Submodular Functions, Optimization, and Applications to Machine Learning

— Fall Quarter, Lecture 11 —

http://www.ee.washington.edu/people/faculty/bilmes/classes/ee563_spring_2018/

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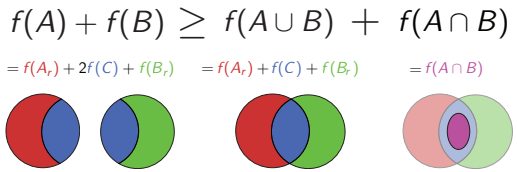
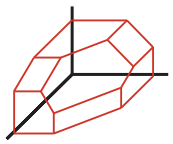
Nov 4th, 2020





Announcements, Assignments, and Reminders

- Homework 3, out soon.
- Reminder, all lectures are being recorded and posted to youtube. To get the links, see our announcements (`https://canvas.uw.edu/courses/1397085/announcements`).
- Office hours, Wed & Thur, 10:00pm at our class zoom link.

Class Road Map - EE563

- L1(9/30): Motivation, Applications, Definitions, Properties
- L2(10/5): Sums concave(modular), uses (diversity/costs, feature selection), information theory
- L3(10/7): Monge, More Definitions, Graph and Combinatorial Examples,
- L4(10/12): Graph & Combinatorial Examples, Matrix Rank, Properties, Other Defs, Independence
- L5(10/14): Properties, Defs of Submodularity, Independence
- L6(10/19): Matroids, Matroid Examples, Matroid Rank,
- L7(10/21): Matroid Rank, More on Partition Matroid, Laminar Matroids, System of Distinct Reps, Transversals
- L8(10/26): Transversal Matroid, Matroid and representation, Dual Matroid
- L9(10/28): Other Matroid Properties, Combinatorial Geometries, Matroid and Greedy, Polyhedra, Matroid Polytopes
- L10(11/2): Matroid Polytopes, Matroids $\rightarrow$ Polymatroids
- L11(11/4): Matroids $\rightarrow$ Polymatroids, Polymatroids
- L12(11/9):
- L–(11/11): Veterans Day, Holiday
- L13(11/16):
- L14(11/18):
- L15(11/23):
- L16(11/25):
- L17(11/30):
- L18(12/2):
- L19(12/7):
- L20(12/9): maximization.

Last day of instruction, Fri. Dec 11th. Finals Week: Dec 12-18, 2020

Summary of Important (for us) Matroid Definitions

Given an independence system, matroids are defined equivalently by any of the following:

- All maximally independent sets have the same size.
- A normalized monotone non-decreasing submodular integral rank function with unit increments.
- The greedy algorithm achieves the maximum weight independent set for all non-negative weight functions.

Independence Polyhedra

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I \in \{0,1\}^E \subset [0,1]^E \subset \mathbb{R}_+^E$.
- Taking the convex hull, we get the independent set polytope, that is

$$P_{\text{ind. set}} = \text{conv}\left\{\bigcup_{I \in \mathcal{I}} \{\mathbf{1}_I\}\right\} \subseteq [0,1]^E \tag{11.1}$$

- Now take the rank function r of M, and define the following polyhedron:

$$P_r^+ \triangleq \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\right\} \tag{11.2}$$

  Examples of P_r^+ are forthcoming.
- Now, take any $x \in P_{\text{ind. set}}$, then we will show that that $x \in P_r^+$ (or $P_{\text{ind. set}} \subseteq P_r^+$). We show this after a few examples of P_r^+.

Matroid Polyhedron in 2D

$$P_r^+ = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\right\} \tag{11.1}$$

- Consider this in two dimensions. We have equations of the form:

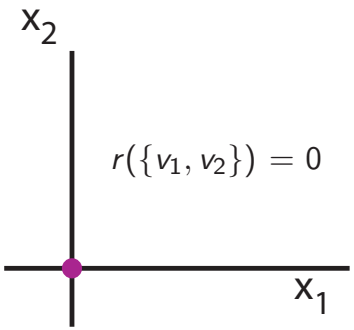
$$x_1 \geq 0 \text{ and } x_2 \geq 0 \tag{11.2}$$
$$x_1 \leq r(\{v_1\}) \in \{0, 1\} \tag{11.3}$$
$$x_2 \leq r(\{v_2\}) \in \{0, 1\} \tag{11.4}$$
$$x_1 + x_2 \leq r(\{v_1, v_2\}) \in \{0, 1, 2\} \tag{11.5}$$

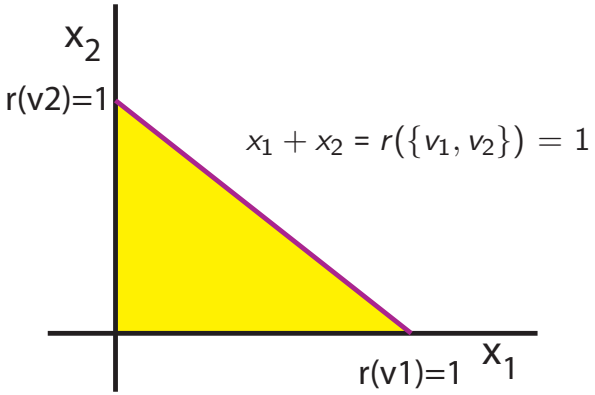
- Because r is submodular, we have

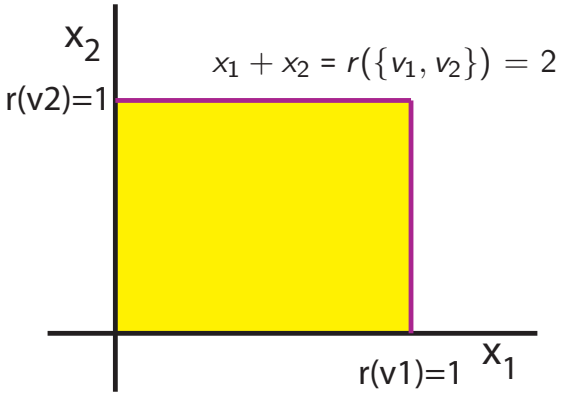
$$r(\{v_1\}) + r(\{v_2\}) \geq r(\{v_1, v_2\}) + r(\emptyset) \tag{11.6}$$

so since $r(\{v_1, v_2\}) \leq r(\{v_1\}) + r(\{v_2\})$, the last inequality is either superfluous ($r(v_1, v_2) = r(v_1) + r(v_2)$, "inactive") or "active."

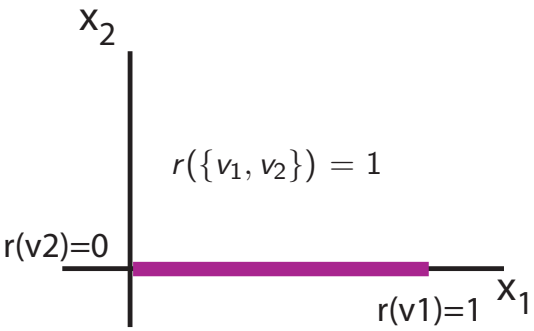
Matroid Polyhedron in 2D

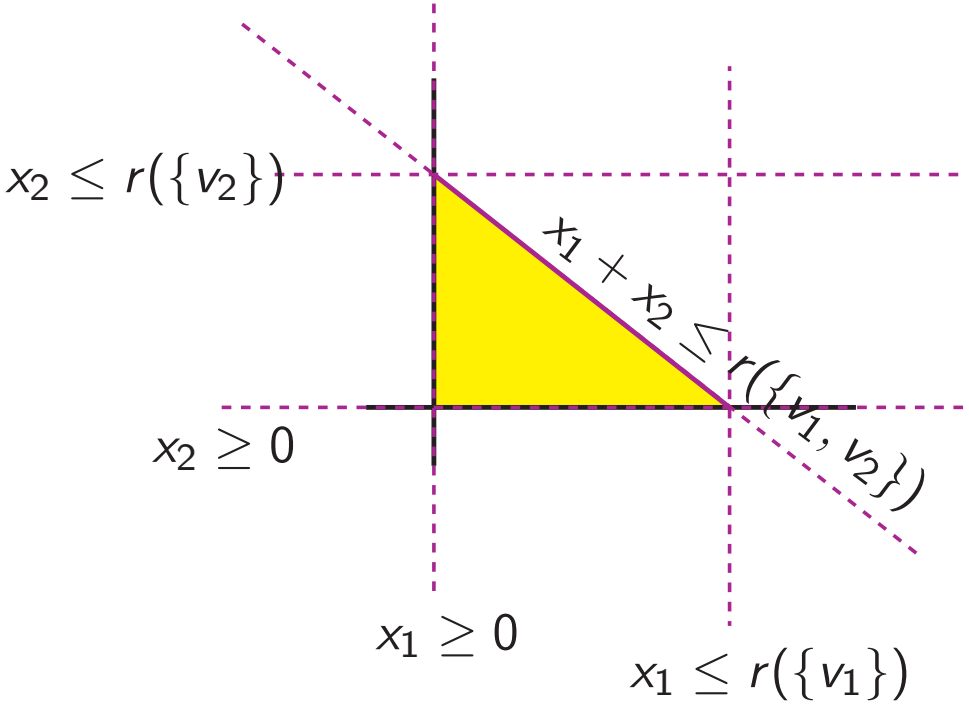






And, if v2 is a loop ...





Matroid Polyhedron in 3D

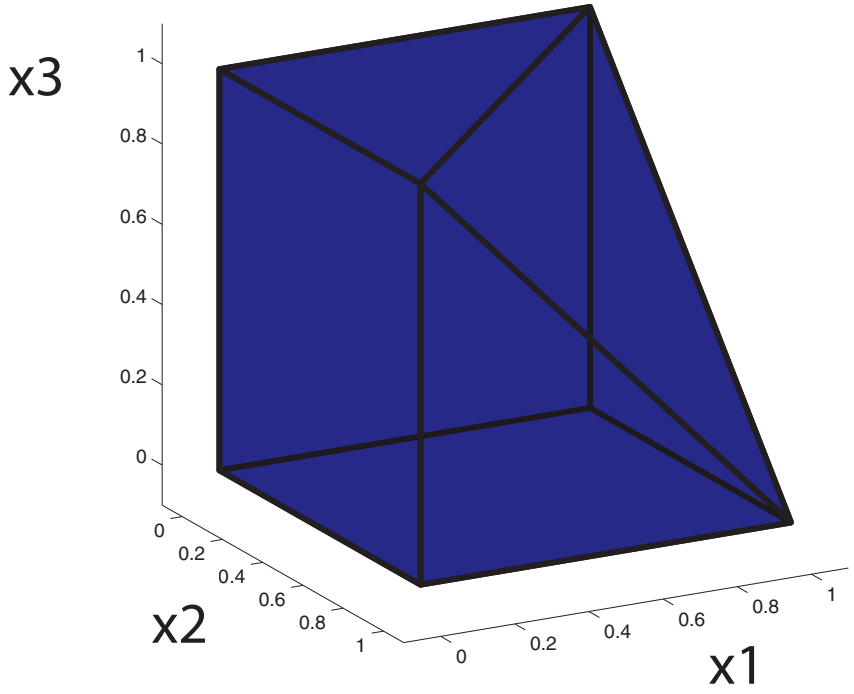
$$P_r^+ = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\right\} \tag{11.1}$$

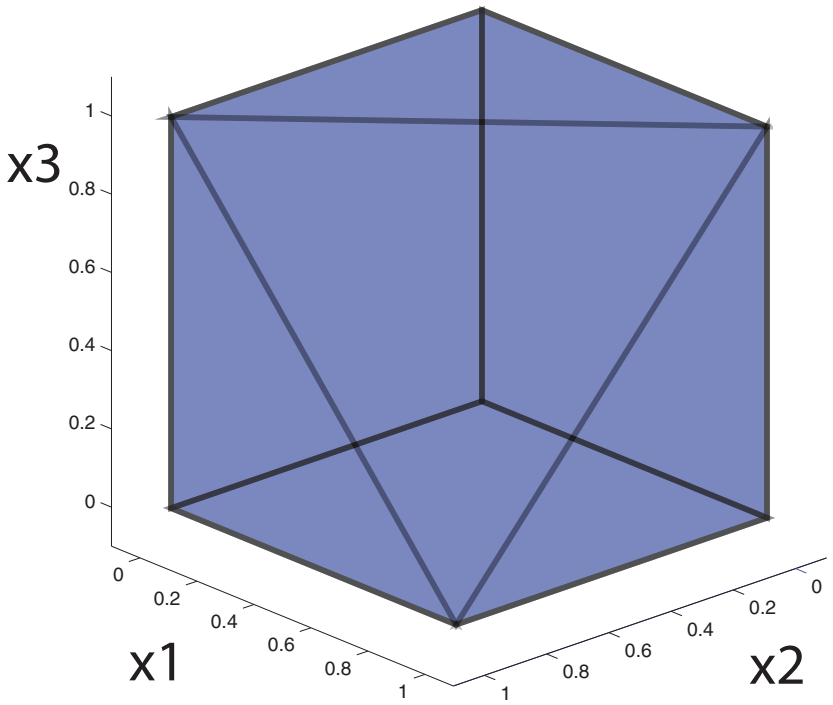
- Consider three dimensions, $E = \{1, 2, 3\}$. Get equations of the form:

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \text{ and } x_3 \geq 0 \tag{11.2}$$
$$x_1 \leq r(\{v_1\}) \tag{11.3}$$
$$x_2 \leq r(\{v_2\}) \tag{11.4}$$
$$x_3 \leq r(\{v_3\}) \tag{11.5}$$
$$x_1 + x_2 \leq r(\{v_1, v_2\}) \tag{11.6}$$
$$x_2 + x_3 \leq r(\{v_2, v_3\}) \tag{11.7}$$
$$x_1 + x_3 \leq r(\{v_1, v_3\}) \tag{11.8}$$
$$x_1 + x_2 + x_3 \leq r(\{v_1, v_2, v_3\}) \tag{11.9}$$

Matroid Polyhedron in 3D

Two view of P_r^+ associated with a matroid $(\{e_1, e_2, e_3\}, \{\emptyset, \{e_1\}, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\})$.





Maximum weight independent set via greedy weighted rank

Theorem 11.2.1

Let $M = (V, \mathcal{I})$ be a matroid, with rank function r, then for any weight function $w \in \mathbb{R}_+^V$, there exists a chain of sets $U_1 \subset U_2 \subset \cdots \subset U_n \subseteq V$ such that

$$\max \{w(I) | I \in \mathcal{I}\} = \sum_{i=1}^{n} \lambda_i r(U_i) \tag{11.4}$$

where $\lambda_i \geq 0$ satisfy

$$w = \sum_{i=1}^{n} \lambda_i \mathbf{1}_{U_i} \tag{11.5}$$

Polytope Equivalence (Summarizing the above)

- For each $I \in \mathcal{I}$ of a matroid $M = (E, \mathcal{I})$, we can form the incidence vector $\mathbf{1}_I$.
- Taking the convex hull, we get the independent set polytope, that is

$$P_{\text{ind. set}} = \text{conv}\left\{\cup_{I \in \mathcal{I}} \{\mathbf{1}_I\}\right\} \tag{11.7}$$

- Now take the rank function r of M, and define the following polytope:

$$P_r^+ = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq r(A), \forall A \subseteq E\right\} \tag{11.8}$$

Theorem 11.2.1

$$P_r^+ = P_{\textit{ind. set}} \tag{11.9}$$

Greedy solves a linear programming problem

- So we can describe the independence polytope of a matroid using the set of inequalities (an exponential number of them).
- In fact, considering equations starting at Eq ??, the LP problem with exponential number of constraints $\max\{w^\intercal x : x \in P_r^+\}$ is identical to the maximum weight independent set problem in a matroid, and since greedy solves the latter problem exactly, we have also proven:

Theorem 11.2.1

The LP problem $\max\{w^\intercal x : x \in P_r^+\}$ can be solved exactly using the greedy algorithm.

Note that this LP problem has an exponential number of constraints (since P_r^+ is described as the intersection of an exponential number of half spaces).

- This means that if LP problems have certain structure, they can be solved much easier than immediately implied by the equations.

P-basis of x given compact set $P \subseteq \mathbb{R}_+^E$

Definition 11.2.2 (subvector)

y is a subvector of x if $y \leq x$ (meaning $y(e) \leq x(e)$ for all $e \in E$).

Definition 11.2.3 (P-basis)

Given a compact set $P \subseteq \mathcal{R}_+^E$, for any $x \in \mathbb{R}_+^E$, a subvector y of x is called a P-basis of x if y maximal in P.
In other words, y is a P-basis of x if y is a maximal P-contained subvector of x.

Here, by y being "maximal", we mean that there exists no $z > y$ (more precisely, no $z \geq y + \epsilon \mathbf{1}_e$ for some $e \in E$ and $\epsilon > 0$) having the properties of y (the properties of y being: in P, and a subvector of x).
In still other words: y is a P-basis of x if:

1. $y \leq x$ (y is a subvector of x); and
2. $y \in P$ and $y + \epsilon \mathbf{1}_e \notin P$ for all $e \in E$ where $y(e) < x(e)$ and $\forall \epsilon > 0$ (y is maximal P-contained).

A vector form of rank

- Recall the definition of rank from a matroid $M = (E, \mathcal{I})$.

$$\text{rank}(A) = \max\{|I| : I \subseteq A, I \in \mathcal{I}\} = \max_{I \in \mathcal{I}} |A \cap I| \tag{11.19}$$

- vector rank: Given a compact set $P \subseteq \mathbb{R}_+^E$, define a form of "vector rank" relative to P: Given an $x \in \mathbb{R}^E$:

$$\text{rank}(x) = \max(y(E) : y \leq x, y \in P) = \max_{y \in P} (x \wedge y)(E) \tag{11.20}$$

  where $y \leq x$ is componentwise inequality ($y_i \leq x_i, \forall i$), and where $(x \wedge y) \in \mathbb{R}_+^E$ has $(x \wedge y)(i) = \min(x(i), y(i))$.
- Sometimes use $\text{rank}_P(x)$ to make P explicit.
- If $\mathcal{B}_x$ is the set of P-bases of x, than $\text{rank}(x) = \max_{y \in \mathcal{B}_x} y(E)$.
- If $x \in P$, then $\text{rank}(x) = x(E)$ (x is its own unique self P-basis).
- If $x_{\text{min}} \in \text{argmin}_{x \in P} x(E)$, and $x \leq x_{\text{min}}$ what then? Then $\text{rank}(x)$ is either $x(E)$ (if $x = x_{\text{min}}$) or otherwise $\text{rank}(x) = -\infty$.
- In general, might be hard to compute and/or have ill-defined properties. Next, we look at an object that restrains and cultivates this form of rank.

Polymatroidal polyhedron (or a "polymatroid")

Definition 11.3.1 (polymatroid)

A polymatroid is a compact set $P \subseteq \mathbb{R}_+^E$ satisfying

1. $0 \in P$
2. If $y \leq x \in P$ then $y \in P$ (called down monotone).
3. For every $x \in \mathbb{R}_+^E$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any P-basis of x), has the same component sum $y(E)$

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- Condition 3 restated: That is for any two distinct maximal vectors $y^1, y^2 \in P$, with $y^1 \leq x$ & $y^2 \leq x$, with $y^1 \neq y^2$, we must have $y^1(E) = y^2(E)$.

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- Condition 3 restated (again): For every vector $x \in \mathbb{R}_+^E$, every maximal independent (i.e., $\in P$) subvector y of x has the same component sum $y(E) = \text{rank}(x)$.

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- Condition 3 restated (again): For every vector $x \in \mathbb{R}_+^E$, every maximal independent (i.e., $\in P$) subvector y of x has the same component sum $y(E) = \text{rank}(x)$.
- Condition 3 restated (yet again): All P-bases of x have the same component sum.

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- Vectors within P (i.e., any $y \in P$) are called independent, and any vector outside of P is called dependent.

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3. For every $x \in \mathbb{R}_+^E$, any maximal vector $y \in P$ with $y \leq x$ (i.e., any P-basis of x), has the same component sum $y(E)$

- Vectors within P (i.e., any $y \in P$) are called independent, and any vector outside of P is called dependent.
- Since all P-bases of x have the same component sum, if $\mathcal{B}_x$ is the set of P-bases of x, than $\text{rank}(x) = y(E)$ for any $y \in \mathcal{B}_x$.

Matroid and Polymatroid: side-by-side

A Matroid is:

A Polymatroid is:

Matroid and Polymatroid: side-by-side

A Matroid is:

1. a set system $(E, \mathcal{I})$

A Polymatroid is:

1. a compact set $P \subseteq \mathbb{R}_+^E$

Matroid and Polymatroid: side-by-side

A Matroid is:

1. a set system $(E, \mathcal{I})$
2. empty-set containing $\emptyset \in \mathcal{I}$

A Polymatroid is:

1. a compact set $P \subseteq \mathbb{R}_+^E$
2. zero containing, $\mathbf{0} \in P$

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3. down closed, $\emptyset \subseteq I' \subseteq I \in \mathcal{I} \Rightarrow I' \in \mathcal{I}$.

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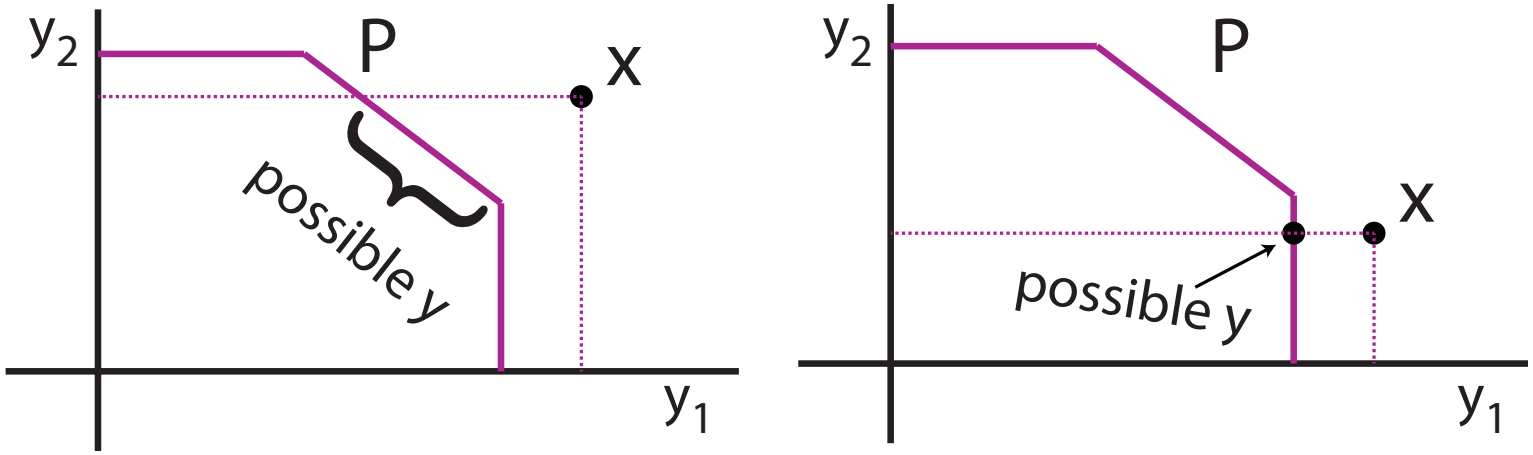
A Matroid is:

1. a set system $(E, \mathcal{I})$
2. empty-set containing $\emptyset \in \mathcal{I}$
3. down closed, $\emptyset \subseteq I' \subseteq I \in \mathcal{I} \Rightarrow I' \in \mathcal{I}$.
4. any maximal set I in $\mathcal{I}$, bounded by another set A, has the same matroid rank (any maximal independent subset $I \subseteq A$ has same size $|I|$).

A Polymatroid is:

1. a compact set $P \subseteq \mathbb{R}_+^E$
2. zero containing, $\mathbf{0} \in P$
3. down monotone, $0 \leq y \leq x \in P \Rightarrow y \in P$
4. any maximal vector y in P, bounded by another vector x, has the same vector rank (any maximal independent subvector $y \leq x$ has same sum $y(E)$).

Polymatroidal polyhedron (or a "polymatroid")



Left: $\exists$ multiple maximal $y \leq x$ Right: $\exists$ only one maximal $y \leq x$,

- Polymatroid condition here: $\forall$ maximal $y \in P$, with $y \leq x$ (which here means $y_1 \leq x_1$ and $y_2 \leq x_2$), we just have $y(E) = y_1 + y_2 =$ const.

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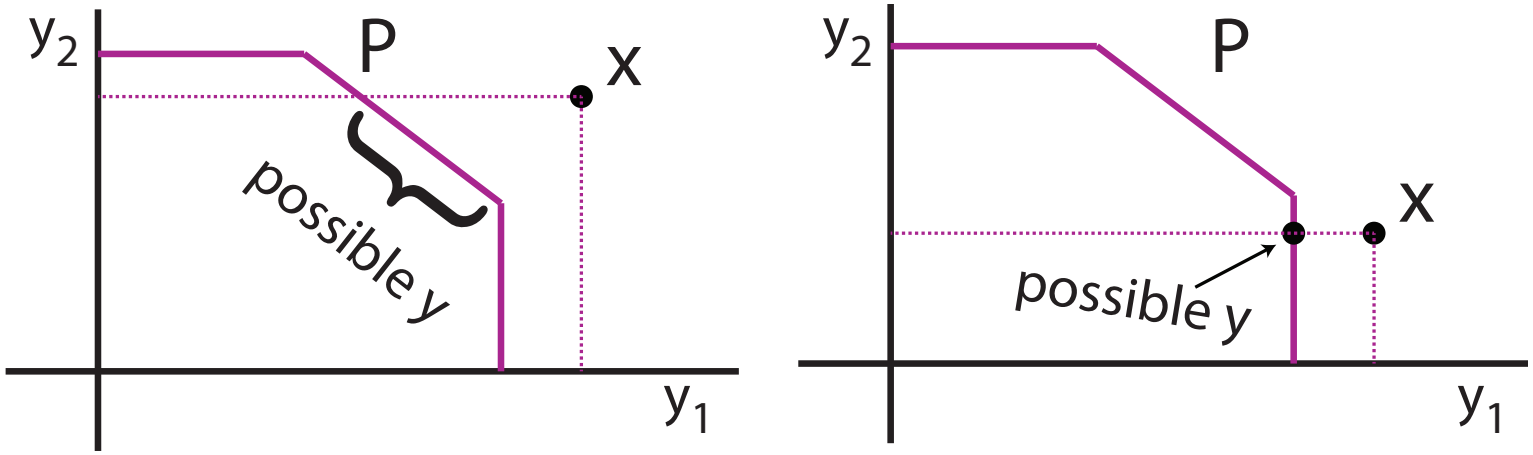

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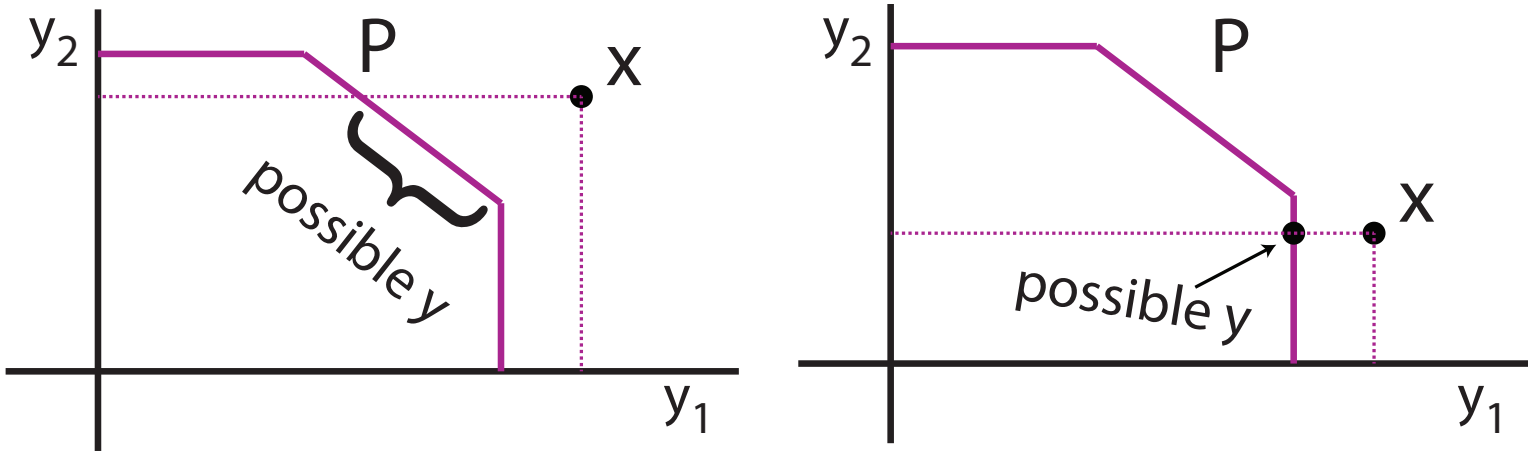
- On the left, we see there are multiple possible maximal $y \in P$ such that $y \leq x$. Each such y must have the same value $y(E)$.

Polymatroidal polyhedron (or a "polymatroid")



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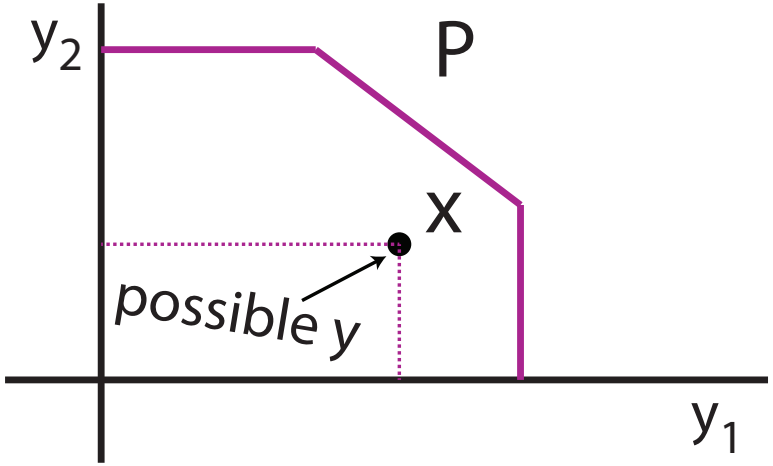
- On the left, we see there are multiple possible maximal $y \in P$ such that $y \leq x$. Each such y must have the same value $y(E)$.
- On the right, there is only one maximal $y \in P$. Since there is only one, the condition on the same value of $y(E), \forall y$ is vacuous.

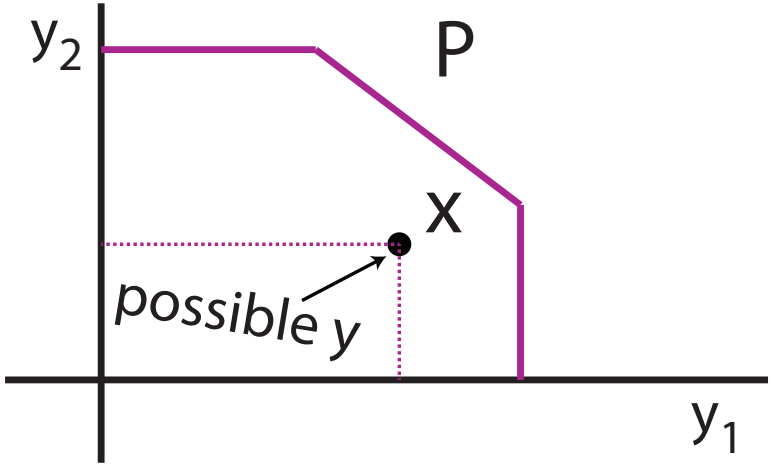
Polymatroidal polyhedron (or a "polymatroid")



$\exists$ only one maximal $y \leq x$.

- If $x \in P$ already, then x is its own P-basis, i.e., it is a self P-basis.

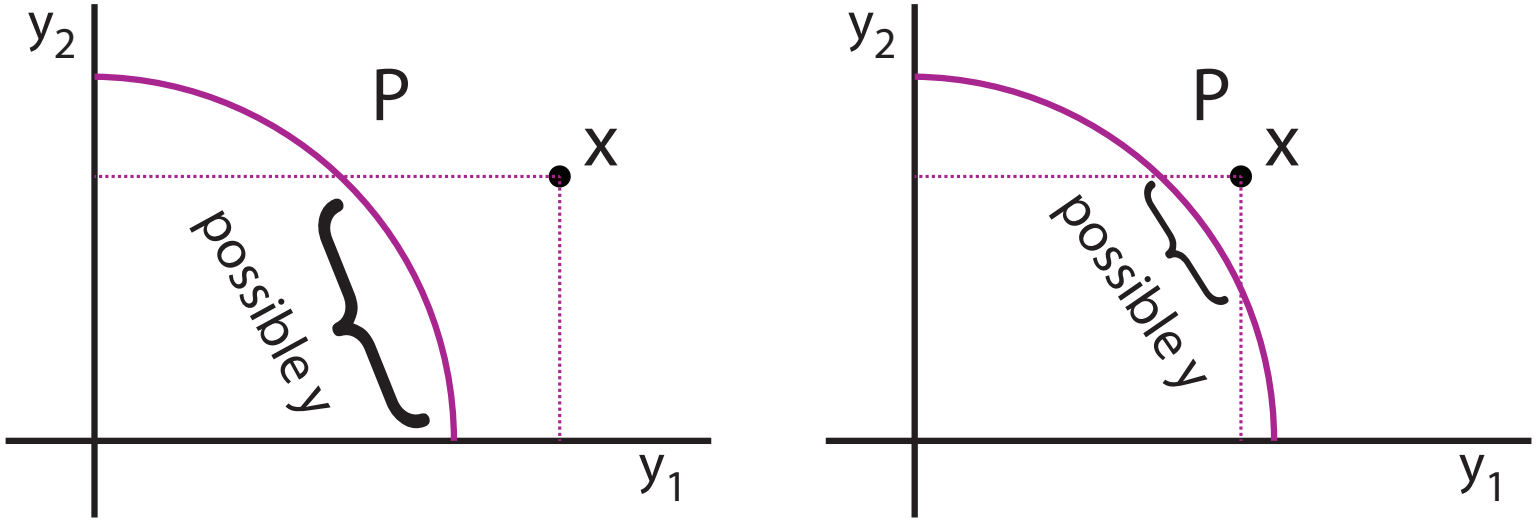
Polymatroidal polyhedron (or a "polymatroid")



$\exists$ only one maximal $y \leq x$.

- If $x \in P$ already, then x is its own P-basis, i.e., it is a self P-basis.
- In a matroid, a base of A is the maximally contained independent set. If A is already independent, then A is a self-base of A (as we saw in previous Lectures)

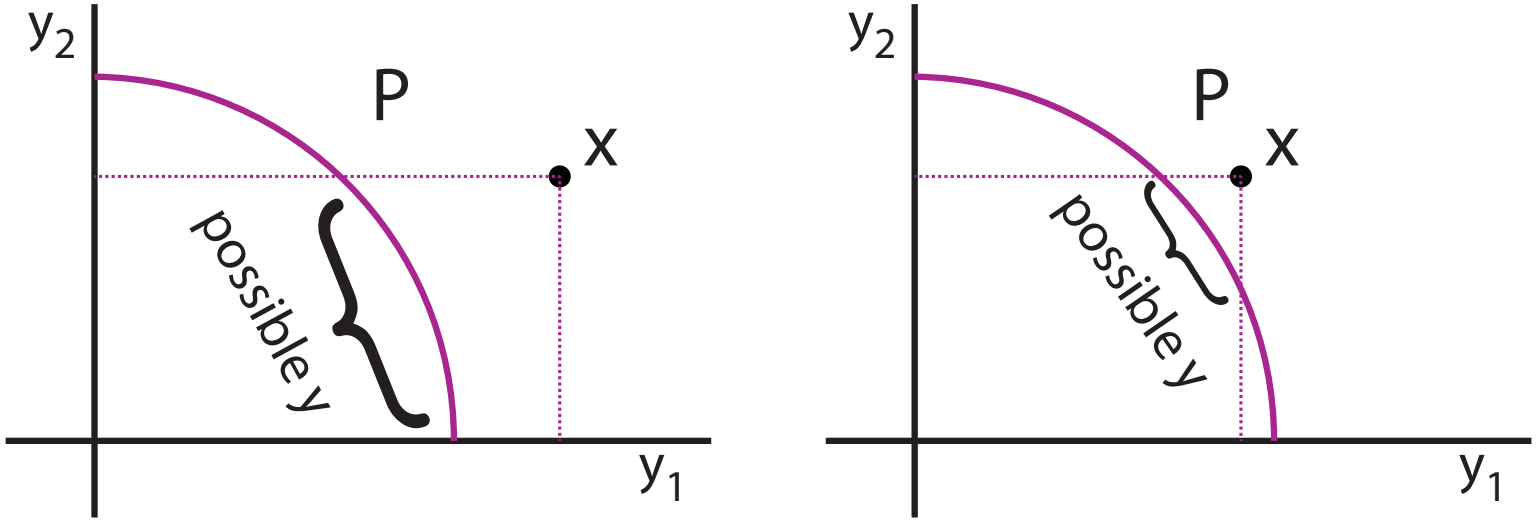
Polymatroid as well?



Left and right: $\exists$ multiple maximal $y \leq x$ as indicated.

- On the left, we see there are multiple possible maximal such $y \in P$ that are $y \leq x$. Each such y must have the same value $y(E)$, but since the equation for the curve is $y_1^2 + y_2^2 =$ const. $\neq y_1 + y_2$, we see this is not a polymatroid.

Polymatroid as well? no



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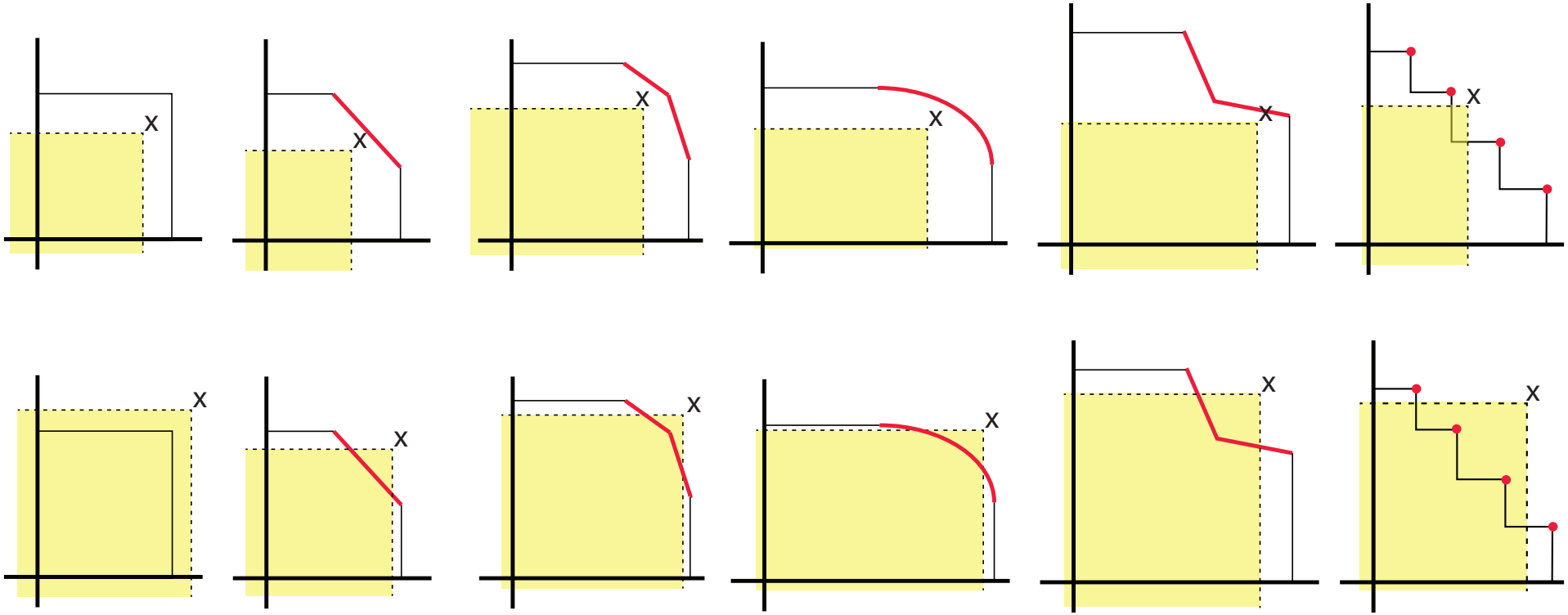
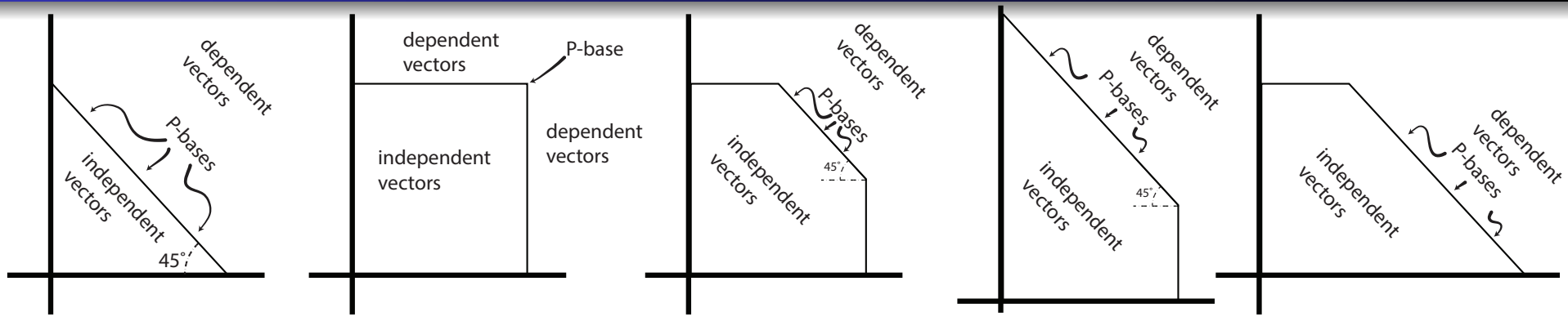
- On the right, we have a similar situation, just the set of potential values that must have the $y(E)$ condition changes, but the values of course are still not constant.

Other examples: Polymatroid or not?



Some possible polymatroid forms in 2D



It appears that we have five possible forms of polymatroid in 2D, when neither of the elements $\{v_1, v_2\}$ are self-dependent.

1. On the left: full dependence between v_1 and v_2

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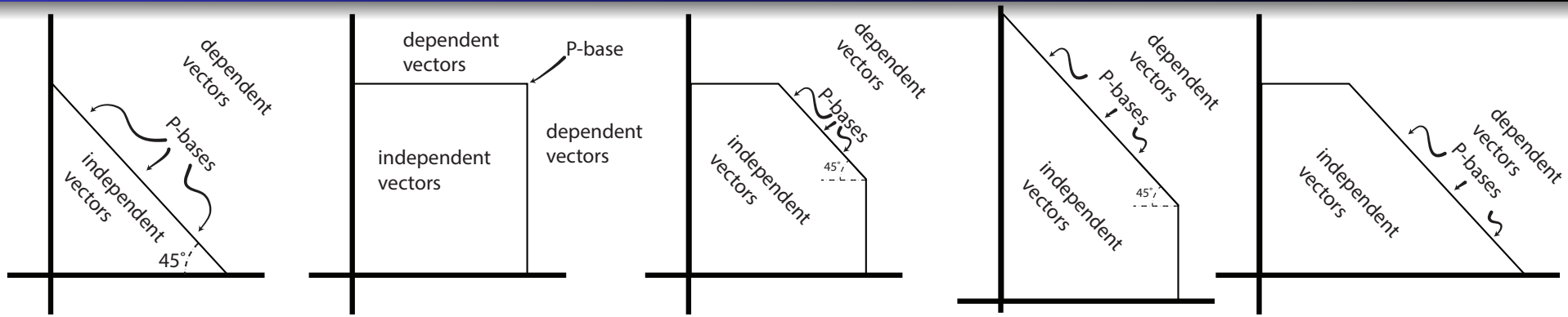

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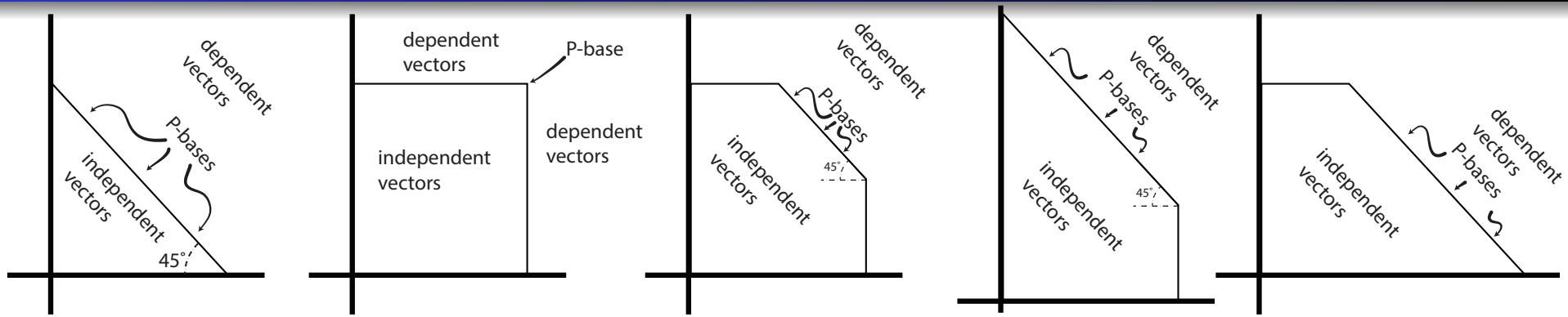

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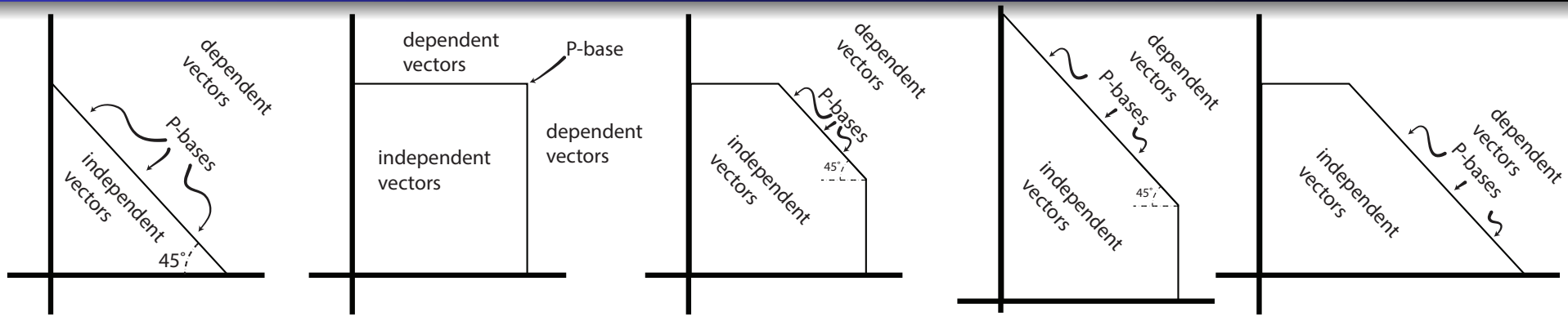

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4. Right two: other forms of partial independence between v_1 and v_2

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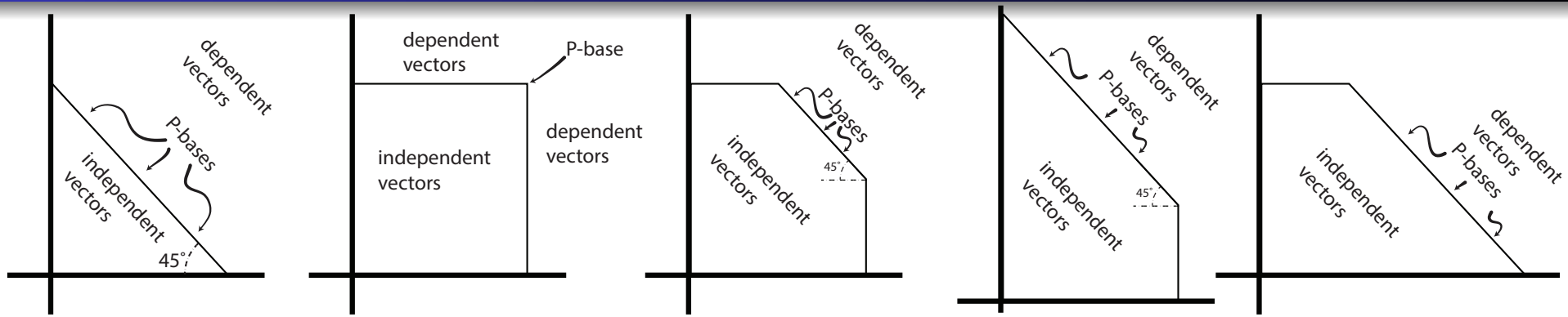

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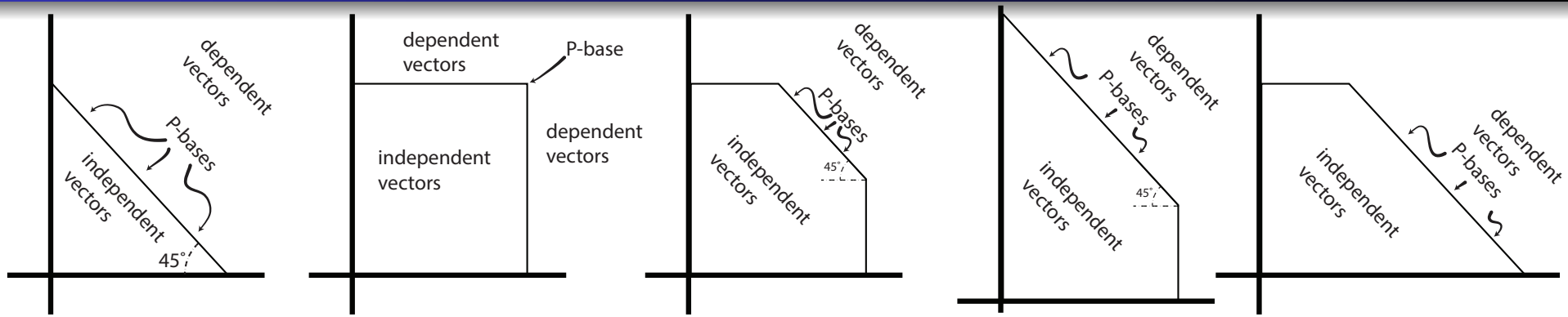

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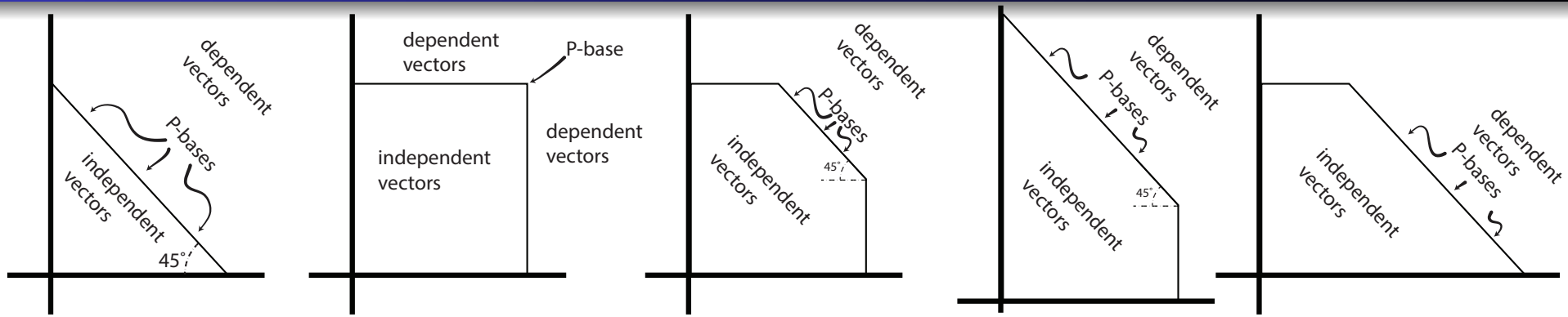

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- The set of P-bases for a polytope is called the base polytope.

Polymatroidal polyhedron (or a "polymatroid")

- Suppose $x, y \in \mathbb{R}_+^E$ with $y \leq x$. If x contains any zeros (i.e., x has $E \setminus S$ s.t. $x(E \setminus S) = 0$, so S indicates the non-zero elements, or $S = \text{supp}(x)$), then this also forces $y(E \setminus S) = 0$, so that $y(E) = y(S)$. This is true either for $x \in P$ or $x \notin P$.

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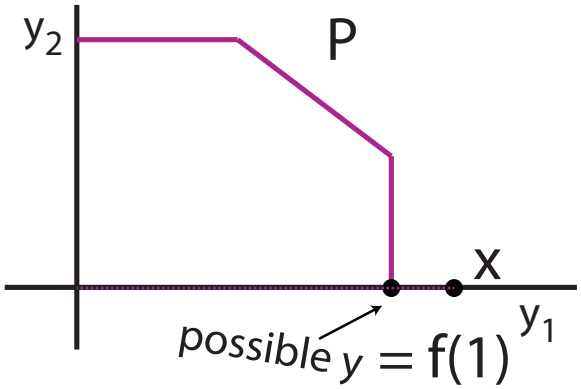
- Therefore, in this case, it is the non-zero elements of x, i.e., support $\text{supp}(x)$ of$x$, are the elements that determine the common component sum of a P-basis of x.
- For the case of either $x \notin P$ or right at the boundary of P, we might give a "name" to this component sum, lets say $f(S)$ for any given set S of non-zero elements of x. We could name rank$(\frac{1}{\epsilon}\mathbf{1}_S) \triangleq f(S)$ for ϵ small enough. What kind of function might f be?



Polymatroid function and its polyhedron.

Definition 11.3.2

A polymatroid function is a real-valued function f defined on subsets of E which is normalized, non-decreasing, and submodular. That is we have

1. $f(\emptyset) = 0$ (normalized)
2. $f(A) \leq f(B)$ for any $A \subseteq B \subseteq E$ (monotone non-decreasing)
3. $f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$ for any $A, B \subseteq E$ (submodular)

We can define the polyhedron P_f^+ associated with a polymatroid function as follows

$$P_f^+ = \{y \in \mathbb{R}_+^E : y(A) \leq f(A) \text{ for all } A \subseteq E\} \tag{11.1}$$

$$= \{y \in \mathbb{R}^E : y \geq 0, y(A) \leq f(A) \text{ for all } A \subseteq E\} \tag{11.2}$$

Associated polyhedron with a polymatroid function

$$P_f^+ = \left\{x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\right\} \tag{11.3}$$

- Consider this in three dimensions. We have equations of the form:

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \text{ and } x_3 \geq 0 \tag{11.4}$$
$$x_1 \leq f(\{v_1\}) \tag{11.5}$$
$$x_2 \leq f(\{v_2\}) \tag{11.6}$$
$$x_3 \leq f(\{v_3\}) \tag{11.7}$$
$$x_1 + x_2 \leq f(\{v_1, v_2\}) \tag{11.8}$$
$$x_2 + x_3 \leq f(\{v_2, v_3\}) \tag{11.9}$$
$$x_1 + x_3 \leq f(\{v_1, v_3\}) \tag{11.10}$$
$$x_1 + x_2 + x_3 \leq f(\{v_1, v_2, v_3\}) \tag{11.11}$$

Associated polyhedron with a polymatroid function

- Consider the asymmetric graph cut function on the simple chain graph $v_1 - v_2 - v_3$. That is, $f(S) = |\{(v, s) \in E(G) : v \in V, s \in S\}|$ is count of any edges within S or between S and $V \setminus S$, so that $\delta(S) = f(S) + f(V \setminus S) - f(V)$ is the standard graph cut.

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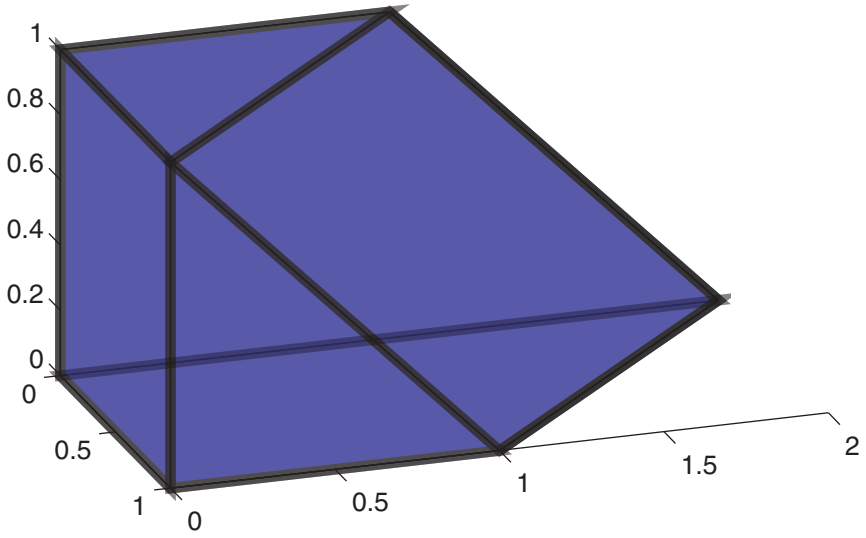
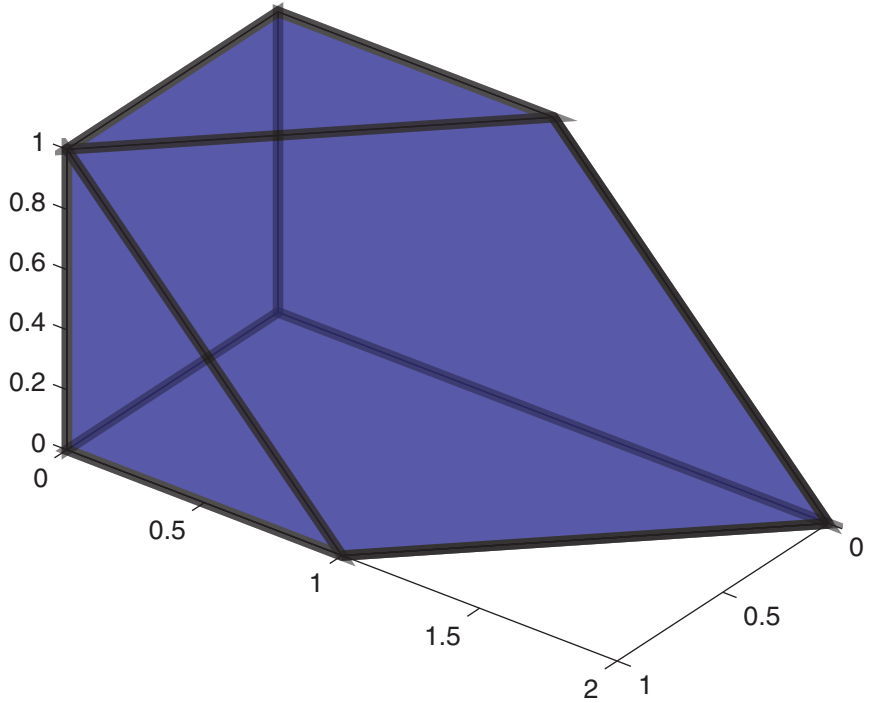
- Observe: P_f^+ (at two views):





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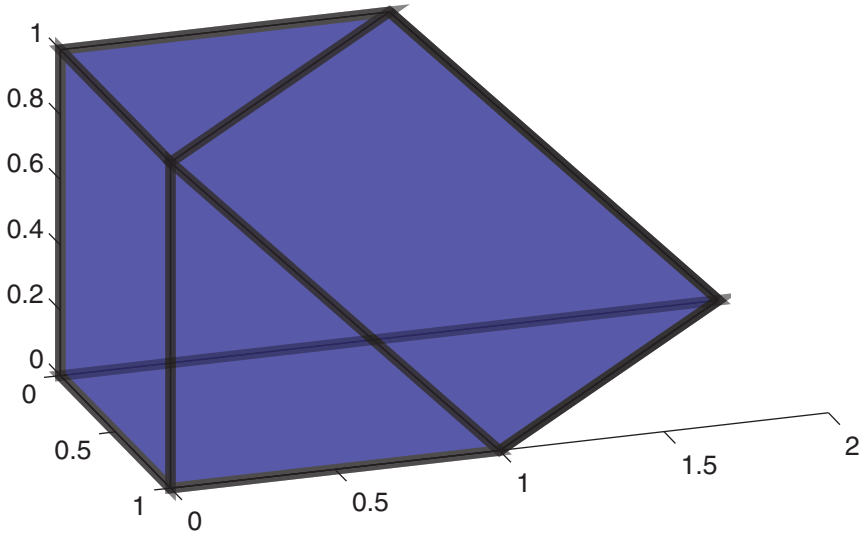

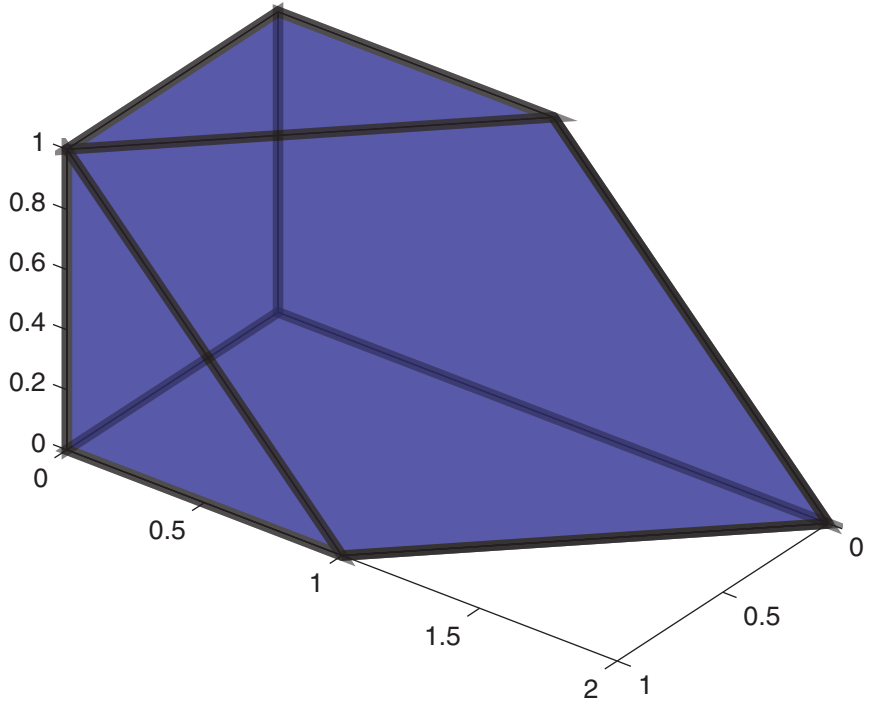

- which axis is which?

Associated polyhedron with a polymatroid function

- Consider: $f(\emptyset) = 0$, $f(\{v_1\}) = 1.5$, $f(\{v_2\}) = 2$, $f(\{v_1, v_2\}) = 2.5$, $f(\{v_3\}) = 3$, $f(\{v_3, v_1\}) = 3.5$, $f(\{v_3, v_2\}) = 4$, $f(\{v_3, v_2, v_1\}) = 4.3$.

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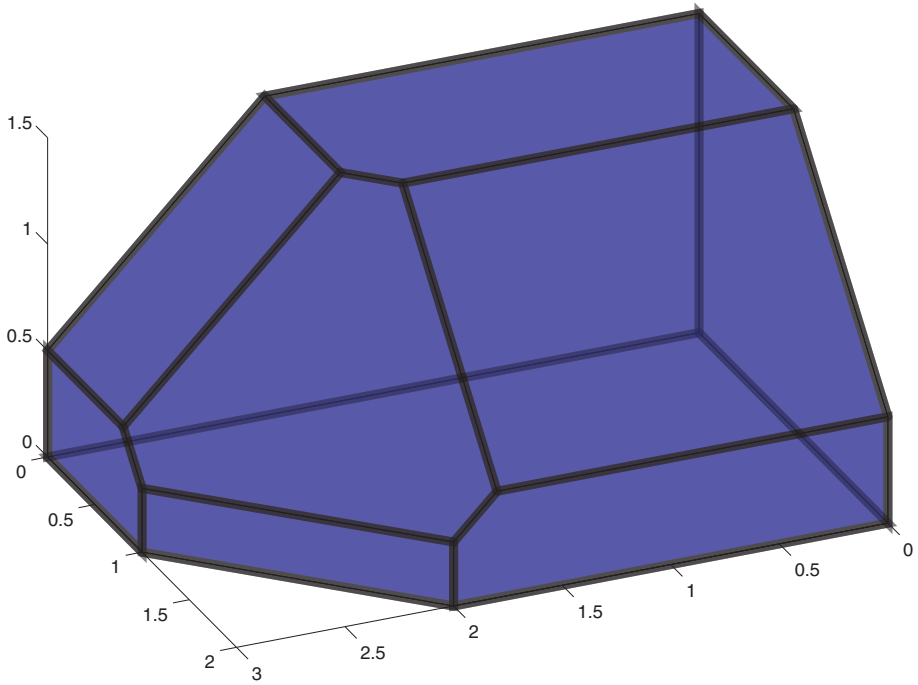

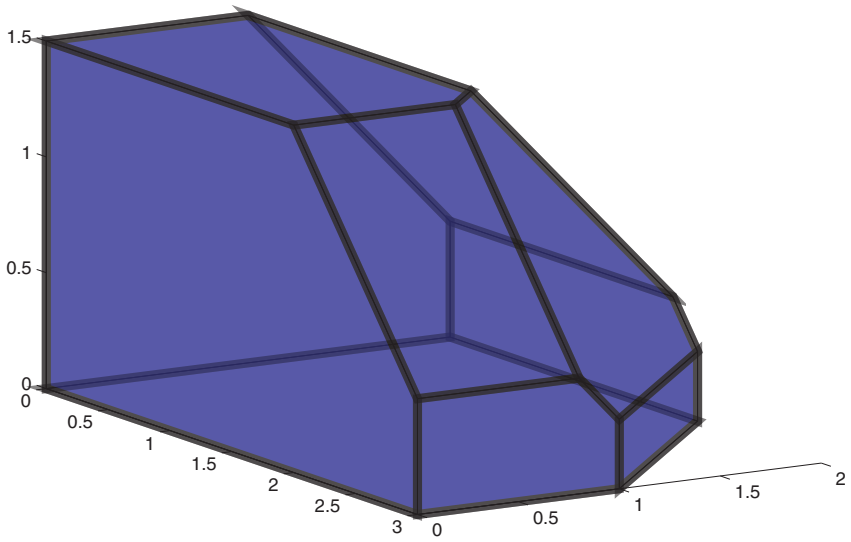

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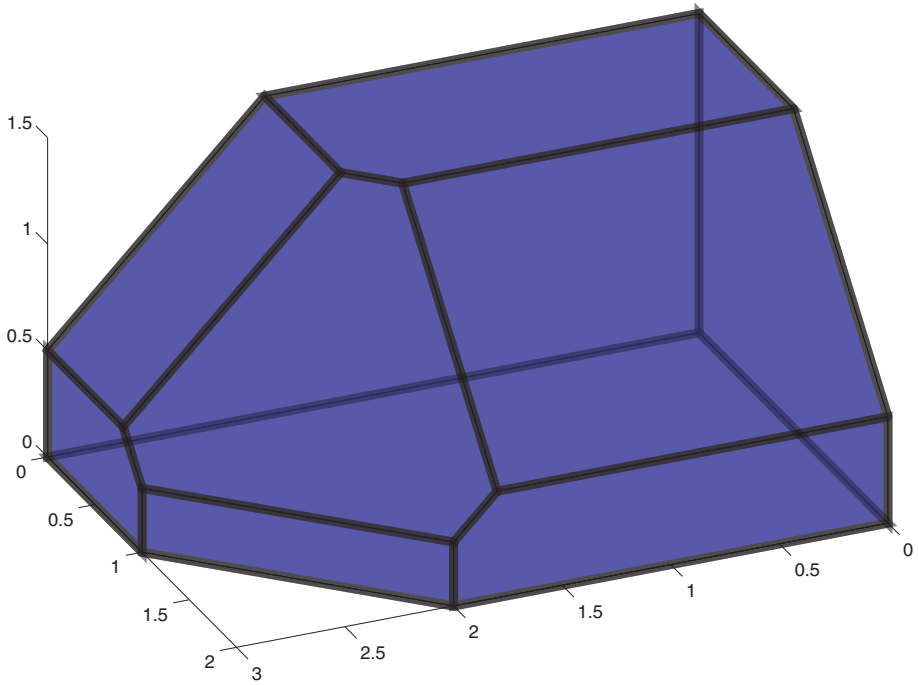

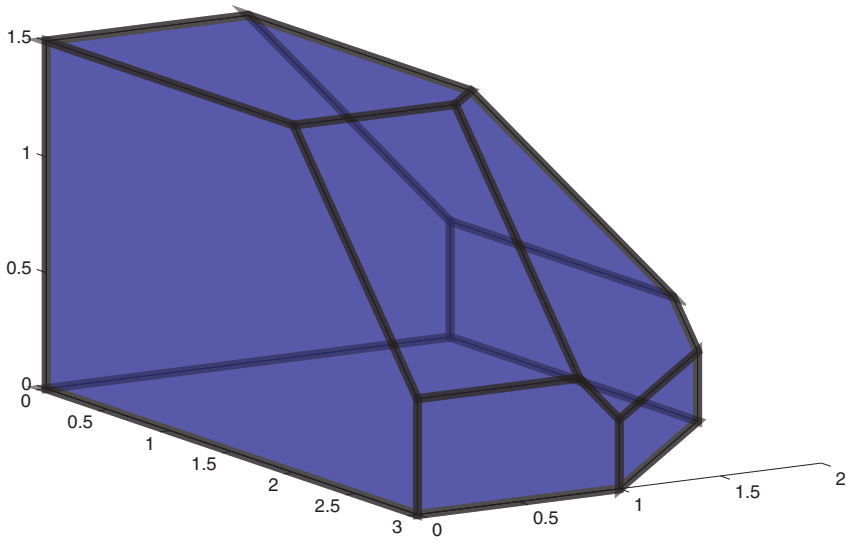

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Associated polyhedron with a polymatroid function

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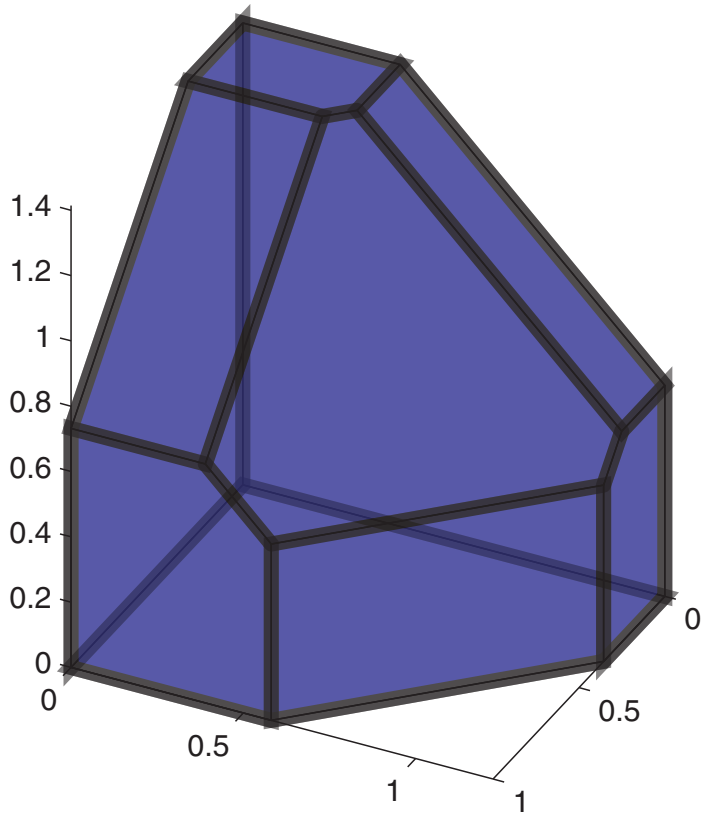

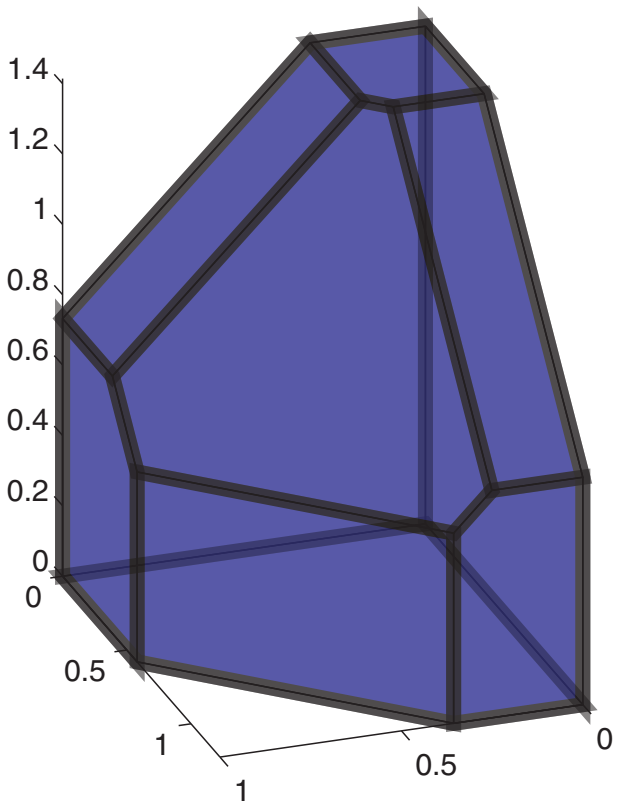

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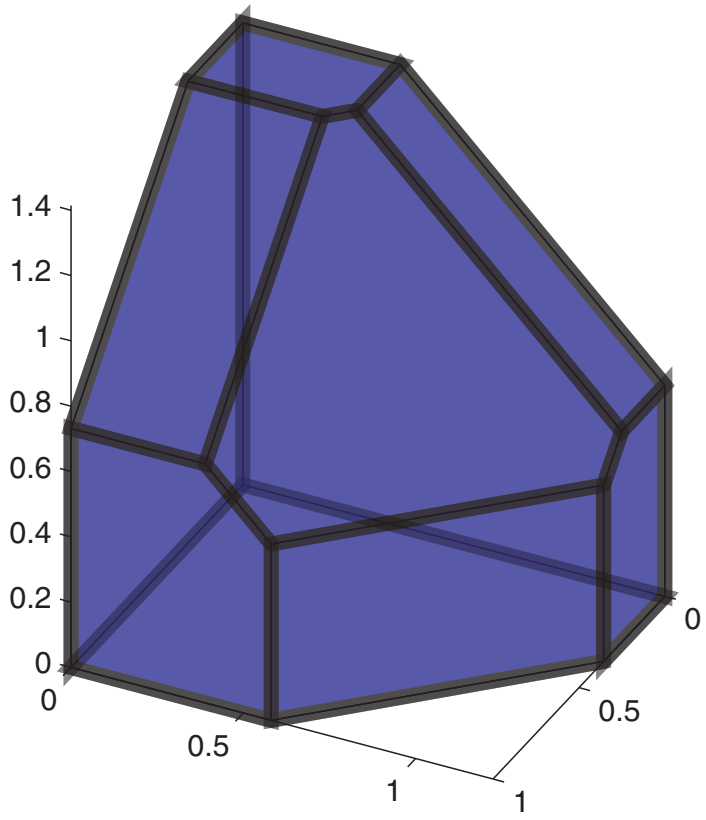

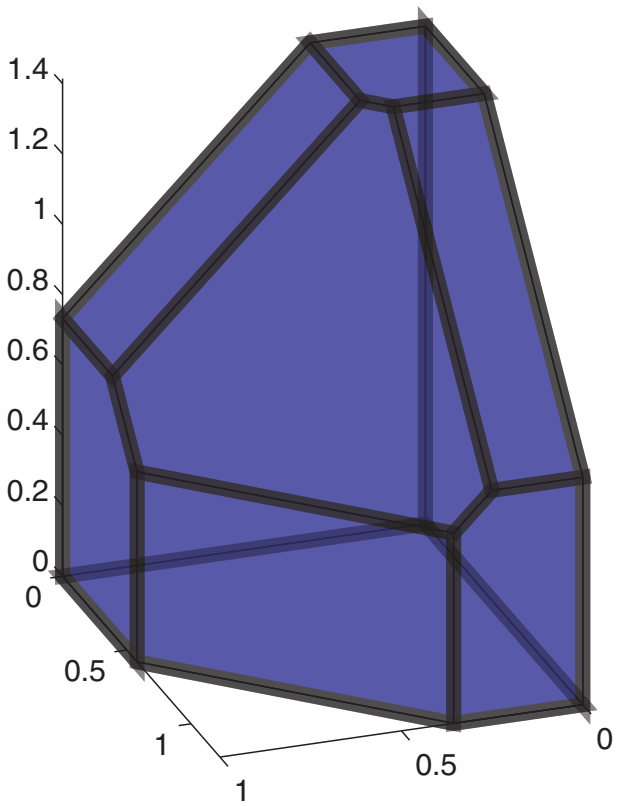

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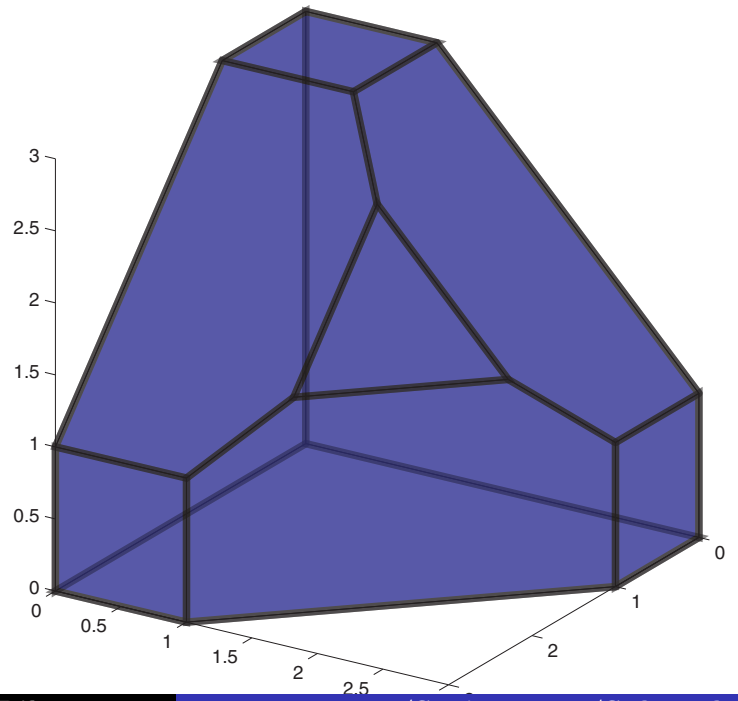
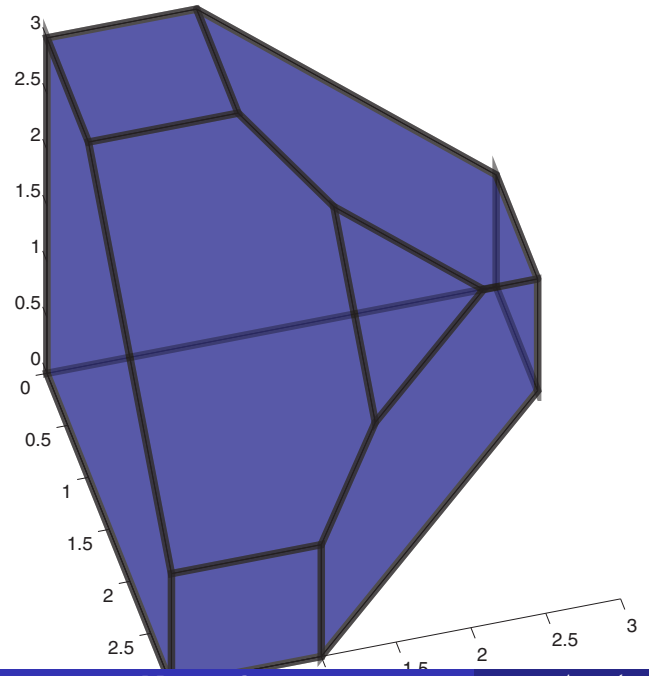
- Observe: P_f^+ (at two views), maximal independent subvectors not constant rank, hence not a polymatroid.





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- In the next theorem, we show that any P_f^+-basis has the same component sum, when f is a polymatroid function, and P_f^+ satisfies the other properties so that P_f^+ is a polymatroid.

A polymatroid function's polyhedron is a polymatroid.

Theorem 11.4.1

Let f be a polymatroid function defined on subsets of E. For any $x \in \mathbb{R}_+^E$, and <u>any</u> P_f^+-basis $y^x \in \mathbb{R}_+^E$ of x, the component sum of y^x is

$$\begin{aligned} y^x(E) = \textit{rank}(x) &\triangleq \max\left(y(E) : y \leq x, y \in P_f^+\right) \\ &= \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \end{aligned} \tag{11.13}$$

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Taking $E \setminus B = \mathrm{supp}(x)$ (so elements B are all zeros in x), and for $b \notin B$ we make $x(b)$ is big enough, the r.h.s. min has solution $A^* = B$. We recover submodular function from the polymatroid polyhedron via the following:

$$\text{rank}\left(\frac{1}{\epsilon}\mathbf{1}_{E\setminus B}\right) = f(E \setminus B) = \max\left\{y(E \setminus B) : y \in P_f^+\right\} \tag{11.14}$$

A polymatroid function's polyhedron is a polymatroid.

Theorem 11.4.1

Let f be a polymatroid function defined on subsets of E. For any $x \in \mathbb{R}_+^E$, and <u>any</u> P_f^+-basis $y^x \in \mathbb{R}_+^E$ of x, the component sum of y^x is

$$\begin{aligned} y^x(E) = \textit{rank}(x) &\triangleq \max\left(y(E) : y \leq x, y \in P_f^+\right) \\ &= \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \end{aligned} \tag{11.13}$$

As a consequence, P_f^+ is a polymatroid, since r.h.s. is constant w.r.t. y^x.

Taking $E \setminus B = \text{supp}(x)$ (so elements B are all zeros in x), and for $b \notin B$ we make $x(b)$ is big enough, the r.h.s. min has solution $A^* = B$. We recover submodular function from the polymatroid polyhedron via the following:

$$\text{rank}\left(\frac{1}{\epsilon}\mathbf{1}_{E\setminus B}\right) = f(E \setminus B) = \max\left\{y(E \setminus B) : y \in P_f^+\right\} \tag{11.14}$$

In fact, we will ultimately see a number of important consequences of this theorem (other than just that P_f^+ is a polymatroid)

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Proof of Thm 11.4.1.

- Clearly $0 \in P_f^+$ since f is non-negative.

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- Goal is to show that any such y^x has $y^x(E) = \text{const}$, dependent <u>only</u> on x and also f (which defines the polytope) but not dependent on y^x, the particular P_f^+-basis.

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- Doing so will thus establish that P_f^+ is a polymatroid.

...

A polymatroid function's polyhedron is a polymatroid.

. . . proof of Thm 11.4.1 continued.

- First trivial case: could have $y^x = x$, which happens if $x(A) \leq f(A), \forall A \subseteq E$ (i.e., $x \in P_f^+$ strictly). In such case,

$$\min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{11.15}$$
$$= x(E) + \min\left(f(E \setminus A) - x(E \setminus A) : A \subseteq E\right) \tag{11.16}$$
$$= x(E) + \min\left(f(A) - x(A) : A \subseteq E\right) \tag{11.17}$$
$$= x(E) \tag{11.18}$$

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- When $x \in P_f^+$, $y = x$ is clearly the solution to $\max\left(y(E) : y \leq x, y \in P_f^+\right)$, so this is tight, and rank$(x) = x(E)$.

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- When $x \in P_f^+$, $y = x$ is clearly the solution to $\max\left(y(E) : y \leq x, y \in P_f^+\right)$, so this is tight, and rank$(x) = x(E)$.
- This is a value dependent only on x, a self basis, unique P_f^+-base.

...

A polymatroid function's polyhedron is a polymatroid.

...proof of Thm 11.4.1 continued.

- 2nd trivial case: $x(A) > f(A), \forall A \subseteq E$ (i.e., $x \notin P_f^+$ every direction),

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- 2nd trivial case: $x(A) > f(A), \forall A \subseteq E$ (i.e., $x \notin P_f^+$ every direction),
- Then for any order $(a_1, a_2, \dots)$ of the elements and $A_i \triangleq (a_1, a_2, \dots, a_i)$, we have $x(a_i) \geq f(a_i) \geq f(a_i | A_{i-1})$, the second inequality by submodularity.

A polymatroid function's polyhedron is a polymatroid.

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$$\min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{11.19}$$

$$= x(E) + \min\left(f(A) - x(A) : A \subseteq E\right) \tag{11.20}$$

$$= x(E) + \min\left(\sum_i f(a_i|A_{i-1}) - \sum_i x(a_i) : A \subseteq E\right) \tag{11.21}$$

$$= x(E) + \min\left(\sum_i \underbrace{\Big(f(a_i|A_{i-1}) - x(a_i)\Big)}_{\leq 0} : A \subseteq E\right) \tag{11.22}$$

$$= x(E) + f(E) - x(E) = f(E) = \max(y(E) : y \in P_f^+). \tag{11.23}$$

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. . . proof of Thm 11.4.1 continued.

- Assume neither trivial case. Because $y^x \in P_f^+$, we have that $y^x(A) \leq f(A)$ for all $A \subseteq E$.

. . .

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- Assume neither trivial case. Because $y^x \in P_f^+$, we have that $y^x(A) \leq f(A)$ for all $A \subseteq E$.
- We show that the constant is given by

$$y^x(E) = \min\left(x(A) + f(E \setminus A) : A \subseteq E\right) \tag{11.24}$$

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- For any P_f^+-basis y^x of x, and any $A \subseteq E$, we have weak relationship:

$$y^x(E) = y^x(A) + y^x(E \setminus A) \tag{11.25}$$

$$\leq x(A) + f(E \setminus A). \tag{11.26}$$

This follows since $y^x \leq x$ and since $y^x \in P_f^+$.

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- Given an A where equality in Eqn. (11.26) holds, above min result follows. ...

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. . . proof of Thm 11.4.1 continued.

- For any $y \in P_f^+$, call a set $B \subseteq E$ tight if $y(B) = f(B)$. The union (and intersection) of tight sets B, C is again tight, since

$$f(B) + f(C)$$

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which requires equality at the two inequalities above.

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  which requires equality at the two inequalities above.
- Because $y(A) \leq f(A), \forall A$, this means $y(B \cap C) = f(B \cap C)$ and $y(B \cup C) = f(B \cup C)$, so both also are tight.
- For $y \in P_f^+$, it will be ultimately useful to define this lattice family of tight sets: $\mathcal{D}(y) \triangleq \{A : A \subseteq E,\ y(A) = f(A)\}$.

...

A polymatroid function's polyhedron is a polymatroid.

...proof of Thm 11.4.1 continued.

- Also, we define $\text{sat}(y) \stackrel{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\}$, so $y(\text{sat}(y)) = f(\text{sat}(y))$.

$\square$

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- Given a $e \in E$, either $y^x(e)$ is cut off due to x (so $y^x(e) = x(e)$) or e is saturated by f, meaning it is an element of some tight set and $e \in \text{sat}(y^x)$ (since if $e \in T \in \mathcal{D}(y^x)$, then $e \in \text{sat}(y^x)$).

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- Let $E \setminus A = \text{sat}(y^x)$ be the union of all such tight sets (which is also tight, so $y^x(E \setminus A) = f(E \setminus A)$).

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- Hence, we have

$$y^x(E) = y^x(A) + y^x(E \setminus A) = x(A) + f(E \setminus A) \tag{11.32}$$

□

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- Let $E \setminus A = \text{sat}(y^x)$ be the union of all such tight sets (which is also tight, so $y^x(E \setminus A) = f(E \setminus A)$).
- Hence, we have

$$y^x(E) = y^x(A) + y^x(E \setminus A) = x(A) + f(E \setminus A) \tag{11.32}$$

- So we identified the A to be the elements that are non-tight, and achieved the min, as desired. $\square$

A polymatroid is a polymatroid function's polytope

- So, when f is a polymatroid function, P_f^+ is a polymatroid.

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- Is it the case that, conversely, for any polymatroid P, there is an associated polymatroidal function f such that $P = P_f^+$?

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Theorem 11.4.2

For any polymatroid P (compact subset of $\mathbb{R}_+^E$, zero containing, down-monotone, and $\forall x \in \mathbb{R}_+^E$ any maximal independent subvector $y \leq x$ has same component sum $y(E) = \text{rank}(x)$), there is a polymatroid function $f : 2^E \to \mathbb{R}$ (normalized, monotone non-decreasing, submodular) such that $P = P_f^+$ where $P_f^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\}$.

Tight sets $\mathcal{D}(y)$ are closed, and max tight set $\mathrm{sat}(y)$

Recall the definition of the set of tight sets at $y \in P_f^+$:

$$\mathcal{D}(y) \triangleq \{A : A \subseteq E,\ y(A) = f(A)\} \tag{11.33}$$

Theorem 11.4.3

For any $y \in P_f^+$, with f a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

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Proof.

We have already proven this as part of Theorem 11.4.1 □

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Theorem 11.4.3

For any $y \in P_f^+$, with f a polymatroid function, then $\mathcal{D}(y)$ is closed under union and intersection.

Proof.

We have already proven this as part of Theorem 11.4.1 □

Also recall the definition of $\mathrm{sat}(y)$, the maximal set of tight elements relative to $y \in \mathbb{R}_+^E$.

$$\mathrm{sat}(y) \stackrel{\text{def}}{=} \bigcup \{T : T \in \mathcal{D}(y)\} \tag{11.34}$$

Join $\vee$ and meet $\wedge$ for $x, y \in \mathbb{R}_+^E$

- For $x, y \in \mathbb{R}_+^E$, define vectors $x \wedge y \in \mathbb{R}_+^E$ and $x \vee y \in \mathbb{R}_+^E$ such that, for all $e \in E$

$$(x \vee y)(e) = \max(x(e), y(e)) \tag{11.35}$$

$$(x \wedge y)(e) = \min(x(e), y(e)) \tag{11.36}$$

Hence,

$$x \vee y \triangleq \Big(\max\Big(x(e_1), y(e_1)\Big), \max\Big(x(e_2), y(e_2)\Big), \ldots, \max\Big(x(e_n), y(e_n)\Big)\Big)$$

and similarly

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- From this, we can define things like an lattices, and other constructs.

Vector rank, rank(x), is submodular

- Recall that the matroid rank function is submodular.

Vector rank, rank(x), is submodular

- Recall that the matroid rank function is submodular.
- The vector rank function rank(x) also satisfies a form of submodularity, namely one defined on the real lattice.

Vector rank, rank(x), is submodular

- Recall that the matroid rank function is submodular.
- The vector rank function rank(x) also satisfies a form of submodularity, namely one defined on the real lattice.

Theorem 11.4.4 (vector rank and submodularity)

Let P be a polymatroid polytope. The vector rank function rank $: \mathbb{R}_+^E \to \mathbb{R}$ *with rank*$(x) = \max\left(y(E) : y \leq x, y \in P\right)$ *satisfies, for all* $u, v \in \mathbb{R}_+^E$

$$\textit{rank}(u) + \textit{rank}(v) \geq \textit{rank}(u \vee v) + \textit{rank}(u \wedge v) \tag{11.37}$$

Vector rank rank(x) is submodular, proof

Proof of Theorem 11.4.4.

- Let $a \in \mathbb{R}_+^E$ be a P-basis of $u \wedge v$, so rank$(u \wedge v) = a(E)$.

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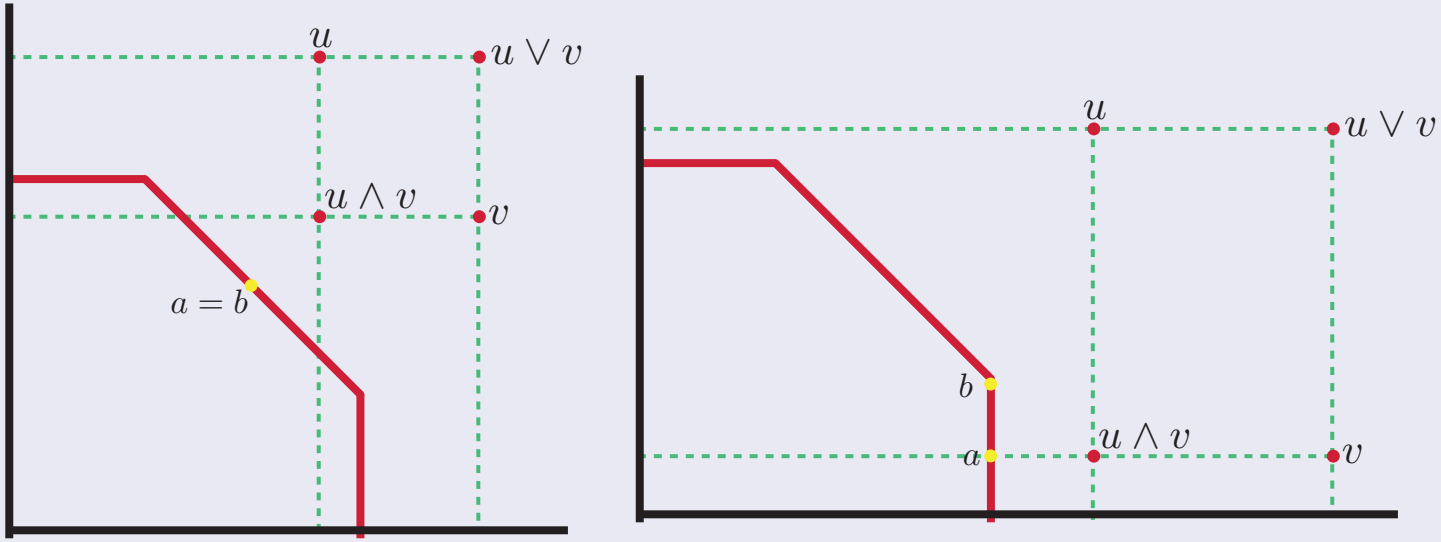

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- Given any $e \in E$: if $a(e)$ is maximal due to P, then $a(e) = b(e) \leq \min(u(e), v(e)) \ldots$

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- $\ldots$ and since $b \leq u \vee v$, we get

$$a + b \tag{11.38}$$

$\ldots$

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$$a + b = b \wedge u \wedge v + b = b \wedge u + b \wedge v \tag{11.38}$$

How? With $b \leq u \vee v$, three cases: 1) b is minimum ($a + b = b + b$); 2) u is minimum with $b \leq v$ ($a + b = u + b$); 3) v is minimum with $b \leq u$ ($a + b = v + b$). ...

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- b is independent, and $b \wedge u$ and $b \wedge v$ are independent subvectors of u and v respectively, so $(b \wedge u)(E) \leq \text{rank}(u)$ and $(b \wedge v)(E) \leq \text{rank}(v)$.

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- Hence,
$$\text{rank}(u \wedge v) + \text{rank}(u \vee v)$$

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$$= (b \wedge u)(E) + (b \wedge v)(E) \tag{11.40}$$

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- Hence,

$$\begin{aligned} \text{rank}(u \wedge v) + \text{rank}(u \vee v) &= a(E) + b(E) && (11.39) \\ &= (b \wedge u)(E) + (b \wedge v)(E) && (11.40) \\ &\leq \text{rank}(u) + \text{rank}(v) && (11.41) \end{aligned}$$

□

A polymatroid function's polyhedron vs. a polymatroid.

- Note the remarkable similarity between the proof of Theorem 11.4.4 and the proof of Theorem 6.5.1 that the standard matroid rank function is submodular.

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- Next, we prove Theorem 11.4.2, that any polymatroid polytope P has a polymatroid function f such that $P = P_f^+$.
- Given this result, we can conclude that a polymatroid is really an extremely natural polyhedral generalization of a matroid. This was all realized by Jack Edmonds in the mid 1960s (and published in 1969 in his landmark paper "Submodular Functions, Matroids, and Certain Polyhedra").

A polymatroid is a polymatroid function's polytope

- So, when f is a polymatroid function, P_f^+ is a polymatroid.
- Is it the case that, conversely, for any polymatroid P, there is an associated polymatroidal function f such that $P = P_f^+$?

Theorem 11.4.2

For any polymatroid P (compact subset of $\mathbb{R}_+^E$, zero containing, down-monotone, and $\forall x \in \mathbb{R}_+^E$ any maximal independent subvector $y \leq x$ has same component sum $y(E) = \text{rank}(x)$), there is a polymatroid function $f : 2^E \to \mathbb{R}$ (normalized, monotone non-decreasing, submodular) such that $P = P_f^+$ where $P_f^+ = \{x \in \mathbb{R}^E : x \geq 0, x(A) \leq f(A), \forall A \subseteq E\}$.

Method to prove Theorem 11.4.2

- To show Theorem 11.4.2, we will first define a function f, show that that it is monotone non-decreasing submodular, which allows us to define P_f^+, and that we show that $P \subseteq P_f^+$.

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- Next, will show that $P_f^+ \subseteq P$
- This results in that $P_f^+ = P$ to complete the proof.

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Proof of Theorem 11.4.2 ($\forall P, \exists f$ s.t. $P = P_f^+$).

- We are given a polymatroid P.

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- Define $\alpha_{\text{max}} \triangleq \max\{x(E) : x \in P\}$, and note that $\alpha_{\text{max}} > 0$ when P is non-empty, and $\alpha_{\text{max}} = \lim_{\alpha\to\infty} \text{rank}(\alpha \mathbf{1}_E) = \text{rank}(\alpha_{\text{max}} \mathbf{1}_E)$.

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- Hence, for any $x \in P$, and $\forall e \in E$, we have $x(e) \leq x(E) \leq \alpha_{\text{max}}$.
- Define a function $f : 2^V \to \mathbb{R}$ as, for any $A \subseteq E$,

$$f(A) \triangleq \text{rank}(\alpha_{\text{max}} \mathbf{1}_A) \tag{11.42}$$

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$$\geq \text{rank}(\alpha_{\text{max}} \mathbf{1}_A \vee \alpha_{\text{max}} \mathbf{1}_B) + \text{rank}(\alpha_{\text{max}} \mathbf{1}_A \wedge \alpha_{\text{max}} \mathbf{1}_B) \tag{11.44}$$

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- Moreover, we have that f is non-negative, normalized with $f(\emptyset) = 0$, and monotone non-decreasing (since rank is monotone).

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$$x_A(e) = \begin{cases} x(e) & \text{if } e \in A \\ 0 & \text{else} \end{cases} \tag{11.47}$$

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note this is an analogous definition to $\mathbf{1}_A$ but for a not necessarily unity vector x.

- Hence $x_A(A) = x(A)$ and $x_A(E \setminus A) = 0$.
- Using $f(A) \triangleq \text{rank}(\alpha_{\text{max}} \mathbf{1}_A)$, consider the polytope P_f^+ defined as:

$$P_f^+ = \left\{ x \in \mathbb{R}_+^E : x(A) \leq f(A), \ \forall A \subseteq E \right\} \tag{11.48}$$

Proof of Theorem 11.4.2

Proof of Theorem 11.4.2 ($\forall P, \exists f$ s.t. $P = P_f^+$).

- Given an $x \in P$, then for any $A \subseteq E$, $x_A \leq \alpha_{\mathsf{max}} \mathbf{1}_A$, and thus $x(A) \leq \alpha_{\mathsf{max}} |A|$.

...

Proof of Theorem 11.4.2

Proof of Theorem 11.4.2 ($\forall P, \exists f$ s.t. $P = P_f^+$).

- Given an $x \in P$, then for any $A \subseteq E$, $x_A \leq \alpha_{\mathsf{max}} \mathbf{1}_A$, and thus $x(A) \leq \alpha_{\mathsf{max}}|A|$.
- Therefore,

$$
\begin{aligned}
x(A) &\leq \max\{z(A) : z \in P, z_A \leq \alpha_{\mathsf{max}} \mathbf{1}_A\} && (11.49)\\
&= \max\{z(A) : z \in P, z \leq \alpha_{\mathsf{max}} \mathbf{1}_A\} && (11.50)\\
&\leq \max\{z(E) : z \in P, z \leq \alpha_{\mathsf{max}} \mathbf{1}_A\} && (11.51)\\
&= \mathsf{rank}(\alpha_{\mathsf{max}} \mathbf{1}_A) && (11.52)\\
&= f(A) && (11.53)
\end{aligned}
$$

Therefore $x \in P_f^+$.

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&= \max\{z(A) : z \in P, z \leq \alpha_{\mathsf{max}} \mathbf{1}_A\} && (11.50)\\
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  Therefore $x \in P_f^+$.
- Hence, $P \subseteq P_f^+$.
- We will next show that $P_f^+ \subseteq P$ to complete the proof.

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$$N(y) = \{e \in E : y(e) < x(e)\} \tag{11.54}$$

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- Choose w between y and x, so that

$$y \leq w \triangleq (y + x)/2 \leq x \tag{11.55}$$

so y is also a P-basis of w. Thus, $\forall e \in N(y)$, $y(e) < w(e) < x(e)$.

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  so y is also a P-basis of w. Thus, $\forall e \in N(y)$, $y(e) < w(e) < x(e)$.
- Hence, rank(x) = rank$(w) = y(E)$, and the set of P-bases of w are also P-bases of x.

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  the last inequality follows since $w \leq x \in P_f^+$, and $y \leq w$.
- Thus, $y \wedge x_{N(y)}$ is not a P-basis of $w \wedge x_{N(y)}$ since, over $N(y)$, it is neither tight at w nor tight at the rank (i.e., not a maximal independent subvector on $N(y)$).

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- We can extend $y \wedge x_{N(y)}$ to be a P-basis of $w \wedge x_{N(y)}$ since $y \wedge x_{N(y)} < w \wedge x_{N(y)}$.

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- Thus, $\hat{y}$ is a base of x, which violates the maximality of $|N(y)|$.
- This contradiction means that we must have had $x \in P$.
- Therefore, $P_f^+ = P$. $\square$

More on polymatroids

Theorem 11.4.5

A polymatroid can equivalently be defined as a pair (E, P) where E is a finite ground set and $P \subseteq R_+^E$ is a compact non-empty set of independent vectors such that

1. *every subvector of an independent vector is independent (if $x \in P$ and $y \leq x$ then $y \in P$, i.e., down closed)*

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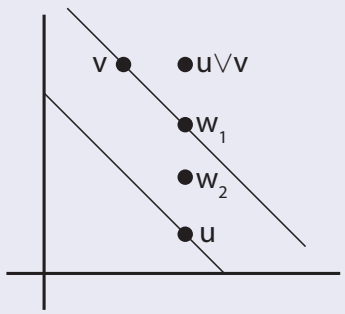
2. *If $u, v \in P$ (i.e., are independent) and $u(E) < v(E)$, then there exists a vector $w \in P$ such that*

$$u < w \leq u \vee v \qquad (11.57)$$



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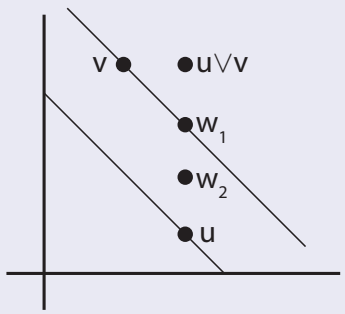

Corollary 11.4.6

The independent vectors of a polymatroid form a convex polyhedron in $\mathbb{R}_+^E$.

Review

- The next slide comes from lecture 6.

Matroids by bases

In general, besides independent sets and rank functions, there are other equivalent ways to characterize matroids.

Theorem 11.4.3 (Matroid (by bases))

Let E be a set and $\mathcal{B}$ be a nonempty collection of subsets of E. Then the following are equivalent.

1. *$\mathcal{B}$ is the collection of bases of a matroid;*
2. *if $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B' - x + y \in \mathcal{B}$ for some $y \in B \setminus B'$.*
3. *If $B, B' \in \mathcal{B}$, and $x \in B' \setminus B$, then $B - y + x \in \mathcal{B}$ for some $y \in B \setminus B'$.*

Properties 2 and 3 are called "exchange properties."
Proof here is omitted but think about this for a moment in terms of linear spaces and matrices, and (alternatively) spanning trees.

More on polymatroids

For any compact set P, b is a base of P if it is a maximal subvector within P. Recall the bases of matroids. In fact, we can define a polymatroid via vector bases (analogous to how a matroid can be defined via matroid bases).

Theorem 11.4.7

A polymatroid can equivalently be defined as a pair (E, P) where E is a finite ground set and $P \subseteq R_+^E$ is a compact non-empty set of independent vectors such that

1. *every subvector of an independent vector is independent (if $x \in P$ and $y \leq x$ then $y \in P$, i.e., down closed)*
2. *if b, c are bases of P and d is such that $b \wedge c < d < b$, then there exists an f, with $d \wedge c < f \leq c$ such that $d \vee f$ is a base of P*
3. *All of the bases of P have the same rank.*

Note, all three of the above are required for a polymatroid (a matroid analogy would require the equivalent of only the first two).

A word on terminology & notation

- Recall how a matroid is sometimes given as (E, r) where r is the rank function.

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A word on terminology & notation

- Recall how a matroid is sometimes given as (E, r) where r is the rank function.
- We mention also that the term "polymatroid" is sometimes not used for the polytope itself, but instead but for the pair (E, f),
- But now we see that (E, f) is equivalent to a polymatroid polytope, so this is sensible.

Where are we going with this?

- Consider the right hand side of Theorem 11.4.1: $\min\left(x(A) + f(E \setminus A) : A \subseteq E\right)$

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- We are going to study this problem, and approaches that address it, as part of our ultimate goal which is to present strategies for submodular function minimization (that we will ultimately get to, in near future lectures).
- As a bit of a hint on what's to come, recall that we can write it as: $x(E) + \min\left(f(A) - x(A) : A \subseteq E\right)$ where f is a polymatroid function.

Another Interesting Fact: Matroids from polymatroid functions

Theorem 11.4.8

Given integral polymatroid function f, let $(E, \mathcal{F})$ be a set system with ground set E and set of subsets $\mathcal{F}$ such that

$$\forall F \in \mathcal{F}, \ \ \forall \emptyset \subset S \subseteq F, |S| \leq f(S) \tag{11.58}$$

Then $M = (E, \mathcal{F})$ is a matroid.

Proof.

Exercise □

And its rank function is Exercise.

Matroid instance of Theorem 11.4.1

- Considering Theorem 11.4.1, the matroid case is now a special case, where we have that:

Corollary 11.4.9

We have that:

$$\max\{y(E) : y \in P_{\textit{ind. set}}(M), y \leq x\} = \min\{r_M(A) + x(E \setminus A) : A \subseteq E\} \tag{11.59}$$

where r_M is the matroid rank function of some matroid.