring (as we will soon see in Theorem 12.4.1) that the resulting $x \in P_f$.
- This process continues, using the next largest possible gain of e_i for $x(e_i)$ while ensuring (as we will show) we do not leave the polytope, given the values we've already chosen for $x(e_{i'})$ for $i' < i$.

Polymatroidal polyhedron and greedy

Theorem 12.4.1

The vector $x \in \mathbb{R}_+^E$ as previously defined using the greedy algorithm maximizes wx over P_f^+, with $w \in \mathbb{R}_+^E$, if f is submodular.

Proof.

Polymatroidal polyhedron and greedy

Theorem 12.4.1

The vector $x \in \mathbb{R}_+^E$ as previously defined using the greedy algorithm maximizes wx over P_f^+, with $w \in \mathbb{R}_+^E$, if f is submodular.

Proof.

- Consider the LP strong duality equation:

$$\max(wx : x \in P_f^+) = \min\Big(\sum_{A \subseteq E} y_A f(A) : y \in \mathbb{R}_+^{2^E}, \sum_{A \subseteq E} y_A \mathbf{1}_A \geq w\Big) \tag{12.33}$$

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$$y_{E_i} \leftarrow w(e_i) - w(e_{i+1}) \text{ for } i = 1 \ldots (m-1), \tag{12.34}$$
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$E_i = \{e_1, e_2, \ldots, e_i\}$

Polymatroidal polyhedron and greedy

Proof.

- We first will see that greedy $x \in P_f^+$ (that is $x(A) \leq f(A), \forall A$).

...

Polymatroidal polyhedron and greedy

Proof.

- We first will see that greedy $x \in P_f^+$ (that is $x(A) \leq f(A), \forall A$).
- Order $A = (a_1, a_2, \ldots, a_k)$ based on order $(e_1, e_2, \ldots, e_m)$.

		a_1		a_2	a_3			a_4		a_5	$\ldots$	
e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	$\ldots$	e_m

$\ldots$

Polymatroidal polyhedron and greedy

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- Define $e^{-1} : E \to \{1, \ldots, m\}$ so that $e^{-1}(e_i) = i$.
  This means that with $A = \{a_1, a_2, \ldots, a_k\}$, and $\forall j \leq k$

$$\{a_1, a_2, \ldots, a_j\} \subseteq \left\{e_1, e_2, \ldots, e_{e^{-1}(a_j)}\right\} \tag{12.37}$$

and

$$\{a_1, a_2, \ldots, a_{j-1}\} \subseteq \left\{e_1, e_2, \ldots, e_{e^{-1}(a_j)-1}\right\} \tag{12.38}$$

Also recall matlab notation: $a_{1:j} \equiv \{a_1, a_2, \ldots, a_j\}$.
E.g., with $j = 4$ we get $e^{-1}(a_4) = 9$, and

$$\{a_1, a_2, a_3, a_4\} \subseteq \{e_1, e_2, \ldots, e_9\} \tag{12.39}$$

Polymatroidal polyhedron and greedy

Proof.

- We first will see that greedy $x \in P_f^+$ (that is $x(A) \leq f(A), \forall A$).
- Order any $A = (a_1, a_2, \ldots, a_k)$ based on order $(e_1, e_2, \ldots, e_m)$.

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e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	$\ldots$	e_m

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- Then, we have $x \in P_f^+$ since for all A:

$$f(A) = \sum_{i=1}^{k} f(a_i | a_{1:i-1}) \tag{12.37}$$

$$\geq \sum_{i=1}^{k} f(a_i | e_{1:e^{-1}(a_i)-1}) \tag{12.38}$$

$$= \sum_{a \in A} f(a | e_{1:e^{-1}(a)-1}) = x(A) \tag{12.39}$$

Polymatroidal polyhedron and greedy

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Polymatroidal polyhedron and greedy

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- y being dual feasible in Eq. 12.33 means: $y \geq 0$ and $\sum_{A \subseteq E} y_A \mathbf{1}_A \geq w$.

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Polymatroidal polyhedron and greedy

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- Next, we check that y is dual feasible. Clearly, $y \geq 0$,

that we chose

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- Next, we check that y is dual feasible. Clearly, $y \geq 0$,
- and also, considering y component wise, for any i, we have that

$$\sum_{A: e_i \in A} y_A = \sum_{j \geq i} y_{E_j} = \sum_{j=i}^{m-1} (w(e_j) - w(e_{j+1})) + w(e_m) = w(e_i).$$

$\geq w(e_i)$

$i = 4$

$E_0 = \emptyset$

$E_1 = \{e_1\}$

$E_2 = \{e_1, e_2\}$

$E_3 = \{e_1, e_2, e_3\}$

$E_4 = \{e_1, e_2, e_3, e_4\}$ ...

$E_5 = \{ \ldots e_4 \ldots \}$

Proof.

- Consider the LP strong duality equation:

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Polymatroidal polyhedron and greedy

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$$\sum_{A: e_i \in A} y_A = \sum_{j \geq i} y_{E_j} = \sum_{j=i}^{m-1} (w(e_j) - w(e_{j+1})) + w(e_m) = w(e_i).$$

- Now optimality for x and y follows from strong duality, i.e.:

$$wx = \sum_{e \in E} w(e)x(e) = \sum_{i=1}^{m} w(e_i) f(e_i | E_{i-1}) = \sum_{i=1}^{m} w(e_i) \Big(f(E_i) - f(E_{i-1}) \Big)$$

for the sets that are chosen

$$= \sum_{i=1}^{m-1} f(E_i) \Big(w(e_i) - w(e_{i+1}) \Big) + f(E) w(e_m) = \sum_{A \subseteq E} y_A f(A)$$

...

Polymatroidal polyhedron and greedy

Proof.

- The equality in prev. Eq. follows via Abel summation:

$$
\begin{aligned}
wx &= \sum_{i=1}^{m} w_i x_i && (12.40)\\
&= \sum_{i=1}^{m} w_i \Big(f(E_i) - f(E_{i-1}) \Big) && (12.41)\\
&= \sum_{i=1}^{m} w_i f(E_i) - \sum_{i=1}^{m-1} w_{i+1} f(E_i) && (12.42)\\
&= w_m f(E_m) + \sum_{i=1}^{m-1} \big(w_i - w_{i+1} \big) f(E_i) && (12.43)
\end{aligned}
$$

□

What about $w \in \mathbb{R}^E$

- When w contains negative elements, we have $x(e_i) = 0$ for $i = k+1, \ldots, m$, where k is the last positive element of w when it is sorted in decreasing order.

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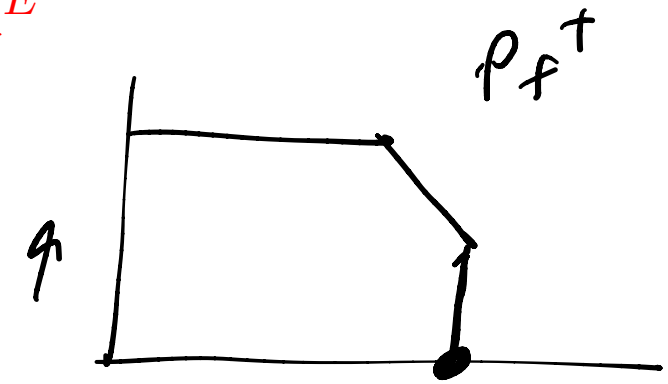
- When w contains negative elements, we have $x(e_i) = 0$ for $i = k+1, \ldots, m$, where k is the last positive element of w when it is sorted in decreasing order.
- Exercise: show a modification of the previous proof that works for arbitrary $w \in \mathbb{R}^E$



Polymatroidal polyhedron and greedy

Theorem 12.4.1

Conversely, suppose P_f^+ is a polytope of form $P_f^+ = \{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\}$, then the greedy solution to $\max(wx : x \in P_f^+)$ is optimum only if f is submodular.

Proof.

- Choose A and B arbitrarily, and then order elements of E as $(e_1, e_2, \ldots, e_m)$, with $E_i = (e_1, e_2, \ldots, e_i)$, so the following is true:

Polymatroidal polyhedron and greedy

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- Choose A and B arbitrarily, and then order elements of E as $(e_1, e_2, \ldots, e_m)$, with $E_i = (e_1, e_2, \ldots, e_i)$, so the following is true:
- For $1 \leq p \leq q \leq m$, $A = \{e_1, e_2, \ldots, e_k, e_{k+1}, \ldots, e_p\} = E_p$ and $B = \{e_1, e_2, \ldots, e_k, e_{p+1}, \ldots, e_q\} = E_k \cup (E_q \setminus E_p) = (A \cap B) \cup (B \setminus A)$

$|A| = p \quad |A \cup B| = q \geq p$

$|A \cap B| = k$

Polymatroidal polyhedron and greedy

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- Note, then we have $A \cap B = \{e_1, \ldots, e_k\} = E_k$, and $A \cup B = E_q$.

Polymatroidal polyhedron and greedy

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- Note, then we have $A \cap B = \{e_1, \ldots, e_k\} = E_k$, and $A \cup B = E_q$.
- Define $w \in \{0,1\}^m$ as:

$$w \stackrel{\text{def}}{=} \sum_{i=1}^{q} \mathbf{1}_{e_i} = \mathbf{1}_{A \cup B} \tag{12.44}$$

Polymatroidal polyhedron and greedy

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- Suppose optimum solution x is given by the greedy procedure.

Polymatroidal polyhedron and greedy

Proof.

- Then

$$\sum_{i=1}^{k} x_i = f(E_1) + \sum_{i=2}^{k} (f(E_i) - f(E_{i-1})) = f(E_k) = f(A \cap B) \tag{12.45}$$

...

Polymatroidal polyhedron and greedy

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- Then

$$\sum_{i=1}^{k} x_i = f(E_1) + \sum_{i=2}^{k} (f(E_i) - f(E_{i-1})) = f(E_k) = f(A \cap B) \tag{12.45}$$

- and

$$\sum_{i=1}^{p} x_i = f(E_1) + \sum_{i=2}^{p} (f(E_i) - f(E_{i-1})) = f(E_p) = f(A) \tag{12.46}$$

...

Polymatroidal polyhedron and greedy

Proof.

- Then

$$\sum_{i=1}^{k} x_i = f(E_1) + \sum_{i=2}^{k} (f(E_i) - f(E_{i-1})) = f(E_k) = f(A \cap B) \tag{12.45}$$

- and

$$\sum_{i=1}^{p} x_i = f(E_1) + \sum_{i=2}^{p} (f(E_i) - f(E_{i-1})) = f(E_p) = f(A) \tag{12.46}$$

- and

$$\sum_{i=1}^{q} x_i = f(E_1) + \sum_{i=2}^{q} (f(E_i) - f(E_{i-1})) = f(E_q) = f(A \cup B) \tag{12.47}$$

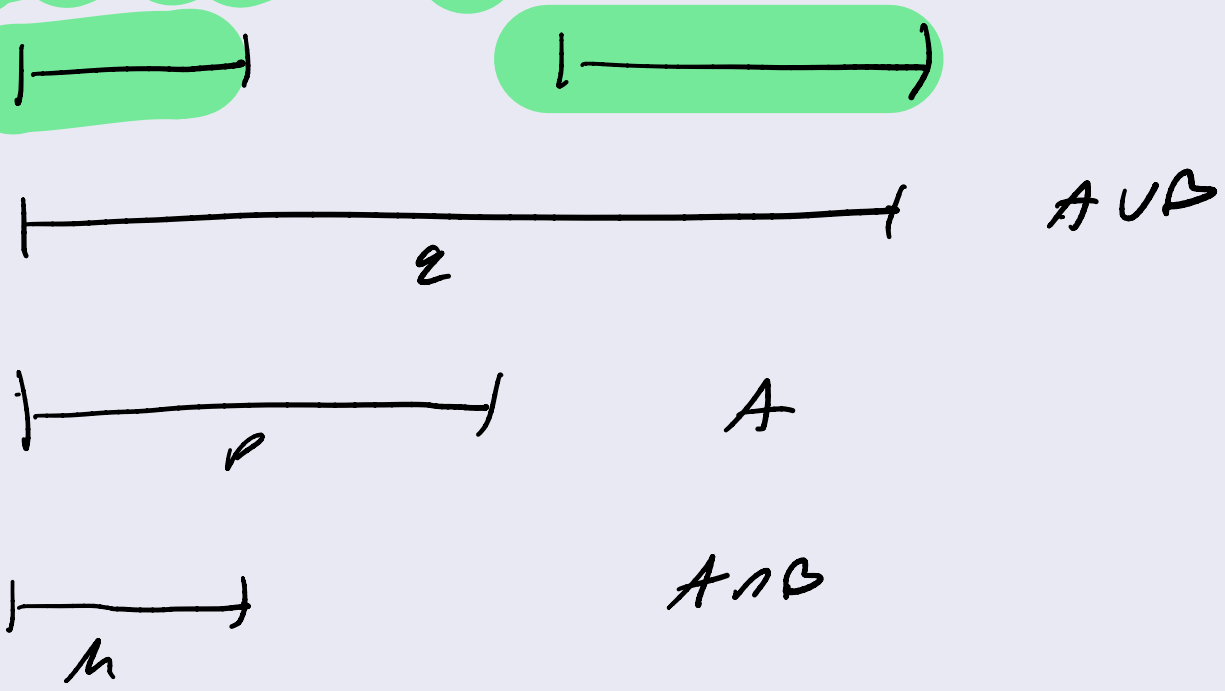
...

Polymatroidal polyhedron and greedy

Proof.

- Thus, we have

$$x(B) = \sum_{i \in 1,\ldots,k,p+1,\ldots,q} x_i = \sum_{i:e_i \in B} x_i = f(A \cup B) + f(A \cap B) - f(A) \tag{12.48}$$



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Polymatroidal polyhedron and greedy

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- But given that the greedy algorithm gives the optimal solution to $\max(wx : x \in P_f^+)$, we have that $x \in P_f^+$ and thus $x(B) \leq f(B)$.

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Polymatroidal polyhedron and greedy

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- But given that the greedy algorithm gives the optimal solution to $\max(wx : x \in P_f^+)$, we have that $x \in P_f^+$ and thus $x(B) \leq f(B)$.
- Thus,

$$x(B) = f(A \cup B) + f(A \cap B) - f(A) = \sum_{i:e_i \in B} x_i \leq f(B) \tag{12.49}$$

ensuring the submodularity of f, since A and B are arbitrary.

□

Review from Lecture 8

- The next slide comes from lecture 8.

Matroid and the greedy algorithm

- Let $(E, \mathcal{I})$ be an independence system, and we are given a non-negative modular weight function $w : E \to \mathbb{R}_+$.

Algorithm 1: The Matroid Greedy Algorithm

Set $X \leftarrow \emptyset$;
while $\exists v \in E \setminus X$ *s.t.* $X \cup \{v\} \in \mathcal{I}$ **do**
  $v \in \operatorname{argmax} \{w(v) : v \in E \setminus X,\ X \cup \{v\} \in \mathcal{I}\}$;
  $X \leftarrow X \cup \{v\}$;

- Same as sorting items by decreasing weight w, and then choosing items in that order that retain independence.

Theorem 12.4.4

Let $(E, \mathcal{I})$ be an independence system. Then the pair $(E, \mathcal{I})$ is a matroid if and only if for each weight function $w \in \mathcal{R}_+^E$, Algorithm ?? above leads to a set $I \in \mathcal{I}$ of maximum weight $w(I)$.

Polymatroidal polyhedron and greedy

- Thus, restating the above results into a single complete theorem, we have a result very similar to what we saw for matroids (i.e., Theorem ??)

Theorem 12.4.1

If $f : 2^E \to \mathbb{R}_+$ is given, and P is a polytope in $\mathbb{R}_+^E$ of the form $P = \left\{x \in \mathbb{R}_+^E : x(A) \leq f(A), \forall A \subseteq E\right\}$, then the greedy solution to the problem $\max(w^\intercal x : x \in P)$ is $\forall w$ optimum iff f is monotone non-decreasing submodular (i.e., iff P is a polymatroid).