Submodular Functions, Optimization, and Applications to Machine Learning
— Fall Quarter, Lecture 11 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/

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\[ f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \]

- \[ f(A) + 2f(C) + f(B) \]
- \[ f(A) + f(C) + f(B) \]
- \[ f(A \cap B) \]
Homework 3, out soon.

Reminder, all lectures are being recorded and posted to youtube. To get the links, see our announcements (https://canvas.uw.edu/courses/1397085/announcements).

Office hours, Wed & Thur, 10:00pm at our class zoom link.
# Class Road Map - EE563

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**Last day of instruction, Fri. Dec 11th. Finals Week: Dec 12-18, 2020**
Summary of Important (for us) Matroid Definitions

Given an independence system, matroids are defined equivalently by any of the following:

- All maximally independent sets have the same size.
- A normalized monotone non-decreasing submodular integral rank function with unit increments.
- The greedy algorithm achieves the maximum weight independent set for all non-negative weight functions.
Independence Polyhedra

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $1_I \in \{0, 1\}^E \subset [0, 1]^E \subset \mathbb{R}_+^E$.
- Taking the convex hull, we get the independent set polytope, that is
  \[
  P_{\text{ind. set}} = \text{conv} \left\{ \bigcup_{I \in \mathcal{I}} \{1_I\} \right\} \subseteq [0, 1]^E
  \] (11.1)

- Now take the rank function $r$ of $M$, and define the following polyhedron:
  \[
  P^+_r \triangleq \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \}
  \] (11.2)

Examples of $P^+_r$ are forthcoming.

- Now, take any $x \in P_{\text{ind. set}}$, then we will show that that $x \in P^+_r$ (or $P_{\text{ind. set}} \subseteq P^+_r$). We show this after a few examples of $P^+_r$. 
Matroid Polyhedron in 2D

\[ P_r^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \} \quad (11.1) \]

- Consider this in two dimensions. We have equations of the form:

\[ x_1 \geq 0 \text{ and } x_2 \geq 0 \quad (11.2) \]
\[ x_1 \leq r(\{v_1\}) \in \{0, 1\} \quad (11.3) \]
\[ x_2 \leq r(\{v_2\}) \in \{0, 1\} \quad (11.4) \]
\[ x_1 + x_2 \leq r(\{v_1, v_2\}) \in \{0, 1, 2\} \quad (11.5) \]

- Because \( r \) is submodular, we have

\[ r(\{v_1\}) + r(\{v_2\}) \geq r(\{v_1, v_2\}) + r(\emptyset) \quad (11.6) \]

so since \( r(\{v_1, v_2\}) \leq r(\{v_1\}) + r(\{v_2\}) \), the last inequality is either superfluous \((r(v_1, v_2) = r(v_1) + r(v_2), \text{ “inactive”})\) or “active.”
Matroid Polyhedron in 2D

And, if \( v_2 \) is a loop ...

\[
x_2 \leq r(\{v_2\})
\]

\[
x_2 \geq 0
\]

\[
x_1 \geq 0
\]

\[
x_1 \leq r(\{v_1\})
\]
Matroid Polyhedron in 3D

\[ P_r^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \} \]  \hspace{1cm} (11.1)

- Consider three dimensions, \( E = \{1, 2, 3\} \). Get equations of the form:

\[
\begin{align*}
    x_1 &\geq 0 \text{ and } x_2 \geq 0 \text{ and } x_3 \geq 0 \quad \text{(11.2)} \\
    x_1 &\leq r(\{v_1\}) \quad \text{(11.3)} \\
    x_2 &\leq r(\{v_2\}) \quad \text{(11.4)} \\
    x_3 &\leq r(\{v_3\}) \quad \text{(11.5)} \\
    x_1 + x_2 &\leq r(\{v_1, v_2\}) \quad \text{(11.6)} \\
    x_2 + x_3 &\leq r(\{v_2, v_3\}) \quad \text{(11.7)} \\
    x_1 + x_3 &\leq r(\{v_1, v_3\}) \quad \text{(11.8)} \\
    x_1 + x_2 + x_3 &\leq r(\{v_1, v_2, v_3\}) \quad \text{(11.9)}
\end{align*}
\]
Matroid Polyhedron in 3D

Two view of $P^+_r$ associated with a matroid
($\{e_1, e_2, e_3\}, \{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}$).
Theorem 11.2.1

Let $M = (V, \mathcal{I})$ be a matroid, with rank function $r$, then for any weight function $w \in \mathbb{R}^V_+$, there exists a chain of sets $U_1 \subset U_2 \subset \cdots \subset U_n \subseteq V$ such that

$$\max \{w(I) | I \in \mathcal{I} \} = \sum_{i=1}^{n} \lambda_i r(U_i)$$

(11.4)

where $\lambda_i \geq 0$ satisfy

$$w = \sum_{i=1}^{n} \lambda_i 1_{U_i}$$

(11.5)
Polytope Equivalence (Summarizing the above)

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I$.
- Taking the convex hull, we get the independent set polytope, that is

$$P_{\text{ind. set}} = \text{conv} \left\{ \bigcup_{I \in \mathcal{I}} \{ \mathbf{1}_I \} \right\} \quad (11.7)$$

- Now take the rank function $r$ of $M$, and define the following polytope:

$$P^+_r = \left\{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E \right\} \quad (11.8)$$

**Theorem 11.2.1**

$$P^+_r = P_{\text{ind. set}} \quad (11.9)$$
Greedy solves a linear programming problem

- So we can describe the independence polytope of a matroid using the set of inequalities (an exponential number of them).
- In fact, considering equations starting at Eq ??, the LP problem with exponential number of constraints \( \max \{ w^\top x : x \in P_r^+ \} \) is identical to the maximum weight independent set problem in a matroid, and since greedy solves the latter problem exactly, we have also proven:

**Theorem 11.2.1**

*The LP problem \( \max \{ w^\top x : x \in P_r^+ \} \) can be solved exactly using the greedy algorithm.*

Note that this LP problem has an exponential number of constraints (since \( P_r^+ \) is described as the intersection of an exponential number of half spaces).

- This means that if LP problems have certain structure, they can be solved much easier than immediately implied by the equations.
**Definition 11.2.2 (subvector)**

*y* is a subvector of *x* if *y* ≤ *x* (meaning *y*(*)e* *) ≤ *x*(*)e* *) for all *e* ∈ *E*).

**Definition 11.2.3 (P-basis)**

Given a compact set *P* ⊆ *R*^E_+, for any *x* ∈ *R*^E_+, a subvector *y* of *x* is called a *P*-basis of *x* if *y* maximal in *P*.

In other words, *y* is a *P*-basis of *x* if *y* is a maximal *P*-contained subvector of *x*.

Here, by *y* being “maximal”, we mean that there exists no *z* > *y* (more precisely, no *z* ≥ *y* + *ϵ*1,*e* for some *e* ∈ *E* and *ϵ* > 0) having the properties of *y* (the properties of *y* being: in *P*, and a subvector of *x*).

In still other words: *y* is a *P*-basis of *x* if:

1. *y* ≤ *x* (*y* is a subvector of *x*); and
2. *y* ∈ *P* and *y* + *ϵ*1,*e* ∉ *P* for all *e* ∈ *E* where *y*(*e*) < *x*(*e*) and ∀*ϵ* > 0 (*y* is maximal *P*-contained).
A vector form of rank

- Recall the definition of rank from a matroid $M = (E, \mathcal{I})$.

  $$\text{rank}(A) = \max \{|I| : I \subseteq A, I \in \mathcal{I}\} = \max_{I \in \mathcal{I}} |A \cap I| \quad (11.19)$$

- **vector rank**: Given a compact set $P \subseteq \mathbb{R}^E_+$, define a form of “vector rank” relative to $P$: Given an $x \in \mathbb{R}^E$:

  $$\text{rank}(x) = \max (y(E) : y \leq x, y \in P) = \max_{y \in P} (x \wedge y)(E) \quad (11.20)$$

  where $y \leq x$ is componentwise inequality ($y_i \leq x_i, \forall i$), and where $(x \wedge y) \in \mathbb{R}^E_+$ has $(x \wedge y)(i) = \min(x(i), y(i))$.

- Sometimes use $\text{rank}_P(x)$ to make $P$ explicit.

- If $B_x$ is the set of $P$-bases of $x$, then $\text{rank}(x) = \max_{y \in B_x} y(E)$.

- If $x \in P$, then $\text{rank}(x) = x(E)$ ($x$ is its own unique self $P$-basis).

- If $x_{\text{min}} \in \text{argmin}_{x \in P} x(E)$, and $x \leq x_{\text{min}}$ what then? Then $\text{rank}(x)$ is either $x(E)$ (if $x = x_{\text{min}}$) or otherwise $\text{rank}(x) = -\infty$.

- In general, might be hard to compute and/or have ill-defined properties. Next, we look at an object that restrains and cultivates this form of rank.
A **polymatroid** is a compact set \( P \subseteq \mathbb{R}^E_+ \) satisfying

1. \( 0 \in P \)
2. If \( y \leq x \in P \) then \( y \in P \) (called **down monotone**).
3. For every \( x \in \mathbb{R}^E_+ \), any maximal vector \( y \in P \) with \( y \leq x \) (i.e., any \( P \)-basis of \( x \)), has the same component sum \( y(E) \).
Polymatroidal polyhedron (or a “polymatroid”)

**Definition 11.3.1 (polymatroid)**

A polymatroid is a compact set $P \subseteq \mathbb{R}^E_+$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For every $x \in \mathbb{R}^E_+$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any $P$-basis of $x$), has the same component sum $y(E)$.

Condition 3 restated: That is for any two distinct maximal vectors $y^1, y^2 \in P$, with $y^1 \leq x$ & $y^2 \leq x$, with $y^1 \neq y^2$, we must have $y^1(E) = y^2(E)$. 

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Condition 3 restated (again): For every vector $x \in \mathbb{R}_+^E$, every maximal independent (i.e., $\in P$) subvector $y$ of $x$ has the same component sum $y(E) = \text{rank}(x)$. 
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Condition 3 restated: That is for any two distinct maximal vectors $y^1, y^2 \in P$, with $y^1 \leq x$ & $y^2 \leq x$, with $y^1 \neq y^2$, we must have $y^1(E) = y^2(E)$.

Condition 3 restated (again): For every vector $x \in \mathbb{R}^E_+$, every maximal independent (i.e., $\in P$) subvector $y$ of $x$ has the same component sum $y(E) = \text{rank}(x)$.

Condition 3 restated (yet again): All $P$-bases of $x$ have the same component sum.
Polymatroidal polyhedron (or a “polymatroid’’)

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A polymatroid is a compact set \( P \subseteq \mathbb{R}_+^E \) satisfying

1. \( 0 \in P \)
2. If \( y \leq x \in P \) then \( y \in P \) (called down monotone).
3. For every \( x \in \mathbb{R}_+^E \), any maximal vector \( y \in P \) with \( y \leq x \) (i.e., any \( P \)-basis of \( x \)), has the same component sum \( y(E) \)

- Vectors within \( P \) (i.e., any \( y \in P \)) are called independent, and any vector outside of \( P \) is called dependent.
Polymatroidal polyhedron (or a “polymatroid’’)

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A polymatroid is a compact set $P \subseteq \mathbb{R}_+^E$ satisfying

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2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For every $x \in \mathbb{R}_+^E$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any $P$-basis of $x$), has the same component sum $y(E)$.

- Vectors within $P$ (i.e., any $y \in P$) are called independent, and any vector outside of $P$ is called dependent.
- Since all $P$-bases of $x$ have the same component sum, if $\mathcal{B}_x$ is the set of $P$-bases of $x$, than $\text{rank}(x) = y(E)$ for any $y \in \mathcal{B}_x$. 
Matroid and Polymatroid: side-by-side

A Matroid is:

A Polymatroid is:
Matroid and Polymatroid: side-by-side

A Matroid is:

1. a set system \((E, \mathcal{I})\)

A Polymatroid is:

1. a compact set \(P \subseteq \mathbb{R}^E_+\)
Matroid and Polymatroid: side-by-side

A Matroid is:
1. a set system $(E, \mathcal{I})$
2. empty-set containing $\emptyset \in \mathcal{I}$

A Polymatroid is:
1. a compact set $P \subseteq \mathbb{R}_+^E$
2. zero containing, $\mathbf{0} \in P$
Matroid and Polymatroid: side-by-side

A Matroid is:
1. a set system \((E, \mathcal{I})\)
2. empty-set containing \(\emptyset \in \mathcal{I}\)
3. down closed, \(\emptyset \subseteq I' \subseteq I \in \mathcal{I} \Rightarrow I' \in \mathcal{I}\).

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1. a set system \((E, \mathcal{I})\)
2. empty-set containing \(\emptyset \in \mathcal{I}\)
3. down closed, \(\emptyset \subseteq I' \subseteq I \in \mathcal{I} \Rightarrow I' \in \mathcal{I}\).
4. any maximal set \(I\) in \(\mathcal{I}\), bounded by another set \(A\), has the same matroid rank (any maximal independent subset \(I \subseteq A\) has same size \(|I|\)).

A Polymatroid is:

1. a compact set \(P \subseteq \mathbb{R}^E_+\)
2. zero containing, \(0 \in P\)
3. down monotone, \(0 \leq y \leq x \in P \Rightarrow y \in P\)
4. any maximal vector \(y\) in \(P\), bounded by another vector \(x\), has the same vector rank (any maximal independent subvector \(y \leq x\) has same sum \(y(E)\)).
Polymatroidal polyhedron (or a “polymatroid’’)

Left: \( \exists \) multiple maximal \( y \leq x \)  
Right: \( \exists \) only one maximal \( y \leq x \),

- Polymatroid condition here: \( \forall \) maximal \( y \in P \), with \( y \leq x \) (which here means \( y_1 \leq x_1 \) and \( y_2 \leq x_2 \)), we just have \( y(E) = y_1 + y_2 = \text{const.} \)
Polymatroidal polyhedron (or a "polymatroid")

Left: $\exists$ multiple maximal $y \leq x$ Right: $\exists$ only one maximal $y \leq x$,

- Polymatroid condition here: $\forall$ maximal $y \in P$, with $y \leq x$ (which here means $y_1 \leq x_1$ and $y_2 \leq x_2$), we just have $y(E) = y_1 + y_2 = \text{const}$.
- On the left, we see there are multiple possible maximal $y \in P$ such that $y \leq x$. Each such $y$ must have the same value $y(E)$. 
Polymatroidal polyhedron (or a "polymatroid")

Left: $\exists$ multiple maximal $y \leq x$ Right: $\exists$ only one maximal $y \leq x$,

- Polymatroid condition here: $\forall$ maximal $y \in P$, with $y \leq x$ (which here means $y_1 \leq x_1$ and $y_2 \leq x_2$), we just have $y(E) = y_1 + y_2 = \text{const.}$
- On the left, we see there are multiple possible maximal $y \in P$ such that $y \leq x$. Each such $y$ must have the same value $y(E)$.
- On the right, there is only one maximal $y \in P$. Since there is only one, the condition on the same value of $y(E)$, $\forall y$ is vacuous.
$\exists$ only one maximal $y \leq x$.

- If $x \in P$ already, then $x$ is its own $P$-basis, i.e., it is a self $P$-basis.
∃ only one maximal $y \leq x$.

- If $x \in P$ already, then $x$ is its own $P$-basis, i.e., it is a **self $P$-basis**.
- In a matroid, a base of $A$ is the maximally contained independent set. If $A$ is already independent, then $A$ is a self-base of $A$ (as we saw in previous Lectures)
Polymatroid as well?

Left and right: $\exists$ multiple maximal $y \leq x$ as indicated.

- On the left, we see there are multiple possible maximal such $y \in P$ that are $y \leq x$. Each such $y$ must have the same value $y(E)$, but since the equation for the curve is $y_1^2 + y_2^2 = \text{const.} \neq y_1 + y_2$, we see this is not a polymatroid.
Polymatroid as well? no

Left and right: \( \exists \) multiple maximal \( y \leq x \) as indicated.

- On the left, we see there are multiple possible maximal such \( y \in P \) that are \( y \leq x \). Each such \( y \) must have the same value \( y(E) \), but since the equation for the curve is \( y_1^2 + y_2^2 = \text{const.} \neq y_1 + y_2 \), we see this is not a polymatroid.

- On the right, we have a similar situation, just the set of potential values that must have the \( y(E) \) condition changes, but the values of course are still not constant.
Other examples: Polymatroid or not?
It appears that we have five possible forms of polymatroid in 2D, when neither of the elements \( \{v_1, v_2\} \) are self-dependent.

1. **On the left:** full dependence between \( v_1 \) and \( v_2 \)
Some possible polymatroid forms in 2D

It appears that we have five possible forms of polymatroid in 2D, when neither of the elements \( \{v_1, v_2\} \) are self-dependent.

1. On the left: full dependence between \( v_1 \) and \( v_2 \)
2. Next: full independence between \( v_1 \) and \( v_2 \)
It appears that we have five possible forms of polymatroid in 2D, when neither of the elements \( \{v_1, v_2\} \) are self-dependent.

1. On the left: full dependence between \( v_1 \) and \( v_2 \)
2. Next: full independence between \( v_1 \) and \( v_2 \)
3. Next: partial independence between \( v_1 \) and \( v_2 \)
Some possible polymatroid forms in 2D

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2. Next: full independence between \( v_1 \) and \( v_2 \)

3. Next: partial independence between \( v_1 \) and \( v_2 \)

4. Right two: other forms of partial independence between \( v_1 \) and \( v_2 \)
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4. Right two: other forms of partial independence between \( v_1 \) and \( v_2 \)
   - The \( P \)-bases (or single \( P \)-base in the middle case) are as indicated.
It appears that we have five possible forms of polymatroid in 2D, when neither of the elements \( \{v_1, v_2\} \) are self-dependent.

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- The \( P \)-bases (or single \( P \)-base in the middle case) are as indicated.
- Independent vectors are those within or on the boundary of the polytope. Dependent vectors are exterior to the polytope.
Some possible polymatroid forms in 2D

It appears that we have five possible forms of polymatroid in 2D, when neither of the elements \( \{v_1, v_2\} \) are self-dependent.

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4. Right two: other forms of partial independence between \( v_1 \) and \( v_2 \)
   - The \( P \)-bases (or single \( P \)-base in the middle case) are as indicated.
   - Independent vectors are those within or on the boundary of the polytope. Dependent vectors are exterior to the polytope.
   - The set of \( P \)-bases for a polytope is called the base polytope.
Polymatroidal polyhedron (or a “polymatroid’”) 

Suppose $x, y \in \mathbb{R}_+^E$ with $y \leq x$. If $x$ contains any zeros (i.e., $x$ has $E \setminus S$ s.t. $x(E \setminus S) = 0$, so $S$ indicates the non-zero elements, or $S = \text{supp}(x)$), then this also forces $y(E \setminus S) = 0$, so that $y(E) = y(S)$. This is true either for $x \in P$ or $x \notin P$. 
Polymatroidal polyhedron (or a “polymatroid’’)

• Suppose \( x, y \in \mathbb{R}_+^E \) with \( y \leq x \). If \( x \) contains any zeros (i.e., \( x \) has \( E \setminus S \) s.t. \( x(E \setminus S) = 0 \), so \( S \) indicates the non-zero elements, or \( S = \text{supp}(x) \)), then this also forces \( y(E \setminus S) = 0 \), so that \( y(E) = y(S) \). This is true either for \( x \in P \) or \( x \not\in P \).

• Therefore, in this case, it is the non-zero elements of \( x \), i.e., support \( \text{supp}(x) \) of \( x \), are the elements that determine the common component sum of a \( P \)-basis of \( x \).
Polymatroidal polyhedron (or a “polymatroid’”)

- Suppose $x, y \in \mathbb{R}_{E}^{E}$ with $y \leq x$. If $x$ contains any zeros (i.e., $x$ has $E \setminus S$ s.t. $x(E \setminus S) = 0$, so $S$ indicates the non-zero elements, or $S = \text{supp}(x)$), then this also forces $y(E \setminus S) = 0$, so that $y(E) = y(S)$. This is true either for $x \in P$ or $x \notin P$.

- Therefore, in this case, it is the non-zero elements of $x$, i.e., support $\text{supp}(x)$ of $x$, are the elements that determine the common component sum of a $P$-basis of $x$.

- For the case of either $x \notin P$ or right at the boundary of $P$, we might give a “name” to this component sum, lets say $f(S)$ for any given set $S$ of non-zero elements of $x$. We could name $\text{rank}(\frac{1}{\epsilon}1_{S}) \triangleq f(S)$ for $\epsilon$ small enough. What kind of function might $f$ be?
Matroids $\rightarrow$ Polymatroids

Polymatroid function and its polyhedron.

Definition 11.3.2

A **polymatroid function** is a real-valued function $f$ defined on subsets of $E$ which is normalized, non-decreasing, and submodular. That is we have

1. $f(\emptyset) = 0$ (normalized)
2. $f(A) \leq f(B)$ for any $A \subseteq B \subseteq E$ (monotone non-decreasing)
3. $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$ for any $A, B \subseteq E$ (submodular)

We can define the polyhedron $P_f^+$ associated with a polymatroid function as follows

$$P_f^+ = \{ y \in \mathbb{R}^E_+ : y(A) \leq f(A) \text{ for all } A \subseteq E \}$$  \hspace{1cm} (11.1)

$$= \{ y \in \mathbb{R}^E : y \geq 0, y(A) \leq f(A) \text{ for all } A \subseteq E \}$$  \hspace{1cm} (11.2)
Associated polyhedron with a polymatroid function

\[ P_f^+ = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E \} \]

Consider this in three dimensions. We have equations of the form:

\[ x_1 \geq 0 \text{ and } x_2 \geq 0 \text{ and } x_3 \geq 0 \]  \hspace{1em} (11.4)

\[ x_1 \leq f(\{v_1\}) \]  \hspace{1em} (11.5)

\[ x_2 \leq f(\{v_2\}) \]  \hspace{1em} (11.6)

\[ x_3 \leq f(\{v_3\}) \]  \hspace{1em} (11.7)

\[ x_1 + x_2 \leq f(\{v_1, v_2\}) \]  \hspace{1em} (11.8)

\[ x_2 + x_3 \leq f(\{v_2, v_3\}) \]  \hspace{1em} (11.9)

\[ x_1 + x_3 \leq f(\{v_1, v_3\}) \]  \hspace{1em} (11.10)

\[ x_1 + x_2 + x_3 \leq f(\{v_1, v_2, v_3\}) \]  \hspace{1em} (11.11)
Consider the asymmetric graph cut function on the simple chain graph \( v_1 - v_2 - v_3 \). That is, \( f(S) = |\{(v, s) \in E(G) : v \in V, s \in S\}| \) is count of any edges within \( S \) or between \( S \) and \( V \setminus S \), so that \( \delta(S) = f(S) + f(V \setminus S) - f(V) \) is the standard graph cut.
Consider the asymmetric graph cut function on the simple chain graph $v_1 - v_2 - v_3$. That is, $f(S) = |\{(v, s) \in E(G) : v \in V, s \in S\}|$ is count of any edges within $S$ or between $S$ and $V \setminus S$, so that $\delta(S) = f(S) + f(V \setminus S) - f(V)$ is the standard graph cut.

Observe: $P_f^+$ (at two views):
Consider the asymmetric graph cut function on the simple chain graph $v_1 - v_2 - v_3$. That is, $f(S) = |\{(v, s) \in E(G) : v \in V, s \in S\}|$ is count of any edges within $S$ or between $S$ and $V \setminus S$, so that
\[\delta(S) = f(S) + f(V \setminus S) - f(V)\]
is the standard graph cut.

Observe: $P^+_f$ (at two views):

- which axis is which?
Associated polyhedron with a polymatroid function

Consider: \( f(\emptyset) = 0 \), \( f(\{v_1\}) = 1.5 \), \( f(\{v_2\}) = 2 \), \( f(\{v_1, v_2\}) = 2.5 \), \( f(\{v_3\}) = 3 \), \( f(\{v_3, v_1\}) = 3.5 \), \( f(\{v_3, v_2\}) = 4 \), \( f(\{v_3, v_2, v_1\}) = 4.3 \).
Associated polyhedron with a polymatroid function

- Consider: \( f(\emptyset) = 0, \ f(\{v_1\}) = 1.5, \ f(\{v_2\}) = 2, \ f(\{v_1, v_2\}) = 2.5, \ f(\{v_3\}) = 3, \ f(\{v_3, v_1\}) = 3.5, \ f(\{v_3, v_2\}) = 4, \ f(\{v_3, v_2, v_1\}) = 4.3. \)

- Observe: \( P^+_f \) (at two views):
Consider: $f(\emptyset) = 0$, $f(\{v_1\}) = 1.5$, $f(\{v_2\}) = 2$, $f(\{v_1, v_2\}) = 2.5$, $f(\{v_3\}) = 3$, $f(\{v_3, v_1\}) = 3.5$, $f(\{v_3, v_2\}) = 4$, $f(\{v_3, v_2, v_1\}) = 4.3$.

Observe: $P_f^+$ (at two views):

- which axis is which?
Consider modular function $w : V \rightarrow \mathbb{R}_+$ as $w = (1, 1.5, 2)^T$, and then the submodular function $f(S) = \sqrt{w(S)}$. 

Observe: 

Which axis is which?
Consider modular function $w : V \rightarrow \mathbb{R}_+^+$ as $w = (1, 1.5, 2)^\top$, and then the submodular function $f(S) = \sqrt{w(S)}$.

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Consider modular function $w : V \rightarrow \mathbb{R}_+$ as $w = (1, 1.5, 2)\top$, and then the submodular function $f(S) = \sqrt{w(S)}$.

Observe: $P_f^+$ (at two views):
Consider function on integers: \( g(0) = 0, \ g(1) = 3, \ g(2) = 4, \) and \( g(3) = 5.5. \)
Consider function on integers: \( g(0) = 0, g(1) = 3, g(2) = 4, \) and \( g(3) = 5.5. \) Is \( f(S) = g(|S|) \) submodular?
Consider function on integers: $g(0) = 0$, $g(1) = 3$, $g(2) = 4$, and $g(3) = 5.5$. Is $f(S) = g(|S|)$ submodular? $f(S) = g(|S|)$ is not submodular since $f(\{e_1, e_3\}) + f(\{e_1, e_2\}) = 4 + 4 = 8$ but $f(\{e_1, e_2, e_3\}) + f(\{e_1\}) = 5.5 + 3 = 8.5$. 

Alternatively, consider concavity violation, $1 = g(1 + 1) - g(1) < g(2 + 1) - g(2) = 1.5$.
Consider function on integers: \( g(0) = 0, \ g(1) = 3, \ g(2) = 4, \) and \( g(3) = 5.5. \) Is \( f(S) = g(|S|) \) submodular? \( f(S) = g(|S|) \) is not submodular since \( f(\{e_1, e_3\}) + f(\{e_1, e_2\}) = 4 + 4 = 8 \) but \( f(\{e_1, e_2, e_3\}) + f(\{e_1\}) = 5.5 + 3 = 8.5. \) Alternatively, consider concavity violation, \( 1 = g(1 + 1) - g(1) < g(2 + 1) - g(2) = 1.5. \)
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Observe: $P_f^+$ (at two views), maximal independent subvectors not constant rank, hence not a polymatroid.
A polymatroid vs. a polymatroid function’s polyhedron

Summarizing the above, we have:

- Given a polymatroid function $f$, its associated polytope is given as
  $$P^+ f = \{ y \in \mathbb{R}^E : y(A) \leq f(A) \text{ for all } A \subseteq E \} \quad (11.12)$$

- We also have the definition of a polymatroidal polytope $P$ (compact subset, zero containing, down-monotone, and $\forall x$ any maximal in independent subvector $y \leq x$ has same component sum $y(E)$).

Is there any relationship between these two polytopes?

In the next theorem, we show that any $P^+ f$-basis has the same component sum, when $f$ is a polymatroid function, and $P^+ f$ satisfies the other properties so that $P^+ f$ is a polymatroid.
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Is there any relationship between these two polytopes?

In the next theorem, we show that any $P_f^+$-basis has the same component sum, when $f$ is a polymatroid function, and $P_f^+$ satisfies the other properties so that $P_f^+$ is a polymatroid.
A polymatroid function’s polyhedron is a polymatroid.

Theorem 11.4.1

Let $f$ be a polymatroid function defined on subsets of $E$. For any $x \in \mathbb{R}^E_+$, and any $P_f^+$-basis $y^x \in \mathbb{R}^E_+$ of $x$, the component sum of $y^x$ is

$$y^x(E) = \text{rank}(x) \triangleq \max \left( y(E) : y \leq x, y \in P_f^+ \right)$$

$$= \min (x(A) + f(E \setminus A) : A \subseteq E) \quad (11.13)$$

As a consequence, $P_f^+$ is a polymatroid, since r.h.s. is constant w.r.t. $y^x$. 
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Taking $E \setminus B = \text{supp}(x)$ (so elements $B$ are all zeros in $x$), and for $b \notin B$ we make $x(b)$ is big enough, the r.h.s. min has solution $A^* = B$. We recover submodular function from the polymatroid polyhedron via the following:

$$\text{rank} \left( \frac{1}{\epsilon} \mathbf{1}_{E \setminus B} \right) = f(E \setminus B) = \max \left\{ y(E \setminus B) : y \in P_f^+ \right\} \quad (11.14)$$
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In fact, we will ultimately see a number of important consequences of this theorem (other than just that \( P_f^+ \) is a polymatroid).
A polymatroid function’s polyhedron is a polymatroid.

Proof of Thm 11.4.1.

- Clearly $0 \in P_f^+$ since $f$ is non-negative.
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- Now suppose that we are given an $x \in \mathbb{R}_+^E$, and maximal $y^x \in P_f^+$ with $y^x \leq x$ (i.e., $y^x$ is a $P_f^+$-basis of $x$).
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- Goal is to show that any such $y^x$ has $y^x(E) = \text{const}$, dependent only on $x$ and also $f$ (which defines the polytope) but not dependent on $y^x$, the particular $P_f^+$-basis.
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- Doing so will thus establish that $P_f^+$ is a polymatroid.
A polymatroid function's polyhedron is a polymatroid.

... proof of Thm 11.4.1 continued.

- First trivial case: could have $y^x = x$, which happens if $x(A) \leq f(A), \forall A \subseteq E$ (i.e., $x \in P^+_f$ strictly). In such case,

\[
\begin{align*}
\min (x(A) + f(E \setminus A) : A \subseteq E) &= x(E) + \min (f(E \setminus A) - x(E \setminus A) : A \subseteq E) \\
&= x(E) + \min (f(A) - x(A) : A \subseteq E) \\
&= x(E)
\end{align*}
\]
A polymatroid function’s polyhedron is a polymatroid.

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$$\min (x(A) + f(E \setminus A) : A \subseteq E)$$

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$$= x(E)$$

- When $x \in P_f^+$, $y = x$ is clearly the solution to

$$\max (y(E) : y \leq x, y \in P_f^+)$$

so this is tight, and $\text{rank}(x) = x(E)$. 

...
A polymatroid function’s polyhedron is a polymatroid.

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- First trivial case: could have \( y^x = x \), which happens if 
  \[ x(A) \leq f(A), \forall A \subseteq E \] (i.e., \( x \in P_f^+ \) strictly). In such case,

\[
\min (x(A) + f(E \setminus A) : A \subseteq E) \\
= x(E) + \min (f(E \setminus A) - x(E \setminus A) : A \subseteq E) \\
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= x(E)
\] (11.15) (11.16) (11.17) (11.18)

- When \( x \in P_f^+ \), \( y = x \) is clearly the solution to 
  \[ \max \left( y(E) : y \leq x, y \in P_f^+ \right) \], so this is tight, and \( \text{rank}(x) = x(E) \).

- This is a value dependent only on \( x \), a self basis, unique \( P_f^+ \)-base.
A polymatroid function’s polyhedron is a polymatroid.

...proof of Thm 11.4.1 continued.

- 2nd trivial case: $x(A) > f(A), \forall A \subseteq E$ (i.e., $x \notin P^+_f$ every direction),
A polymatroid function’s polyhedron is a polymatroid.

...proof of Thm 11.4.1 continued.

- 2nd trivial case: \( x(A) > f(A), \forall A \subseteq E \) (i.e., \( x \notin P_f^+ \) every direction),
- Then for any order \( (a_1, a_2, \ldots) \) of the elements and \( A_i \triangleq (a_1, a_2, \ldots, a_i) \), we have \( x(a_i) \geq f(a_i) \geq f(a_i | A_{i-1}) \), the second inequality by submodularity.
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... proof of Thm 11.4.1 continued.

- 2nd trivial case: \( x(A) > f(A), \forall A \subseteq E \) (i.e., \( x \notin P_f^+ \) every direction),

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\[
\min (x(A) + f(E \setminus A) : A \subseteq E) = x(E) + \min (f(A) - x(A) : A \subseteq E) \tag{11.19}
\]

\[
= x(E) + \min \left( \sum_i f(a_i | A_{i-1}) - \sum_i x(a_i) : A \subseteq E \right) \tag{11.20}
\]

\[
= x(E) + \min \left( \sum_i \left( f(a_i | A_{i-1}) - x(a_i) \right) : A \subseteq E \right) \tag{11.21}
\]

\[
= x(E) + \min \left( \sum_i \left( f(a_i | A_{i-1}) - x(a_i) \right) : A \subseteq E \right) \leq 0 \tag{11.22}
\]

\[
= x(E) + f(E) - x(E) = f(E) = \max (y(E) : y \in P_f^+) \tag{11.23}
\]
A polymatroid function’s polyhedron is a polymatroid.

\[ \text{... proof of Thm 11.4.1 continued.} \]

- Assume neither trivial case. Because \( y^x \in P_f^+ \), we have that \( y^x(A) \leq f(A) \) for all \( A \subseteq E \).
... proof of Thm 11.4.1 continued.

- Assume neither trivial case. Because $y^x \in P^+_f$, we have that $y^x(A) \leq f(A)$ for all $A \subseteq E$.
- We show that the constant is given by
  \[
  y^x(E) = \min \left( x(A) + f(E \setminus A) : A \subseteq E \right) \tag{11.24}
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A polymatroid function’s polyhedron is a polymatroid.

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$$y^x(E) = \min (x(A) + f(E \setminus A) : A \subseteq E) \quad (11.24)$$

- For any $P_f^+$-basis $y^x$ of $x$, and any $A \subseteq E$, we have weak relationship:

$$y^x(E) = y^x(A) + y^x(E \setminus A) \quad (11.25)$$

$$\leq x(A) + f(E \setminus A). \quad (11.26)$$

This follows since $y^x \leq x$ and since $y^x \in P_f^+$. 

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A polymatroid function’s polyhedron is a polymatroid.

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  \]  
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This follows since \( y^x \leq x \) and since \( y^x \in P_f^+ \).

- This ensures
  \[
  \max \left( y(E) : y \leq x, y \in P_f^+ \right) \leq \min (x(A) + f(E \setminus A) : A \subseteq E)
  \]  
  (11.27)
... proof of Thm 11.4.1 continued.

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  \[
y^x(E) = y^x(A) + y^x(E \setminus A) \leq x(A) + f(E \setminus A). \tag{11.25}
  \]
  \[\text{This follows since } y^x \leq x \text{ and since } y^x \in P_f^+. \tag{11.26}\]

- This ensures
  \[
  \max \{ y(E) : y \leq x, y \in P_f^+ \} \leq \min \{ x(A) + f(E \setminus A) : A \subseteq E \} \tag{11.27}
  \]

- Given an $A$ where equality in Eqn. (11.26) holds, above min result follows.
A polymatroid function’s polyhedron is a polymatroid.

... proof of Thm 11.4.1 continued.

For any \( y \in P_f^+ \), call a set \( B \subseteq E \) tight if \( y(B) = f(B) \). The union (and intersection) of tight sets \( B, C \) is again tight, since

\[
f(B) + f(C)
\]

which requires equality at the two inequalities above.

Because \( y(A) \leq f(A) \), \( \forall A \), this means \( y(B \cap C) = f(B \cap C) \) and \( y(B \cup C) = f(B \cup C) \), so both also are tight.

For \( y \in P_f^+ \), it will be ultimately useful to define this lattice family of tight sets:

\[
D(y) \equiv \{ A : A \subseteq E, y(A) = f(A) \}
\]
...proof of Thm 11.4.1 continued.

- For any \( y \in \mathcal{P}^+_f \), call a set \( B \subseteq E \) tight if \( y(B) = f(B) \). The union (and intersection) of tight sets \( B, C \) is again tight, since

\[
f(B) + f(C) = y(B) + y(C)
\]  

(11.28)
For any \( y \in P_f^+ \), call a set \( B \subseteq E \) tight if \( y(B) = f(B) \). The union (and intersection) of tight sets \( B, C \) is again tight, since

\[
f(B) + f(C) = y(B) + y(C) = y(B \cap C) + y(B \cup C)
\]

(11.28)  

(11.29)

...
A polymatroid function’s polyhedron is a polymatroid.

...proof of Thm 11.4.1 continued.

For any $y \in P^+_f$, call a set $B \subseteq E$ **tight** if $y(B) = f(B)$. The union (and intersection) of tight sets $B, C$ is again tight, since

$$f(B) + f(C) = y(B) + y(C) = y(B \cap C) + y(B \cup C) \leq f(B \cap C) + f(B \cup C)$$

(11.28) \hspace{1cm} (11.29) \hspace{1cm} (11.30)
A polymatroid function’s polyhedron is a polymatroid.

... proof of Thm 11.4.1 continued.

For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets $B, C$ is again tight, since

\[
\begin{align*}
f(B) + f(C) &= y(B) + y(C) \\
&= y(B \cap C) + y(B \cup C) \\
&\leq f(B \cap C) + f(B \cup C) \\
&\leq f(B) + f(C)
\end{align*}
\]
A polymatroid function’s polyhedron is a polymatroid.

...proof of Thm 11.4.1 continued.

For any \( y \in P^+_f \), call a set \( B \subseteq E \) tight if \( y(B) = f(B) \). The union (and intersection) of tight sets \( B, C \) is again tight, since

\[
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f(B) + f(C) &= y(B) + y(C) \\
&= y(B \cap C) + y(B \cup C) \\
&\leq f(B \cap C) + f(B \cup C) \\
&\leq f(B) + f(C)
\end{align*}
\]

which requires equality at the two inequalities above.
A polymatroid function’s polyhedron is a polymatroid.

...proof of Thm 11.4.1 continued.

- For any $y \in P_f^+$, call a set $B \subseteq E$ **tight** if $y(B) = f(B)$. The union (and intersection) of tight sets $B, C$ is again tight, since

\[
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  f(B) + f(C) &= y(B) + y(C) \\
  &= y(B \cap C) + y(B \cup C) \\
  &\leq f(B \cap C) + f(B \cup C) \\
  &\leq f(B) + f(C)
\end{align*}
\]

which requires equality at the two inequalities above.

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A polymatroid function’s polyhedron is a polymatroid.

...proof of Thm 11.4.1 continued.

- For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets $B, C$ is again tight, since

\[
\begin{align*}
    f(B) + f(C) &= y(B) + y(C) \\
    &= y(B \cap C) + y(B \cup C) \\
    &\leq f(B \cap C) + f(B \cup C) \\
    &\leq f(B) + f(C)
\end{align*}
\]

(11.28) (11.29) (11.30) (11.31)

which requires equality at the two inequalities above.

- Because $y(A) \leq f(A), \forall A$, this means $y(B \cap C) = f(B \cap C)$ and $y(B \cup C) = f(B \cup C)$, so both also are tight.

- For $y \in P_f^+$, it will be ultimately useful to define this lattice family of tight sets: $\mathcal{D}(y) \triangleq \{A : A \subseteq E, \ y(A) = f(A)\}$. 

...
A polymatroid function’s polyhedron is a polymatroid.

...proof of Thm 11.4.1 continued.

- Also, we define $\text{sat}(y) \overset{\text{def}}{=} \bigcup \{ T : T \in \mathcal{D}(y) \}$, so $y(\text{sat}(y)) = f(\text{sat}(y))$. 

Let $E \setminus A = \text{sat}(y)$ be the union of all such tight sets (which is also tight, so $y(x(E)) = f(E \setminus A)$).

Hence, we have $y(x(E)) = y(x(A)) + y(x(E \setminus A)) = x(A) + f(E \setminus A)$ (11.32)

So we identified the $A$ to be the elements that are non-tight, and achieved the min, as desired.
A polymatroid function’s polyhedron is a polymatroid.

...proof of Thm 11.4.1 continued.

- Also, we define $\text{sat}(y) \overset{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\}$, so $y(\text{sat}(y)) = f(\text{sat}(y))$.
- Consider again a $P_f^+$-basis $y^x$ (so maximal).
A polymatroid function's polyhedron is a polymatroid.

...proof of Thm 11.4.1 continued.

- Also, we define \( \text{sat}(y) \overset{\text{def}}{=} \bigcup \{ T : T \in \mathcal{D}(y) \} \), so \( y(\text{sat}(y)) = f(\text{sat}(y)) \).
- Consider again a \( P_f^+ \)-basis \( y^x \) (so maximal).
- Given a \( e \in E \), either \( y^x(e) \) is cut off due to \( x \) (so \( y^x(e) = x(e) \)) or \( e \) is saturated by \( f \), meaning it is an element of some tight set and \( e \in \text{sat}(y^x) \) (since if \( e \in T \in \mathcal{D}(y^x) \), then \( e \in \text{sat}(y^x) \)).
A polymatroid function’s polyhedron is a polymatroid.

...proof of Thm 11.4.1 continued.

- Also, we define $\text{sat}(y) \overset{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\}$, so $y(\text{sat}(y)) = f(\text{sat}(y))$.
- Consider again a $P_f^+$-basis $y^x$ (so maximal).
- Given a $e \in E$, either $y^x(e)$ is cut off due to $x$ (so $y^x(e) = x(e)$) or $e$ is saturated by $f$, meaning it is an element of some tight set and $e \in \text{sat}(y^x)$ (since if $e \in T \in \mathcal{D}(y^x)$, then $e \in \text{sat}(y^x)$).
- Let $E \setminus A = \text{sat}(y^x)$ be the union of all such tight sets (which is also tight, so $y^x(E \setminus A) = f(E \setminus A)$).
A polymatroid function’s polyhedron is a polymatroid.

...proof of Thm 11.4.1 continued.

- Also, we define $\text{sat}(y) \overset{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\}$, so $y(\text{sat}(y)) = f(\text{sat}(y))$.
- Consider again a $P_f^+$-basis $y^x$ (so maximal).
- Given a $e \in E$, either $y^x(e)$ is cut off due to $x$ (so $y^x(e) = x(e)$) or $e$ is saturated by $f$, meaning it is an element of some tight set and $e \in \text{sat}(y^x)$ (since if $e \in T \in \mathcal{D}(y^x)$, then $e \in \text{sat}(y^x)$).
- Let $E \setminus A = \text{sat}(y^x)$ be the union of all such tight sets (which is also tight, so $y^x(E \setminus A) = f(E \setminus A)$).
- Hence, we have

$$y^x(E) = y^x(A) + y^x(E \setminus A) = x(A) + f(E \setminus A) \quad (11.32)$$
A polymatroid function’s polyhedron is a polymatroid.

...proof of Thm 11.4.1 continued.

- Also, we define \( \text{sat}(y) \overset{\text{def}}{=} \bigcup \{ T : T \in D(y) \} \), so \( y(\text{sat}(y)) = f(\text{sat}(y)) \).
- Consider again a \( P_f^+ \)-basis \( y^x \) (so maximal).
- Given a \( e \in E \), either \( y^x(e) \) is cut off due to \( x \) (so \( y^x(e) = x(e) \)) or \( e \) is saturated by \( f \), meaning it is an element of some tight set and \( e \in \text{sat}(y^x) \) (since if \( e \in T \in D(y^x) \), then \( e \in \text{sat}(y^x) \)).
- Let \( E \setminus A = \text{sat}(y^x) \) be the union of all such tight sets (which is also tight, so \( y^x(E \setminus A) = f(E \setminus A) \)).
- Hence, we have

\[
y^x(E) = y^x(A) + y^x(E \setminus A) = x(A) + f(E \setminus A)
\]

(11.32)

- So we identified the \( A \) to be the elements that are non-tight, and achieved the min, as desired.
A polymatroid is a polymatroid function’s polytope

- So, when \( f \) is a polymatroid function, \( P_f^+ \) is a polymatroid.
A polymatroid is a polymatroid function’s polytope

- So, when \( f \) is a polymatroid function, \( P_f^+ \) is a polymatroid.
- Is it the case that, conversely, for any polymatroid \( P \), there is an associated polymatroidal function \( f \) such that \( P = P_f^+ \)?
A polymatroid is a polymatroid function’s polytope

- So, when $f$ is a polymatroid function, $P_f^+$ is a polymatroid.
- Is it the case that, conversely, for any polymatroid $P$, there is an associated polymatroidal function $f$ such that $P = P_f^+$?

**Theorem 11.4.2**

For any polymatroid $P$ (compact subset of $\mathbb{R}_+^E$, zero containing, down-monotone, and $\forall x \in \mathbb{R}_+^E$ any maximal independent subvector $y \leq x$ has same component sum $y(E) = \text{rank}(x)$), there is a polymatroid function $f : 2^E \to \mathbb{R}$ (normalized, monotone non-decreasing, submodular) such that $P = P_f^+$ where $P_f^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\}$. 
**Tight sets** $\mathcal{D}(y)$ **are closed, and max tight set** $\text{sat}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{A : A \subseteq E, y(A) = f(A)\}$$  \hspace{1cm} (11.33)

**Theorem 11.4.3**

*For any* $y \in P_f^+$, *with* $f$ *a polymatroid function, then* $\mathcal{D}(y)$ *is closed under union and intersection.*
Tight sets $\mathcal{D}(y)$ are closed, and max tight set $\text{sat}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{ A : A \subseteq E, \; y(A) = f(A) \}$$  \hspace{1cm} (11.33)

**Theorem 11.4.3**

For any $y \in P_f^+$, with $f$ a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

**Proof.**

We have already proven this as part of Theorem 11.4.1
Tight sets $D(y)$ are closed, and max tight set $\text{sat}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$D(y) \triangleq \{ A : A \subseteq E, \ y(A) = f(A) \}$$  \hspace{1cm} (11.33)

Theorem 11.4.3

For any $y \in P_f^+$, with $f$ a polymatroid function, then $D(y)$ is closed under union and intersection.

Proof.
We have already proven this as part of Theorem 11.4.1

Also recall the definition of $\text{sat}(y)$, the maximal set of tight elements relative to $y \in \mathbb{R}_E^+$.

$$\text{sat}(y) \overset{\text{def}}{=} \bigcup \{ T : T \in D(y) \}$$  \hspace{1cm} (11.34)
Join $\lor$ and meet $\land$ for $x, y \in \mathbb{R}_+^E$

- For $x, y \in \mathbb{R}_+^E$, define vectors $x \land y \in \mathbb{R}_+^E$ and $x \lor y \in \mathbb{R}_+^E$ such that, for all $e \in E$

  \[
  (x \lor y)(e) = \max(x(e), y(e)) \quad (11.35)
  
  (x \land y)(e) = \min(x(e), y(e)) \quad (11.36)
  \]

  Hence,

  \[
  x \lor y \triangleq \left( \max(x(e_1), y(e_1)), \max(x(e_2), y(e_2)), \ldots, \max(x(e_n), y(e_n)) \right)
  \]

  and similarly

  \[
  x \land y \triangleq \left( \min(x(e_1), y(e_1)), \min(x(e_2), y(e_2)), \ldots, \min(x(e_n), y(e_n)) \right)
  \]
Join $\vee$ and meet $\wedge$ for $x, y \in \mathbb{R}^E_+$

For $x, y \in \mathbb{R}^E_+$, define vectors $x \wedge y \in \mathbb{R}^E_+$ and $x \vee y \in \mathbb{R}^E_+$ such that, for all $e \in E$

\[
(x \vee y)(e) = \max(x(e), y(e)) \quad (11.35)
\]
\[
(x \wedge y)(e) = \min(x(e), y(e)) \quad (11.36)
\]

Hence,

\[
x \vee y \triangleq \left( \max(x(e_1), y(e_1)), \max(x(e_2), y(e_2)), \ldots, \max(x(e_n), y(e_n)) \right)
\]

and similarly

\[
x \wedge y \triangleq \left( \min(x(e_1), y(e_1)), \min(x(e_2), y(e_2)), \ldots, \min(x(e_n), y(e_n)) \right)
\]

From this, we can define things like an lattices, and other constructs.
Vector rank, $\text{rank}(x)$, is submodular

- Recall that the matroid rank function is submodular.
Vector rank, $\text{rank}(x)$, is submodular

- Recall that the matroid rank function is submodular.
- The vector rank function $\text{rank}(x)$ also satisfies a form of submodularity, namely one defined on the real lattice.
Recall that the matroid rank function is submodular.

The vector rank function $\text{rank}(x)$ also satisfies a form of submodularity, namely one defined on the real lattice.

**Theorem 11.4.4 (vector rank and submodularity)**

Let $P$ be a polymatroid polytope. The vector rank function $\text{rank} : \mathbb{R}_+^E \to \mathbb{R}$ with $\text{rank}(x) = \max \{ y(E) : y \leq x, y \in P \}$ satisfies, for all $u, v \in \mathbb{R}_+^E$,

$$\text{rank}(u) + \text{rank}(v) \geq \text{rank}(u \vee v) + \text{rank}(u \wedge v) \quad (11.37)$$
Vector rank $\text{rank}(x)$ is submodular, proof

Proof of Theorem 11.4.4.

- Let $a \in \mathbb{R}^E_+$ be a $P$-basis of $u \land v$, so $\text{rank}(u \land v) = a(E)$. 

...
Vector rank \( \text{rank}(x) \) is submodular, proof

**Proof of Theorem 11.4.4.**

- Let \( a \in \mathbb{R}_+^E \) be a \( P \)-basis of \( u \land v \), so \( \text{rank}(u \land v) = a(E) \).
- Claim: By the polymatroid property, \( \exists \) an independent \( b \in P \) such that:
  \[ a \leq b \leq u \lor v \]
Vector rank \( \text{rank}(x) \) is submodular, proof

Proof of Theorem 11.4.4.

- Let \( a \in \mathbb{R}^E_+ \) be a \( P \)-basis of \( u \wedge v \), so \( \text{rank}(u \wedge v) = a(E) \).
- Claim: By the polymatroid property, \( \exists \) an independent \( b \in P \) such that: \( a \leq b \leq u \vee v \) and also such that \( \text{rank}(b) = b(E) = \text{rank}(u \vee v) \), so \( b \) is a \( P \)-basis of \( u \vee v \), and thus \( b \leq u \vee v \).
Vector rank $\text{rank}(x)$ is submodular, proof

Proof of Theorem 11.4.4.

- Let $a \in \mathbb{R}_+^E$ be a $P$-basis of $u \land v$, so $\text{rank}(u \land v) = a(E)$.
- Claim: By the polymatroid property, $\exists$ an independent $b \in P$ such that: $a \leq b \leq u \lor v$ and also such that $\text{rank}(b) = b(E) = \text{rank}(u \lor v)$, so $b$ is a $P$-basis of $u \lor v$, and thus $b \leq u \lor v$.
- Given any $e \in E$: if $a(e)$ is maximal due to $P$, then $a(e) = b(e) \leq \min(u(e), v(e))$ ...
Vector rank $\text{rank}(x)$ is submodular, proof

Proof of Theorem 11.4.4.

- Let $a \in \mathbb{R}^E_+$ be a $P$-basis of $u \land v$, so $\text{rank}(u \land v) = a(E)$.
- Claim: By the polymatroid property, $\exists$ an independent $b \in P$ such that: $a \leq b \leq u \lor v$ and also such that $\text{rank}(b) = b(E) = \text{rank}(u \lor v)$, so $b$ is a $P$-basis of $u \lor v$, and thus $b \leq u \lor v$.
- Given any $e \in E$: if $a(e)$ is maximal due to $P$, then $a(e) = b(e)$ \leq \min(u(e), v(e)) \ldots$
- otherwise, if $a(e)$ is maximal due to $(u \land v)(e)$, then $a(e) = \min(u(e), v(e)) \leq b(e)$. 

\ldots
Vector rank \( \text{rank}(x) \) is submodular, proof

**Proof of Theorem 11.4.4.**

- Let \( a \in \mathbb{R}^E_+ \) be a \( P \)-basis of \( u \land v \), so \( \text{rank}(u \land v) = a(E) \).

- Claim: By the polymatroid property, \( \exists \) an independent \( b \in P \) such that:
  \( a \leq b \leq u \lor v \) and also such that \( \text{rank}(b) = b(E) = \text{rank}(u \lor v) \), so \( b \) is a \( P \)-basis of \( u \lor v \), and thus \( b \leq u \lor v \).

- Given any \( e \in E \): if \( a(e) \) is maximal due to \( P \), then \( a(e) = b(e) \leq \min(u(e), v(e)) \) ...

- otherwise, if \( a(e) \) is maximal due to \( (u \land v)(e) \), then
  \( a(e) = \min(u(e), v(e)) \leq b(e) \).

- Therefore, in either case, \( a = b \land (u \land v) \) ...

...
Vector rank $\text{rank}(x)$ is submodular, proof

Proof of Theorem 11.4.4.

- Let $a \in \mathbb{R}_+^E$ be a $P$-basis of $u \land v$, so $\text{rank}(u \land v) = a(E)$.
- Claim: By the polymatroid property, $\exists$ an independent $b \in P$ such that: $a \leq b \leq u \lor v$ and also such that $\text{rank}(b) = b(E) = \text{rank}(u \lor v)$, so $b$ is a $P$-basis of $u \lor v$, and thus $b \leq u \lor v$.
- Given any $e \in E$: if $a(e)$ is maximal due to $P$, then $a(e) = b(e) \leq \min(u(e), v(e))$...
- otherwise, if $a(e)$ is maximal due to $(u \land v)(e)$, then $a(e) = \min(u(e), v(e)) \leq b(e)$.
- Therefore, in either case, $a = b \land (u \land v)$...
- ...and since $b \leq u \lor v$, we get $a + b$ (11.38)
**Proof of Theorem 11.4.4.**

- Let $a \in \mathbb{R}^E_+$ be a $P$-basis of $u \land v$, so $\text{rank}(u \land v) = a(E)$.

- **Claim:** By the polymatroid property, $\exists$ an independent $b \in P$ such that:
  
  \[ a \leq b \leq u \lor v \]
  
  and also such that $\text{rank}(b) = b(E) = \text{rank}(u \lor v)$, so $b$ is a $P$-basis of $u \lor v$, and thus $b \leq u \lor v$.

- Given any $e \in E$: if $a(e)$ is maximal due to $P$, then
  
  \[ a(e) = b(e) \leq \min(u(e), v(e)) \]
  
  otherwise, if $a(e)$ is maximal due to $(u \land v)(e)$, then
  
  \[ a(e) = \min(u(e), v(e)) \leq b(e) \]

- Therefore, in either case, $a = b \land (u \land v)$.

- ... and since $b \leq u \lor v$, we get

\[ a + b = b \land u \land v + b \]

(11.38)
Vector rank \( \text{rank}(x) \) is submodular, proof

Proof of Theorem 11.4.4.

- Let \( a \in \mathbb{R}_+^E \) be a \( P \)-basis of \( u \land v \), so \( \text{rank}(u \land v) = a(E) \).

- Claim: By the polymatroid property, \( \exists \) an independent \( b \in P \) such that: \( a \leq b \leq u \lor v \) and also such that \( \text{rank}(b) = b(E) = \text{rank}(u \lor v) \), so \( b \) is a \( P \)-basis of \( u \lor v \), and thus \( b \leq u \lor v \).

- Given any \( e \in E \): if \( a(e) \) is maximal due to \( P \), then \( a(e) = b(e) \leq \min(u(e), v(e)) \ldots \)

- otherwise, if \( a(e) \) is maximal due to \( (u \land v)(e) \), then \( a(e) = \min(u(e), v(e)) \leq b(e) \).

- Therefore, in either case, \( a = b \land (u \land v) \ldots \)

- \( \ldots \) and since \( b \leq u \lor v \), we get

\[
a + b = b \land u \land v + b = b \land u + b \land v \quad (11.38)
\]
Vector rank $\text{rank}(x)$ is submodular, proof

Proof of Theorem 11.4.4.

- Let $a \in \mathbb{R}_{E}^{+}$ be a $P$-basis of $u \land v$, so $\text{rank}(u \land v) = a(E)$.
- Claim: By the polymatroid property, $\exists$ an independent $b \in P$ such that: $a \leq b \leq u \lor v$ and also such that $\text{rank}(b) = b(E) = \text{rank}(u \lor v)$, so $b$ is a $P$-basis of $u \lor v$, and thus $b \leq u \lor v$.
- Given any $e \in E$: if $a(e)$ is maximal due to $P$, then $a(e) = b(e) \leq \min(u(e), v(e))$...
- otherwise, if $a(e)$ is maximal due to $(u \land v)(e)$, then $a(e) = \min(u(e), v(e)) \leq b(e)$.
- Therefore, in either case, $a = b \land (u \land v)$...
- ...and since $b \leq u \lor v$, we get

\[
a + b = b \land u \land v + b = b \land u + b \land v \tag{11.38}
\]

How? With $b \leq u \lor v$, three cases: 1) $b$ is minimum ($a + b = b + b$); 2) $u$ is minimum with $b \leq v$ ($a + b = u + b$); 3) $v$ is minimum with $b \leq u$ ($a + b = v + b$).
Vector rank $\text{rank}(x)$ is submodular, proof

...proof of Theorem 11.4.4.

- $b$ is independent, and $b \wedge u$ and $b \wedge v$ are independent subvectors of $u$ and $v$ respectively, so $(b \wedge u)(E) \leq \text{rank}(u)$ and $(b \wedge v)(E) \leq \text{rank}(v)$. 
...proof of Theorem 11.4.4.

- $b$ is independent, and $b \land u$ and $b \land v$ are independent subvectors of $u$ and $v$ respectively, so $(b \land u)(E) \leq \text{rank}(u)$ and $(b \land v)(E) \leq \text{rank}(v)$.

- Hence,
  \[
  \text{rank}(u \land v) + \text{rank}(u \lor v)
  \]
Vector rank $\text{rank}(x)$ is submodular, proof

...proof of Theorem 11.4.4.

- $b$ is independent, and $b \wedge u$ and $b \wedge v$ are independent subvectors of $u$ and $v$ respectively, so $(b \wedge u)(E) \leq \text{rank}(u)$ and $(b \wedge v)(E) \leq \text{rank}(v)$.
- Hence,
  
  $$\text{rank}(u \wedge v) + \text{rank}(u \vee v) = a(E) + b(E) \quad (11.39)$$
Vector rank $\text{rank}(\mathbf{x})$ is submodular, proof

...proof of Theorem 11.4.4.

- $b$ is independent, and $b \wedge u$ and $b \wedge v$ are independent subvectors of $u$ and $v$ respectively, so $(b \wedge u)(E) \leq \text{rank}(u)$ and $(b \wedge v)(E) \leq \text{rank}(v)$.

- Hence,

\[
\text{rank}(u \wedge v) + \text{rank}(u \vee v) = a(E) + b(E) \\
= (b \wedge u)(E) + (b \wedge v)(E)
\] (11.39) (11.40)
Vector rank \( \text{rank}(x) \) is submodular, proof

... proof of Theorem 11.4.4.

- \( b \) is independent, and \( b \wedge u \) and \( b \wedge v \) are independent subvectors of \( u \) and \( v \) respectively, so \( (b \wedge u)(E) \leq \text{rank}(u) \) and \( (b \wedge v)(E) \leq \text{rank}(v) \).

- Hence,

\[
\text{rank}(u \wedge v) + \text{rank}(u \vee v) = a(E) + b(E) = (b \wedge u)(E) + (b \wedge v)(E) \leq \text{rank}(u) + \text{rank}(v)
\]

(11.39) (11.40) (11.41)
Note the remarkable similarity between the proof of Theorem 11.4.4 and the proof of Theorem 6.5.1 that the standard matroid rank function is submodular.
Note the remarkable similarity between the proof of Theorem 11.4.4 and the proof of Theorem 6.5.1 that the standard matroid rank function is submodular.

Next, we prove Theorem 11.4.2, that any polymatroid polytope $P$ has a polymatroid function $f$ such that $P = P_f^+$. 
Note the remarkable similarity between the proof of Theorem 11.4.4 and the proof of Theorem 6.5.1 that the standard matroid rank function is submodular.

Next, we prove Theorem 11.4.2, that any polymatroid polytope $P$ has a polymatroid function $f$ such that $P = P_f^+$.

Given this result, we can conclude that a polymatroid is really an extremely natural polyhedral generalization of a matroid. This was all realized by Jack Edmonds in the mid 1960s (and published in 1969 in his landmark paper “Submodular Functions, Matroids, and Certain Polyhedra”).
A polymatroid is a polymatroid function’s polytope

- So, when $f$ is a polymatroid function, $P^+_f$ is a polymatroid.
- Is it the case that, conversely, for any polymatroid $P$, there is an associated polymatroidal function $f$ such that $P = P^+_f$?

**Theorem 11.4.2**

For any polymatroid $P$ (compact subset of $\mathbb{R}^E_+$, zero containing, down-monotone, and $\forall x \in \mathbb{R}^E_+$ any maximal independent subvector $y \leq x$ has same component sum $y(E) = \text{rank}(x)$), there is a polymatroid function $f : 2^E \rightarrow \mathbb{R}$ (normalized, monotone non-decreasing, submodular) such that $P = P^+_f$ where $P^+_f = \{ x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E \}$. 
To show Theorem 11.4.2, we will first define a function \( f \), show that it is monotone non-decreasing submodular, which allows us to define \( P_f^+ \), and that we show that \( P \subseteq P_f^+ \).
Method to prove Theorem 11.4.2

- To show Theorem 11.4.2, we will first define a function $f$, show that that it is monotone non-decreasing submodular, which allows us to define $P^+_f$, and that we show that $P \subseteq P^+_f$.

- Next, will show that $P^+_f \subseteq P$
To show Theorem 11.4.2, we will first define a function $f$, show that it is monotone non-decreasing submodular, which allows us to define $P_f^+$, and that we show that $P \subseteq P_f^+$.

Next, will show that $P_f^+ \subseteq P$

This results in that $P_f^+ = P$ to complete the proof.
Proof of Theorem 11.4.2

Proof of Theorem 11.4.2 (∀P, ∃f s.t. \( P = P_f^+ \)).

- We are given a polymatroid \( P \).
Proof of Theorem 11.4.2 (
\(\forall P, \exists f\) s.t. \(P = P_f^+\)).

- We are given a polymatroid \(P\).
- Define \(\alpha_{\text{max}} \triangleq \max \{x(E) : x \in P\}\), and note that \(\alpha_{\text{max}} > 0\) when \(P\) is non-empty, and \(\alpha_{\text{max}} = \lim_{\alpha \to \infty} \text{rank}(\alpha 1_E) = \text{rank}(\alpha_{\text{max}} 1_E)\).
Proof of Theorem 11.4.2

We are given a polymatroid $P$.

Define $\alpha_{\max} \triangleq \max \{x(E) : x \in P\}$, and note that $\alpha_{\max} > 0$ when $P$ is non-empty, and $\alpha_{\max} = \lim_{\alpha \to \infty} \text{rank}(\alpha 1_E) = \text{rank}(\alpha_{\max} 1_E)$.

Hence, for any $x \in P$, and $\forall e \in E$, we have $x(e) \leq x(E) \leq \alpha_{\max}$.
Proof of Theorem 11.4.2

Proof of Theorem 11.4.2 \((\forall P, \exists f \text{ s.t. } P = P_f^+)\).

- We are given a polymatroid \(P\).
- Define \(\alpha_{\text{max}} \triangleq \max \{x(E) : x \in P\}\), and note that \(\alpha_{\text{max}} > 0\) when \(P\) is non-empty, and \(\alpha_{\text{max}} = \lim_{\alpha \to \infty} \text{rank}(\alpha 1_E) = \text{rank}(\alpha_{\text{max}} 1_E)\).
- Hence, for any \(x \in P\), and \(\forall e \in E\), we have \(x(e) \leq x(E) \leq \alpha_{\text{max}}\).
- Define a function \(f : 2^V \to \mathbb{R}\) as, for any \(A \subseteq E\),

\[
f(A) \triangleq \text{rank}(\alpha_{\text{max}} 1_A) \quad (11.42)
\]
Proof of Theorem 11.4.2 ($\forall P, \exists f$ s.t. $P = P^+_f$).

- We are given a polymatroid $P$.
- Define $\alpha_{\text{max}} \triangleq \max \{ x(E) : x \in P \}$, and note that $\alpha_{\text{max}} > 0$ when $P$ is non-empty, and $\alpha_{\text{max}} = \lim_{\alpha \to \infty} \text{rank}(\alpha 1_E) = \text{rank}(\alpha_{\text{max}} 1_E)$.
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$$f(A) \triangleq \text{rank}(\alpha_{\text{max}} 1_A) \quad (11.42)$$

- Then $f$ is submodular since

$$f(A) + f(B)$$
Proof of Theorem 11.4.2

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- Define a function \(f : 2^V \to \mathbb{R}\) as, for any \(A \subseteq E\),

\[
 f(A) \triangleq \text{rank}(\alpha_{\text{max}} 1_A) \tag{11.42}
\]

- Then \(f\) is submodular since

\[
 f(A) + f(B) = \text{rank}(\alpha_{\text{max}} 1_A) + \text{rank}(\alpha_{\text{max}} 1_B) \geq \text{rank}(\alpha_{\text{max}} 1_A \lor \alpha_{\text{max}} 1_B) + \text{rank}(\alpha_{\text{max}} 1_A \land \alpha_{\text{max}} 1_B) \tag{11.44}
\]
We are given a polymatroid $P$.

Define $\alpha_{\text{max}} \triangleq \max \{ x(E) : x \in P \}$, and note that $\alpha_{\text{max}} > 0$ when $P$ is non-empty, and $\alpha_{\text{max}} = \lim_{\alpha \to \infty} \text{rank}(\alpha \mathbf{1}_E) = \text{rank}(\alpha_{\text{max}} \mathbf{1}_E)$.

Hence, for any $x \in P$, and $\forall e \in E$, we have $x(e) \leq x(E) \leq \alpha_{\text{max}}$.

Define a function $f : 2^V \to \mathbb{R}$ as, for any $A \subseteq E$,

$$f(A) \triangleq \text{rank}(\alpha_{\text{max}} \mathbf{1}_A) \quad (11.42)$$

Then $f$ is submodular since

$$f(A) + f(B) = \text{rank}(\alpha_{\text{max}} \mathbf{1}_A) + \text{rank}(\alpha_{\text{max}} \mathbf{1}_B) \quad (11.43)$$

$$\geq \text{rank}(\alpha_{\text{max}} \mathbf{1}_A \lor \alpha_{\text{max}} \mathbf{1}_B) + \text{rank}(\alpha_{\text{max}} \mathbf{1}_A \land \alpha_{\text{max}} \mathbf{1}_B) \quad (11.44)$$

$$= \text{rank}(\alpha_{\text{max}} \mathbf{1}_{A \cup B}) + \text{rank}(\alpha_{\text{max}} \mathbf{1}_{A \cap B}) \quad (11.45)$$
Proof of Theorem 11.4.2 (\( \forall P, \exists f \) s.t. \( P = P_f^+ \)).

- We are given a polymatroid \( P \).
- Define \( \alpha_{\text{max}} \triangleq \max \{ x(E) : x \in P \} \), and note that \( \alpha_{\text{max}} > 0 \) when \( P \) is non-empty, and \( \alpha_{\text{max}} = \lim_{\alpha \to \infty} \text{rank}(\alpha 1_E) = \text{rank}(\alpha_{\text{max}} 1_E) \).
- Hence, for any \( x \in P \), and \( \forall e \in E \), we have \( x(e) \leq x(E) \leq \alpha_{\text{max}} \).
- Define a function \( f : 2^V \to \mathbb{R} \) as, for any \( A \subseteq E \),

\[
f(A) \triangleq \text{rank}(\alpha_{\text{max}} 1_A)
\]  

(11.42)

- Then \( f \) is submodular since

\[
f(A) + f(B) = \text{rank}(\alpha_{\text{max}} 1_A) + \text{rank}(\alpha_{\text{max}} 1_B)
\]

(11.43)

\[\geq \text{rank}(\alpha_{\text{max}} 1_A \lor \alpha_{\text{max}} 1_B) + \text{rank}(\alpha_{\text{max}} 1_A \land \alpha_{\text{max}} 1_B)
\]

(11.44)

\[= \text{rank}(\alpha_{\text{max}} 1_{A \cup B}) + \text{rank}(\alpha_{\text{max}} 1_{A \cap B})
\]

(11.45)

\[= f(A \cup B) + f(A \cap B)
\]

(11.46)
Proof of Theorem 11.4.2

Proof of Theorem 11.4.2 ($\forall P, \exists f \text{ s.t. } P = P_f^+$).

Moreover, we have that $f$ is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).
Proof of Theorem 11.4.2

Moreover, we have that $f$ is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).

Hence, $f$ is a polymatroid function.
Proof of Theorem 11.4.2

Proof of Theorem 11.4.2 ($\forall P, \exists f$ s.t. $P = P_f^+$).

Moreover, we have that $f$ is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).

Hence, $f$ is a polymatroid function.

Definition: for any $A \subseteq E$, define $x_A \in \mathbb{R}_E^+$ as

$$x_A(e) = \begin{cases} x(e) & \text{if } e \in A \\ 0 & \text{else} \end{cases} \quad (11.47)$$

Note this is an analogous definition to $1_A$ but for a not necessarily unity vector $x$. 

...
Proof of Theorem 11.4.2 (\(\forall P, \exists f\) s.t. \(P = P_f^+\)).

- Moreover, we have that \(f\) is non-negative, normalized with \(f(\emptyset) = 0\), and monotone non-decreasing (since rank is monotone).
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- Definition: for any \(A \subseteq E\), define \(x_A \in \mathbb{R}^E_+\) as
  \[
  x_A(e) = \begin{cases} 
  x(e) & \text{if } e \in A \\
  0 & \text{else}
  \end{cases} \tag{11.47}
  \]

  note this is an analogous definition to \(1_A\) but for a not necessarily unity vector \(x\).

- Hence \(x_A(A) = x(A)\) and \(x_A(E \setminus A) = 0\).
Proof of Theorem 11.4.2

Moreover, we have that $f$ is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).

Hence, $f$ is a polymatroid function.

Definition: for any $A \subseteq E$, define $x_A \in \mathbb{R}^E_+$ as

$$
x_A(e) = \begin{cases} 
x(e) & \text{if } e \in A \\
0 & \text{else}
\end{cases}
$$

(11.47)

Note this is an analogous definition to $1_A$ but for a not necessarily unity vector $x$.

Hence $x_A(A) = x(A)$ and $x_A(E \setminus A) = 0$.

Using $f(A) \triangleq \text{rank}(\alpha_{\text{max}} 1_A)$, consider the polytope $P_f^+$ defined as:

$$
P_f^+ = \{ x \in \mathbb{R}^E_+ : x(A) \leq f(A), \ \forall A \subseteq E \}
$$

(11.48)
Proof of Theorem 11.4.2

Proof of Theorem 11.4.2 (\(\forall P, \exists f \text{ s.t. } P = P_f^+\)).

- Given an \(x \in P\), then for any \(A \subseteq E\), \(x_A \leq \alpha_{\max} 1_A\), and thus
  \[x(A) \leq \alpha_{\max}|A|\.]
Proof of Theorem 11.4.2

Proof of Theorem 11.4.2 \((\forall P, \exists f \text{ s.t. } P = P_f^+)\).

- Given an \(x \in P\), then for any \(A \subseteq E\), \(x_A \leq \alpha_{\text{max}} 1_A\), and thus \(x(A) \leq \alpha_{\text{max}} |A|\).
- Therefore,

\[
x(A) \leq \max \left\{ z(A) : z \in P, z_A \leq \alpha_{\text{max}} 1_A \right\} \quad (11.49)
\]

\[
= \max \left\{ z(A) : z \in P, z \leq \alpha_{\text{max}} 1_A \right\} \quad (11.50)
\]

\[
\leq \max \left\{ z(E) : z \in P, z \leq \alpha_{\text{max}} 1_A \right\} \quad (11.51)
\]

\[
= \text{rank}(\alpha_{\text{max}} 1_A) \quad (11.52)
\]

\[
= f(A) \quad (11.53)
\]

Therefore \(x \in P_f^+\).
Proof of Theorem 11.4.2

Given an $x \in P$, then for any $A \subseteq E$, $x_A \leq \alpha_{\text{max}} 1_A$, and thus $x(A) \leq \alpha_{\text{max}} |A|$.

Therefore,

$$x(A) \leq \max \left\{ z(A) : z \in P, z_A \leq \alpha_{\text{max}} 1_A \right\}$$  \hspace{1cm} (11.49)

$$= \max \left\{ z(A) : z \in P, z \leq \alpha_{\text{max}} 1_A \right\}$$  \hspace{1cm} (11.50)

$$\leq \max \left\{ z(E) : z \in P, z \leq \alpha_{\text{max}} 1_A \right\}$$  \hspace{1cm} (11.51)

$$= \text{rank}(\alpha_{\text{max}} 1_A)$$  \hspace{1cm} (11.52)

$$= f(A)$$  \hspace{1cm} (11.53)

Therefore $x \in P_f^+$.  

Hence, $P \subseteq P_f^+$.  

...
Proof of Theorem 11.4.2

Given an \( x \in P \), then for any \( A \subseteq E \), \( x_A \leq \alpha_{\text{max}}1_A \), and thus
\[
x(A) \leq \alpha_{\text{max}}|A|.
\]
Therefore,
\[
x(A) \leq \max \{ z(A) : z \in P, z_A \leq \alpha_{\text{max}}1_A \} \tag{11.49}
= \max \{ z(A) : z \in P, z \leq \alpha_{\text{max}}1_A \} \tag{11.50}
\leq \max \{ z(E) : z \in P, z \leq \alpha_{\text{max}}1_A \} \tag{11.51}
= \text{rank}(\alpha_{\text{max}}1_A) \tag{11.52}
= f(A) \tag{11.53}
\]

Therefore \( x \in P_f^+ \).

Hence, \( P \subseteq P_f^+ \).

We will next show that \( P_f^+ \subseteq P \) to complete the proof.

\[ \ldots \]
Proof of Theorem 11.4.2

Let $x \in P_f^+$ be chosen arbitrarily (goal is to show that $x \in P$).

Suppose $x \not\in P$. Then, choose $y$ to be a $P$-basis of $x$ that maximizes the number of $y$ elements strictly less than the corresponding $x$ element. I.e., that maximizes $|N(y)|$, where $N(y) = \{ e \in E : y(e) < x(e) \}$.

Choose $w$ between $y$ and $x$, so that $y \leq w = (y + x)/2 \leq x$.

Hence, $\text{rank}(x) = \text{rank}(w) = y(E)$, and the set of $P$-bases of $w$ are also $P$-bases of $x$. Therefore, $x \in P$.

...
Proof of Theorem 11.4.2

Proof of Theorem 11.4.2 ($\forall P, \exists f$ s.t. $P = P_f^+$).

- Let $x \in P_f^+$ be chosen arbitrarily (goal is to show that $x \in P$).
- Suppose $x \notin P$. 

...
Proof of Theorem 11.4.2

Proof of Theorem 11.4.2 \((\forall P, \exists f \text{ s.t. } P = P_f^+)\).

- Let \(x \in P_f^+\) be chosen arbitrarily (goal is to show that \(x \in P\)).
- Suppose \(x \not\in P\). Then, choose \(y\) to be a \(P\)-basis of \(x\) that maximizes the number of \(y\) elements strictly less than the corresponding \(x\) element. I.e., that maximizes \(|N(y)|\), where

\[
N(y) = \{ e \in E : y(e) < x(e) \}
\]

(11.54)
Proof of Theorem 11.4.2 \((\forall P, \exists f \text{ s.t. } P = P_f^+)\).

- Let \(x \in P_f^+\) be chosen arbitrarily (goal is to show that \(x \in P\)).
- Suppose \(x \notin P\). Then, choose \(y\) to be a \(P\)-basis of \(x\) that maximizes the number of \(y\) elements strictly less than the corresponding \(x\) element. I.e., that maximizes \(|N(y)|\), where

\[
N(y) = \{e \in E : y(e) < x(e)\} \tag{11.54}
\]

- Choose \(w\) between \(y\) and \(x\), so that

\[
y \leq w \triangleq (y + x)/2 \leq x \tag{11.55}
\]

so \(y\) is also a \(P\)-basis of \(w\). Thus, \(\forall e \in N(y), y(e) < w(e) < x(e)\).
Proof of Theorem 11.4.2 (\(\forall P, \exists f \text{ s.t. } P = P_f^+\)).

- Let \(x \in P_f^+\) be chosen arbitrarily (goal is to show that \(x \in P\)).
- Suppose \(x \notin P\). Then, choose \(y\) to be a \(P\)-basis of \(x\) that maximizes the number of \(y\) elements strictly less than the corresponding \(x\) element. I.e., that maximizes \(|N(y)|\), where

\[
N(y) = \{e \in E : y(e) < x(e)\} \tag{11.54}
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- Choose \(w\) between \(y\) and \(x\), so that

\[
y \leq w \triangleq (y + x)/2 \leq x \tag{11.55}
\]

so \(y\) is also a \(P\)-basis of \(w\). Thus, \(\forall e \in N(y), y(e) < w(e) < x(e)\).

- Hence, \(\text{rank}(x) = \text{rank}(w) = y(E)\), and the set of \(P\)-bases of \(w\) are also \(P\)-bases of \(x\).
Proof of Theorem 11.4.2 \( (\forall P, \exists f \text{ s.t. } P = P^+_f) \).

Now, we have

\[
y(N(y)) < w(N(y)) \leq f(N(y)) = \text{rank}(\alpha_{\text{max}}1_{N(y)}) \tag{11.56}
\]

the last inequality follows since \( w \leq x \in P^+_f \), and \( y \leq w \).
Proof of Theorem 11.4.2 ($\forall P, \exists f \text{ s.t. } P = P^+_f$).

- Now, we have

$$y(N(y)) < w(N(y)) \leq f(N(y)) = \text{rank}(\alpha_{\max}1_{N(y)}) \quad (11.56)$$

the last inequality follows since $w \leq x \in P^+_f$, and $y \leq w$.

- Thus, $y \land x_{N(y)}$ is not a $P$-basis of $w \land x_{N(y)}$ since, over $N(y)$, it is neither tight at $w$ nor tight at the rank (i.e., not a maximal independent subvector on $N(y)$).
Proof of Theorem 11.4.2 (\(\forall P, \exists f \text{ s.t. } P = P_f^+\)).

- We can extend \(y \land x_{N(y)}\) to be a \(P\)-basis of \(w \land x_{N(y)}\) since \(y \land x_{N(y)} < w \land x_{N(y)}\).
Proof of Theorem 11.4.2 ($\forall P, \exists f \text{ s.t. } P = P_f^+$).

- We can extend $y \land x_{N(y)}$ to be a $P$-basis of $w \land x_{N(y)}$ since $y \land x_{N(y)} < w \land x_{N(y)}$.
- This $P$-basis, in turn, can be extended to be a $P$-basis $\hat{y}$ of $w \land x$. 

This contradiction means that we must have had $x \in P$. Therefore, $P + f = P$. 

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Proof of Theorem 11.4.2

We can extend $y \land x_{N(y)}$ to be a $P$-basis of $w \land x_{N(y)}$ since

$y \land x_{N(y)} < w \land x_{N(y)}$.

This $P$-basis, in turn, can be extended to be a $P$-basis $\hat{y}$ of $w \& x$.

Now, we have $\hat{y}(N(y)) > y(N(y))$, which violates the maximality of $|N(y)|$.
Proof of Theorem 11.4.2

Proof of Theorem 11.4.2 ($\forall P, \exists f$ s.t. $P = P_f^+$).

- We can extend $y \land x_{N(y)}$ to be a $P$-basis of $w \land x_{N(y)}$ since $y \land x_{N(y)} < w \land x_{N(y)}$.
- This $P$-basis, in turn, can be extended to be a $P$-basis $\hat{y}$ of $w \land x$.
- Now, we have $\hat{y}(N(y)) > y(N(y))$,
- and also that $\hat{y}(E) = y(E)$ (since both are $P$-bases),
Proof of Theorem 11.4.2

Proof of Theorem 11.4.2 ($\forall P, \exists f$ s.t. $P = P_f^+$).

- We can extend $y \land x_{N(y)}$ to be a $P$-basis of $w \land x_{N(y)}$ since $y \land x_{N(y)} < w \land x_{N(y)}$.
- This $P$-basis, in turn, can be extended to be a $P$-basis $\hat{y}$ of $w \land x$.
- Now, we have $\hat{y}(N(y)) > y(N(y))$, and also that $\hat{y}(E) = y(E)$ (since both are $P$-bases),
- hence $\hat{y}(e) < y(e)$ for some $e \notin N(y)$.

□
Proof of Theorem 11.4.2

Proof of Theorem 11.4.2 ($\forall P, \exists f$ s.t. $P = P_f^+$).

- We can extend $y \wedge x_{N(y)}$ to be a $P$-basis of $w \wedge x_{N(y)}$ since $y \wedge x_{N(y)} < w \wedge x_{N(y)}$.
- This $P$-basis, in turn, can be extended to be a $P$-basis $\hat{y}$ of $w \& x$.
- Now, we have $\hat{y}(N(y)) > y(N(y))$,
- and also that $\hat{y}(E) = y(E)$ (since both are $P$-bases),
- hence $\hat{y}(e) < y(e)$ for some $e \notin N(y)$.
- Thus, $\hat{y}$ is a base of $x$, which violates the maximality of $|N(y)|$. 

\[\square\]
Proof of Theorem 11.4.2

We can extend \( y \land x_{N(y)} \) to be a \( P \)-basis of \( w \land x_{N(y)} \) since \( y \land x_{N(y)} < w \land x_{N(y)} \).

This \( P \)-basis, in turn, can be extended to be a \( P \)-basis \( \hat{y} \) of \( w \& x \).

Now, we have \( \hat{y}(N(y)) > y(N(y)) \),

and also that \( \hat{y}(E) = y(E) \) (since both are \( P \)-bases),

hence \( \hat{y}(e) < y(e) \) for some \( e \notin N(y) \).

Thus, \( \hat{y} \) is a base of \( x \), which violates the maximality of \( |N(y)| \).

This contradiction means that we must have had \( x \in P \).
Proof of Theorem 11.4.2

Proof of Theorem 11.4.2 (∀P, ∃f s.t. P = P_f^+).

- We can extend \( y \land x_{N(y)} \) to be a \( P \)-basis of \( w \land x_{N(y)} \) since \( y \land x_{N(y)} < w \land x_{N(y)} \).
- This \( P \)-basis, in turn, can be extended to be a \( P \)-basis \( \hat{y} \) of \( w \& x \).
- Now, we have \( \hat{y}(N(y)) > y(N(y)) \),
- and also that \( \hat{y}(E) = y(E) \) (since both are \( P \)-bases),
- hence \( \hat{y}(e) < y(e) \) for some \( e \notin N(y) \).
- Thus, \( \hat{y} \) is a base of \( x \), which violates the maximality of \( |N(y)| \).
- This contradiction means that we must have had \( x \in P \).
- Therefore, \( P_f^+ = P \).
Theorem 11.4.5

A polymatroid can equivalently be defined as a pair \((E, P)\) where \(E\) is a finite ground set and \(P \subseteq \mathbb{R}^E_+\) is a compact non-empty set of independent vectors such that

1. every subvector of an independent vector is independent (if \(x \in P\) and \(y \leq x\) then \(y \in P\), i.e., down closed)
More on polymatroids

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1. **every subvector of an independent vector is independent** (if \(x \in P\) and \(y \leq x\) then \(y \in P\), i.e., down closed)

2. **if** \(u, v \in P\) (i.e., are independent) and \(u(E) < v(E)\), **then there exists a vector** \(w \in P\) **such that**

\[ u < w \leq u \lor v \quad (11.57) \]
More on polymatroids

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  u < w \leq u \vee v \quad (11.57)
\]

**Corollary 11.4.6**

The independent vectors of a polymatroid form a convex polyhedron in \(\mathbb{R}^E_+\).
The next slide comes from lecture 6.
In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

**Theorem 11.4.3 (Matroid (by bases))**

Let $E$ be a set and $\mathcal{B}$ be a nonempty collection of subsets of $E$. Then the following are equivalent.

1. $\mathcal{B}$ is the collection of bases of a matroid;
2. if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.
3. If $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.

Properties 2 and 3 are called “exchange properties.”

Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.
More on polymatroids

For any compact set $P$, $b$ is a base of $P$ if it is a maximal subvector within $P$. Recall the bases of matroids. In fact, we can define a polymatroid via vector bases (analogous to how a matroid can be defined via matroid bases).

**Theorem 11.4.7**

A polymatroid can equivalently be defined as a pair $(E, P)$ where $E$ is a finite ground set and $P \subseteq \mathbb{R}_+^E$ is a compact non-empty set of independent vectors such that

1. every subvector of an independent vector is independent (if $x \in P$ and $y \leq x$ then $y \in P$, i.e., down closed)

2. if $b, c$ are bases of $P$ and $d$ is such that $b \land c < d < b$, then there exists an $f$, with $d \land c < f \leq c$ such that $d \lor f$ is a base of $P$

3. All of the bases of $P$ have the same rank.

Note, all three of the above are required for a polymatroid (a matroid analogy would require the equivalent of only the first two).
A word on terminology & notation

- Recall how a matroid is sometimes given as $(E, r)$ where $r$ is the rank function.
A word on terminology & notation

- Recall how a matroid is sometimes given as \((E, r)\) where \(r\) is the rank function.
- We mention also that the term “polymatroid” is sometimes not used for the polytope itself, but instead but for the pair \((E, f)\),
A word on terminology & notation

- Recall how a matroid is sometimes given as \((E, r)\) where \(r\) is the rank function.
- We mention also that the term “polymatroid” is sometimes not used for the polytope itself, but instead but for the pair \((E, f)\),
- But now we see that \((E, f)\) is equivalent to a polymatroid polytope, so this is sensible.
Where are we going with this?

Consider the right hand side of Theorem 11.4.1:
\[
\min (x(A) + f(E \setminus A) : A \subseteq E)
\]
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- We are going to study this problem, and approaches that address it, as part of our ultimate goal which is to present strategies for submodular function minimization (that we will ultimately get to, in near future lectures).
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- Consider the right hand side of Theorem 11.4.1:
  \[ \min (x(A) + f(E \setminus A) : A \subseteq E) \]

- We are going to study this problem, and approaches that address it, as part of our ultimate goal which is to present strategies for submodular function minimization (that we will ultimately get to, in near future lectures).

- As a bit of a hint on what’s to come, recall that we can write it as:
  \[ x(E) + \min (f(A) - x(A) : A \subseteq E) \] where \( f \) is a polymatroid function.
Another Interesting Fact: Matroids from polymatroid functions

**Theorem 11.4.8**

Given integral polymatroid function $f$, let $(E, \mathcal{F})$ be a set system with ground set $E$ and set of subsets $\mathcal{F}$ such that

$$\forall F \in \mathcal{F}, \quad \forall \emptyset \subset S \subseteq F, |S| \leq f(S) \quad (11.58)$$

Then $M = (E, \mathcal{F})$ is a matroid.

**Proof.**

Exercise

And its rank function is Exercise.
Matroid instance of Theorem 11.4.1

- Considering Theorem 11.4.1, the matroid case is now a special case, where we have that:

**Corollary 11.4.9**

*We have that:*

\[
\max \{ y(E) : y \in P_{\text{ind. set}}(M), y \leq x \} = \min \{ r_M(A) + x(E \setminus A) : A \subseteq E \}
\]  
(11.59)

where \( r_M \) is the matroid rank function of some matroid.